POSITIVE REPRESENTATIONS OF $C_0(X)$, I

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ABSTRACT. We introduce the notion of a *positive spectral measure* on a σ-algebra, taking values in the positive projections on a Banach lattice. Such a measure generates a bounded positive representation of the bounded measurable functions. If X is a locally compact Hausdorff space and if π is a positive representation of $C_0(X)$ on a KB-space, then π is the restriction to $C_0(X)$ of such a representation generated by a unique regular positive spectral measure on the Borel σ-algebra of X. The relation between a positive representation of $C_0(X)$ on a Banach lattice and—if it exists—a generating positive spectral measure on the Borel σ-algebra are further investigated; here and elsewhere, phenomena occur that are specific for the ordered context.

1. INTRODUCTION AND OVERVIEW

Suppose that X is a locally compact Hausdorff space, with $C_0(X)$ denoting the ordered Banach algebra of all real-valued continuous functions vanishing at infinity. Positive representations of such an algebra $C_0(X)$ on a Banach lattice E, that is, representations such that positive functions act as positive operators on E, are quite common. Rather trivially, $C_0(X)$ acts positively by multiplication on many Banach lattices of (equivalence classes of) functions on X. Somewhat in disguise, since the center $Z(E)$ of an arbitrary Banach lattice is such a space for compact X, there is also a positive representation of this type associated with

every Banach lattice. Such positive representations also occur in a very general context where ordering is not present from the start: if E is an arbitrary Banach space, X is compact, and π is a bounded unital representation of $C_0(X)$ on E, then every cyclic closed subspace of E can be supplied with an ordering and an equivalent norm so that it becomes a Banach lattice on which $C_0(X)$ acts positively (see [2, Lemma 4.6] or [16, Proposition 2.5]); according to [2, proof of Lemma 4.6] this result goes back to [13].

Given the ubiquity of their occurrence, it is natural to ask which positive representations of $C_0(X)$ on a Banach lattice E are generated by a (regular) positive spectral measure on the Borel σ-algebra Ω of X; here *positive* refers to the fact that the measure takes values in the positive projections on E. It is one of our main results that this is always the case if E is a KB-space (see Theorem 5.6).

The study of positive spectral measures and of positive representations of $C_0(X)$ (and also that of positive representations of algebras of bounded measurable functions) appears to be new. The main related results in the literature that we are aware of are concerned with bounded unital representations of $C_0(X)$ for compact X on a Banach space. Thus the class of representation spaces is more general, but the class of representations is more restrictive: our X need not be compact, and, even if it is, our representation need not be unital. Of course, one can extend a bounded representation π of $C_0(X)$ for noncompact X to a bounded unital representation π_∞ of $C_0(X_\infty)$, where X_∞ is the one-point compactification of X. Thus one places oneself in the convenient unital and compact framework, but removing (and hence also masking) all possible nondegeneracy of the original representation from the very start seemed counterproductive to us. Hence we have considered the most general case, which for positive representations in Banach lattices turns out to be feasible.

The presence of ordered structures brings about several new phenomena as compared to the Banach space context, some of them already at a very basic level. For example, boundedness of positive representations tends to be automatic, and it is often possible to give exact expressions for their norms (see Propositions 4.5, 5.1, and 5.10). Furthermore, relations of a type that is particular to the ordered context make their appearance. For example, if P is a positive spectral measure on Ω that is regular in the (usual) sense of Definition 5.2, and if Δ is a Borel set, then

$$P(\Delta) = \inf\big\{P(V) : V \text{ open and } \Delta \subset V\big\}$$

in the regular operators on E (see Proposition 5.3). Therefore P is outer regular as a map from the Ω to the regular operators on E; likewise, P is inner regular in this sense. In a similar vein, if π is a positive representation of $C_0(X)$ on E that has a generating regular positive spectral measure P on Ω, and V is open, then

$$P(V) = \sup\big\{\pi(\phi) : \phi \in C_c(X), \operatorname{supp}\phi \subset V, 0 \le \phi \le \mathbf{1}_X\big\}$$

in the regular operators on E (see Theorem 5.12 for this and related results). As in the previous example, this statement, which is reminiscent of (and stems from) a well-known formula in the circle of ideas around the Riesz representation

theorem, is only meaningful in an ordered context, as it tells us how to determine P from π in a purely order-theoretical manner.

The reader who is familiar with the theory of spectral measures in Banach spaces will without difficulty recognize several ideas and techniques employed in this article. We have, nevertheless, still included the detailed proofs of our results, as these proofs are inspired by, but not identical to, those in the general case. The ordered context has its own peculiarities from the very start, and it seemed to us that the only way to exploit this extra information to its full potential, while still producing a readable summary was to make a fresh start and give precise arguments. This way has the additional advantage that it makes the paper essentially self-contained. We have given pertinent references to the general theory to the best of our abilities.

This paper is organized in the following form.

Section 2 contains the notation and terminology, as well as an overview of material around the Riesz representation theorem. We need more than the mere existence of measures representing bounded linear functionals, and we have included what we need for the ease of reference, and also to establish terminology (which is not uniform throughout the literature).

Section 3 introduces positive spectral measures on arbitrary σ-algebras and contains some first basic results.

In Section 4 a bounded positive representation of the bounded measurable functions on a measurable space is constructed from a positive spectral measure. Any positive representation of this algebra is, in fact, always bounded, but when is there an underlying positive spectral measure? If E has σ-order continuous norm, then we can characterize the positive representations thus obtained: they are precisely the σ-order continuous ones. It is not true that an arbitrary positive representation of the bounded measurable functions has a generating positive spectral measure on the pertinent σ-algebra; this will be taken up in [7].

Section 5 focuses on the topological context of a locally compact Hausdorff space X. We are concerned with automatic continuity of positive representations of $C_0(X)$, and we show that every positive representation of $C_0(X)$ on a KB-space has a generating regular positive spectral measure on Ω (see Theorem 5.6). It is indicated in Remark 5.7 how this result can also be derived using Banach space results and Banach space properties of KB-spaces. This alternative approach is more involved and lacks the order-theoretical flavor of ours. Additional regularity properties of a regular positive spectral measures are investigated, as well as the relation between a positive representation of $C_0(X)$ and—if there is one—a generating possibly regular positive spectral measure on Ω for that representation.

This paper, which we consider to be part of the groundwork for the theory of positive representations of $C_0(X)$, is expected to have a sequel [7]. There are still several issues to be investigated. The relation between the degeneracy of the representation of π and the projection corresponding to the element X of Ω is one of these, and Remarks 2.2 and 4.6 contain additional ones. Furthermore, it is interesting to turn the tables: if E is a Banach lattice such that every positive representation of every $C_0(X)$-space (where X is a locally compact Hausdorff space) on E has a generating regular positive spectral measure on the pertinent

Borel σ-algebra, then what can one say about E? Must E be a KB-space; that is, is Theorem 5.6 optimal?

For a currently, more distant perspective, we recall that the spectral theorem for representations of commutative C^*-algebras on Hilbert spaces underlies, in the end, a structure theorem (even a classification theorem) for an arbitrary normal operator on a separable Hilbert space (see [6, Theorem IX.10.1]). Although still a considerable effort would be needed, and success is not guaranteed (presumably one would need a reasonable analogue of von Neumann algebras), it could be hoped that, analogously, Theorem 5.6 could be the basis for a structure theorem for an arbitrary orthomorphism on suitable KB-spaces.

2. Preliminaries

We start by collecting some basic notation and facts, as well as giving a precise overview of the Riesz representation theorem and the facts surrounding it.

2.1. **Basics.** All vector spaces in this paper are over the real numbers. If V is an ordered vector space, then we write V_+ and V_- for its positive and negative cones, respectively.

If E is a Banach space, then E' denotes its norm dual. We write $\mathcal{L}(E)$ for the bounded linear operators on E, and $\mathbf{id}_E$ for the identity map on E. If E is a Banach lattice, then $\mathcal{L}_r(E)$ will denote the vector space of regular operators on E, that is, the linear operators on E that can be written as a difference of two positive linear operators on E. It is well known that every positive linear operator on E is bounded, so that $\mathcal{L}_r(E) \subset \mathcal{L}(E)$, and that, for $T \geq 0$, $\|T\| = \sup\{\|Tx\| : x \in E_+, \|x\| \leq 1\}$, where E_+ is the positive cone of E (see [14, Proposition 1.3.5]). Consequently, the operator norm is monotone on the positive cone $\mathcal{L}_r(E)_+$ of $\mathcal{L}_r(E)$: if $T_1, T_2 \in \mathcal{L}_r(E)$ and $0 \leq T_1 \leq T_2$, then $\|T_1\| \leq \|T_2\|$.

If E is a Dedekind complete Banach lattice, then $\mathcal{L}_r(E)$ is a Dedekind complete Banach lattice when supplied with the natural ordering and the regular norm $\| \cdot \|_r$, defined by $\|T\|_r = \||T|\|$ ($T \in \mathcal{L}_r(E)$) (see [5, Theorem 4.74]). If $E \neq \{0\}$, then the inclusion map $(\mathcal{L}_r(E), \| \cdot \|_r) \hookrightarrow (\mathcal{L}(E), \| \cdot \|)$ has norm 1 (see [5, p. 255]).

If F is a normed space and E is a Banach lattice, then the norm of a bounded linear map $\pi \colon F \to (\mathcal{L}(E), \| \cdot \|)$ or $\pi \colon F \to (\mathcal{L}_r(E), \| \cdot \|)$ will be denoted by $\|\pi\|$. If E is Dedekind complete, then the norm of a bounded linear map $\pi \colon F \to (\mathcal{L}_r(E), \| \cdot \|_r)$ will be denoted by $\|\pi\|_r$.

Definition 2.1. If A is an ordered algebra and E is a Banach lattice, then a *positive representation of A on E* is an algebra homomorphism $\pi \colon A \to \mathcal{L}(E)$ such that $\pi(A_+) \subset \mathcal{L}_r(E)_+$. If A is normed, then we do not require π to be bounded. If A is unital, then we do not require π to be unital. The linear span of $\{\pi(a)x : a \in A, x \in E\}$ need not be dense in E; that is, the representation can be degenerate.

Remark 2.2. In the cases of our interest, the ordered algebra A is in fact a lattice, so that a positive representation of A on E maps A into $\mathcal{L}_r(E)$. If E is Dedekind complete, then $\mathcal{L}_r(E)$ is also a lattice, and it is meaningful to require that $\pi \colon A \to \mathcal{L}_r(E)$ is a lattice homomorphism. We emphasize that this is not required in the

present paper, but in [7] we shall investigate to which extent this is automatically the case.

If V is a not necessarily order complete ordered vector space, and if $(v_n)_{n=1}^\infty \subset V$, then we will write $v_n \uparrow v$ if $v_n \leq v_{n+1}$ $(n \geq 1)$ and $v = \sup_n v_n$.

If X is a set, then $\mathbf{1}_X$ will denote the constant function 1 on X. If Ω is a σ-algebra of subsets of X, then $\mathcal{B}(X)$ will denote the bounded Ω-measurable functions on X. It is a Banach lattice algebra, and we let $\mathcal{S}(X)$ denote the lattice subalgebra of simple functions. The order bounded subsets of $\mathcal{B}(X)$ are precisely the norm-bounded ones. If $(\phi_n)_{n=1}^\infty \subset \mathcal{B}(X)$ and $\phi \in \mathcal{B}(X)$, then $\phi_n \uparrow \phi$ in $\mathcal{B}(X)$ if and only if $\phi_n(\xi) \uparrow \phi(\xi)$ for all $\xi \in X$.

The supremum norm of $\phi \in \mathcal{B}(X)$ is written as $\|\phi\|$.

2.2. Regular Borel measures.
If X is a locally compact Hausdorff space, then we let Ω denote the Borel σ-algebra generated by the open sets and we let $\mathcal{B}(X)$ be the Banach lattice algebra of bounded Borel measurable functions on X. We write $C_c(X)$ for the normed lattice algebra of continuous functions on X with compact support, and we write $C_0(X)$ for the Banach lattice algebra of continuous functions vanishing at infinity. If X is compact, then we still write $C_0(X)$ rather than $C(X)$ for the sake of uniform terminology and notation.

