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SOME MATRIX INEQUALITIES FOR WEIGHTED POWER MEAN

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Abstract. In this paper, we prove that, for any positive definite matrices A, B, and real numbers ν, μ, p with $-1 \le p < 1$ and $0 < \nu \le \mu < 1$, we have

$$\frac{\nu}{\mu}(A\nabla_\mu B - A\sharp_{p,\mu}B) \le A\nabla_\nu B - A\sharp_{p,\nu}B \le \frac{1-\nu}{1-\mu}(A\nabla_\mu B - A\sharp_{p,\mu}B),$$

where ∇_ν and $\sharp_{p,\nu}$ stand for weighted arithmetic and power mean, respectively. In the special cases when $p = 0, 1$, this inequality can be considered as a generalization of harmonic-arithmetic and geometric-arithmetic means inequalities and their reverses.

Applying this inequality, some inequalities for the Heinz mean and determinant inequalities related to weighted power means are obtained.

1. Introduction

Let $\mathbb{B}(H)$ be the algebra of bounded linear operators on a Hilbert space H, and let $\mathbb{B}(H)^+$ stand for the strictly positive operators. A binary operation $(A, B) \mapsto A\sigma B$, from $\mathbb{B}^+(H) \times \mathbb{B}^+(H)$ to $\mathbb{B}^+(H)$, is called a *connection* if the following conditions are satisfied:

- $A \le C$ and $B \le D$ imply $A\sigma B \le C\sigma D$,
- $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$ in the strong operator topology,
- $T^*(A\sigma B)T \le (T^*AT)\sigma(T^*BT)$ for every operator T.

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An operator mean is a connection with normalized condition $I\sigma I = I$. An easy calculation shows that if T is an invertible operator, then $T^*(A\sigma B)T = (T^*AT)\sigma(T^*BT)$.

For $\nu \in [0,1]$ and $p \in [-1,1]$, we can define the weighted power mean

$$A\sharp_{p,\nu}B = A^{\frac{1}{2}}\big((1-\nu)I + \nu(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^p\big)^{\frac{1}{p}}A^{\frac{1}{2}}$$

corresponding to the operator monotone function $f(t) = [(1-\nu)+\nu t^p]^{\frac{1}{p}}$. For the case $p=0$, we take the limit as $p \to 0$ and

$$A\sharp_{0,\nu}B = A\sharp_\nu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu A^{\frac{1}{2}},$$

which is known as the *weighted geometric mean*. (Note that $A\sharp_{1,\nu}B = (1-\nu)A + \nu B$ and $A\sharp_{-1,\nu}B = ((1-\nu)A^{-1} + \nu B^{-1})^{-1}$ are called *weighted arithmetic* and *harmonic means* and are denoted by ∇_ν and $!_\nu$ in the literature.) It can be easily seen that $A\sharp_{p,\nu}B \le A\sharp_{q,\nu}B$ if and only if $p \le q$. In addition, an easy calculation shows that $A\sharp_{p,\nu}B = B\sharp_{p,1-\nu}A$.

Another important mean which interpolates arithmetic and geometric means is the Heinz mean:

$$H_\nu(A,B) = \frac{A\sharp_\nu B + B\sharp_\nu A}{2} = \frac{A\sharp_\nu B + A\sharp_{1-\nu}B}{2}.$$

(For more details on operator means, see [8].)

Throughout this paper, we identify the matrix algebra $M_n(\mathbb{C})$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$ with the space of $\mathbb{B}(\mathbb{C}^n)$, and by *positive definite matrices* we mean the strictly positive operators on $\mathbb{B}(\mathbb{C}^n)$.

In [4] and [5], Kittaneh and Manasrah found an improvement of arithmetic-geometric mean inequality and its reverse, and obtained the following results:

$$\begin{aligned}\max\{\nu,1-\nu\}(\sqrt{a}-\sqrt{b})^2 &\ge (1-\nu)a + \nu b - a^{1-\nu}b^\nu \\ &\ge \min\{\nu,1-\nu\}(\sqrt{a}-\sqrt{b})^2,\end{aligned}$$

where a and b are positive real numbers and $\nu \in [0,1]$. On the other hand, in [4], they obtained the following result:

