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A BOUNDED TRANSFORM APPROACH TO SELF-ADJOINT OPERATORS: FUNCTIONAL CALCULUS AND AFFILIATED VON NEUMANN ALGEBRAS

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Abstract. Spectral theory and functional calculus for unbounded self-adjoint operators on a Hilbert space are usually treated through von Neumann's Cayley transform. Using ideas of Woronowicz, we redevelop this theory from the point of view of multiplier algebras and the so-called *bounded transform* (which establishes a bijective correspondence between closed operators and pure contractions). This also leads to a simple account of the affiliation relation between von Neumann algebras and self-adjoint operators.

1. Introductory overview

The theory of unbounded self-adjoint operators on a Hilbert space was initiated by von Neumann [7], partly motivated by mathematical problems of quantum mechanics. The monograph by Schmüdgen [10] presents an excellent survey of the present state of the art.

Von Neumann's approach was based on the Cayley transform, and in its subsequent development the notion of a spectral measure played an important role, especially in defining a functional calculus. We consider this route a bit indirect and will avoid both by first invoking the *bounded transform* instead of the Cayley transform; that is, the formal expressions

$$S = T\sqrt{I+T^2}^{-1}, \tag{1.1}$$

$$T = S\sqrt{I-S^2}^{-1}, \tag{1.2}$$

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make rigorous sense and provide a bijective correspondence between self-adjoint operators T and self-adjoint pure contractions S (i.e., $\|Sx\| < \|x\|$ for each $x \in \mathcal{H} \setminus \{0\}$; see [3], [4], [10]).

Note that the bounded transform $T \mapsto S$ is an operatorial version of the homeomorphism $\mathbb{R} \cong (-1,1)$ given by the function $b : \mathbb{R} \to (-1,1)$ and its inverse $u : (-1,1) \to \mathbb{R}$ defined by

$$b(x) = \frac{x}{\sqrt{1+x^2}}, \tag{1.3}$$

$$u(x) = \frac{x}{\sqrt{1-x^2}}. \tag{1.4}$$

Second, we replace spectral measures by simple arguments using multiplier algebras. Our approach is based on Woronowicz's work (see [12] and [13]), whose functional calculus we adopt and to some extent complete, at least in the usual context of operators on a Hilbert space (Woronowicz's work was mainly intended to deal with problems involving multiplier algebras and, even more generally, with operators on Hilbert C^*-modules, as in [5]).

If T is bounded (and, by standing assumption, self-adjoint), then it is easy to prove the equality

$$C^*(T) = C^*(S), \tag{1.5}$$

where $C^*(S)$ is the C^*-algebra generated within $B(\mathcal{H})$ by S and the unit, and so forth. Furthermore, the spectral mapping theorem implies that the spectra of S and T are related by

$$\sigma(T) = \big\{\mu(1-\mu^2)^{-1/2} \bigm| \mu \in \sigma(S)\big\}, \tag{1.6}$$

$$\sigma(S) = \big\{\lambda(1+\lambda^2)^{-1/2} \bigm| \lambda \in \sigma(T)\big\}, \tag{1.7}$$

preserving point spectra. As to the continuous functional calculus, for $S = S^* \in B(\mathcal{H})$ we have the familiar isomorphism $C(\sigma(S)) \xrightarrow{\cong} C^*(S)$, written $g \mapsto g(S)$, given by the spectral theorem. Assuming that $T = T^* \in B(\mathcal{H})$, the same applies to T. These calculi are related by

$$f(T) = (f \circ u)(S), \tag{1.8}$$

where $f \in C(\sigma(T))$ so that $f \circ u \in C(\sigma(S))$. Self-adjointness is preserved in that

$$f(T)^* = f^*(T), \tag{1.9}$$

where $f^*(x) = \overline{f(x)}$. In particular, if f is real-valued, then $f(T)$ is self-adjoint. At the level of von Neumann algebras, defining $W^*(S) = C^*(S)''$ and similarly for T, equation (1.5) gives

$$W^*(T) = W^*(S). \tag{1.10}$$

The functional calculus $f \mapsto f(T)$ may then be extended to bounded Borel functions f on $\sigma(T)$, in which case it is still given by (1.8). We then have $f(T) \in W^*(T)$, while (1.9) remains valid; however, instead of the isometric property $\|f(T)\| = \|f\|_\infty$ for continuous f, we now have $\|f(T)\| \le \|f\|_\infty$ (where $\|\cdot\|_\infty$ is the supremum-norm) (see, e.g., [8]).

