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REALIZATION OF COMPACT SPACES AS CB-HELSON SETS

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Dedicated to Professor Anthony To-Ming Lau, with thanks for all his work on behalf of the international community in abstract harmonic analysis

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Abstract. We show that, given a compact Hausdorff space Ω, there is a compact group $\mathbb{G}$ and a homeomorphic embedding of Ω into $\mathbb{G}$, such that the restriction map $\mathrm{A}(\mathbb{G}) \to C(\Omega)$ is a complete quotient map of operator spaces. In particular, this shows that there exist compact groups which contain infinite cb-Helson subsets, answering a question raised by Choi and Samei. A negative result from that paper is also improved.

1. Introduction

The notion of a cb-Helson subset of a locally compact group was introduced in [4], in connection with the study of quotients of Fourier algebras. For instance, the following result can be obtained by an easy modification of the proof of [4, Theorem B].

Theorem 1.1 (Corollary of work in [4])**.** *Let G be a SIN group, and let J be a closed ideal in the Fourier algebra $\mathrm{A}(G)$. Suppose that $\mathrm{A}(G)/J$ is completely boundedly isomorphic to a closed subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then there is a cb-Helson subset $E \subset G$ such that $J = \{f \in \mathrm{A}(G) \colon f|_E = 0\}$.*

We defer the definition of a cb-Helson set to Section 2. For now, we note that such sets appear to be rather hard to come by: finite subsets of locally compact groups have the cb-Helson property, but hitherto no infinite cb-Helson sets were

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known to exist. Uncountable Helson sets can be found in $\mathbb{T}$ and $\mathbb{R}$, using classical constructions of Fourier analysis (see, e.g., [12, Theorem 5.6.6]). By Herz's restriction theorem, these give uncountable Helson sets in any locally compact group that contains a closed copy of either $\mathbb{T}$ or $\mathbb{R}$. In contrast, every cb-Helson subset of a locally compact abelian group must be finite (see [4, Theorem C]).

The purpose of this note is twofold. First, we strengthen [4, Theorem C] as follows.

Theorem 1.2. *Let G be a locally compact group, and let G_d be the same group equipped with the discrete topology. Suppose that G_d is amenable. Then every cb-Helson subset of G is finite.*

(This applies, in particular, to any locally compact G which contains a solvable subgroup of finite index.) Second, answering a question raised in [4], we show that infinite cb-Helson sets do exist, albeit inside some "pathologically large" groups.

Theorem 1.3. *Let Ω be a compact Hausdorff space. Then there is a compact, connected group $\mathbb{G}$ and a cb-Helson subset of $\mathbb{G}$ that is homeomorphic to Ω.*

Theorem 1.3 can be thought of as an analogue of a folklore result for "classical" Helson sets, which, roughly speaking, says that each compact Hausdorff space Ω can be homeomorphically embedded as a Helson set inside *some* compact abelian group. Note that if we restrict attention to particular compact abelian groups, then there can be topological restrictions on the possible Helson subsets: for instance, a Helson subset of $\mathbb{T}$ cannot contain any closed subset homeomorphic to $[0, 1]$ (cf. [12, Remark 5.6.8]).

One drawback of Theorem 1.3 is that we create groups specially for the purpose of containing certain spaces as cb-Helson subsets, rather than finding "natural" examples of groups that contain infinite cb-Helson subsets. In particular, the following questions remain open (the second of which was suggested by the referee).

Questions.

(1) Is there a connected, linear Lie group that contains an infinite cb-Helson subset?

(2) Does there exist an infinite index set $\mathbb{I}$ and some $n \in \mathbb{N}$ such that $\prod_{i\in\mathbb{I}} \mathbb{U}_n$ contains an infinite cb-Helson subset?

Let us say something about the proofs. Theorem 1.2 is a quick consequence of some hard (but standard) results in the theory of operator spaces; its proof will be given in Section 3. In contrast, the proof of Theorem 1.3 is much longer, but only uses basic tools. The arguments are not difficult once one thinks of the right idea; the key part is based on the standard embedding of a dual operator space inside a product of matrix algebras. We will actually prove a more precise statement, given as Theorem 4.1.