The results in Section 5 rely heavily on the Riesz representation theorem and the general theory of regular Borel measures on locally compact Hausdorff spaces. Since the terminology in this field is not entirely standardized and we need a bit more than the bare minimum, we give precise definitions and an overview of what we need.

Adapting the terminology from [3, p. 352], we say that a measure $\mu\colon \Omega \to [0, \infty]$ is

(1) a *Borel measure* if $\mu(K) < \infty$ for all compact $K \subset X$;
(2) *outer regular on* $\Delta \in \Omega$ if $\mu(\Delta) = \inf\{\mu(V) : V \text{ open and } \Delta \subset V\}$;
(3) *inner regular on* $\Delta \in \Omega$ if $\mu(\Delta) = \sup\{\mu(K) : K \text{ compact and } K \subset \Delta\}$;
(4) a *regular Borel measure* if it is a Borel measure that is outer regular on all $\Delta \in \Omega$ and inner regular on all open subsets of X.

The nomenclature is not uniform in the literature; sometimes the inner regularity on all elements of Ω rather than on just the compact subsets is incorporated in the definition of a regular Borel measure, as in [12, p. 212]. In [12, p. 212] our regular Borel measures are Radon measures.

We let $M_b(X)$ be the regular finite signed Borel measures on Ω, that is, the finite signed measures on Ω that can be written as a difference of two regular finite Borel measures. Then $M_b(X)$ is a Banach lattice when supplied with the natural ordering and the total variation norm. If $\mu \in M_b(X)$, then the linear functional $I_\mu\colon C_0(X) \to \mathbb{R}$, defined by $I_\mu(\phi) = \int_X \phi \, d\mu$ $(\phi \in C_0(X))$, is bounded.

Most of the results we need are collected in the following overview theorem. Part (1) is the combination of [3, Theorem 38.7] and [12, Theorem 7.17]; part (2) follows from [12, Corollary 7.6]; and part (3) is contained in [12, Theorem 7.2].

Theorem 2.3. *Let X be a locally compact Hausdorff space with Borel σ-algebra Ω. Then we have the following:*

(1) *(Riesz representation theorem) The map $\mu \mapsto I_\mu$ is an isometric order isomorphism between the Banach lattices $M_{\mathrm{b}}(X)$ and $C_0(X)'$.*
(2) *If $\mu \in M_{\mathrm{b}}(X)_+$, then μ is inner regular on all elements of Ω.*
(3) *If $\mu \colon \Omega \to [0, \infty]$ is a regular Borel measure on Ω and if V is an open subset of X, then*

$$\mu(V) = \sup\left\{ \int_X \phi \, d\mu : \phi \in C_{\mathrm{c}}(X), \operatorname{supp} \phi \subset V, 0 \le \phi \le \mathbf{1}_X \right\}, \qquad (2.1)$$

and if K is a compact subset of X, then

$$\mu(K) = \inf\left\{ \int_X \phi \, d\mu : \phi \in C_{\mathrm{c}}(X), \phi \ge \chi_K \right\}. \qquad (2.2)$$

In (2.2), and analogously elsewhere, χ_K denotes the characteristic function of the subset K of X.

The following fact is implied by [4, Problem 38.12].

Lemma 2.4. *Let X be a locally compact Hausdorff space with Borel σ-algebra Ω. Let $\mu \in M_{\mathrm{b}}(X)$ and $\psi \in \mathcal{B}(X)$, and define $\mu^\psi \colon \Omega \to \mathbb{R}$ by $\mu^\psi(\Delta) = \int_\Delta \psi \, d\mu$ ($\Delta \in \Omega$). Then $\mu^\psi \in M_{\mathrm{b}}(X)$.*

The following consequence of Lemma 2.4 will be used in the proof of Theorem 5.6.

Lemma 2.5. *Let X be a locally compact Hausdorff space with Borel σ-algebra Ω, and let $\psi \in \mathcal{B}(X)$. Suppose that $\mu, \nu \in M_{\mathrm{b}}(X)$ are such that*

$$\int_X \phi \psi \, d\mu = \int \phi \, d\nu \qquad (2.3)$$

for all $\phi \in C_0(X)$. Then (2.3) holds for all $\phi \in \mathcal{B}(X)$.

Proof. If we let $\mu^\psi(\Delta) = \int_\Delta \psi \, d\mu$ ($\Delta \in \Omega$), then (2.3) implies that $\int_X \phi \, d\mu^\psi = \int_X \phi \psi \, d\mu = \int_X \phi \, d\nu$ for all $\phi \in C_0(X)$. Since Lemma 2.4 asserts that μ^ψ is again a regular finite signed Borel measure, the uniqueness statement in the Riesz representation theorem shows that $\mu^\psi = \nu$. But then $\int_X \phi \, d\mu^\psi = \int_X \phi \, d\nu$ for all $\phi \in \mathcal{B}(X)$; that is, (2.3) holds for all $\phi \in \mathcal{B}(X)$. $\qquad \square$

2.3. **Consequence of a monotone class theorem.** In the proof of Theorem 5.6 we will need the following special case of a monotone class theorem from measure theory (see [20, Theorem 3.14]).

Theorem 2.6. *Let X be a locally compact Hausdorff space with Borel σ-algebra Ω. Suppose that L is a vector space of bounded functions on X such that*

(1) *$\chi_V \in L$ for all open subsets V of X;*
(2) *if ϕ_n is a sequence of functions in L, and if ϕ is a bounded function on X such that $0 \le \phi_n(\xi) \uparrow \phi(\xi)$ for all $\xi \in X$, then $\phi \in L$.*

Then $\mathcal{B}(X) \subset L$.

3. Positive spectral measures

In this section we introduce the notion of a *positive spectral measure* on a general σ-algebra and establish some basic properties.

Definition 3.1. Let X be a set, let Ω be a σ-algebra of subsets of X, and let E be a Banach lattice. A map $P \colon \Omega \to \mathcal{L}_{\mathrm{r}}(E) \subset \mathcal{L}(E)$ is called a *positive spectral measure on* Ω when it has the following properties:

(1) for each Δ in Ω, $P(\Delta)$ is a positive projection;
(2) $P(\emptyset) = 0$;
(3) $P(\Delta_1 \cap \Delta_2) = P(\Delta_1)P(\Delta_2)$ for $\Delta_1, \Delta_2 \in \Omega$;
(4) P is σ-additive for the strong operator topology on $\mathcal{L}(E)$, that is, if $(\Delta_n)_{n=1}^\infty$ are pairwise disjoint elements of Ω, and $x \in E$, then

$$P\Big(\bigcup_{n=1}^\infty \Delta_n\Big)x = \sum_{n=1}^\infty P(\Delta_n)x,$$

where the series converges in the norm topology.

If $P(X) = \mathbf{id}_E$, then P is called *unital*.

Remark 3.2. It is instrumental for the proof of Theorems 4.8 and 5.6 that, as an immediate consequence of a theorem of Pettis [9, Theorem IV.10.1] (see also [17, Lemma III.2]), the combination of (1), (2), (3), and (4) is equivalent to the combination of (1), (2), (3), and (4′), where

(4′) P is σ-additive for the weak operator topology on $\mathcal{L}(E)$; that is, if $(\Delta_n)_{n=1}^\infty$ are pairwise disjoint elements of Ω, and $x \in E, x' \in E'$, then

$$\Big\langle P\Big(\bigcup_{n=1}^\infty \Delta_n\Big)x, x'\Big\rangle = \sum_{n=1}^\infty \langle P(\Delta_n)x, x'\rangle.$$

Remark 3.3.

(1) There are several variants of the definition of spectral measures (or resolutions of the identity) to be found in the literature.
   (a) In [6, Definition IX.1.1], [18, Definition 12.17] (both in the context of Hilbert spaces), and [17, Definition III.4] (in the context of Banach spaces), a spectral measure is required to be σ-additive in the strong operator topology (or, equivalently, in the weak operator topology), as we do. Contrary to our definition, however, it is required to be unital.
   (b) In the context of Banach spaces, a spectral measure is defined on a Boolean algebra of subsets of a set in [11, Definition XV.2.1]. A spectral measure in that sense on a σ-algebra is required to be only finitely additive, whereas we require σ-additivity. Contrary to our definition, it is required to be unital.
(2) Using the terminology of [8, p. 1], a positive spectral measure on Ω as in our definition is a map $P \colon \Omega \to \mathcal{L}_{\mathrm{r}}(E)$ such that it takes values in the positive projections on E, $P(\Delta_1 \cap \Delta_2) = P(\Delta_1)P(\Delta_2)$ $(\Delta_1, \Delta_2 \in \Omega)$, and

such that, for all $x \in E$, the map $\Delta \mapsto P(\Delta)x$ ($\Delta \in \Omega$) is a countably additive vector measure on Ω with values in E.

It follows easily from Definition 3.1 that a positive spectral measure is finitely additive, and the following lemma on monotonicity and uniform boundedness is then clear.

Lemma 3.4. *Let $P \colon \Omega \to \mathcal{L}_{\mathrm{r}}(E)$ be a positive spectral measure. Then P is monotone; that is, if $\Delta_1, \Delta_2 \in \Omega$ and $\Delta_1 \subset \Delta_2$, then $0 \leq P(\Delta_1) \leq P(\Delta_2)$. Consequently, $\|P(\Delta_1)\| \leq \|P(\Delta_2)\|$ for such Δ_1, Δ_2, and in particular $\|P(\Delta)\| \leq \|P(X)\|$ for all $\Delta \in \Omega$.*

Remark 3.5.

(1) As a consequence of the uniform boundedness principle, a (σ-additive) spectral measure taking values in the bounded projections on a Banach space is always uniformly bounded (see [17, Lemma III.3]; the proof also applies if the spectral measure is not unital). The point in Lemma 3.4 is, therefore, the explicit and sharp uniform upper bound $\|P(X)\|$.

(2) If P is unital, then $P(\Omega)$ is a σ-complete Boolean algebra of bounded projections on the Banach space E. In that case, [11, Lemma XVII.3.3] also implies that $P(\Omega)$ is uniformly bounded.

If $P \colon \Omega \to \mathcal{L}_{\mathrm{r}}(E)$ is a positive spectral measure, and $x \in E, x' \in E'$, we define $\mu_{x,x'} \colon \Omega \to \mathbb{R}$ by $\mu_{x,x'}(\Delta) = \langle P(\Delta)x, x' \rangle$ ($x \in E, x' \in E'$); our notation $\mu_{x,x'}$ follows Schaefer (see [19, proof of Proposition V.3.2]). It is clear that $\mu_{x,x'}$ is a finite signed measure and that $\mu_{x,x'}$ is positive when $x \in E_+, x' \in E'_+$. By a standard argument (see [9, p. 97]), we have $\|\mu_{x,x'}\| \leq 2 \sup_{\Delta \in \Omega} |\mu_{x,x'}(\Delta)|$, so that $\|\mu_{x,x'}\| \leq 2\|P(X)\|\|x\|\|x'\|$ ($x \in E, x' \in E'$). The factor 2 can be removed here, as is stated in the following result, which is even more precise.

Lemma 3.6. *Let $P \colon \Omega \to \mathcal{L}_{\mathrm{r}}(E)$ be a positive spectral measure, and let $x \in E, x' \in E'$. Then $\|\mu_{x,x'}\| \leq \langle P(X)|x|, |x'| \rangle$. Equality holds if $x \in E_+ \cup E_-$ and $x' \in E'_+ \cup E'_-$.*

For all $x \in E, x' \in E'$, $\|\mu_{x,x'}\| \leq \|P(X)\|\|x\|\|x'\|$.