Our aim here is to generalize these results to the case where T is unbounded. This indeed turns out to be possible so that our main results are as follows. Throughout the remainder of this article, we assume that $T^* = T$ is possibly unbounded with bounded transform S.

Theorem 1.1. *The spectra of T and its bounded transform S are related by*

$$\sigma(T) = \big\{\mu(1-\mu^2)^{-1/2} : \mu \in \tilde{\sigma}(S)\big\}, \tag{1.11}$$

$$\sigma(S) = \big\{\lambda(1+\lambda^2)^{-1/2} : \lambda \in \sigma(T)\big\}^-, \tag{1.12}$$

where $^-$ denotes the closure in $\mathbb{R}$, and we abbreviate

$$\tilde{\sigma}(S) = \sigma(S) \cap (-1,1). \tag{1.13}$$

Note that $\tilde{\sigma}(S) = \sigma(S)$ if and only if T is bounded (in which case $\sigma(S)$ is a compact subset of $(-1,1)$ since $\pm 1 \in \sigma(S)$ if and only if T is unbounded). We define the following operator algebras within $B(\mathcal{H})$:

$$C^*_\bullet(S) = \big\{g(S) : g \in C_\bullet\big(\tilde{\sigma}(S)\big)\big\}, \tag{1.14}$$

where $\bullet$ is b, c, or 0 so that we have defined $C^*_c(S)$, $C^*_0(S)$, and $C^*_b(S)$. Notice that $C(\sigma(S))$ consists of all $g \in C_b(\tilde{\sigma}(S))$ for which $\lim_{y\to\pm 1} g(y)$ exists, where this limit is 0 if and only if $g \in C_0(\tilde{\sigma}(S))$; hence, we have the inclusions (of which the first set implies the second)

$$C_c\big(\tilde{\sigma}(S)\big) \subseteq C_0\big(\tilde{\sigma}(S)\big) \subseteq C\big(\sigma(S)\big) \subseteq C_b\big(\tilde{\sigma}(S)\big), \tag{1.15}$$

$$C^*_c(S) \subseteq C^*_0(S) \subseteq C^*(S) \subseteq C^*_b(S), \tag{1.16}$$

with equalities if and only if T is bounded. This means that $g(S)$ is defined for $g \in C_0(\tilde{\sigma}(S))$, and hence *a fortiori* also for $g \in C_c(\tilde{\sigma}(S))$. Consequently, $f(T)$ may be defined by (1.8) whenever $f \in C_0(\sigma(T))$, including $f \in C_c(\sigma(T))$. To pass to the larger class $f \in C_b(\sigma(T))$, we define $C^*_0(S)\mathcal{H}$ as the linear span of all vectors of the form $g(S)\psi$, where $g \in C_0(\tilde{\sigma}(S))$ and $\psi \in \mathcal{H}$. Then $C^*_0(S)\mathcal{H}$ is dense in $\mathcal{H}$ (see Lemma 2.1). In the spirit of Woronowicz (see [5, Chapter 10], [12]), we then initially define $f(T)$ for $f \in C_b(\sigma(T))$ on the domain $C^*_0(S)\mathcal{H}$ by linear extension of the formula

$$f(T)_0 h(T)\psi = (fh)(T)\psi, \tag{1.17}$$

where $h \in C_0(\sigma(T))$, and hence also $fh \in C_0(\sigma(T))$ since $C_b(\sigma(T))$ is the mutiplier algebra of $C_0(\sigma(T))$. Then $f(T)_0$ is bounded (see Lemma 2.2), and we define $f(T)$ as its closure; that is,

$$f(T) = f(T)_0^-. \tag{1.18}$$

This also works for $f \in C(\sigma(T))$, in which case $f(T)_0$ may no longer be bounded, but remains closable (see Lemma 2.3) so that we may once again define $f(T)$ as its closure (cf. (1.18)). We have the following theorem (see also Theorem 1.4).