$$H_\nu(a,b) + \min\{\nu,1-\nu\}(\sqrt{a}-\sqrt{b})^2 \le \frac{a+b}{2}.$$

Further, they obtained a matrix version of these inequalities; that is, if $B, C \in M_n(\mathbb{C})$ such that B is positive definite and C is invertible, then, for $0 \le \nu \le 1$,

$$\begin{aligned}&2\min\{\nu,1-\nu\}\big(A\nabla B - C^*(C^{*^{-1}}BC^{-1})^{1/2}C\big) \\ &\quad \le A\nabla_\nu B - C^*(C^{*^{-1}}BC^{-1})^\nu C,\end{aligned} \tag{1.1}$$

and

$$\begin{aligned}&2\max\{\nu,1-\nu\}\big(A\nabla B - C^*(C^{*^{-1}}BC^{-1})^{1/2}C\big) \\ &\quad \ge A\nabla_\nu B - C^*(C^{*^{-1}}BC^{-1})^\nu C,\end{aligned} \tag{1.2}$$

where $A = C^*C$ (see [5]).

Recently, Alzer, da Fonseca, and Kovačec in [1] compared the difference of weighted arithmetic and geometric means with respect to two different weights. They showed that, for positive definite matrices $A, B \in M_n(\mathbb{C})$,

$$\frac{v}{\tau}(A\nabla_\tau B - A\sharp_\tau B) \leq A\nabla_v B - A\sharp_v B \leq \frac{1-v}{1-\tau}(A\nabla_\tau B - A\sharp_\tau B), \tag{1.3}$$

where $0 < v \leq \tau < 1$, and they obtained generalizations of some results about the Young inequality and its reverse in [2], [4], and [5] as consequences.

One should note that

$$C^*(C^{*^{-1}}BC^{-1})^\nu C = C^*(I\sharp_\nu C^{*^{-1}}BC^{-1})C = C^*C\sharp_\nu B = A\sharp_\nu B.$$

Thus, we can consider the inequality (1.3) as a generalization of inequalities (1.1) and (1.2). In addition, some mathematicians investigated the difference of weighted arithmetic and harmonic means, and proved some similar results which can be found in [6], [7], and [9]. In this article, we present a version of (1.3) for the difference of weighted arithmetic and power means, namely,

$$\frac{\nu}{\mu}(A\nabla_\mu B - A\sharp_{p,\mu} B) \leq A\nabla_\nu B - A\sharp_{p,\nu} B \leq \frac{1-\nu}{1-\mu}(A\nabla_\mu B - A\sharp_{p,\mu} B)$$

for any positive definite matrices A, B, and real numbers ν, μ, p with $-1 \leq p < 1$ and $0 < \nu \leq \mu < 1$.

These results are proved in [1] and [7] for two special cases of the geometric mean and the harmonic mean. Also, similar inequalities for the Heinz mean and some trace inequalities are obtained.

2. Numerical inequalities

To prove our main inequality, we need the following lemma.

Lemma 2.1. *Let t be a real number and let $y > -1$. Then*

$$(1+y)^t(ty-1)+1 \geq 0$$

for $t \in \mathbb{R} \setminus [-1, 0]$. If $0 < t < 1$, then the inequality yields its converse.

Proof. Let t be a fixed number, and let $g(y) = (1+y)^t(ty-1)$. Then $g'(y) = t(1+y)^{t-1}(ty-1) + t(1+y)^t = t(t+1)y(1+y)^{t-1}$. If $t \in \mathbb{R} \setminus [-1, 0]$, then $t(t+1)(1+y)^{t-1} > 0$. It follows that g attains its minimum at $y = 0$; hence, $g(y) \geq -1$ for all $y > -1$. If $0 < t < 1$, then we have $t(t+1)(1+y)^{t-1} < 0$. Therefore, g attains its maximum at $y = 0$ so that $g(y) \leq -1$. □

Note that in the numerical case, we can define weighted power mean $\sharp_{p,\nu}$ for each real number p.