Proof. Let $x \in E_+, x' \in E'_+$. If $X = \bigcup_{i=1}^{n} \Delta_i$ is a measurable disjoint partition of X, then, using the fact that the $P(\Delta_i)$ are positive, we see that

$$\sum_{i=1}^{n} \left| \mu_{x,x'}(\Delta_i) \right| = \sum_{i=1}^{n} \left| \langle P(\Delta_i)x, x' \rangle \right|$$
$$= \sum_{i=1}^{n} \langle P(\Delta_i)x, x' \rangle$$
$$= \left\langle \sum_{i=1}^{n} P(\Delta_i)x, x' \right\rangle$$
$$= \langle P(X)x, x' \rangle.$$

Hence $\|\mu_{x,x'}\| = \langle P(X)x, x' \rangle$ for $x \in E_+$ and $x' \in E'_+$. This implies that $\|\mu_{x,x'}\| = \langle P(X)|x|, |x'| \rangle$ if $x \in E_+ \cup E_-$ and $x' \in E'_+ \cup E'_-$. The rest of the lemma follows by splitting arbitrary $x \in E$ and $x' \in E'$ into their positive and negative parts. $\square$

The following fact will be needed in the proof of Proposition 5.3, where it is shown that regular positive spectral measures on the Borel σ-algebra of a locally compact Hausdorff space are also regular in a natural sense that is specific for the ordered context.

Lemma 3.7. *Let Ω be a σ-algebra of subsets of a set X, let E be a Banach lattice, and let $P \colon \Omega \to \mathcal{L}_{\mathrm{r}}(E)$ be a positive spectral measure with associated finite signed measures $\mu_{x,x'}$ $(x \in E, x' \in E')$ on Ω.*

(1) Suppose that $\Delta, \Delta_i \in I$ $(i \in I)$ are elements of Ω such that
    (a) $\Delta \subset \Delta_i$ $(i \in I)$;
    (b) $\mu_{x,x'}(\Delta) = \inf_{i \in I} \mu_{x,x'}(\Delta_i)$ for all $x \in E_+, x' \in E'_+$.
    Then $P(\Delta) = \inf_{i \in I} P(\Delta_i)$ in $\mathcal{L}_{\mathrm{r}}(E)$.
(2) Suppose that $\Delta, \Delta_i \in I$ $(i \in I)$ are elements of Ω such that
    (a) $\Delta \supset \Delta_i$ $(i \in I)$;
    (b) $\mu_{x,x'}(\Delta) = \sup_{i \in I} \mu_{x,x'}(\Delta_i)$ for all $x \in E_+, x' \in E'_+$.
    Then $P(\Delta) = \sup_{i \in I} P(\Delta_i)$ in $\mathcal{L}_{\mathrm{r}}(E)$.

Note that we do not assume that E is Dedekind complete, and hence the existence of the infimum and supremum is not automatic.

Proof. We prove part (1); the proof of part (2) is similar. Since P is monotone, $P(\Delta) \leq P(\Delta_i)$ $(i \in I)$. Let $T \in \mathcal{L}_{\mathrm{r}}(E)$ and $T \leq P(\Delta_i)$ $(i \in I)$. Let $x \in E_+, x' \in E'_+$. Then $\langle Tx, x' \rangle \leq \langle P(\Delta_i)x, x' \rangle = \mu_{x,x'}(\Delta_i)$ $(i \in I)$, so that $\langle Tx, x' \rangle \leq \inf_{i \in I} \mu_{x,x'}(\Delta_i) = \mu_{x,x'}(\Delta) = \langle P(\Delta)x, x' \rangle$. Hence $T \leq P(\Delta)$. $\square$

4. Positive $\mathcal{B}(X)$-representations generated by positive spectral measures

Since we know from Lemma 3.4 that a positive spectral measure is uniformly bounded, one can employ a standard method (see [10, p. 891–892], [17, p. 13–14]) to construct a representation of the bounded measurable functions $\mathcal{B}(X)$ on X from this measure. In the first part of this section, we study the basic properties of the representations thus obtained. In the second part, we turn the tables and ask ourselves which positive representations of $\mathcal{B}(X)$ arise in this fashion.

Starting with a positive spectral measure $P \colon \Omega \to \mathcal{L}_{\mathrm{r}}(E)$ on a σ-algebra Ω of subsets of a set X, the associated representation $\pi_P \colon \mathcal{B}(X) \to \mathcal{L}_{\mathrm{r}}(E)$ is constructed as follows. If $\phi = \sum_{i=1}^n \alpha_i \chi_{\Delta_i} \in \mathcal{B}(X)$ is a simple function, with the Δ_i not necessarily disjoint, then let $\pi_P(\phi) = \sum_{i=1}^n \alpha_i P(\Delta_i) \in \mathcal{L}_{\mathrm{r}}(E)$. This is well defined, and one thus obtains a representation π_P of the simple functions $\mathcal{S}(X)$ on E that is clearly a positive representation of the ordered algebra $\mathcal{S}(X)$. Taking the Δ_i in $\phi = \sum_{i=1}^n \alpha_i \chi_{\Delta_i}$ to be a measurable disjoint partition of X, and

invoking Lemma 3.6 in the penultimate step, we see that

$$\left\|\pi_P(\phi)\right\| = \sup\left\{\left|\left\langle\sum_{i=1}^n \alpha_i P(\Delta_i)x, x'\right\rangle\right| : x \in E, \|x\| \leq 1, x' \in E', \|x'\| \leq 1\right\}$$

$$\leq \sup\left\{\sum_{i=1}^n |\alpha_i|\left|\left\langle P(\Delta_i)x, x'\right\rangle\right| : x \in E, \|x\| \leq 1, x' \in E', \|x'\| \leq 1\right\}$$

$$\leq \|\phi\| \sup\left\{\sum_{i=1}^n \left|\left\langle P(\Delta_i)x, x'\right\rangle\right| : x \in E, \|x\| \leq 1, x' \in E', \|x'\| \leq 1\right\}$$

$$\leq \|\phi\| \sup\left\{\|\mu_{x,x'}\| : x \in E, \|x\| \leq 1, x' \in E', \|x'\| \leq 1\right\}$$

$$\leq \|\phi\|\|P(X)\|$$

$$= \|\phi\|\|\pi(\mathbf{1}_X)\|.$$

We conclude that $\pi_P \colon \mathcal{S}(X) \to (\mathcal{L}_r(E), \|\cdot\|) \subset (\mathcal{L}(E), \|\cdot\|)$ is bounded with norm $\|P(X)\|$. Since $\mathcal{S}(X)$ is dense in $\mathcal{B}(X)$, π_P extends uniquely to a bounded representation $\pi_P \colon \mathcal{B}(X) \to (\mathcal{L}(E), \|\cdot\|)$. Furthermore, since $\mathcal{S}(X)_+$ is dense in $\mathcal{B}(X)_+$ and E_+ is closed in E, we see that actually $\pi_P(\mathcal{B}(X)_+) \subset \mathcal{L}_r(E)_+$. Hence $\pi_P \colon \mathcal{B}(X) \to (\mathcal{L}_r(E), \|\cdot\|)$ is a positive representation with norm $\|P(X)\|$.

If E is Dedekind complete, then the regular norm is defined on $\mathcal{L}_r(E)$. Since π_P is positive, $\|\cdot\|$ is monotone on $\mathcal{L}_r(E)_+$, and $|\phi| \leq \|\phi\|\mathbf{1}_X$ for $\phi \in \mathcal{B}(X)$, we have $\|\pi_P(\phi)\|_r = \||\pi_P(\phi)|\| \leq \|\pi_P(|\phi|)\| \leq \|\pi_P(\|\phi\|\mathbf{1}_X)\| = \|P(X)\|\|\phi\|$. We see that $\pi_P \colon \mathcal{B}(X) \to (\mathcal{L}_r(E), \|\cdot\|_r)$ is also bounded and that $\|\pi_P\|_r = \|P(X)\|$.

We collect our first results and a few more or less standard properties of π_P in the following theorem. At the moment of writing, it is an open question whether the common subalgebra in part (7)(b) can be a proper subalgebra of the common subalgebra in part (5).

Theorem 4.1. *Let Ω be a σ-algebra of subsets of a set X, let E be a Banach lattice, and let $P \colon \Omega \to \mathcal{L}_r(E)$ be a positive spectral measure.*

(1) *The map $\pi_P \colon \mathcal{S}(X) \to (\mathcal{L}_r(E), \|\cdot\|)$, defined on the simple functions by $\pi_P(\sum_{i=1}^n \alpha_j \chi_{\Delta_i}) = \sum_{i=1}^n \alpha_j P(\Delta_i)$, extends uniquely to a bounded linear map $\pi_P \colon \mathcal{B}(X) \to (\mathcal{L}(E), \|\cdot\|)$. This extension is a positive representation $\pi_P \colon \mathcal{B}(X) \to (\mathcal{L}_r(E), \|\cdot\|)$ with norm $\|\pi_P\| = \|P(X)\| = \|\pi_P(\mathbf{1}_X)\|$.*

(2) *Also, π_P is unital if and only if P is unital.*

(3) *For $\phi \in \mathcal{B}(X)$, $\pi_P(\phi) \in \mathcal{L}_r(E)$ is the unique element of $\mathcal{L}(E)$ such that, for all $x \in E, x' \in E'$,*

$$\left\langle\pi_P(\phi)x, x'\right\rangle = \int_X \phi \, d\mu_{x,x'}. \tag{4.1}$$

(4) *If $\phi \in \mathcal{B}(X)$ and if $(\phi)_{n=1}^\infty \subset \mathcal{B}(X)$ is a bounded sequence in $\mathcal{B}(X)$ such that $\lim_{n\to\infty} \phi_n(\xi) = \phi(\xi)$ for all $\xi \in X$, then $\pi_P(\phi) = \text{WOT-}\lim_n \pi_P(\phi_n)$.*

(5) *The closed subalgebras of $(\mathcal{L}(E), \|\cdot\|)$ generated by $P(\Omega)$, $\pi_P(\mathcal{S}(X))$, and $\pi_P(\mathcal{B}(X))$ are equal.*

(6) *The commutants $P(\Omega)'$, $\pi_P(\mathcal{S}(X))'$, and $\pi_P(\mathcal{B}(X))'$ in $\mathcal{L}(E)$ are equal.*

(7) *If E is Dedekind complete, then*

(a) $\pi_P \colon \mathcal{B}(X) \to (\mathcal{L}_r(E), \| \cdot \|_r)$ *is bounded, and* $\|\pi_P\|_r = \|P(X)\| = \|\pi_P(\mathbf{1}_X)\|$;

(b) *the closed subalgebras of* $(\mathcal{L}_r(E), \| \cdot \|_r)$ *generated by* $P(\Omega)$, $\pi_P(\mathcal{S}(X))$, *and* $\pi_P(\mathcal{B}(X))$ *are equal, and this common subalgebra is contained in the common subalgebra in part (5).*

Proof. For part (3), we note that (4.1) holds by construction if $\phi \in \mathcal{S}(X)$. Since, for fixed $x \in E, x' \in E'$, both sides in (4.1) are bounded linear functionals on $\mathcal{B}(X)$, the general case follows by continuity. The uniqueness statement in part (3) is clear, and part (4) is immediate from an application of the dominated convergence theorem. The remaining statements follow easily from the discussion preceding the theorem.   □

Remark 4.2.

(1) The standard estimate (for the real case; see [10, p. 892]) yields that $\|\pi_P\| \leq 2 \sup_{\Delta \in \Omega} \|P(\Delta)\|$, and combination with Lemma 3.4 then implies that we know a priori that $\|\pi_P\| \leq 2\|P(X)\|$. Part (1) therefore shows that, in the ordered context, one can remove the factor 2 and obtain equality.

(2) If E is Dedekind complete, then the positive map $\pi \colon \mathcal{B}(X) \to (\mathcal{L}_r(E), \| \cdot \|_r)$ is automatically bounded (and then so is $\pi \colon \mathcal{B}(X) \to (\mathcal{L}_r(E), \| \cdot \|)$). The point in part (7)(a) is the value of $\|\pi_P\|_r$.