Theorem 1.2. *If $f \in C(\sigma(T))$ is real-valued, then $f(T)$ is self-adjoint; that is, $f(T)_0^- = f(T)_0^*$, and, more generally, $f(T)^* = f^*(T)$. Furthermore, the continuous functional calculus $f \mapsto f(T)$ restricts to an isometric *-homomorphism from $C_0(\sigma(T))$ (with supremum-norm) to $C^*(S)$.*

In addition, the map $f \mapsto f(T)$ has the reassuring special cases

$$\mathbf{1}_{\sigma(T)}(T) = I, \tag{1.19}$$

$$\mathrm{id}(T) = T, \tag{1.20}$$

$$(\mathrm{id} - z)^{-1}(T) = (T - z)^{-1}, \quad z \in \rho(T), \tag{1.21}$$

where $\mathbf{1}_{\sigma(T)}(x) = 1$ and $\mathrm{id}(x) = x$ ($x \in \sigma(T)$), and therefore it does what it is supposed to do.

Finding the right analogue of (1.10) for unbounded $T = T^*$ first requires a redefinition of $W^*(T)$, which is standard (see [8]). If T is unbounded and $R \in B(\mathcal{H})$, then we say that R and T *commute*, written $TR \subset RT$, if $R\psi \in \mathcal{D}(T)$ and $RT\psi = TR\psi$ for any $\psi \in \mathcal{D}(T)$. Let $\{T\}'$ be the set of all bounded operators that commute with T. If $T^* = T$, then $\{T\}'$ is a unital, strongly closed $*$-subalgebra of $B(\mathcal{H})$, and hence a von Neumann algebra (see [8]). Its commutant

$$W^*(T) = \{T\}'' \tag{1.22}$$

is a von Neumann algebra, too. If T is bounded, then $W^*(T)$ is the von Neumann algebra generated by T, which coincides with $C^*(T)''$. As usual, we call a closed unbounded operator X *affiliated* to a von Neumann algebra $A \subset B(H)$, written $X\eta A$, if and only if $XR \subset RX$ for each $R \in A'$. For example, if $T^* = T$, then $T\eta W^*(T)$, and if $T\eta A$, then $W^*(T) \subseteq A$; in other words, $W^*(T)$ is the smallest von Neumann algebra such that T is affiliated to it.

As a result of independent interest as well as a lemma for Theorem 1.4, we may then adapt [8, Lemma 5.2.8] to the bounded transform, as in this theorem.

Theorem 1.3. *Let $A \subset B(H)$ be a von Neumann algebra. Then $T\eta A$ if and only if $S \in A$.*

Denoting the (Banach) space of (bounded) Borel functions on $\sigma(T)$ (equipped with the supremum-norm) by $\mathcal{B}_{(b)}(\sigma(T))$, we may still define $f(T)$ by (1.8) and the usual Borel functional calculus for the bounded transform S.

Theorem 1.4. *The map $f \mapsto f(T)$ is a norm-decreasing *-homomorphism from $\mathcal{B}_b(\sigma(T))$ to*

$$W^*(T) = W^*(S). \tag{1.23}$$

More generally, if $f \in \mathcal{B}(\sigma(T))$, then $f(T)$ is affiliated with $W^*(T)$.

The remainder of this paper simply consists of the proofs of these theorems.

2. Proofs

This section contains all proofs. We will not repeat the theorems.