Lemma 2.2. *Let $p \in \mathbb{R}$, let $p \neq 1$, and let $0 < \nu \leq \mu < 1$. Then for all positive real number $x > 0$, we have*

$$\frac{\nu}{\mu} \leq \frac{1\nabla_\nu x - 1\sharp_{p,\nu} x}{1\nabla_\mu x - 1\sharp_{p,\mu} x} \leq \frac{1-\nu}{1-\mu}. \tag{2.1}$$

Proof. First let $p \neq 0$. Put

$$F(\nu) = \frac{1\sharp_{p,\nu}x - 1\nabla_\nu x}{\nu} = \frac{(1-\nu+\nu x^p)^{1/p} - 1 + \nu - \nu x}{\nu}.$$

Then

$$\begin{aligned}\frac{\partial}{\partial\nu}F &= \frac{1}{\nu}\big[1/p(-1+x^p)(1-\nu+\nu x^p)^{1/p-1} - (-1+x)\big] \\ &\quad - \frac{1}{\nu^2}\big[(1-\nu+\nu x^p)^{1/p} - (1-\nu+\nu x)\big] \\ &= \frac{1}{\nu^2}\Big[(1-\nu+\nu x^p)^{1/p-1}\Big[\Big(\frac{\nu}{p}(x^p-1) - 1 + \nu - \nu x^p\Big)\Big] + 1\Big] \\ &= \frac{1}{\nu^2}\Big[(1+y)^{1/p-1}\Big[\Big(y\Big(\frac{1}{p}-1\Big) - 1\Big)\Big] + 1\Big],\end{aligned}$$

where $y = \nu x^p - \nu$. Using Lemma 2.1, if $p < 1$, then we get $\frac{\partial}{\partial\nu}F \geq 0$, and so F is increasing. Therefore, $F(\nu) \leq F(\mu)$. Since F is negative, the first inequality follows. Similarly, for $p > 1$, since $\frac{\partial}{\partial\nu}F \leq 0$ and F is a positive function, the inequality follows.

For the second inequality, we use the fact that $1\sharp_{p,\nu}x = x(1\sharp_{p,1-\nu}x^{-1})$. Now applying the first inequality for $1-\nu$ and $1-\mu$, the result follows.

To prove the result for $p = 0$, let $p \to 0$ and use the continuity of $\sharp_{p,\nu}$. □

If we consider $x = \frac{b}{a}$ in (2.1) and we multiply both the numerator and denominator of the fraction by b, then we get the following result.

Theorem 2.3. *Let $p \in \mathbb{R}$, let $p \neq 1$, and let $0 < \nu \leq \mu < 1$. Then for all positive real numbers a, b,*

$$\frac{\nu}{\mu} \leq \frac{a\nabla_\nu b - a\sharp_{p,\nu}b}{a\nabla_\mu b - a\sharp_{p,\mu}b} \leq \frac{1-\nu}{1-\mu}.$$

We can state a version of this theorem for the Heinz mean.

Corollary 2.4. *Let $0 < \nu \leq \mu < 1$. Then for all positive real numbers a, b,*

$$\frac{\nu}{\mu} \leq \frac{a\nabla b - H_\nu(a,b)}{a\nabla b - H_\mu(a,b)} \leq \frac{1-\nu}{1-\mu}.$$

Proof. Applying Theorem 2.3, for $\nu \leq \mu$, we have

$$\frac{\nu}{\mu}(a\nabla_\mu b - a\sharp_\mu b) \leq a\nabla_\nu b - a\sharp_\nu b \leq \frac{1-\nu}{1-\mu}(a\nabla_\mu b - a\sharp_\mu b).$$

In addition, due to $1-\mu \leq 1-\nu$, we have

$$\frac{1-\mu}{1-\nu} \leq \frac{a\nabla_{1-\mu}b - a\sharp_{1-\mu}b}{a\nabla_{1-\nu}b - a\sharp_{1-\nu}b} \leq \frac{\mu}{\nu},$$

and therefore

$$\frac{\nu}{\mu}(a\nabla_{1-\mu}b - a\sharp_{1-\mu}b) \leq a\nabla_{1-\nu}b - a\sharp_{1-\nu}b \leq \frac{1-\nu}{1-\mu}(a\nabla_{1-\mu}b - a\sharp_{1-\mu}b).$$

Therefore,

$$\begin{aligned}&\frac{\nu}{\mu}\big((a\nabla_\mu b + a\nabla_{1-\mu} b) - (a\sharp_\mu b + a\sharp_{1-\mu} b)\big)\\ &\quad\le (a\nabla_\nu b + a\nabla_{1-\nu} b) - (a\sharp_\nu b + a\sharp_{1-\nu} b)\\ &\quad\le \frac{1-\nu}{1-\mu}\big((a\nabla_\mu b + a\nabla_{1-\mu} b) - (a\sharp_\mu b + a\sharp_{1-\mu} b)\big).\end{aligned}$$

Thus,

$$\frac{\nu}{\mu}\big(a\nabla b - H_\mu(a,b)\big) \le a\nabla b - H_\nu(a,b) \le \frac{1-\nu}{1-\mu}\big(a\nabla b - H_\mu(a,b)\big). \qquad \square$$

Using Theorem 2.3 and Lemma 2.5, we can find lower and upper bounds for the difference of weighted arithmetic and power means.