(3) If E is Dedekind complete, then there is a seemingly alternative way of obtaining a positive representation of $\mathcal{B}(X)$ on E. Indeed, one can also view π_P on $\mathcal{S}(X)$ as a bounded map $\pi_P \colon \mathcal{S}(X) \to (\mathcal{L}_r(E), \| \cdot \|_r)$, where the codomain is now likewise a Banach space. Extending by continuity, we obtain a bounded positive representation $\pi_P^r \colon \mathcal{S}(X) \to (\mathcal{L}_r(E), \| \cdot \|_r)$. Since the inclusion map $(\mathcal{L}_r(E), \| \cdot \|_r) \hookrightarrow (\mathcal{L}(E), \| \cdot \|)$ is bounded, a moment's thought shows that actually $\pi_P = \pi_P^r$. Hence there is no ambiguity.

The first statement in the next result is specific for the ordered context. We do not assume that E is Dedekind complete, and hence the existence of $\sup_n \pi_P(\phi_n)$ is not automatic. By [1, Lemma 1.24], it implies that π_P is a σ-order continuous map between the ordered vector spaces $\mathcal{B}(X)$ and $\mathcal{L}_r(E)$.

Proposition 4.3. *Let Ω be a σ-algebra of subsets of a set X, let E be a Banach lattice, and let $P \colon \Omega \to \mathcal{L}_r(E)$ be a positive spectral measure. If $\phi \in \mathcal{B}(X)$ and $(\phi)_{n=1}^\infty \subset \mathcal{B}(X)$ is a bounded sequence in $\mathcal{B}(X)$ such that $\phi_n \uparrow \phi$ in $\mathcal{B}(X)$, then $\pi_P(\phi_n) \uparrow \pi_P(\phi)$ in $\mathcal{L}_r(E)$, and $\pi_P(\phi) = \text{WOT-}\lim_n \pi_P(\phi_n)$.*

Proof. Clearly $\pi_P(\phi) \geq \pi_P(\phi_n)$ for all n. Suppose $T \in \mathcal{L}_r(E)$ and $T \geq \pi_P(\phi_n)$ for all n. Then, for $x \in E_+, x' \in E'_+$, we have $\langle Tx, x' \rangle \geq \langle \pi_P(\phi_n)x, x' \rangle = \int_X \phi_n \, d\mu_{x,x'}$ for all n. The dominated convergence theorem yields $\langle Tx, x' \rangle \geq \int_X \phi \, d\mu_{x,x'} = \langle \pi_P(\phi)x, x' \rangle$. Hence $T \geq \pi_P(\phi)$. We have shown that $\pi(\phi_n) \uparrow \pi(\phi)$; the second statement follows from part (4) of Theorem 4.1.   □

Since $P(\Delta) = \pi_P(\chi_\Delta)$ $(\Delta \in \Omega)$, the map $P \mapsto \pi_P$ is injective. This validates the choice of the definite article in the following definition.

Definition 4.4. Let Ω be a σ-algebra of subsets of a set X, let E be a Banach lattice, let $P \colon \Omega \to \mathcal{L}_{\mathrm{r}}(E)$ be a positive spectral measure, and let $\pi_P \colon \mathcal{B}(X) \to \mathcal{L}_{\mathrm{r}}(E)$ be the positive representation of $\mathcal{B}(X)$ on E as constructed above. Then we shall say that *π_P is generated by P* and that *P is the generating positive spectral measure of π_P on Ω*.

We will now concentrate on the question as to which positive representations of $\mathcal{B}(X)$ on Banach lattices have a generating positive spectral measure on Ω. This is not always the case (see [7]). A positive representation that has a generating positive spectral measure is bounded according to Theorem 4.1, but this is not necessarily a distinguishing feature, as is shown by the next result on automatic boundedness.

Proposition 4.5. *Let Ω be a σ-algebra of subsets of a set X, let E be a Banach lattice, and let $\pi \colon \mathcal{B}(X) \to \mathcal{L}_{\mathrm{r}}(E)$ be a positive representation. Then*

(1) *$\pi \colon \mathcal{B}(X) \to (\mathcal{L}_{\mathrm{r}}(E), \| \cdot \|)$ is bounded, and $\|\pi\| = \|\pi(\mathbf{1}_X)\|$;*
(2) *if E is Dedekind complete, then $\pi \colon \mathcal{B}(X) \to (\mathcal{L}_{\mathrm{r}}(E), \| \cdot \|_{\mathrm{r}})$ is bounded, and $\|\pi\| = \|\pi\|_{\mathrm{r}} = \|\pi(\mathbf{1}_X)\|$.*

Proof. As to (1), let $\phi \in \mathcal{B}(X)_+$. Since $0 \le \phi \le \|\phi\|\mathbf{1}_X$ and π is positive, we have $0 \le \pi(\phi) \le \|\phi\|\pi(\mathbf{1}_X)$. Hence $\|\pi(\phi)\| \le \|\pi(\mathbf{1}_X)\|\|\phi\|$. For general $\phi = \phi_+ - \phi_- \in \mathcal{B}(X)$, this implies that $\|\pi(\phi)\| \le \|\pi(\mathbf{1}_X)\|(\|\phi_+\| + \|\phi_-\|) \le 2\|\pi(\mathbf{1}_X)\|\|\phi\|$. In particular, $\pi \colon \mathcal{B}(X) \to (\mathcal{L}_{\mathrm{r}}(E), \| \cdot \|)$ is bounded. For the statement concerning $\|\pi\|$, it is sufficient to show that $\|\pi(\phi)\| \le \|\pi(\mathbf{1}_X)\|\|\phi\|$ for all ϕ in the dense subspace $\mathcal{S}(X)$ of $\mathcal{B}(X)$. As to this, we first note that the map $\Delta \mapsto \pi(\chi_\Delta)$ is a positive operator-valued finitely additive measure on Ω. Proceeding as in the proof of Lemma 3.6, one then sees that $\sum_{i=1}^n |\langle \pi(\chi_{\Delta_i})x, x' \rangle| \le \langle \pi(\mathbf{1}_X)|x|, |x'| \rangle \le \|\pi(\mathbf{1}_X)\|\|x\|\|x'\|$ for all measurable disjoint partitions $X = \bigcup_{i=1}^n \Delta_i$ of X and all $x \in E, x' \in E'$. Taking the Δ_i in $\phi = \sum_{i=1}^n \alpha_i \chi_{\Delta_i} \in \mathcal{B}(X)$ to be a measurable disjoint partition of X, this implies, as in the discussion preceding Theorem 4.1, that

$$
\begin{aligned}
\|\pi(\phi)\| &= \sup\left\{ \left| \left\langle \sum_{i=1}^n \alpha_i \pi(\chi_{\Delta_i})x, x' \right\rangle \right| : x \in E, \|x\| \le 1, x' \in E', \|x'\| \le 1 \right\} \\
&\le \sup\left\{ \sum_{i=1}^n |\alpha_i| \left| \left\langle \pi(\chi_{\Delta_i})x, x' \right\rangle \right| : x \in E, \|x\| \le 1, x' \in E', \|x'\| \le 1 \right\} \\
&\le \|\phi\| \sup\left\{ \sum_{i=1}^n \left| \left\langle \pi(\chi_{\Delta_i})x, x' \right\rangle \right| : x \in E, \|x\| \le 1, x' \in E', \|x'\| \le 1 \right\} \\
&\le \|\phi\|\|\pi(\mathbf{1}_X)\|.
\end{aligned}
$$

For part (2), let $\phi \in \mathcal{B}(X)$. Since $|\phi| \le \|\phi\|\mathbf{1}_X$, we have $\pi(|\phi|) \le \|\phi\|\pi(\mathbf{1}_X)$. Therefore $0 \le |\pi(\phi)| \le \pi(|\phi|) \le \|\phi\|\pi(\mathbf{1}_X)$, implying that $\|\pi(\phi)\|_{\mathrm{r}} = \||\pi(\phi)|\| \le \|\|\phi\|\pi(\mathbf{1}_X)\| = \|\pi(\mathbf{1}_X)\|\|\phi\|$. Hence $\pi \colon \mathcal{B}(X) \to (\mathcal{L}_{\mathrm{r}}(E), \| \cdot \|_{\mathrm{r}})$ is bounded (which also follows from the automatic continuity of positive maps between Banach lattices), and $\|\pi\|_{\mathrm{r}} = \|\pi(\mathbf{1}_X)\|$. $\square$

Remark 4.6. If E is Dedekind complete, then it is well known that the operator norm and the regular norm coincide on the center $Z(E)$ of E. If $\pi(\mathcal{B}(X)) \subset Z(E)$, the equality of $\|\pi\|$ and $\|\pi\|_{\mathrm{r}}$, as asserted in Proposition 4.5, is then a priori clear. For general (not necessarily central) positive representations of $\mathcal{B}(X)$, this equality may seem somewhat surprising, but also in this case more can be said (see [7]).

With the automatic boundedness available from Proposition 4.5, we can now give a description of the positive representations of $\mathcal{B}(X)$ that have a generating positive spectral measure on Ω.

Proposition 4.7. *Let Ω be a σ-algebra of subsets of a set X, let E be a Banach lattice, and let $\pi\colon \mathcal{B}(X) \to \mathcal{L}_{\mathrm{r}}(E)$ be a positive representation. Then the following are equivalent:*

    (1) *π has a generating positive spectral measure on Ω;*
    (2) *if $(\Delta_n)_{n=1}^{\infty}$ are pairwise disjoint elements of Ω, and $x \in E_+$, then*

$$\pi(\chi_{\bigcup_{n=1}^{\infty} \Delta_n})x = \sum_{n=1}^{\infty} \pi(\chi_{\Delta_n})x,$$

       where the series converges in the norm topology;
    (3) *if $(\Delta_n)_{n=1}^{\infty}$ are pairwise disjoint elements of Ω, and $x \in E_+, x' \in E'_+$, then*

$$\left\langle \pi(\chi_{\bigcup_{n=1}^{\infty} \Delta_n})x, x' \right\rangle = \sum_{n=1}^{\infty} \left\langle \pi(\chi_{\Delta_n})x, x' \right\rangle.$$

In that case, the generating positive spectral measure P on Ω of π is given by $P(\Delta) = \pi(\chi_{\Delta})$ $(\Delta \in \Omega)$.

Proof. Since the only possible generating positive spectral measure P for π on Ω must be given by $P(\Delta) = \pi(\chi_{\Delta})$ $(\Delta \in \Omega)$, it is clear that (1) implies (2). Clearly (2) implies (3). Assuming (3), we first observe that the equality in (3) then holds for all $x \in E, x' \in E'$. We define $P(\Delta) = \pi(\chi_{\Delta})$ $(\Delta \in \Omega)$. Then $P\colon \Omega \to \mathcal{L}_{\mathrm{r}}(E)$ satisfies (1), (2), (3), and (4') in Definition 3.1 and Remark 3.2, and hence is a positive spectral measure on Ω. It is clear that the positive representation π_P of $\mathcal{B}(X)$ on E that is generated by P agrees with π on $\mathcal{S}(X)$. Since both are bounded according to Proposition 4.5, $\pi_P = \pi$. $\qquad\qquad\square$

For a fairly large practical class of lattices, the criterion for the existence of a generating positive spectral measure is particularly easy, and completely order-theoretical.