2.1. **Proof of Theorem 1.1.** The operator $\sqrt{1 - S^2}$ is a bijection from $\mathcal{H}$ to $\mathcal{R}(\sqrt{1 - S^2}) = \mathcal{D}(T)$ (see [4], proof of Theorem 1). Let $\lambda \in \rho(T) \equiv \mathbb{C} \setminus \sigma(T)$ so that $T - \lambda I$ is a bijection from $\mathcal{D}(T)$ to $\mathcal{H}$. Thus, by composition, we have a bijection $\mathcal{H} \to \mathcal{H}$; equivalently, $(T - \lambda I)(\sqrt{I - S^2})$ is invertible, which in turn is equivalent to invertibility of $S - \lambda\sqrt{I - S^2}$. Thus, $\lambda \in \rho(T) \iff S - \lambda\sqrt{I - S^2}$

is a bijection, or, expressed contrapositively, $\lambda \in \sigma(T) \iff S - \lambda\sqrt{I - S^2}$ is not invertible in $B(\mathcal{H})$. This is the case if and only if $S - \lambda\sqrt{I - S^2}$ is not invertible in $C^*(S)$, which, using the Gelfand isomorphism $C^*(S) \cong C(\sigma(S))$, in turn is true if and only if the function $k_\lambda(x) = x - \lambda\sqrt{1 - x^2}$ is not invertible in $C(\sigma(S))$; that is, if and only if $0 \in \sigma(k_\lambda)$. Since in $C(X)$ we have $\sigma(f) = \mathcal{R}(f)$ (with X a compact Hausdorff space), and since $\sigma(S)$ is indeed compact and Hausdorff because S is bounded, we obtain $\lambda \in \sigma(T)$ if and only if $0 \in \mathcal{R}(k_\lambda)$. If ± 1 lie in $\sigma(S)$, then they cannot give rise to $0 \in \mathcal{R}(k_\lambda)$ since $k_\lambda(\pm 1) = \pm 1$ for each λ; hence, $0 \in \mathcal{R}(k_\lambda)$ if and only if $\lambda = \mu(1 - \mu^2)^{-1/2}$ for some $\mu \in \sigma(S) \cap (-1, 1)$, which yields (1.11).

The same argument shows that $\mu \in \sigma(S) \cap (-1, 1)$ comes from $\lambda \in \sigma(T)$. But since $\sigma(S)$ is compact and hence closed in $[-1, 1]$, we obtain (1.12).

2.2. **Proof of Theorem 1.2.** This proof relies on three lemmas.

Lemma 2.1. *Let $C_c^*(S)\mathcal{H}$ be the linear span of all vectors of the form $g(S)\psi$, where $g \in C_c(\tilde{\sigma}(S))$ and $\psi \in \mathcal{H}$. Then $C_c^*(S)\mathcal{H}$ is dense in H.*

Proof. Define $g_n : (-1, 1) \to [0, 1]$ by putting $g_n(x) = 0$ for $x \in (-1, \frac{1}{n} - 1] \cup [1 - \frac{1}{n}, 1)$, $g_n(x) = 1$ if $x \in [\frac{2}{n} - 1, 1 - \frac{2}{n}]$, and linear interpolation in between. The ensuing sequence converges pointwise to the unit $\mathbf{1}$ on $(-1, 1)$. Restricting each g_n to $\tilde{\sigma}(S)$, the continuous functional calculus gives $g_n(S) \to \mathbf{1}_{\tilde{\sigma}(S)}$ strongly. Therefore, for any $\psi \in \mathcal{H}$, we have a sequence $\psi_n = g_n(S)\psi$ in $C_c^*(S)\mathcal{H}$ such that $\psi_n \to \psi$. □

Lemma 2.2. *For $f \in C_b(\sigma(T))$, define an operator $f(T)_0$ on the domain $C_0^*(S)\mathcal{H}$ by (1.17). Then $f(T)_0$ is bounded with bound*

$$\|f(T)_0\| \le \|f\|_\infty. \tag{2.1}$$

Proof. Let $\varepsilon > 0$. If $h \in C_0(\sigma(T))$, then $fh \in C_0(\sigma(T))$ so that we can find a compact subset $K \subset \sigma(T)$ such that $|h(x)f(x)| < \varepsilon$ for each $x \notin K$. Let $\tilde{h} = h \circ u$ (see (1.4)). Then $\tilde{h} \in C_0(\tilde{\sigma}(S))$ whenever $h \in C_0(\sigma(T))$; in fact, we have an isometric isomorphism

$$C_0\big(\sigma(T)\big) \xrightarrow{\cong} C_0\big(\tilde{\sigma}(S)\big), \qquad h \mapsto h \circ u. \tag{2.2}$$