2. Notation and preliminaries

2.1. **Notation and terminology.** If X is a Banach space, then $\mathsf{ball}(X)$ denotes its *closed* unit ball. If B is a unital C*-algebra, then $\mathbf{U}(B)$ denotes its unitary group. If $\mathcal{H}$ is a Hilbert space, then we write $\mathcal{U}(\mathcal{H})$ instead of $\mathbf{U}(\mathcal{B}(\mathcal{H}))$. We use $\mathbb{U}_n$ as an abbreviation for $\mathcal{U}(\mathbb{C}^n)$, that is, the usual group of $n \times n$ unitary matrices. Similarly, we write $\mathbb{M}_n$ as an abbreviation for $M_n(\mathbb{C})$.

Given a family $(X_i)_{i\in\mathbb{I}}$ of Banach spaces, we shall write $\prod^{\mathsf{B}}_{i\in\mathbb{I}} X_i$ for the Banach space direct product of this family, which is sometimes called the ℓ_∞-*direct sum*. This is just to avoid ambiguity since we shall also be dealing with sets or groups that arise as Cartesian products (in the usual sense) of sets or groups.

All topological groups are assumed to be a priori Hausdorff; if G is a topological group, then G_d denotes the same group equipped with the discrete topology.

The abbreviation WOT stands for *weak operator topology* (on $\mathcal{B}(\mathcal{H})$ for some given Hilbert space $\mathcal{H}$). Given a topological group G and a Hilbert space $\mathcal{H}$, a representation $\pi : G \curvearrowright \mathcal{H}$ is said to be *unitary* if $\pi(G) \subseteq \mathcal{U}(\mathcal{H})$, and said to be *WOT-continuous* if $\pi : G \to (\mathcal{B}(\mathcal{H}), \mathrm{WOT})$ is continuous as a map between topological spaces. For most of what we do, G is locally compact, but it is useful to have terminology which is well defined, without having to establish local compactness.

Given a family $(\sigma_i)_{i\in\mathbb{I}}$ of unitary representations of a common group G, with $\sigma_i : G \curvearrowright \mathcal{H}_i$, we define the *direct product* of this family (sometimes called the *direct sum*) as follows. Let $\mathcal{H} := \ell^2\text{-}\bigoplus_{i\in\mathbb{I}} \mathcal{H}_i$ be the ℓ^2-direct sum of the representation spaces. Given $\mathbf{x} = (x_i)_{i\in\mathbb{I}} \in G$ and $\boldsymbol{\xi} = (\xi_i)_{i\in\mathbb{I}} \in \mathcal{H}$, let

$$\left[\sigma(\mathbf{x})(\boldsymbol{\xi})\right]_i := \sigma_i(x_i)(\xi_i) \quad (i \in \mathbb{I}).$$

Clearly, σ is also a unitary representation: we often denote it by $\prod_{i\in\mathbb{I}} \sigma_i$.

Remark 2.1. Let G be a topological group. Then the direct product of a family of WOT-continuous unitary representations of G is itself WOT-continuous. In the locally compact setting, this is often deduced as an application of the correspondence between WOT-continuous representations of a locally compact group G and nondegenerate $*$-representations of $L^1(G)$. However, it is not hard to give a proof which works for arbitrary topological groups; since we did not see this proof in the sources we consulted, it is included in the Appendix for the reader's interest.

If $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is a unitary representation of a (topological) group, we denote by $\mathrm{VN}_\pi(G)$ the WOT-closed subalgebra of $\mathcal{B}(\mathcal{H}_\pi)$ generated by the subset $\pi(G)$. Given $\xi, \eta \in \mathcal{H}_\pi$ we denote by $\xi *_\pi \eta$ the function $g \mapsto (\pi(g)\xi \mid \eta)$. If π is furthermore WOT-continuous, then each function of the form $\xi *_\pi \eta$ belongs to $C_b(G)$, and the map $\xi \otimes \overline{\eta} \mapsto \xi *_\pi \eta$ gives rise to a bounded linear map $\mathcal{H}_\pi \widehat{\otimes} \overline{\mathcal{H}_\pi} \to C_b(G)$. If, furthermore, we assume that G is locally compact, then we can appeal to the machinery of coefficient spaces as in, for example, Arsac's thesis [1]: in particular, $\mathrm{VN}_\pi(G)$ has a canonical predual $\mathrm{A}_\pi(G)$, which is the image of the map $\mathcal{H}_\pi \widehat{\otimes} \overline{\mathcal{H}_\pi} \to C_b(G)$ equipped with the quotient norm.