Theorem 4.8. *Let Ω be a σ-algebra of subsets of a set X, let E be a σ-Dedekind complete Banach lattice, and let $\pi\colon \Omega \to \mathcal{L}_{\mathrm{r}}(E)$ be a positive representation. If E' consists of σ-order continuous linear functionals only (equivalently: if E has σ-order continuous norm), then the following are equivalent:*

    (1) *π has a generating positive spectral measure on Ω,*
    (2) *π is σ-order continuous.*

Proof. We had already observed preceding Proposition 4.3 that part (1) implies part (2), even without any further assumptions on E. For the converse implication, we verify the condition in part (3) of Proposition 4.7. In the pertinent notation, we let $\psi = \chi_{\bigcup_{n=1}^{\infty} \Delta_n}$ and $\psi_N = \sum_{n=1}^{N} \chi_{\Delta_n}$ $(N \geq 1)$. Then $\psi_N \uparrow \psi$ in $\mathcal{B}(X)$, so $\pi(\psi_N) \uparrow \pi(\psi)$ in $\mathcal{L}_r(E)$ by the σ-order continuity of π. By a straightforward modification of the proof of [5, Theorem 1.18], the σ-Dedekind completeness of E implies that $\pi(\psi_N)x \uparrow \pi(\psi)x$ for all $x \in E_+$. By the assumption on E' this implies that $\langle \pi(\psi_N)x, x' \rangle \uparrow \langle \pi(\psi)x, x' \rangle$ for all $x \in E_+, x' \in E'_+$. That is, the condition in part (3) of Proposition 4.7 is satisfied. $\square$

Remark 4.9.

(1) Theorem 4.8 applies, in particular, to all Dedekind complete Banach lattices with order continuous norm. A class of Banach lattice with a σ-order continuous norm that is not order continuous can be found in [21, Example 7, p. 46].
(2) The equivalence involving the σ-order continuity can be found in [22, p. 336].

5. Positive $C_0(X)$-representations generated by positive spectral measures

In this section, X is a locally compact Hausdorff space. We will investigate the relation between positive representations of $C_0(X)$ on a Banach lattice E and $\mathcal{L}_r(E)$-valued positive spectral measures on the Borel σ-algebra Ω of X. One of our main results here, Theorems 5.6, is concerned with the existence of a generating regular positive spectral measure for such a representation. In most of the other results, such as Theorem 5.12, the existence of a generating (regular) positive spectral measure is merely assumed, and the relation between the representation and the spectral measure is studied.

We start with the following result on automatic boundedness of positive representations of $C_0(X)$, in the same vein as Proposition 4.5.

Proposition 5.1. *Let X be a locally compact Hausdorff space, let E be a Banach lattice, and let $\pi \colon C_0(X) \to \mathcal{L}_r(E)$ be a positive representation.*

(1) *If X is compact, and E is not necessarily Dedekind complete, then*
  (a) *$\|\pi(\phi)\| \leq \|\pi(\mathbf{1}_X)\|(\|\phi_+\| + \|\phi_-\|)$ $(\phi \in C_0(X))$;*
  (b) *$\pi \colon C_0(X) \to (\mathcal{L}_r(E), \| \cdot \|)$ is bounded, and $\|\pi\| \leq 2\|\pi(\mathbf{1}_X)\|$.*
(2) *If X is not necessarily compact, and E is Dedekind complete, then the maps $\pi \colon C_0(X) \to (\mathcal{L}_r(E), \| \cdot \|)$ and $\pi \colon C_0(X) \to (\mathcal{L}_r(E), \| \cdot \|_r)$ are both bounded, and $\|\pi\| \leq \|\pi\|_r$.*
(3) *If X is compact, and E is Dedekind complete, then the maps $\pi \colon C_0(X) \to (\mathcal{L}_r(E), \| \cdot \|)$ and $\pi \colon C_0(X) \to (\mathcal{L}_r(E), \| \cdot \|_r)$ are both bounded, and $\|\pi\| = \|\pi\|_r = \|\pi(\mathbf{1}_X)\|$.*

Proof. Part (1) follows as in the beginning of the proof of part (1) of Proposition 4.5.

As to part (2), if E is Dedekind complete, then $(\mathcal{L}_r(E), \| \cdot \|_r)$ is a Banach lattice. Hence the positive map $\pi \colon C_0(X) \to (\mathcal{L}_r(E), \| \cdot \|_r)$ is automatically

bounded. Using the contractivity of the inclusion map $(\mathcal{L}_{\mathrm{r}}(E), \| \cdot \|_{\mathrm{r}}) \hookrightarrow (\mathcal{L}_{\mathrm{r}}(E), \| \cdot \|)$ completes the proof of part (2).

As to part (3), it follows as in the proof of part (2) of Proposition 4.5 that $\pi\colon C_0(X) \to (\mathcal{L}_{\mathrm{r}}(E), \| \cdot \|_{\mathrm{r}})$ is bounded and that $\|\pi\|_{\mathrm{r}} = \|\pi(\mathbf{1}_X)\|$. The contractivity of the inclusion map $(\mathcal{L}_{\mathrm{r}}(E), \| \cdot \|_{\mathrm{r}}) \hookrightarrow (\mathcal{L}_{\mathrm{r}}(E), \| \cdot \|)$ then implies that $\|\pi\| = \|\pi(\mathbf{1}_X)\|$. $\qquad\square$

At the time of writing we have no information for the case where X is not compact and E is not Dedekind complete, unless we assume that π has a generating positive spectral measure (see Proposition 5.10).

The following definition is the usual one.

Definition 5.2. Let X be a locally compact Hausdorff space with Borel σ-algebra Ω, let E be a Banach lattice, and let $P\colon \Omega \to \mathcal{L}_{\mathrm{r}}(E)$ be a positive spectral measure. Then P is *regular* if the finite signed measure $\mu_{x,x'}\colon \Omega \to \mathbb{R}$, defined by $\mu_{x,x'}(\Delta) = \langle P(\Delta)x, x'\rangle$ ($\Delta \in \Omega$), is a regular finite signed Borel measure for all $x \in E, x' \in E'$.

Interestingly enough, a regular positive spectral measure is also inner and outer regular on all elements of Ω in a natural sense that is meaningful only in an ordered context.

Proposition 5.3. *Let X be a locally compact Hausdorff space with Borel σ-algebra Ω, let E be a Banach lattice, and let $P\colon \Omega \to \mathcal{L}_{\mathrm{r}}(E)$ be a regular positive spectral measure. Then, for all $\Delta \in \Omega$,*

(1) *$P(\Delta) = \inf\{P(V) : V \text{ open and } \Delta \subset V\}$ in $\mathcal{L}_{\mathrm{r}}(E)$;*
(2) *$P(\Delta) = \sup\{P(K) : K \text{ compact and } K \subset \Delta\}$ in $\mathcal{L}_{\mathrm{r}}(E)$.*

We do not assume that E is Dedekind complete, and hence the existence of the infimum and supremum is not automatic.

Proof. If $x \in E_+, x' \in E'_+$, then the finite measure $\mu_{x,x'}$ is a regular Borel measure by assumption. Part (2) of Theorem 2.3 shows that it is not only outer regular on all elements of Ω, but also inner regular on all elements of Ω. An appeal to Lemma 3.7 then finishes the proof. $\qquad\square$

In view of the results in Section 4, the following usual definition is natural.

Definition 5.4. Let X be a locally compact Hausdorff space with Borel σ-algebra Ω, let E be a Banach lattice, and let $\pi\colon C_0(X) \to \mathcal{L}_{\mathrm{r}}(E)$ be a positive representation of $C_0(X)$ on E. If $P\colon \Omega \to \mathcal{L}_{\mathrm{r}}(E)$ is a positive spectral measure on Ω, then Section 4 furnishes the positive representation $\pi_P\colon \mathcal{B}(X) \to \mathcal{L}_{\mathrm{r}}(E)$ of $\mathcal{B}(X)$ on E that is generated by P. We say that P *generates* π if π is the restriction of π_P to $C_0(X)$. If P is a regular positive spectral measure on Ω generating π, we say that π *has a generating regular positive spectral measure on Ω*.

Remark 5.5.

(1) If π has a generating regular positive spectral measure on Ω, then it is unique. This is immediate from (4.1) and the uniqueness statement in the Riesz representation theorem.

(2) For the sake of completeness, we note that every Borel measure is automatically both outer and inner regular on all elements of Ω if every open subset of X is σ-compact (i.e., if it is the countable union of compact subsets of X; see [12, Theorem 7.8]). For such spaces (in particular, for all second countable spaces), every positive spectral measure on Ω is therefore automatically regular.

We will now establish an existence result for generating regular positive spectral measures.

Recall that a Banach lattice is a KB-space if every nonnegative increasing norm-bounded sequence is norm-convergent. Reflexive Banach lattices and AL-spaces are KB-spaces (see [5, p. 232]). A KB-space has order continuous norm (see [14, Theorem 2.4.2]), and hence is Dedekind complete, and a Banach lattice E is a KB-space if and only if E is a band of E'' (see [5, Theorem 4.60]). The latter property is what makes our proof of the following theorem work.

Theorem 5.6. *Let X be a locally compact Hausdorff space with Borel σ-algebra Ω, let E be a KB-space, and let $\pi\colon C_0(X) \to \mathcal{L}_\mathrm{r}(E)$ be a positive representation. Then π has a unique generating regular positive spectral measure P on Ω.*

Proof. The uniqueness of a generating regular positive spectral measure was already observed in the first part of Remark 5.5.

For its existence, we combine Theorem 2.6 with ideas employed in the literature for unital representations of commutative C^*-algebras on Hilbert spaces (see [6, Theorem IX.1.4], [18, Theorem 12.22]), and for bounded unital representations of $C_0(X)$ (where X is compact) on Banach spaces [17, Theorem III.3]. The strategy is, as usual, to construct a (positive) representation $\tilde{\pi}\colon \mathcal{B}(X) \to \mathcal{L}_\mathrm{r}(E)$ that extends π, and then show that $\tilde{\pi}$ has a generating regular (positive) spectral measure P on Ω, so that one can actually write $\tilde{\pi} = \pi_P$.

We start with the construction of $\tilde{\pi}$.

Since we had already observed that the KB-space E is Dedekind complete, part (2) of Proposition 5.1 implies that $\pi\colon C_0(X) \to (\mathcal{L}_\mathrm{r}(E), \|\cdot\|)$ is bounded.

Let $x \in E, x' \in E'$, and consider the linear functional $\phi \mapsto \langle \pi(\phi)x, x'\rangle$ on $C_0(X)$. We have

$$\left|\langle \pi(\phi)x, x'\rangle\right| \le \|\pi\|\|\phi\|\|x\|\|x'\| \quad (\phi \in C_0(X), x \in E, x' \in E').$$

Consequently, this functional is bounded and has norm at most $\|\pi\|\|x\|\|x'\|$. The Riesz representation theorem furnishes a regular finite signed Borel measure $\mu_{x,x'}$ such that

$$\langle \pi(\phi)x, x'\rangle = \int_X \phi \, d\mu_{x,x'} \quad (\phi \in C_0(X), x \in E, x' \in E'). \tag{5.1}$$

Moreover, $\|\mu_{x,x'}\| \le \|\pi\|\|x\|\|x'\|$, and if $x \in E_+, x' \in E'_+$, then $\mu_{x,x'} \ge 0$ as a consequence of the positivity of π. As a consequence of the uniqueness statement in the Riesz representation theorem, the map $(x, x') \mapsto \mu_{x,x'}$ is bilinear. This implies that, for fixed $\phi \in \mathcal{B}(X)$, the form $[\cdot, \cdot]_\phi$ on $E \times E'$, defined by

$$[x, x']_\phi = \int_X \phi \, d\mu_{x,x'} \quad (x \in E, x' \in E'),$$

is also bilinear. Moreover, $|[x, x']_\phi| \le \|\phi\|\|\mu_{x,x'}\| \le \|\phi\|\|\pi\|\|x\|\|x'\|$. Hence $[\cdot, \cdot]_\phi$ is a bounded bilinear form on $E \times E'$, and this implies that there exists a unique operator $\tilde{\pi}(\phi) \in \mathcal{L}(E, E'')$ such that $\langle \tilde{\pi}(\phi)x, x' \rangle = [x, x']_\phi$ for all $x \in E, x' \in E'$. Hence

$$\langle \tilde{\pi}(\phi)x, x' \rangle = \int_X \phi \, d\mu_{x,x'} \quad \left(\phi \in \mathcal{B}(X), x \in E, x' \in E' \right). \tag{5.2}$$

We now use the fact that E is a band of E'' and Theorem 2.6 to see that actually $\tilde{\pi}(\phi)(E) \subset E$ (rather than E'') for all $\phi \in \mathcal{B}(X)$.