Contractivity of the Borel functional calculus for bounded operators on $\mathcal{H}$ gives

$$\big\|(\widetilde{\mathbf{1}_{K^c} fh})(S)\psi\big\| \le \big\|(\widetilde{\mathbf{1}_{K^c} fh})(S)\big\|\|\psi\| \le \|\widetilde{\mathbf{1}_{K^c} fh}\|_\infty\|\psi\| < \varepsilon\|\psi\|.$$

Using also the homomorphism property of the Borel functional calculus, we then find that

$$\begin{aligned}
\big\|(fh)(T)\psi\big\| &= \big\|(\widetilde{fh})(S)\psi\big\| \\
&= \big\|(\widetilde{\mathbf{1}_K fh})(S) + (\widetilde{fh} - \widetilde{\mathbf{1}_K fh})(S)\psi\big\| \\
&\le \big\|(\widetilde{\mathbf{1}_K fh})(S)\psi\big\| + \big\|(\widetilde{\mathbf{1}_{K^c} fh})(S)\psi\big\| \\
&= \big\|(\widetilde{\mathbf{1}_K f})(S)\tilde{h}(S)\psi\big\| + \big\|(\widetilde{\mathbf{1}_{K^c} fh})(S)\psi\big\|
\end{aligned}$$

$$< \|\widetilde{(\mathbf{1}_K f)}\|_\infty \|h(T)\psi\| + \varepsilon\|\psi\|$$
$$\leq \|f\|_\infty \|h(T)\psi\| + \varepsilon\|\psi\|$$

since $\|\widetilde{(\mathbf{1}_K f)}\|_\infty \leq \|\tilde{f}\|_\infty = \|f\|_\infty$. Since the last expression above is independent of K, we may let $\varepsilon \to 0$, obtaining boundedness of $f(T)$ as well as (2.1). □

The last claim in Theorem 1.2 now follows from the continuous functional calculus for S and the isometric isomorphism (2.2). Although isometry may be lost if we go from $C_0(\sigma(T))$ to $C_b(\sigma(T))$, it easily follows from (1.17)–(1.18) that the map $f \mapsto f(T)$ at least defines a *-homomorphism $C_b(\sigma(T)) \to B(H)$. This property will be used after Lemma 2.4 below.

Lemma 2.3. *For $f \in C(\sigma(T))$, define an operator $f(T)_0$ on the domain $C_c^*(S)\mathcal{H}$ by (1.17). Then $f(T)_0$ is closable. Moreover, if f is real-valued ($f^* = f$), then $f(T)_0$ is symmetric.*

Proof. Suppose that $h_1(T)\psi_1$ and $h_2(T)\psi_2$ lie in $\mathcal{D}(f(T)_0)$. Then we may compute

$$\begin{aligned}\langle h_2(T)\psi_2, f(T)_0 h_1(T)\psi_1\rangle &= \langle \psi_2, h_2(T)^*(fh_1)(T)\psi_1\rangle \\ &= \langle \psi_2, (\overline{h_2} f h_1)(T)\psi_1\rangle; \end{aligned} \tag{2.3}$$

$$\begin{aligned}\langle (h_2\overline{f})(T)\psi_2, h_1(T)\psi_1\rangle &= \langle \psi_2, h_2\overline{f})(T)^* h_1(T)\psi_1\rangle \\ &= \langle \psi_2, (\overline{h_2} f h_1)(T)\psi_1\rangle, \end{aligned} \tag{2.4}$$

where in the first equality in (2.3) we have $h_2 \in C_0(\sigma(T))$ so that the operator $h_2(T) = h_2 \circ u(S)$ is defined by (1.8), and hence is bounded (see Section 1). The continuous functional calculus for S then gives $h_2(T)^* = \overline{h_2}(T)$ as well as $\overline{h_2}(T)(fh_1)(T) = (\overline{h_2} f h_1)(T)$, and similarly in (2.4).