2.2. **Helson sets and cb-Helson sets.** Let E be a closed subset of a locally compact group G; let $\mathrm{A}(G)$ denote the Fourier algebra of G, as defined in [6]. (Recall that if G is locally compact and abelian, then $\mathrm{A}(G)$ is naturally identified with the space of Fourier transforms of integrable functions on the dual group $\widehat{G}$.)

Restriction of functions defines a contractive algebra homomorphism $\mathrm{A}(G) \to C_0(E)$, whose kernel we denote by $I_G(E)$; let $\mathrm{A}_G(E)$ be the quotient algebra $\mathrm{A}(G)/I_G(E)$. There is a natural, injective homomorphism $\mathsf{inc}_{G,E} : \mathrm{A}_G(E) \to C_0(E)$, which in general need not be surjective.

Definition 2.2. We say that E is a *Helson subset of* G or a *Helson set* for short, if $\mathsf{inc}_{G,E} : \mathrm{A}(G) \to C_0(E)$ is surjective, and hence an isomorphism of Banach spaces.

Equivalently: E is a Helson subset of G if and only if each continuous function on E that vanishes at infinity can be extended to some function in the Fourier algebra of G.

Most of the operator space theory needed for this paper will be explained as we need it: all necessary background that is not explained here can be found in the standard references [2], [5], and [10]. Let us recall briefly what is meant by the *natural* operator space structures on the spaces $C_0(G)$ and $\mathrm{A}_G(E)$. Since $C_0(E)$ and $\mathrm{VN}(G)$ are C^*-algebras, they have canonical operator space structures; $\mathrm{VN}(G)$ is a weak-star closed subspace of $\mathcal{B}(L^2(G))$, and hence its predual $\mathrm{A}(G)$ has an operator space structure, whose dual (as an operator space) coincides with the original operator space structure on $\mathrm{VN}(G)$; and finally, $\mathrm{A}_G(E)$ is a quotient of $\mathrm{A}(G)$, so it carries a natural quotient operator space structure.

Definition 2.3. We say that E is a *cb-Helson subset of* G, or a *cb-Helson set* for short, if $\mathsf{inc}_{G,E} : \mathrm{A}_G(E) \to C_0(E)$ is an isomorphism of operator spaces.

Remark 2.4. Since the natural operator space structure on $C_0(E)$ is minimal, the map $\mathsf{inc}_{G,E} : \mathrm{A}_G(E) \to C_0(E)$ is automatically completely bounded. (In fact, it is completely *contractive*.) Thus E is a cb-Helson set if and only if $\mathsf{inc}_{G,E} : \mathrm{A}_G(E) \to C_0(E)$ is surjective and has completely bounded inverse.

2.3. **Warning remarks on terminology.** There seems to be some disagreement in the literature over the precise definition of Helson sets in noncompact groups. The original result of Helson that motivated the term *Helson set* was in the context of closed subsets of compact abelian groups. However, once one moves to locally compact abelian groups, terminology seems to differ. Some authors require that Helson subsets be compact as part of the definition; our convention, that they need not be compact, is in accordance with [7].

There is also the notion of a *Sidon subset* E in a discrete abelian group Γ, which, roughly speaking, requires that functions on E extend to elements of the Fourier–Stieltjes algebra $\mathrm{B}(\Gamma)$ with a control of norm. It turns out that every Sidon subset of a discrete abelian group is automatically a Helson subset in our sense (see [12, Theorem 5.7.3]). We prefer to keep the two concepts distinct.

It is actually easy to find examples of infinite cb-Sidon sets (where the definition is the obvious analogue of the cb-Helson condition). Let $\mathbb{F}_\infty$ denote the free group on a countably infinite number of generators, and let E denote the set of generators: then the restriction map $\mathrm{B}(\mathbb{F}_\infty) \to \ell^\infty(E)$ is a complete quotient map, since it can be viewed as the adjoint of the known complete isometry $\mathsf{max}\,\ell^1(E) \hookrightarrow \mathrm{C}^*(\mathbb{F}_\infty)$.

3. Calculations with cb-Helson constants

Classically, a useful tool for studying Helson sets has been to quantify "how Helson they are," by associating a "Helson constant" to each closed subset of a given group. This has an obvious analogue for cb-Helson sets, as given by the following definition from [4].

Definition 3.1 (cb-Helson constants). Let E be a cb-Helson subset of a locally compact group G. The *cb-Helson constant of* E, which we denote by $\mathrm{Hel}_{\mathrm{cb}}(E)$, is defined to be $\| \mathsf{inc}_{G,E}^{-1} \|_{\mathrm{cb}}$.