Lemma 2.5. *(see [1]) Let $0 < \nu < 1$, and let $f : [a,b] \to \mathbb{R}$ be twice differentiable with $-\infty < m \le f''(x) \le M < +\infty$ for all $x \in (a,b)$. Then*

$$\begin{aligned}\frac{\nu(1-\nu)}{2}(b-a)^2 m &\le \nu f(a) + (1-\nu)f(b) - f\big(\nu a + (1-\nu)b\big)\\ &\le \frac{\nu(1-\nu)}{2}(b-a)^2 M.\end{aligned}$$

Theorem 2.6. *Let $0 < \nu < 1$, and let $0 < a < b$. Then*

$$\frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big)a\Big(\Big(\frac{b}{a}\Big)^p - 1\Big)^2 \le a\nabla_\nu b - a\sharp_{p,\nu} b \tag{2.2}$$

$$\le \frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big)b\Big(1 - \Big(\frac{a}{b}\Big)^p\Big)^2 \tag{2.3}$$

for $0 \ne p < \frac{1}{2}$ or $p > 1$. If $\frac{1}{2} \le p \le 1$, then

$$\frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big)b\Big(1 - \Big(\frac{a}{b}\Big)^p\Big)^2 \le a\nabla_\nu b - a\sharp_{p,\nu} b \tag{2.4}$$

$$\le \frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big)a\Big(\Big(\frac{b}{a}\Big)^p - 1\Big)^2. \tag{2.5}$$

In addition,

$$\frac{\nu(1-\nu)}{2} a\ln^2\Big(\frac{b}{a}\Big) \le a\nabla_\nu b - a\sharp_\nu b \le \frac{\nu(1-\nu)}{2} b\ln^2\Big(\frac{a}{b}\Big). \tag{2.6}$$

Proof. Let $f(t) = t^{1/p}$. Then $f''(t) = \frac{1}{p}(\frac{1}{p}-1)t^{\frac{1}{p}-2}$. Assume that $\frac{1}{2} \le p \le 1$. Applying Lemma 2.5 on $[a^p, b^p]$, we have

$$\begin{aligned}\frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big)b\Big(1 - \Big(\frac{a}{b}\Big)^p\Big)^2 &\le b\nabla_\nu a - b\sharp_{p,\nu} a\\ &\le \frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big)a\Big(\Big(\frac{b}{a}\Big)^p - 1\Big)^2.\end{aligned}$$

Replacing ν with $1-\nu$ and employing the fact that $b\sharp_{\alpha,\nu}a = a\sharp_{\alpha,1-\nu}b$, we get the result. If $p \notin [\frac{1}{2},1]$ and $p>0$, then, as in the previous part, apply Lemma 2.5 on $[a^p,b^p]$ and replace ν with $1-\nu$. Also, for $p<0$, use the interval $[b^p,a^p]$ to get the desired result. Finally, for $p=0$, use the above inequalities where p tends to zero. □

In a similar way, we can apply (2.6) for ν and $1-\nu$ to find a bound for the Heinz mean.

Corollary 2.7. *For $0<\nu<1$ and positive numbers a,b, we have*

$$\frac{\nu(1-\nu)}{2}a\ln^2\Big(\frac{b}{a}\Big) \le a\nabla b - H_\nu(a,b) \le \frac{\nu(1-\nu)}{2}b\ln^2\Big(\frac{a}{b}\Big). \tag{2.7}$$

3. Matrix inequalities

Based on the numerical inequalities in the preceding section and the spectral theorem, we obtain the matrix versions of these inequalities. Since $\sharp_{p,\nu}$ defines an operator means only if $-1\le p\le 1$, in this section, we focus on the case $|p|\le 1$.