This implies that $\mathcal{D}(f(T)_0) \subseteq \mathcal{D}(f(T)_0^*)$. Since $\mathcal{D}(f(T)_0)$ is dense, so is $\mathcal{D}(f(T)_0^*)$, which implies that $f(T)_0$ is closable. The second claim is obvious from (2.3)–(2.4). □

Proof of Theorem 1.2. To prove Theorem 1.2, we use a well-known result of Nelson [6] (see also [9]) (this step was suggested to us by Nigel Higson). For convenience we recall this result (without proof).

Lemma 2.4. *Let $\{U(t)\}_{t\in\mathbb{R}}$ be a strongly continuous unitary group of operators on a Hilbert space $\mathcal{H}$. Let $R : \mathcal{D}(R) \to \mathcal{H}$ be densely defined and symmetric. Assume that $\mathcal{D}(R)$ is invariant under $\{U(t)\}_{t\in\mathbb{R}}$; that is, assume that $U(t) : \mathcal{D}(R) \to \mathcal{D}(R)$ for each t, and also that $\{U(t)\}_{t\in\mathbb{R}}$ is strongly differentiable on $\mathcal{D}(R)$. Then $-idU(t)/dt$ is essentially self-adjoint on $\mathcal{D}(R)$, and its closure is the self-adjoint generator of $\{U(t)\}_{t\in\mathbb{R}}$ (given by Stone's theorem). In particular, if $(dU(t)/dt)\psi = iRU(t)\psi$ for each $\psi \in \mathcal{D}(R)$, then R is essentially self-adjoint.*

Set $R = f(T)_0$ for $f \in C(\sigma(T))$ so that

$$\mathcal{D}(R) = C_c^*(S)\mathcal{H}, \tag{2.5}$$

and for each $t \in \mathbb{R}$ define $U(t)$ via the (bounded) function $x \mapsto \exp(itf(x))$ on $\sigma(T)$; that is, for $h \in C_c(\sigma(T))$ and $\psi \in \mathcal{H}$, we initially define

$$U(t)_0 h(T)\psi = (e^{itf}h)(T)\psi. \tag{2.6}$$

Then $U(t)_0$ is bounded by Lemma 2.2, and we define $U(t)$ as the closure of $U(t)_0$. The remark before Lemma 2.3 then implies that $t \mapsto U(t)$ defines a unitary representation of $\mathbb{R}$ on $\mathcal{H}$. Strong continuity of this representation follows from an $\varepsilon/3$ argument. First, for

$$\varphi = h(T)\psi, \tag{2.7}$$

and assuming that $\|\psi\| = 1$ for simplicity, equations (2.6) and (2.1) give

$$\big\|U(t)\varphi - \varphi\big\| \leq \|e^{itf}h - h\|_\infty \leq \|h\|_\infty \|e^{itf} - \mathbf{1}\|_\infty^{(K)}, \tag{2.8}$$

where K is the (compact) support of h in $\sigma(T)$. Since the exponential function is uniformly convergent on any compact set, this gives $\lim_{t\to 0}\|U(t)\varphi - \varphi\| = 0$ for φ of the form (2.7); taking finite linear combinations thereof gives the same result for any $\varphi \in C_c^*(S)\mathcal{H}$. Thus, for any $\varepsilon > 0$, we can find $\delta > 0$ so that $\|U(t)\varphi - \varphi\| < \varepsilon/3$ whenever $|t| < \delta$. For general $\psi' \in H$, we find $\varphi \in C_c^*(S)H$ such that $\|\varphi - \psi'\| < \varepsilon/3$, and we estimate