We adopt the convention that if E is a closed subset of G and not a cb-Helson subset, then $\mathrm{Hel}_{\mathrm{cb}}(E) = +\infty$.

Remark 3.2. Let E be a closed subset of G.

(i) By basic operator space theory, $\mathrm{Hel}_{\mathrm{cb}}(E) = 1$ if and only if the restriction map $\mathrm{A}(G) \to C_0(E)$ is a complete quotient map of operator spaces.

(ii) If F is a closed subset of E, then it follows from Tietze's extension theorem that $\mathrm{Hel}_{\mathrm{cb}}(F) \leq \mathrm{Hel}_{\mathrm{cb}}(E)$ (see [4, Lemma 2.2] for the details).

Now, by Remark 3.2(ii), Theorem 1.2 will follow from the following result on cb-Helson constants of finite subsets in certain groups.

Proposition 3.3. *Let G be a locally compact group such that G_d is amenable. Let $F \subset G$ be a finite subset of size $n \geq 2$. Then* $\mathrm{Hel}_{\mathrm{cb}}(F) \geq n/(2\sqrt{n-1})$.

Proof. To ease notation, we abbreviate $\mathsf{inc}_{G,F}$ to inc_F and $\mathrm{A}_G(F)$ to $\mathrm{A}(F)$. We say *identified* as an abbreviation for "identified completely isometrically as operator spaces."

We identify $C(F)$ with $\mathsf{min}\,\ell_\infty^n$ and hence identify $C(F)^*$ with $\mathsf{max}\,\ell_1^n$. Since $\mathrm{A}(G) \to \mathrm{A}(F)$ is a complete quotient map (by definition), we may identify $\mathrm{A}(F)^*$ with the closed linear span inside $\mathrm{VN}(G)$ of $\{\lambda_x \colon x \in F\}$. Now consider $\mathrm{C}^*_\delta(G)$, the norm-closed subalgebra of $\mathcal{B}(L_2(G))$ generated by the set of left translation operators $\{\lambda_x \colon x \in G\}$. Since G_d is amenable, $\mathrm{C}^*(G_d)$ is nuclear (see [8]); and since $\mathrm{C}^*(G_d)$ quotients onto $\mathrm{C}^*_\delta(G)$, it follows that $\mathrm{C}^*_\delta(G)$ is also nuclear. Thus $\mathrm{A}(F)^*$ is identified with a closed subspace of a nuclear C^*-algebra, and so (see [10, Corollary 12.6])

$$\inf_{V \subset \mathcal{K}(\ell_2)} \mathrm{dist}_{\mathrm{cb}}\big(\mathrm{A}(F)^*, V\big) = 1, \tag{1}$$

where the infimum is taken over all n-dimensional subspaces of $\mathcal{K}(\ell_2)$.

Let $V \subset \mathcal{K}(\ell_2)$ be an n-dimensional subspace of $\mathcal{K}(\ell_2)$. Then by a result of Pisier ([9, Theorem 7]),

$$\mathrm{dist}_{\mathrm{cb}}(\mathsf{max}\,\ell_1^n, V) \geq \frac{n}{2\sqrt{n-1}}. \tag{2}$$

By basic properties of cb-Banach–Mazur distance,

$$\mathrm{dist}_{\mathrm{cb}}(\mathsf{max}\,\ell_1^n, V) \leq \mathrm{dist}_{\mathrm{cb}}\big(\mathsf{max}\,\ell_1^n, \mathrm{A}(F)^*\big)\, \mathrm{dist}_{\mathrm{cb}}\big(\mathrm{A}(F)^*, V\big).$$

Combining this with (1) and (2) yields $n/(2\sqrt{n-1}) \leq \operatorname{dist}_{\mathrm{cb}}(\mathsf{max}\,\ell_1^n, \mathrm{A}(F)^*)$. Since

$$\begin{aligned}\operatorname{dist}_{\mathrm{cb}}\bigl(\mathsf{max}\,\ell_1^n, \mathrm{A}(F)^*\bigr) &= \operatorname{dist}_{\mathrm{cb}}\bigl(\mathsf{min}\,\ell_\infty^n, \mathrm{A}(F)\bigr) \leq \|\,\mathsf{inc}_F\,\|_{\mathrm{cb}} \|\,\mathsf{inc}_F^{-1}\,\|_{\mathrm{cb}} \\ &= \operatorname{Hel}_{\mathrm{cb}}(F),\end{aligned}$$

the proof is complete. □

Remark 3.4. The theme of getting lower bounds on the Helson constants of finite sets is very standard in the study of the classical Helson condition. For instance, to prove that a closed nontrivial arc in $\mathbb{T}$ is not Helson, one considers larger and larger finite subsets, each of which is an "arithmetic progression," and then shows that the Helson constant of a finite arithmetic progression in $\mathbb{T}$ tends to infinity as the number of points in the progression grows.