To start, since $\mu_{x,x'} \ge 0$ for $x \in E_+, x' \in E'_+$, (5.2) shows that $\tilde{\pi}(\phi)x \ge 0$ if $\phi \ge 0$ and $x \in E_+$. This implies that $\tilde{\pi}(\phi) \in \mathcal{L}_{\mathrm{r}}(E, E'')$ $(\phi \in \mathcal{B}(X))$, and that $\tilde{\pi}\colon \mathcal{B}(X) \to \mathcal{L}_{\mathrm{r}}(E, E'')$ is a (clearly linear) positive map. Furthermore, comparing (5.1) and (5.2), we see that $\tilde{\pi}\colon \mathcal{B}(X) \to \mathcal{L}_{\mathrm{r}}(E, E'')$ extends $\pi\colon C_0(X) \to \mathcal{L}_{\mathrm{r}}(E)$, where we identify $\mathcal{L}_{\mathrm{r}}(E)$ with the canonically corresponding subspace of $\mathcal{L}_{\mathrm{r}}(E, E'')$. We let

$$L = \left\{ \phi \in \mathcal{B}(X) : \tilde{\pi}(\phi)(E) \subset E \right\}.$$

Since $\tilde{\pi}$ extends π, we already know that $C_0(X) \subset L$. We proceed to show that L satisfies the two hypotheses in Theorem 2.6, so that actually $L = \mathcal{B}(X)$. As to the first hypothesis, we need to show that $\tilde{\pi}(\chi_V)(E) \subset E$ for all open subsets V of X. We may assume that $V \ne \emptyset$, and in that case we claim that

$$\tilde{\pi}(\chi_V) = \sup\left\{ \tilde{\pi}(\phi) : \phi \in C_{\mathrm{c}}(X), \operatorname{supp}\phi \subset V, 0 \le \phi \le \mathbf{1}_X \right\}, \tag{5.3}$$

where the right-hand side is the supremum in the Dedekind complete Banach lattice $\mathcal{L}_{\mathrm{r}}(E, E'')$. Since $\tilde{\pi} \ge 0$, it is clear that $\tilde{\pi}(\chi_V) \ge \tilde{\pi}(\phi)$ if $\phi \in C_{\mathrm{c}}(X)$, $\operatorname{supp}\phi \subset V$, and $0 \le \phi \le \mathbf{1}_X$. Suppose that $T \in \mathcal{L}_{\mathrm{r}}(E, E'')$ and that $T \ge \tilde{\pi}(\phi)$ for all $\phi \in C_{\mathrm{c}}(X)$ such that $\operatorname{supp}\phi \subset V$ and $0 \le \phi \le \mathbf{1}_X$. Then from (5.2) we have, for all such ϕ, and all $x \in E_+, x' \in E'_+$,

$$\langle Tx, x' \rangle \ge \langle \tilde{\pi}(\phi)x, x' \rangle$$
$$= \int_X \phi \, d\mu_{x,x'}.$$

Therefore, for all $x \in E_+, x' \in E'_+$,

$$\langle Tx, x' \rangle \ge \sup\left\{ \int_X \phi \, d\mu_{x,x'} : \phi \in C_{\mathrm{c}}(X), \operatorname{supp}\phi \subset V, 0 \le \phi \le \mathbf{1}_X \right\}.$$

Now (2.1) shows that the right-hand side in this equation equals $\mu_{x,x'}(V)$. We conclude that, for $x \in E_+, x' \in E'_+$,

$$\langle Tx, x' \rangle \ge \mu_{x,x'}(V)$$
$$= \int_X \chi_V \, d\mu_{x,x'}$$
$$= \langle \tilde{\pi}(\chi_V)x, x' \rangle.$$

Hence $T \ge \tilde{\pi}(\chi_V)$, and our claim in (5.3) has been established.

Since $\{\phi : \phi \in C_{\mathrm{c}}(X), \operatorname{supp}\phi \subset V, 0 \le \phi \le \mathbf{1}_X\}$ is directed upward and $\tilde{\pi}$ is positive, $\{\tilde{\pi}(\phi) : \phi \in C_{\mathrm{c}}(X), \operatorname{supp}\phi \subset V, 0 \le \phi \le \mathbf{1}_X\}$ is a subset of $\mathcal{L}_{\mathrm{r}}(E, E'')$

that is directed upward. Hence its supremum $\tilde{\pi}(\chi_V)$ can be determined pointwise on the positive cone of E (see [5, Theorem 1.19]), and we conclude that

$$\tilde{\pi}(\chi_V)x = \sup\{\tilde{\pi}(\phi)x : \phi \in C_c(X), \operatorname{supp}\phi \subset V, 0 \leq \phi \leq \mathbf{1}_X\} \quad (x \in E_+),$$

where the supremum is in E''. However, since we had already observed that $C_0(X) \subset L$, we know that $\{\tilde{\pi}(\phi)x : \phi \in C_c(X), \operatorname{supp}\phi \subset V, 0 \leq \phi \leq \mathbf{1}_X\} \subset E$ ($x \in E_+$). Since E is a band of E'', the supremum of this set in E'' is actually in E. We conclude that $\tilde{\pi}(\chi_V)x \in E$ ($x \in E_+$), and it follows that $\chi_V \in L$, as required in the first hypothesis in Theorem 2.6.

As to the second hypothesis, suppose that ϕ_n is a sequence of functions in L and that ϕ is a bounded function on X such that $0 \leq \phi_n(\xi) \uparrow \phi(\xi)$ for all $\xi \in X$. Then certainly $\phi \in \mathcal{B}(X)$, and we are left with showing that $\tilde{\pi}(\phi)(E) \subset E$. The proof of this bears some similarity to the above proof that $\tilde{\pi}(\chi_V)(E) \in L$ for every open subset V of X. We claim that $\tilde{\pi}(\phi_n) \uparrow \tilde{\pi}(\phi)$ in $\mathcal{L}_r(E, E'')$. First, it is clear from the positivity of $\tilde{\pi}$ that $\tilde{\pi}(\phi_n) \uparrow$ and that $\tilde{\pi}(\phi) \geq \tilde{\pi}(\phi_n)$ for all n. Second, if $T \in \mathcal{L}_r(E, E'')$ and $T \geq \tilde{\pi}(\phi_n)$ for all n, then, for $x \in E_+, x' \in E'_+$, and $n \in \mathbb{N}$,

$$\begin{aligned}
\langle Tx, x' \rangle &\geq \langle \tilde{\pi}(\phi_n)x, x' \rangle \\
&= \int_X \phi_n \, d\mu_{x,x'}.
\end{aligned} \tag{5.4}$$

Applying the monotone convergence theorem to (5.4) yields $\langle Tx, x' \rangle \geq \langle \tilde{\pi}(\phi)x, x' \rangle$ ($x \in E_+, x' \in E'_+$); hence $T \geq \tilde{\pi}(\phi)$. This establishes our claim. A pointwise argument on E_+ as for $\tilde{\pi}(\chi_V)$ now implies that $\phi \in L$, as required in the second hypothesis in Theorem 2.6.

Theorem 2.6 now shows that $\mathcal{B}(X) \subset L$, and we have finally established that $\tilde{\pi}(\phi)(E) \subset E$ for all $\phi \in \mathcal{B}(X)$. Hence we can view $\tilde{\pi}$ as a positive linear map $\tilde{\pi} : \mathcal{B}(X) \to \mathcal{L}(E)$ with codomain $\mathcal{L}_r(E)$ rather than $\mathcal{L}_r(E, E'')$.

We will now proceed to show that $\tilde{\pi}$ is a positive representation of $\mathcal{B}(X)$ and that it has a generating regular positive spectral measure on Ω.

For the multiplicativity of $\tilde{\pi}$, we argue as follows. Let $\phi, \psi \in C_0(X)$, and let $x \in E, x' \in E'$. Using (5.2) twice, we have

$$\begin{aligned}
\int_X \phi\psi \, d\mu_{x,x'} &= \langle \pi(\phi\psi)x, x' \rangle \\
&= \langle \pi(\phi)\pi(\psi)x, x' \rangle \\
&= \int_X \phi \, d\mu_{\pi(\psi)x,x'}.
\end{aligned}$$

Lemma 2.5 then shows that

$$\int_X \phi\psi \, d\mu_{x,x'} = \int_X \phi \, d\mu_{\pi(\psi)x,x'},$$

for all $\phi \in \mathcal{B}(X), \psi \in C_0(X)$, and $x \in E, x' \in E'$. This implies, using (5.2) in the second and fourth step, that

$$
\begin{aligned}
\int_X \phi\psi \, d\mu_{x,x'} &= \int_X \phi \, d\mu_{\pi(\psi)x,x'} \\
&= \langle \tilde{\pi}(\phi)\pi(\psi)x, x' \rangle \\
&= \langle \pi(\psi)x, \tilde{\pi}(\phi)'x' \rangle \\
&= \int_X \psi \, d\mu_{x,\tilde{\pi}(\phi)'x'},
\end{aligned}
$$

for all $\phi \in \mathcal{B}(X), \psi \in C_0(X)$, and $x \in E, x' \in E'$. Lemma 2.5 now shows that

$$
\int_X \phi\psi \, d\mu_{x,x'} = \int_X \psi \, d\mu_{x,\tilde{\pi}(\phi)'x'},
$$

for all $\phi, \psi \in \mathcal{B}(X)$ and $x \in E, x' \in E'$. Using (5.2) in the second step, we then see that

$$
\begin{aligned}
\int_X \phi\psi \, d\mu_{x,x'} &= \int_X \psi \, d\mu_{x,\tilde{\pi}(\phi)'x'} \\
&= \langle \tilde{\pi}(\psi)x, \tilde{\pi}(\phi)'x' \rangle \\
&= \langle \tilde{\pi}(\phi)\tilde{\pi}(\psi)x, x' \rangle,
\end{aligned}
\tag{5.5}
$$

for all $\phi, \psi \in \mathcal{B}(X)$ and $x \in E, x' \in E'$. On the other hand, (5.2) shows that

$$
\int_X \phi\psi \, d\mu_{x,x'} = \langle \tilde{\pi}(\phi\psi)x, x' \rangle,
$$

for all $\phi, \psi \in \mathcal{B}(X)$ and $x \in E, x' \in E'$. Comparing this with (5.5), we conclude that $\tilde{\pi}(\phi\psi) = \tilde{\pi}(\phi)\tilde{\pi}(\psi)$ for all $\phi, \psi \in \mathcal{B}(X)$. Hence $\tilde{\pi} \colon \mathcal{B}(X) \to \mathcal{L}_r(E)$ is a positive representation of $\mathcal{B}(X)$ on E.