Lemma 3.1. *Let $Q\in M_n(\mathbb{C})$ be positive definite, and let ν, μ, and p be real numbers with $-1\le p<1$ and $0<\nu\le\mu<1$. Then*

$$\frac{\nu}{\mu}(I\nabla_\mu Q - I\sharp_{p,\mu}Q) \le I\nabla_\nu Q - I\sharp_{p,\nu}Q \le \frac{1-\nu}{1-\mu}(I\nabla_\mu Q - I\sharp_{p,\mu}Q). \tag{3.1}$$

Proof. By the spectral theorem, the positive definite matrix Q can be written as $Q=U^*DU$ for some unitary matrix U and diagonal matrix $D=\operatorname{diag}(\lambda_1,\dots,\lambda_n)$, where $\lambda_i>0$ are the eigenvalues of Q. Since $I\sharp_{p,\nu}D = \operatorname{diag}(1\sharp_{p,\nu}\lambda_1,\dots,1\sharp_{p,\nu}\lambda_n)$, applying (2.1) for $x=\lambda_i, i=1,\dots,n$, we have

$$\frac{\nu}{\mu}(I\nabla_\mu D - I\sharp_{p,\mu}D) \le I\nabla_\nu D - I\sharp_{p,\nu}D \le \frac{1-\nu}{1-\mu}(I\nabla_\mu D - I\sharp_{p,\mu}D).$$

Now, by multiplying the inequalities by U from the left and by U^* from the right, the result follows. □

Theorem 3.2. *Let $A,B\in M_n(\mathbb{C})$ be positive definite, and let ν, μ, and p be real numbers with $-1\le p<1$ and $0<\nu\le\mu<1$. Then*

$$\frac{\nu}{\mu}(A\nabla_\mu B - A\sharp_{p,\mu}B) \le A\nabla_\nu B - A\sharp_{p,\nu}B \le \frac{1-\nu}{1-\mu}(A\nabla_\mu B - A\sharp_{p,\mu}B). \tag{3.2}$$

Proof. Put $Q=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (3.1) and multiply the inequalities by $A^{\frac{1}{2}}$ from both the left and right to get the result. □

A similar argument as in Corollary 2.4 leads to the following result for the Heinz mean.

Corollary 3.3. *Let $A,B\in M_n(\mathbb{C})$ be positive definite, and let ν and μ be real numbers with $0<\nu\le\mu<1$. Then*

$$\frac{\nu}{\mu}\big(A\nabla B - H_\mu(A,B)\big) \le A\nabla B - H_\nu(A,B) \le \frac{1-\nu}{1-\mu}\big(A\nabla B - H_\mu(A,B)\big). \tag{3.3}$$

Theorem 3.4. *Let $A, B \in M_n(\mathbb{C})$ be positive definite matrices such that $0 \leq mI \leq A \leq B \leq MI$ and $-1 \leq p \leq 1$. Then, for $0 < \nu < 1$,*

(i) *if $0 \neq p \leq \frac{1}{2}$, then*

$$A\nabla_\nu B - A\sharp_{p,\nu} B \leq \frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big)\Big(1-\Big(\frac{m}{M}\Big)^p\Big)^2 B;$$

(ii) *if $\frac{1}{2} \leq p \leq 1$, then*

$$A\nabla_\nu B - A\sharp_{p,\nu} B \leq \frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big)\Big(\Big(\frac{M}{m}\Big)^p-1\Big)^2 A;$$

(iii) *in addition,*

$$A\nabla_\nu B - A\sharp_\nu B \leq \frac{\nu(1-\nu)}{2} \ln^2\Big(\frac{M}{m}\Big) B.$$

Proof. (i) Let $0 \neq p \leq \frac{1}{2}$. Using (2.2), for $1 \leq t = \frac{b}{a} \leq \frac{M}{m}$, we have

$$\begin{aligned} 1\nabla_\nu t - 1\sharp_{p,\nu} t &\leq \frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big) t(1-t^{-p})^2 \\ &\leq \frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big) t \max_{1\leq t\leq \frac{M}{m}} (1-t^{-p})^2 \\ &= \frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big)\Big(1-\Big(\frac{m}{M}\Big)^p\Big)^2 t. \end{aligned}$$