Remark 3.5. The proof of Proposition 3.3, and hence the proof of Theorem 1.2, works for any locally compact group G such that $\mathrm{C}^*_\delta(G)$ is an exact C^*-algebra. However, I do not know if this actually encompasses any new examples not covered by Theorem 1.2. In fact, after this paper was submitted, I learned of the preprint [11] by Ruan and Wiersma. As a special case of their results, they show that, whenever G is a locally compact amenable group for which G_d is not amenable, such as $\mathrm{SO}(n,\mathbb{R})$ for all $n \geq 3$, the algebra $\mathrm{C}^*_\delta(G)$ is not even locally reflexive, and hence a priori cannot be exact. See Example 4.1(3) and Theorem 4.3 in their paper for further details.

4. Theorem 1.3: A stronger version and some initial reductions

Our goal in the rest of this paper is to prove the following result, which implies Theorem 1.3.

Theorem 4.1 (Embedding theorem, precise version). *Let Ω be a compact Hausdorff space. Then there exists a family $(n(i))_{i\in\mathbb{I}}$ of positive integers such that, when we define $\mathbb{G}$ to be the group $\prod_{i\in\mathbb{I}} \mathbb{U}_{n(i)}$ with the product topology, there is a continuous injection from $\mathbf{j} : \Omega \to \mathbb{G}$ such that* $\operatorname{Hel}_{\mathrm{cb}}(\mathbf{j}(\Omega)) = 1$.

The proof of the theorem will be broken down into two propositions. Before giving all the details, let us provide some motivation. We saw in the previous section that, in order to find large finite sets with small cb-Helson constant, we need to find maps $\mathsf{max}\,\ell_1^n \to \mathrm{VN}(G)$ which are not too far from being embeddings of operator spaces, and which send the standard basis vectors of ℓ_1^n to elements of $\lambda(G) \subset \mathsf{ball}(\mathrm{VN}(G))$. Well, $\mathsf{max}\,\ell_1$ has a completely isometric embedding into $\mathrm{C}^*(\mathbb{F}_\infty)$, which embeds as a C^*-subalgebra of $\prod^{\mathsf{B}}_{n\geq 1} M_{2n}(\mathbb{C})$, which in turn embeds as a C^*-subalgebra of $\mathrm{VN}(\mathrm{SU}(2,\mathbb{C}))$. Although this embedding does not seem to send the standard basis of ℓ_1 to elements of $\lambda(\mathrm{SU}(2,\mathbb{C}))$, it does embed the standard basis of ℓ_1 into $\prod_{n\geq 1} \mathbb{U}_{2n}$, and the latter is a compact group when given the product topology. Since we want an embedding of Ω into $\lambda(\mathbb{G}) \subset \mathsf{ball}(\mathrm{VN}(\mathbb{G}))$ for some compact $\mathbb{G}$, the natural *Ansatz* is to take $\mathbb{G} = \prod_{n\geq 1} \mathbb{U}_{2n}$ and hope that the details work out.

Now we return to the task of proving Theorem 4.1. As we have already seen, a natural way to get complete quotient maps onto $C(\Omega)$ is to dualize and look for completely isometric, w*–w* continuous embeddings of $C(\Omega)^* = M(\Omega)$.

Remark 4.2. We already used, in a small way, the standard result that if V is a minimal operator space, then the dual operator space structure on V^* coincides with the maximal operator space structure. However, for the proofs that follow, we do not actually need to know this.

The first step is to observe that, in some sense, it suffices to get a good embedding of $M(\Omega)$ into $\mathrm{VN}_\pi(\mathbb{G})$ for a suitable WOT-continuous unitary representation $\pi : \mathbb{G} \to \mathcal{H}_\pi$.