To show that $\tilde{\pi}$ has a generating positive spectral measure on Ω, we will verify condition (3) in Proposition 4.7. Let $x \in E_+, x' \in E'_+$, and let $(\Delta_n)_{n=1}^\infty$ be pairwise disjoint elements of Ω. We must show that $\langle \tilde{\pi}(\chi_{\bigcup_{n=1}^\infty \Delta_n})x, x' \rangle = \sum_{n=1}^\infty \langle \tilde{\pi}(\chi_{\Delta_n})x, x' \rangle$. Using the monotone convergence theorem in the second step, we see that

$$
\begin{aligned}
\langle \tilde{\pi}(\chi_{\bigcup_{n=1}^\infty \Delta_n})x, x' \rangle &= \int_X \chi_{\bigcup_{n=1}^\infty \Delta_n} \, d\mu_{x,x'} \\
&= \sum_{n=1}^\infty \int_X \chi_{\Delta_n} \, d\mu_{x,x'} \\
&= \sum_{n=1}^\infty \langle \tilde{\pi}(\chi_{\Delta_n})x, x' \rangle.
\end{aligned}
$$

Hence we conclude from Proposition 4.7 that $\tilde{\pi}$ has a generating positive spectral P measure on Ω that is given by $P(\Delta) = \tilde{\pi}(\chi_\Delta)$. In order to conclude that P is

regular, we consider its associated signed measures, denoted temporarily by $\mu_{x,x'}^P$ ($x \in E, x' \in E'$). For $\Delta \in \Omega$, we have, using (5.2) in the third step,

$$\begin{aligned}
\mu_{x,x'}^P(\Delta) &= \langle P(\Delta)x, x' \rangle \\
&= \langle \tilde{\pi}(\chi_\Delta)x, x' \rangle \\
&= \int_X \chi_\Delta \, d\mu_{x,x'} \\
&= \mu_{x,x'}(\Delta).
\end{aligned}$$

Hence $\mu_{x,x'}^P = \mu_{x,x'}$ ($x \in E, x' \in E'$), which is known to be regular. $\square$

Remark 5.7. The above proof of Theorem 5.6 makes it clear what is the essential feature of a KB-space E in this context: it is a band of E''. Since we are concerned with positive representations, such an order-theoretical property and its ensuing role in the proof seem natural. It should be noted, however, that it is also possible to obtain the existence part in Theorem 5.6 using various Banach space characterizations of KB-spaces, along the following lines.

We first note that, by part (2) of Proposition 5.1, the Dedekind completeness of the KB-space E implies that $\pi \colon C_0(X) \to (\mathcal{L}(E), \|\cdot\|)$ is bounded. Even if X is compact, so that $C_0(X)$ has a unit, we can still consider the augmented algebra $C_0(X)_1 = \mathbb{R} \times C_0(X)$ of $C_0(X)$ (which is the usual unitization of $C_0(X)$ if X is not compact), with norm $\|(\lambda, \phi)\| = |\lambda| + \|\phi\|$ ($\lambda \in \mathbb{R}$, $\phi \in C_0(X)$). Then the representation $\pi_\infty \colon C_0(X)_1 \to (\mathcal{L}(E), \|\cdot\|)$, defined by $\pi_\infty((\lambda, \phi)) = \lambda \, \mathbf{id}_E + \pi(\phi)$ ($\lambda \in \mathbb{R}, \phi \in C_0(X)$), is a unital representation of $C_0(X)_1$ on E that is also bounded.

Next, also if X is already compact, we let $X_\infty = X \cup \{\infty\}$ be the one-point compactification of X. The algebras $C_0(X_\infty)$ and $C_0(X)_1$ are canonically isomorphic as abstract algebras, and although the pertinent isomorphism is not necessarily isometric, it is still a linear homeomorphism. Hence π_∞ can be viewed as a bounded unital representation of $C_0(X_\infty)$ on E that extends π.

There are now various ways to proceed:

(1) A Banach lattice E is a KB-space if and only if it does not contain a subspace linearly homeomorphic to c_0 (see [21, Theorem 7.1]). For Banach spaces that do not contain such a subspace, [17, Theorem III.4] provides a generating regular spectral measure for bounded unital representations of $C_0(X)$-spaces (for compact X) on them. In particular, this applies to the bounded unital representation π_∞ of $C_0(X_\infty)$ on E.

(2) A Banach lattice E is a KB-space if and only if it is weakly sequentially complete (see [21, Theorem 7.1]). For weakly sequentially complete Banach spaces, [11, Theorem XVII.2.5] provides a generating regular spectral measure for bounded unital representations of $C_0(X)$-spaces (for compact X) on them. Again, this applies, in particular, to the bounded unital representation π_∞ of $C_0(X_\infty)$ on E.

(3) Since E does not contain a subspace linearly homeomorphic to c_0, every continuous linear map from $C_0(X)$ (for compact X) into E is weakly compact (see [17, Theorem I.14]). In particular, for all $x \in E$, the map

$f \mapsto \pi_\infty(f)x$ $(f \in C_0(X_\infty))$ is weakly compact. Hence [15, Theorem 3] applies, and provides a generating spectral measure for π_∞.

Once we have concluded that there exists a regular spectral measure P_∞ on the Borel σ-algebra Ω_∞ of X_∞ such that

$$\langle \pi_\infty(\phi)x, x' \rangle = \int_{X_\infty} \phi \, d\mu_{x,x'}^{P_\infty} \quad \left(\phi \in C_0(X_\infty), x \in E, x' \in E'\right), \qquad (5.6)$$

where $\mu_{x,x'}^{P_\infty}(\Delta) = \langle P_\infty(\Delta)x, x' \rangle$ $(\Delta \in \Omega_\infty, x \in E, x' \in E')$, we define $P \colon \Omega \to \mathcal{L}_r(E)$ by $P(\Delta) = P_\infty(\Delta)$ $(\Delta \in \Omega \subset \Omega_\infty)$. It is routine to check that P is a regular spectral measure on Ω, and we let $\mu_{x,x'}$ $(x \in E, x' \in E')$ denote the usual associated regular finite signed Borel measures on Ω. If $\phi \in C_0(X)$, then $\phi(\infty) = 0$, so (5.6) implies that

$$\langle \pi(\phi)x, x' \rangle = \int_X \phi \, d\mu_{x,x'} \quad \left(\phi \in C_0(X), x \in E, x' \in E'\right). \qquad (5.7)$$

Since π is positive, the order statement in the Riesz representation theorem implies that all measures $\mu_{x,x'}$ $(x \in E_+, x' \in E'_+)$ are positive, which shows that P takes its values in the positive projections on E. Comparison of (5.7) with (4.1) yields that the positive representation $\pi_P \colon \mathcal{B}(X) \to \mathcal{L}_r(E)$ that is generated by P restricts to π on $C_0(X)$, as required. This concludes the alternative Banach space proof of Theorem 5.6.

The reader may verify that each of the three aforementioned existence results for generating spectral measures uses a substantial amount of theory of Boolean algebras of projections, as well as general Banach space theory. Since, in addition, these existence results can only be applied once the pertinent Banach space property of a KB-space has been established, we feel that our proof of the existence part in Theorem 5.6, which exploits only order-theoretical properties, is not only more natural in this context, but also considerably simpler than the above alternatives.

Remark 5.8. If X is compact, then additional existence results for regular positive spectral measures generating so-called *R-bounded unital positive representations* of $C_0(X)$ can be obtained using [16, Proposition 2.17].

After these remarks regarding Theorem 5.6, we continue with a consequence thereof that is clear.

Corollary 5.9. *Let X be a locally compact Hausdorff space with Borel σ-algebra Ω, and let E be a KB-space. Then the map $P \to \pi_P \upharpoonright_{C_0(X)}$, sending an $\mathcal{L}_r(E)$-valued positive spectral measure on Ω to the restriction of $\pi_P \colon \mathcal{B}(X) \to \mathcal{L}_r(E)$ to $C_0(X) \subset \mathcal{B}(X)$, restricts to a bijection between the $\mathcal{L}_r(E)$-valued regular positive spectral measures on Ω and the positive representations of $C_0(X)$ on E.*
If X is compact, then $\pi_P \upharpoonright_{C_0(X)}$ is unital if and only if P is unital.

We will now concentrate on the implications of the existence of a generating (regular) positive spectral measure for π. To start with, we have the following result. Note that it also covers the "missing" case in Proposition 5.1, but only under the hypothesis of the existence of a generating positive spectral measure.

Proposition 5.10. *Let X be a locally compact Hausdorff space with Borel σ-algebra Ω, let E be a Banach lattice, and let $\pi\colon C_0(X) \to \mathcal{L}_\mathrm{r}(E)$ be a positive representation. Suppose that π has a generating positive spectral measure P on Ω, and let $\pi_P\colon \mathcal{B}(X) \to \mathcal{L}_\mathrm{r}(E)$ denote the generated bounded positive representation of $\mathcal{B}(X)$ on E extending π. Then we have the following:*

(1) *$\pi\colon C_0(X) \to (\mathcal{L}_\mathrm{r}(E), \|\cdot\|)$ is bounded, and $\|\pi\| \leq \|\pi_P\| = \|P(X)\|$;*
(2) *if P is regular, then $\|\pi\| = \|\pi_P\| = P(X)$;*
(3) *if P is regular and E is Dedekind complete, then the maps $\pi\colon C_0(X) \to (\mathcal{L}_\mathrm{r}(E), \|\cdot\|)$, $\pi\colon C_0(X) \to (\mathcal{L}_\mathrm{r}(E), \|\cdot\|_\mathrm{r})$, $\pi_P\colon \mathcal{B}(X) \to (\mathcal{L}_\mathrm{r}(E), \|\cdot\|)$, and $\pi_P\colon \mathcal{B}(X) \to (\mathcal{L}_\mathrm{r}(E), \|\cdot\|_\mathrm{r})$ are all bounded, and $\|\pi\| = \|\pi\|_\mathrm{r} = \|\pi_P\| = \|\pi_P\|_\mathrm{r} = \|P(X)\|$.*

Proof. We know from part (1) of Theorem 4.1 that $\|\pi_P\| = \|P(X)\|$. Certainly the restriction π of π_P to $C_0(X)$ is also bounded, and $\|\pi\| \leq \|\pi_P\|$; this establishes part (1). For the converse inequality that is needed for part (2) if P is regular, we use (4.1) and the isometry statement in the Riesz representation theorem to see that

$$\|\pi_P\|$$
$$= \sup\big\{\big|\langle \pi(\phi)x, x'\rangle\big| : \phi \in \mathcal{B}(X), \|\phi\| \leq 1, x \in E, \|x\| \leq 1, x' \in E', \|x'\| \leq 1\big\}$$
$$= \sup\Big\{\Big|\int_X \phi\, d\mu_{x,x'}\Big| : \phi \in \mathcal{B}(X), \|\phi\| \leq 1, x \in E, \|x\| \leq 1, x' \in E', \|x'\| \leq 1\Big\}$$
$$\leq \sup\big\{\|\mu_{x,x'}\| : x \in E, \|x\| \leq 1, x' \in E', \|x'\| \leq 1\big\}$$
$$= \sup\Big\{\Big|\int_X \phi\, d\mu_{x,x'}\Big| : \phi \in C_0(X), \|\phi\| \leq 1, x \in E, \|x\| \leq 1, x' \in E', \|x'\| \leq 1\Big\}$$
$$= \sup\big\{\big|\langle \pi(\phi)x, x'\rangle\big| : \phi \in C_0(X), \|\phi\| \leq 1, x \in E, \|x\| \leq 1, x' \in E', \|x'\| \leq 1\big\}$$
$$= \|\pi\|.$$

Turning to (3), we note that for Dedekind complete E, the positive maps $\pi\colon C_0(X) \to (\mathcal{L}_\mathrm{r}(E), \|\cdot\|_\mathrm{r})$ and $\pi_P\colon \mathcal{B}(X) \to (\mathcal{L}_\mathrm{r}(E), \|\cdot\|_\mathrm{r})$ between Banach lattices are bounded; then the contractivity of the inclusion map $(\mathcal{L}_\mathrm{r}(E), \|\cdot\|_\mathrm{r}) \hookrightarrow (\mathcal{L}(E), \|\cdot\|)$ implies that the other two maps are bounded as well. Furthermore, parts (7)(a) and (1) of Theorem 4.1 show that $\|\pi_P\|_\mathrm{r} = \|\pi_P\| = \|P(X)\|$. We also know from part (2) of Proposition 5.1 that $\|\pi\| \leq \|\pi\|_\mathrm{r}$. In addition, we have $\|\pi\|_\mathrm{r} \leq \|\pi_P\|_\mathrm{r}$, since π_P extends π. If P is regular, then we have already established in part (2) that $\|\pi_P\| = \|\pi\|$. Combining all this, we see that, for regular P and Dedekind complete E,

$$\|\pi_P\|_\mathrm{r} = \|\pi_P\| = \|\pi\| \leq \|\pi\|_\mathrm{r} \leq \|\pi_P\|_\mathrm{r},$$

and the proof is complete. $\qquad\square$

We collect a few further consequences (some of them of course familiar from the nonordered context) of the existence of a generating positive spectral measure for $\pi\colon C_0(X) \to \mathcal{L}_\mathrm{r}(E)$ in our next result.