Thus, for the positive definite matrix $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ with $I \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq \frac{M}{m} I$, it can be deduced that

$$\begin{aligned} &I\nabla_\nu A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - I\sharp_{p,\nu} A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \\ &\quad \leq \frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big)\Big(1-\Big(\frac{m}{M}\Big)^p\Big)^2 A^{-\frac{1}{2}}BA^{-\frac{1}{2}}. \end{aligned}$$

Hence,

$$A\nabla_\nu B - A\sharp_{p,\nu} B \leq \frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big)\Big(1-\Big(\frac{m}{M}\Big)^p\Big)^2 B.$$

(ii) Similarly, let $t = \frac{b}{a}$. Then, by inequality (2.4), we get

$$\begin{aligned} 1\nabla_\nu t - 1\sharp_{p,\nu} t &\leq \frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big)(t^p-1)^2 \\ &\leq \frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big) t \max_{1\leq t\leq \frac{M}{m}} (t^p-1)^2 \\ &= \frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big)\Big(\Big(\frac{M}{m}\Big)^p-1\Big)^2. \end{aligned}$$

Thus, for the positive definite matrix $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, it can be deduced that

$$I\nabla_\nu A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - I\sharp_{p,\nu} A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq \frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big)\Big(\Big(\frac{M}{m}\Big)^p-1\Big)^2.$$

Hence,

$$A\nabla_\nu B - A\sharp_{p,\nu} B \leq \frac{\nu(1-\nu)}{2} \times \frac{1}{p}\Big(\frac{1}{p}-1\Big)\Big(\Big(\frac{M}{m}\Big)^p - 1\Big)^2 A.$$

(iii) Using inequality (2.6) and following similar arguments as above, the result follows. □

Corollary 3.5. *Let $A, B \in M_n(\mathbb{C})$ be positive definite matrices such that $0 \leq mI \leq A \leq B \leq MI$ and $0 < \nu < 1$. Then*

$$A\nabla B - H_\nu(A,B) \leq \frac{\nu(1-\nu)}{2} \ln^2\Big(\frac{M}{m}\Big) B.$$

Remark 3.6. Note that for positive definite matrices A, B, with $A \leq B$, we have $0 < \lambda_n(A)I \leq A \leq B \leq \lambda_1(B)I$, where $\lambda_n(A)$ and $\lambda_1(B)$ stand for the minimum of eigenvalues of A and the maximum of eigenvalues of B, respectively; that is, in the preceding theorem, $\frac{M}{m} = \frac{\lambda_1(B)}{\lambda_n(A)}$.

Remark 3.7. Note that one can obtain similar inequalities for strictly positive operators on Hilbert spaces using the following monotonicity property for operator functions: If X is a self-adjoint operator with the spectrum $sp(X)$, then

$$f(t) \geq g(t), \quad t \in sp(X) \Longrightarrow f(X) \geq g(X).$$

For more details about this property, see [8].

4. Determinant inequality

In this section, we state a version of inequality (3.2) for the determinant of the matrices.

Lemma 4.1 (Minkowski's product inequality; [3, p. 560])**.** *Let $a = [a_i]$, $b = [b_i]$, $i = 1, 2, \ldots, n$, such that a_i and b_i are positive real numbers. Then*

$$\Big(\prod_{i=1}^n a_i\Big)^{\frac{1}{n}} + \Big(\prod_{i=1}^n b_i\Big)^{\frac{1}{n}} \leq \Big(\prod_{i=1}^n (a_i+b_i)\Big)^{\frac{1}{n}}.$$

The following lemma can be obtained by an easy calculation.

Lemma 4.2. *Let a and b be positive real numbers. If $\alpha \geq 1$, then*

$$a^\alpha + b^\alpha \leq (a+b)^\alpha.$$

Corollary 4.3. *Let $a = [a_i], b = [b_i], i = 1, 2, \ldots, n$, such that a_i and b_i are positive real numbers. Then, for $\alpha \geq 1$,*

$$\Big(\prod_{i=1}^n a_i\Big)^{\frac{\alpha}{n}} + \Big(\prod_{i=1}^n b_i\Big)^{\frac{\alpha}{n}} \leq \Big(\prod_{i=1}^n (a_i+b_i)\Big)^{\frac{\alpha}{n}}.$$