Proposition 4.3. *Let G be a compact group. Suppose the following exist: a* faithful*, WOT-continuous, unitary representation $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$; and a complete isometry $J : M(\Omega) \to \mathcal{B}(\mathcal{H}_\pi)$ which is* w*–w* *continuous and maps $\{\delta_\omega : \omega \in \Omega\}$ to a subset of $\pi(G)$.*

For each $\omega \in \Omega$, let $\mathbf{j}(\omega)$ be the unique element of G such that $J(\delta_\omega) = \pi(\mathbf{j}(\omega))$. Then

(i) *$\mathbf{j} : \Omega \to G$ is a continuous injection with closed range;*
(ii) *$J_*(h)(\omega) = h(\mathbf{j}(\omega))$ for all $\omega \in \Omega$;*
(iii) $\mathrm{Hel}_{\mathrm{cb}}(\mathbf{j}(\Omega)) = 1$.

Proof. Since J is injective, so is $\mathbf{j}$. Next we show that $\mathbf{j}$ is continuous, which will imply that $\mathbf{j}(\Omega)$ is compact and hence closed in G. Let B denote $\mathsf{ball}(\mathcal{B}(\mathcal{H}_\pi))$ equipped with the relative WOT. Since $\pi : G \to B$ is a continuous map from a compact space to a Hausdorff one, it is a homeomorphism onto its range. Therefore, to show that $\mathbf{j} : \Omega \to G$ is continuous, it suffices to show that $\pi \circ \mathbf{j} : \Omega \to B$ is continuous. Let ev denote the map $\omega \mapsto \delta_\omega$, so that $\pi \circ \mathbf{j} = J \circ \mathrm{ev}$. It is straightforward to check that when $M(\Omega)$ is given the w*-topology, $\mathrm{ev} : \Omega \to M(\Omega)$ is continuous (just look at preimages of sub-basic open sets). Since $J : M(\Omega) \to \mathcal{B}(\mathcal{H}_\pi)$ is w*–w* continuous and contractive, it follows that $J \circ \mathrm{ev} : \Omega \to (\mathsf{ball}(\mathcal{H}_\pi), \mathrm{w}^*)$ is continuous. But the relative w*-topology and the relative WOT coincide on $\mathsf{ball}(\mathcal{H}_\pi)$, so $J \circ \mathrm{ev} : \Omega \to B$ is continuous as required. This completes the proof of part (i).

Part (ii) is a straightforward consequence of the definitions of $\mathrm{A}_\pi(G)$ and $\mathrm{VN}_\pi(G)$ and the pairing between them. In more detail: since $\langle \pi(g), \xi *_\pi \eta \rangle = (\xi *_\pi \eta)(g)$ for each $g \in G$, $\langle \pi(g), h \rangle_{\mathrm{VN}_\pi - \mathrm{A}_\pi} = h(g)$ for all $g \in G$ and all $h \in \mathrm{A}_\pi(G)$. Hence, by the definition of $\mathbf{j}$,

$$h\big(\mathbf{j}(\omega)\big) = \big\langle J(\delta_\omega), h \big\rangle_{\mathrm{VN}_\pi - \mathrm{A}_\pi} = \big\langle \delta_\omega, J_*(h) \big\rangle_{M(\Omega) - C(\Omega)} = J_*(h)(\omega),$$

as required. Thus (ii) is proved.

Finally, let $R : \mathrm{A}(G) \to C(\mathbf{j}(\Omega))$ be the completely contractive map given by restriction of functions, and let $\mathbf{j}^*$ denote the completely isometric isomorphism $C(\mathbf{j}(\Omega)) \to C(\Omega)$ induced by $\mathbf{j}$. Then, by part (ii),

$$\mathbf{j}^* R(h)(\omega) = h\big(\mathbf{j}(\omega)\big) = J_*(h)(\omega) \quad \big(h \in \mathrm{A}(G), \omega \in \Omega\big).$$

Let $\imath : \mathrm{A}_\pi(G) \to \mathrm{A}(G)$ be the natural inclusion map; this is well defined and a complete isometry since G is compact. Then $R \circ \imath = (\mathbf{j}^*)^{-1} \circ J_*$, and $J_* : \mathrm{A}_\pi(G) \to C(\Omega)$ is a complete quotient map (since its adjoint J is a complete isometry); a quick check now shows that R is also a complete quotient map, as required. This finishes the proof of part (iii). □