$$\begin{aligned}\big\|U(t)\psi' - \psi'\big\| &\leq \big\|U(t)\psi' - U(t)\varphi\big\| + \big\|U(t)\varphi - \varphi\big\| + \|\varphi - \psi'\| \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon\end{aligned}$$

since $\|U(t)\psi' - U(t)\varphi\| = \|\psi' - \varphi\|$ by unitarity of $U(t)$. Thus, $\lim_{t\to 0}\|U(t)\psi - \psi\| = 0$ for any $\psi \in \mathcal{H}$ so that the unitary representation $t \mapsto U(t)$ is strongly continuous. Similarly,

$$\left\|\frac{U(t+s)\varphi - U(t)\varphi}{s} - iRU(t)\varphi\right\| \leq \left\|\frac{e^{isf}h - h}{s} - ifh\right\|_\infty, \tag{2.9}$$

assuming (2.7), so that by the same argument as in (2.8) we obtain

$$\frac{dU(t)}{dt}\varphi = iRU(t)\varphi \tag{2.10}$$

initially for any φ of the form (2.7), and hence, taking finite sums, for any $\varphi \in \mathcal{D}(R)$ (see (2.5)). The final part of Lemma 2.4 then shows that $f(T)_0$ is essentially self-adjoint on its domain $C_c^*(S)\mathcal{H}$. Its closure $f(T)$ is therefore self-adjoint, and Theorem 1.2 is proved. □

We now prove the examples (1.19)–(1.21), of which the first is trivial. Writing T_0 for the operator $\mathrm{id}(T)_0$, the definition (1.17) gives

$$T_0\varphi = T\varphi$$

for $\varphi \in \mathcal{D}(T_0) = C_c^*(S)\mathcal{H}$. Let $\psi \in \mathcal{D}(T_0^-)$ so that there is a sequence (φ_n) in $\mathcal{D}(T_0)$ such that $\varphi_n \to \varphi$ and $(T_0\varphi_n)$ converges. Since T is closed, it follows that $T_0\varphi_n = T\varphi_n \to T\varphi$ so that $\varphi \in \mathcal{D}(T)$; hence, $T_0^- \subset T$. Since both operators are self-adjoint, this implies that $T_0^- = T$, which proves (1.20).

The proof of (1.21) is easier since $(T - z)^{-1}$ is bounded: writing

$$f(x) = (x - z)^{-1},$$

where $z \notin \sigma(T)$ is fixed and $x \in \sigma(T)$, we have

$$f(T)_0 h(T)\psi = (fh)(T)\psi = (T - z)^{-1} h(T)\psi,$$

and hence

$$f(T)_0 \varphi = (T - z)^{-1}\varphi$$

for any $\varphi \in \mathcal{D}(f(T)_0) = C_c^*(S)\mathcal{H}$. If $\varphi_n \to \varphi$ for $\varphi \in \mathcal{H}$ and $\varphi_n \in \mathcal{D}(f(T)_0)$, boundedness and hence continuity of the resolvent implies that

$$f(T)\varphi = \lim_{n\to\infty} f(T)_0\varphi_n = \lim_{n\to\infty} (T - z)^{-1}\varphi_n = (T - z)^{-1}\varphi.$$

2.3. **Proof of Theorem 1.3.** The first step consists in the observation that $T\eta A$ if and only if $UT = TU$ (or, equivalently, $UTU^* = T$) for each unitary $U \in A'$ [11, Proposition 5.3.4].

The second step is to show that $UT = TU$ if and only if $SU = US$ for any unitary U. This is a simple computation. First, suppose that $UTU^* = T$. Then

$$\begin{aligned} U(1 + T^2)^{-1}U^* &= \big(U(1 + T^2)U^*\big)^{-1} = \big((U + UT^2)U^*\big)^{-1} \\ &= (UU^* + UT^2U^*)^{-1} = (1 + UTU^*UTU^*)^{-1} \\ &= (1 + T^2)^{-1}. \end{aligned}$$