Theorem 4.4. *Let $A, B \in M_n(\mathbb{C})$ be positive definite. If ν, μ, and α are real numbers with $0 < \nu \leq \mu < 1$ and $\alpha \geq 1$, then, for $-1 \leq p < 1$,*

$$\Big(\frac{\nu}{\mu}\Big)^\alpha \det(A\nabla_\mu B - A\sharp_{p,\mu} B)^{\frac{\alpha}{n}} \leq \det(A\nabla_\nu B)^{\frac{\alpha}{n}} - \det(A\sharp_{p,\nu} B)^{\frac{\alpha}{n}}. \tag{4.1}$$

Proof. Applying the first inequality of (2.1) for the singular values $s_i(T)$ of the positive definite matrix $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we have

$$\frac{\nu}{\mu} \leq \frac{1\nabla_\nu s_i(T) - 1\sharp_{p,\nu} s_i(T)}{1\nabla_\mu s_i(T) - 1\sharp_{p,\mu} s_i(T)}$$

for all $i = 1, 2, \ldots, n$ for which $s_i(T) \neq 1$. We have

$$\begin{aligned}
\det(I\nabla_\nu T)^{\frac{\alpha}{n}} &= \big(\det(I\nabla_\nu T)\big)^{\frac{\alpha}{n}} \\
&= \Big(\prod_{i=1}^{n} 1\nabla_\nu s_i(T)\Big)^{\frac{\alpha}{n}} \\
&\geq \Big(\prod_{i=1}^{n}\Big[\Big(\frac{\nu}{\mu}\Big)\big(1\nabla_\mu s_i(T) - 1\sharp_{p,\mu} s_i(T)\big) + 1\sharp_{p,\nu} s_i(T)\Big]\Big)^{\frac{\alpha}{n}} \\
&\geq \Big(\frac{\nu}{\mu}\Big)^{\alpha} \prod_{i=1}^{n}\big[1\nabla_\mu s_i(T) - 1\sharp_{p,\mu} s_i(T)\big]^{\frac{\alpha}{n}} + \prod_{i=1}^{n}\big[1\sharp_{p,\nu} s_i(T)\big]^{\frac{\alpha}{n}} \\
&= \Big(\frac{\nu}{\mu}\Big)^{\alpha} \det(I\nabla_\mu T - I\sharp_{p,\mu} T)^{\frac{\alpha}{n}} + \det(I\sharp_{p,\nu} T)^{\frac{\alpha}{n}}.
\end{aligned}$$

The second inequality is obtained by Lemma 4.2. Multiplying $(\det A^{1/2})^{\alpha/n}$ to both sides and by the multiplicativity of the determinant, we derive (4.1). □

Remark 4.5. If $\alpha = 1$ in the inequality (4.1), then

$$\frac{\nu}{\mu}\det(A\nabla_\mu B - A\sharp_{p,\mu} B)^{\frac{1}{n}} \leq \det(A\nabla_\nu B)^{\frac{1}{n}} - \det(A\sharp_{p,\nu} B)^{\frac{1}{n}},$$

and if $\alpha = n$ in the inequality (4.1), then

$$\det A\sharp_{p,\nu} B + \Big(\frac{\nu}{\mu}\Big)^{n} \det(A\nabla_\mu B - A\sharp_{p,\mu} B) \leq \det A\nabla_\nu B. \tag{4.2}$$

Similarly, we have the following result.

Corollary 4.6. *Let $A, B \in M_n(\mathbb{C})$ be positive definite. If ν, μ, and α are real numbers with $0 < \nu \leq \mu < 1$ and $\alpha \geq 1$, then*

$$\Big(\frac{\nu}{\mu}\Big)^{\alpha} \det\big(A\nabla B - H_\mu(A,B)\big)^{\frac{\alpha}{n}} \leq \det(A\nabla B)^{\frac{\alpha}{n}} - \det\big(H_\nu(A,B)\big)^{\frac{\alpha}{n}}.$$

In particular,

$$\frac{\nu}{\mu}\det\big(A\nabla B - H_\mu(A,B)\big)^{\frac{1}{n}} \leq \det(A\nabla B)^{\frac{1}{n}} - \det\big(H_\nu(A,B)\big)^{\frac{1}{n}},$$

and

$$\det H_\nu(A,B) + \Big(\frac{\nu}{\mu}\Big)^{n} \det\big(A\nabla B - H_\mu(A,B)\big) \leq \det A\nabla B.$$

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