Proposition 4.4 (An embedding in a product of matrix algebras)**.** *Let Ω be a compact Hausdorff space. Then there exists a family $(\mathcal{H}_i)_{i\in\mathbb{I}}$ of finite-dimensional Hilbert spaces, such that when we take $\mathcal{M} := \prod_i^{\mathsf{B}} \mathcal{B}(\mathcal{H}_i)$, there is a completely isometric,* w*–w* *continuous embedding $J : (\min C(\Omega))^* \to \mathcal{M}$ such that $J(\delta_\omega) \in \prod_i \mathcal{U}(\mathcal{H}_i)$ for all $\omega \in \Omega$.*

The proof of this proposition consists of modifying the construction in [3] of the *standard dual of an operator space*, in the special case of $C(\Omega)$. Since the details are straightforward but somewhat lengthy, the proof of this proposition will be deferred to the next section.

Proof of Theorem 4.1. Let $(\mathcal{H}_i)_{i\in\mathbb{I}}$, $\mathcal{M}$, and J be as provided by Proposition 4.4. Take $\mathbb{G} = \mathcal{U}(\mathcal{M}) = \prod_i \mathcal{U}(\mathcal{H}_i)$ and define $\mathbf{j} : \Omega \to \mathbb{G}$ by $J(\delta_\omega) = \pi(\mathbf{j}(\omega))$. Equip each $\mathcal{U}(\mathcal{H}_i)$ with the relative WOT, and equip $\mathbb{G}$ with the product topology. Since each $\mathcal{H}_i$ is finite-dimensional, each $\mathcal{U}(\mathcal{H}_i)$ is compact, and so $\mathbb{G}$ is compact.

Let $\mathcal{H} = \ell^2\text{-}\bigoplus_{i\in\mathbb{I}} \mathcal{H}_i$, and let $\theta : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ be the natural, w*–w* continuous inclusion. By abuse of notation we also use θ to denote the corresponding representation $\mathbb{G} \to \mathcal{U}(\mathcal{H})$. We claim that (i) $\theta : \mathbb{G} \curvearrowright \mathcal{H}$ is WOT-continuous and (ii) $\mathrm{VN}_\theta(\mathbb{G}) = \theta(\mathcal{M})$. If both of these hold, then the hypotheses of Proposition 4.3 are satisfied, and hence $\mathbf{j}(\Omega)$ is a cb-Helson subset of $\mathbb{G}$ with cb-Helson constant 1.

To prove (i), for each $i \in \mathbb{I}$, let $\theta_i : \mathbb{G} \to \mathcal{U}(\mathcal{H}_i)$ be the coordinate projection: this is WOT-continuous by definition of the product topology. Hence, by Remark 2.1, the direct product $\prod_i \theta_i : \mathbb{G} \curvearrowright \mathcal{H}$ is also WOT-continuous. It is easily checked that $\prod_i \theta_i$ coincides with θ, so we have proved (i).

To prove (ii), note that the inclusion $\mathrm{VN}_\theta(\mathbb{G}) \subseteq \theta(\mathcal{M})$ is trivial. On the other hand, since each element of a unital C*-algebra A is a linear combination of four elements of $\mathcal{U}(A)$, we have $\theta(\mathcal{M}) \subseteq \operatorname{lin} \theta(\mathcal{U}(\mathcal{M})) = \operatorname{lin}(\theta(\mathbb{G})) \subseteq \mathrm{VN}_\theta(\mathbb{G})$, giving the converse inclusion. (As pointed out by the referee, one could instead appeal to more general results on coefficient spaces of products of representations; see, e.g., [1, Corollaire 3.13].) This completes the proof of (ii), and hence completes the proof of the theorem. □

Theorem 4.1 is a cb-analogue of a folklore result in the classical theory of Helson sets, which says that any compact space arises as a Helson subset of a product of (many!) copies of $\mathbb{T}$. To emphasize the analogy, we state a more long-winded version of the classical result.

Proposition 4.5 (Probably folklore)**.** *Let Ω be a compact Hausdorff space. Let $\Gamma = \mathbf{U}(C(\Omega))_d$ (i.e., the unitary group of $C(\Omega)$ equipped with the* discrete *topology), and let G be the Pontryagin dual of Γ, regarded as a subset of $\mathsf{ball}(\ell^\infty(\Gamma))$. Define $J : M(\Omega) \to \ell^\infty(\Gamma)$ by $J(\mu)(\gamma) = \int_\Omega \gamma \, d\mu$, and let $j(\omega) = J(\delta_\omega)$. Then*

(i) *$j(\Omega)$ is a compact subset of G;*
(ii) *$J : M(\Omega) \to \ell^\infty(\Gamma)$ is* w*–w* *continuous and bounded below;*
(iii) *if we identify* $\mathrm{A}(G)$ *with* $\ell^1(\Gamma)$*, then the restriction map* $\mathrm{A}(G) \to C(j(\Omega))$ *is just* J_**.*

In particular, $j(\Omega)$ is a Helson subset of G.