Proposition 5.11. *Let X be a locally compact Hausdorff space with Borel σ-algebra Ω, let E be a Banach lattice, and let $\pi\colon C_0(X) \to \mathcal{L}_{\mathrm{r}}(E)$ be a positive representation. Suppose that π has a generating positive spectral measure P on Ω, and let $\pi_P\colon \mathcal{B}(X) \to \mathcal{L}_{\mathrm{r}}(E)$ denote the generated bounded positive representation of $\mathcal{B}(X)$ on E extending π. Then we have the following.*

(1) *If $\phi \in C_0(X)$, then there is a sequence of linear combinations of elements of $P(\Omega)$ that converges to $\pi(\phi)$ in $(\mathcal{L}_{\mathrm{r}}(E), \|\cdot\|)$. If E is Dedekind complete, then there exists such a sequence converging to $\pi(\phi)$ in $(\mathcal{L}_{\mathrm{r}}(E), \|\cdot\|_{\mathrm{r}})$. If $\phi \in C_0(X)_+$, then the coefficients occurring in these linear combinations can be taken nonnegative.*

(2) *If $\phi \in C_0(X)$, and if $(\phi)_{n=1}^{\infty} \subset C_0(X)$ is a norm-bounded sequence such that $\lim_{n\to\infty} \phi_n(\xi) = \phi(\xi)$ for all $\xi \in X$, then $\pi(\phi) = \mathrm{WOT}\text{-}\lim_n \pi(\phi_n)$.*

(3) *If $\phi \in C_0(X)$ and if $(\phi)_{n=1}^{\infty} \subset C_0(X)$ is a norm-bounded sequence such that $\phi_n(\xi) \uparrow \phi(\xi)$ for all $\xi \in X$, then $\pi(\phi_n) \uparrow \pi(\phi)$ in $\mathcal{L}_{\mathrm{r}}(E)$.*

If P is regular, then

(4) *the commutants $P(\Omega)'$, $\pi_P(\mathcal{S}(X))'$, $\pi_P(\mathcal{B}(X))'$, and $\pi(C_0(X))'$ in $\mathcal{L}(E)$ are equal, and, consequently, $P(\Omega) \subset \pi(C_0(X))''$.*

Proof. For part (1), we take a sequence of simple functions converging uniformly to ϕ in $\mathcal{B}(X)$ and apply parts (1) and (7)(a) of Theorem 4.1.

Part (2) is a specialization of part (4) of Theorem 4.1.

Part (3) is a specialization of Proposition 4.3.

As to part (4), from part (6) of Theorem 4.1 we already know that the commutants $P(\Omega)'$, $\pi_P(\mathcal{S}(X))'$, and $\pi_P(\mathcal{B}(X))'$ in $\mathcal{L}(E)$ are equal. We will show that $\pi(C_0(X))' = P(\Omega)'$. Let $T \in \mathcal{L}(E)$. Then $T \in \pi(C_0(X))'$ if and only if $\langle T\pi(\phi)x, x'\rangle = \langle \pi(\phi)Tx, x'\rangle$ for all $\phi \in C_0(X)$ and $x \in E, x' \in E'$. Now note that

$$\langle T\pi(\phi)x, x'\rangle = \langle \pi(\phi)x, T'x'\rangle$$
$$= \int_X \phi \, d\mu_{x,T'x'},$$

and that

$$\langle \pi(\phi)Tx, x'\rangle = \int_X \phi \, d\mu_{Tx,x'}.$$

Thus $T \in \pi(C_0(X))'$ if and only if

$$\int_X \phi \, d\mu_{x,T'x'} = \int_X \phi \, d\mu_{Tx,x'},$$

for all $\phi \in C_0(X)$ and $x \in E, x' \in E'$. By the uniqueness statement in the Riesz representation theorem, this is the case if and only if $\mu_{x,T'x'} = \mu_{Tx,x'}$ for all $x \in E, x' \in E'$. That is, if and only if $\langle P(\Delta)x, T'x'\rangle = \langle P(\Delta)Tx, x'\rangle$ for all $\Delta \in \Omega, x \in E, x' \in E'$. This, in turn, is equivalent to $T \in P(\Omega)'$.

The folklore final statement is immediate from $P(\Omega) \subset P(\Omega)'' = \pi(C_0(X))''$. $\qquad\square$

We conclude by showing how the generating regular positive spectral measure P of π, if it exists, can be determined directly from π in terms of the ordering on $\mathcal{L}_r(E)$. By the first part of Proposition 5.3, it is sufficient to know $P(V)$ for all open subsets V of X, and part (1) of the next result shows how $P(V)$ can be found from $\pi(C_0(X))$. Likewise, the second part of Proposition 5.3 shows that it is sufficient to know $P(K)$ for all compact subsets K of X, and part (2) of the next result shows how to retrieve these from $\pi(C_0(X))$.

As in similar previous results, we do not assume that E is Dedekind complete, and hence the existence of the various suprema and infima in $\mathcal{L}_r(E)$ is not automatic.

Theorem 5.12. *Let X be a locally compact Hausdorff space with Borel σ-algebra Ω, let E be a Banach lattice, and let $\pi\colon C_0(X) \to \mathcal{L}_r(E)$ be a positive representation. Suppose that π has a generating regular positive spectral measure P on Ω.*

(1) *Let V be an open subset of X. Then*

$$P(V) = \sup\big\{\pi(\phi) : \phi \in C_c(X), \operatorname{supp}\phi \subset V, 0 \leq \phi \leq \mathbf{1}_X\big\}.$$

(2) *Let K be a compact subset of X. Then*

$$P(K) = \inf\big\{\pi(\phi) : \phi \in C_c(X), \phi \geq \chi_K\big\}.$$

(3) *In addition to the expression for $P(X)$ as obtained from part (1), we also have*

$$P(X) = \sup\big\{\pi(\phi) : \phi \in C_0(X), 0 \leq \phi \leq \mathbf{1}_X\big\}.$$

(4) *Let V be an open subset of X. Then V is σ-compact if and only if there exists a sequence $(\phi_n)_{n=1}^\infty$ in $C_c(X)$ such that $\operatorname{supp}\phi_n \subset V$ $(n \geq 1)$ and $\sup_n \phi_n = \chi_V$ in $\mathcal{B}(X)$. In that case, there exists a sequence $(\phi_n)_{n=1}^\infty$ in $C_c(X)$ such that $\operatorname{supp}\phi_n \subset V$ and $0 \leq \phi_n \leq \mathbf{1}_X$ $(n \geq 1)$, and $\phi_n \uparrow \chi_V$ in $\mathcal{B}(X)$. For any norm-bounded sequence $(\phi)_{n=1}^\infty$ in $C_c(X)$ such that $\phi_n \uparrow \chi_V$ in $\mathcal{B}(X)$, we have $\pi(\phi_n) \uparrow P(V)$ in $\mathcal{L}_r(E)$ and $P(V) = \text{WOT-}\lim_n \phi_n$.*

(5) *X is σ-compact if and only if there exists a sequence $(\phi_n)_{n=1}^\infty$ in $C_0(X)$ such that $0 \leq \phi_n \leq \mathbf{1}_X$ $(n \geq 1)$ and $\sup_n \phi_n = \mathbf{1}_X$ in $\mathcal{B}(X)$. In that case, there exists a sequence $(\phi_n)_{n=1}^\infty$ in $C_c(X)$ such that $0 \leq \phi_n \leq \mathbf{1}_X$ $(n \geq 1)$, and $\phi_n \uparrow \mathbf{1}_X$ in $\mathcal{B}(X)$. For any norm-bounded sequence $(\phi_n)_{n=1}^\infty$ in $C_0(X)$ such that $\phi_n \uparrow \mathbf{1}_X$ in $\mathcal{B}(X)$, we have $\pi(\phi_n) \uparrow P(X)$ in $\mathcal{L}_r(E)$ and $P(X) = \text{WOT-}\lim_n \phi_n$.*

Proof. Let $\pi_P\colon \mathcal{B}(X) \to \mathcal{L}_r(E)$ denote the positive representation of $\mathcal{B}(X)$ on E that is generated by P and that extends π, with associated regular finite signed Borel measures $\mu_{x,x'}$ $(x \in E, x' \in E')$. Starting with part (1), if $\phi \in C_c(X)$, $\operatorname{supp}\phi \subset V$, and $0 \leq \phi \leq \mathbf{1}_X$, then $\pi(\phi) = \pi_P(\phi) \leq \pi_P(\chi_V) = P(V)$. Hence $P(V)$ is an upper bound for

$$\big\{\pi(\phi) : \phi \in C_c(X), \operatorname{supp}\phi \subset V, 0 \leq \phi \leq \mathbf{1}_X\big\}.$$

If $T \in \mathcal{L}_{\mathrm{r}}(E)$ is also an upper bound for this set, then, for all $\phi \in C_{\mathrm{c}}(X)$ with $\operatorname{supp} \phi \subset V$ and $0 \leq \phi \leq \mathbf{1}_X$, and all $x \in E_+, x' \in E'_+$, we have

$$\langle Tx, x' \rangle \geq \langle \pi(\phi)x, x' \rangle$$
$$= \int_X \phi \, d\mu_{x,x'},$$

where (4.1) was used. Therefore, for all $x \in E_+, x' \in E'_+$,

$$\langle Tx, x' \rangle \geq \sup\Big\{ \int_X \phi \, d\mu_{x,x'} : \phi \in C_{\mathrm{c}}(X), \operatorname{supp} \phi \subset V, 0 \leq \phi \leq \mathbf{1}_X \Big\}.$$

Since by (2.1) the right-hand side in this equation equals $\mu_{x,x'}(V) = \langle P(V)x, x' \rangle$, we conclude that $\langle Tx, x' \rangle \geq \langle P(V)x, x' \rangle$ for all $x \in E_+, x' \in E'_+$. Hence $T \geq P(V)$.

The proof of part (2) is similar, based on (2.2).

For part (3), the same line of reasoning shows that $P(X) \geq \pi(\phi)$ for all $\phi \in C_0(X)$ with $0 \leq \phi \leq \mathbf{1}_X$. If $T \geq \pi(\phi)$ for all such ϕ, then, for all $x \in E_+, x' \in E'_+$, we find

$$\langle Tx, x' \rangle \geq \sup\Big\{ \int_X \phi \, d\mu_{x,x'} : \phi \in C_0(X), 0 \leq \phi \leq \mathbf{1}_X \Big\}.$$

But the right-hand side is the norm of the positive functional $\phi \mapsto \int_X \phi \, d\mu_{x,x'}$ on $C_0(X)$, which by the isometry statement in the Riesz representation theorem equals $\|\mu_{x,x'}\| = \mu_{x,x'}(X) = \langle P(X)x, x' \rangle$. Therefore $T \geq P(X)$.

For part (4), if such a sequence exists, then $V = \bigcup_n \{\xi \in X : \phi_n(\xi) > 0\}$ is a countable union of σ-compact subsets of X, and hence σ-compact. Conversely, if V is σ-compact, then we may assume that $V = \bigcup_n K_n$ where $K_n \subset K_{n+1}$ for all n. By [12, Corollary 4.32], we can choose $\psi_n \in C_{\mathrm{c}}(X)$ such that $0 \leq \psi_n \leq \mathbf{1}_X$, $\psi_n(\xi) = 1$ for $\xi \in K_n$ and $\operatorname{supp}(\psi_n) \subset V$. Let $\phi_n = \bigvee_{k=1}^n \psi_k$. Then the sequence $(\phi_n)_{n=1}^\infty$ is as required. An appeal to Proposition 4.3 concludes the proof of part (4).

The proof of part (5) is similar to that of part (4). $\square$

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