If R is bounded and positive, then $UR = RU$ if and only if $U \in C^*(R)'$, and since $\sqrt{R} \in C^*(R)$ by the continuous functional calculus, we also have $U\sqrt{R} = \sqrt{R}U$. Consequently,

$$\begin{aligned} USU^* &= U\big(T\sqrt{(1 + T^2)^{-1}}\big)U^* \\ &= (UTU^*)\big(U\sqrt{(1 + T^2)^{-1}}U^*\big) = T\sqrt{(1 + T^2)^{-1}} = S. \end{aligned}$$

Similarly, if $SU = US$, then

$$\begin{aligned} UTU^* &= US\sqrt{1 - S^2}^{-1}U^* \\ &= SU\sqrt{1 - S^2}^{-1}U^* = S(U\sqrt{1 - S^2}U^*)^{-1} = S\sqrt{1 - S^2}^{-1} = T. \end{aligned}$$

Third, as in the first step, $SU = US$ for any unitary $U \in A'$ if and only if $S \in A'' = A$.

2.4. **Proof of Theorem 1.4.** Equation (1.23) in Theorem 1.4 follows from Theorem 1.3: taking $A = W^*(T)$ so that $T\eta A$ yields $S \in W^*(T)$, and hence $W^*(S) \subseteq W^*(T)$. On the other hand, taking $A = W^*(S)$, in which case $S \in A$, gives $T\eta W^*(S)$, and hence $W^*(T) \subseteq W^*(S)$.

Similarly to (2.2), we have an isometric isomorphism

$$\mathcal{B}_b\big(\sigma(T)\big) \stackrel{\cong}{\to} \mathcal{B}_b\big(\tilde{\sigma}(S)\big), \qquad h \mapsto h \circ u \tag{2.11}$$

so that the first claim of Theorem 1.4 follows from the Borel functional calculus for the bounded operator S (see [8]). The proof of the last one is, *mutatis mutandis*, practically the same as in [8, Theorem 5.3.8], so we omit the details (see [2]).

As explained in [8, Section 5.3], there exists a Borel measure μ on $\sigma(T)$ such that the map $f \mapsto f(T)$ may also be seen as a so-called *essential* *-*homomorphism* from $\mathcal{B}(\sigma(T))/\mathcal{N}(\sigma(T))$ into the *-algebra of normal operators affiliated with $W^*(T)$, where $\mathcal{N}(\sigma(T))$ is the set of μ-null functions on $\sigma(T)$. This remains true in our approach with the same proof (see [2]).

3. Epilogue

Let us finally note that, although the present article was inspired by the work of Woronowicz, the C^*-algebraic affiliation relation he defines in [12, Definition 1.1] (as did, independently, also Baaj and Julg in [1]) has not been used here. If we call his relation η' to avoid confusion with the W^*-algebraic relation η we do use, if $A \subset B(\mathcal{H})$, then we have $T\eta' A \Rightarrow T \in A$ (and hence T is bounded) (cf. [12, Proposition 1.3]). Woronowicz does not define a C^*-algebraic counterpart of the von Neumann algebra $W^*(T)$, but it might be reasonable to define $C^*(T)$ as the smallest C^*-algebra A in $B(\mathcal{H})$ such that $T\eta' A$. It follows from [12, Example 4] that this would give $C^*(T) = C_0^*(S)$ as defined in (1.14). This C^*-algebra contains S (and hence T) if and only if T is bounded, in which case $C_0^*(S) = C^*(S)$ and hence $C^*(T) = C^*(S)$, as in our approach (see (1.5)). Also, in general (i.e., if T is possibly unbounded), the bicommutant $C^*(T)''$ coincides with $W^*(T)$ as defined in the usual way (1.22). This follows from $C_0^*(S)'' = C^*(S)'' = W^*(S)$ and (1.10).

Of course, we could also redefine η', now calling it η'', by stipulating that $T\eta'' A$ whenever $S \in A$, and redefine $C^*(T)$ accordingly (i.e., as the smallest C^*-algebra A in $B(\mathcal{H})$ such that $T\eta'' A$). This would give (1.5) even if T is unbounded, though in a somewhat empty way.

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