We omit the proof, which is straightforward bookkeeping (for part (ii) we use the fact that any element of $\mathsf{ball}(C(\Omega))$ can be written as $u_1 + u_2 + i(v_1 + v_2)$ for some $u_1, u_2, v_1, v_2 \in \Gamma$; if we use [12, Lemma 5.5.1], one can actually show that J is an isometry). I do not know an exact reference for the precise statement above, but the construction and the method have appeared independently many times in the literature. For instance, as pointed out in [7, Section 5], the group G can be seen as the *free compact abelian group* generated by Ω, in an appropriate category-theoretic sense.

5. The proof of Proposition 4.4

We must find a w*–w* continuous and completely isometric embedding $J : (\min C(\Omega))^* \to \mathcal{M}$, where $\mathcal{M}$ is a product of matrix algebras, such that

$$J(\delta_\omega) \in \mathbf{U}(\mathcal{M}) \quad \text{for all } \omega \in \Omega. \tag{3}$$

Let us temporarily ignore the requirement (3). Then there is a standard way to embed $(\min C(\Omega))^*$ w*–w* continuously and completely isometrically into a product of matrix algebras: this is a by-product of the definition of the *standard dual* of a given operator space (see [3]).

With minor modifications this embedding procedure can be made to also satisfy (3). Here are the details. Given an operator space V and $p \in \mathbb{N}$, the *standard norm* on $M_p(\mathsf{V}^*)$ is defined as follows: given $\boldsymbol{\mu} = [\mu_{st}] \in M_p(\mathsf{V})$, let $T_{\boldsymbol{\mu}} : \mathsf{V} \to \mathbb{M}_k$ be the linear map $v \mapsto [\mu_{st}(v)]$, and define $\|\boldsymbol{\mu}\|_{(\mathrm{SD})}$ to be $\|T_{\boldsymbol{\mu}}\|_{\mathrm{cb}}$. To get (3), we note that if V is a unital C*-algebra equipped with its canonical operator space structure, the cb-norm of $T_{\boldsymbol{\mu}}$ can be determined by testing on unitary matrices with entries in V. Although this follows from the Russo–Dye theorem, we shall give a more hands-on approach.

Lemma 5.1 (Determining $\|T_{\boldsymbol{\mu}}\|_{\mathrm{cb}}$)**.** *Let V be a unital* C*-*algebra, and let $\boldsymbol{\mu} \in M_p(\mathsf{V}^*)$. Then*

$$\|\boldsymbol{\mu}\|_{(\mathrm{SD})} = \|T_{\boldsymbol{\mu}}\|_{\mathrm{cb}} = \sup_{n\in\mathbb{N}} \sup\left\{ \left\|(T_{\boldsymbol{\mu}})_n(\mathbf{u})\right\|_{M_n(\mathbb{M}_p)} : \mathbf{u} \in \mathcal{U}\big(M_n(\mathsf{V})\big) \right\}.$$

Proof. We use the following well-known trick: since V is a unital C*-algebra, for each $a \in \mathsf{ball}(\mathsf{V})$ the block matrix

$$U_a := \begin{bmatrix} (1-aa^*)^{1/2} & a \\ -a^* & (1-a^*a)^{1/2} \end{bmatrix}$$

is well defined and unitary in $M_2(\mathsf{V})$.

Let $n \in \mathbb{N}$, and let $E_{12} : M_{2n}(\mathsf{V}) \to M_n(\mathsf{V})$ be compression to the (1, 2) entry when we identify $M_{2n}(\mathsf{V})$ with $M_2(M_n(\mathsf{V}))$. By abuse of notation, we also use E_{12} to denote the corresponding compression map $M_{2n}(\mathbb{M}_p) \to M_n(\mathbb{M}_p)$.

Let $\boldsymbol{\mu} \in M_p(\mathsf{V}^*)$. Since $E_{12} \circ (T_{\boldsymbol{\mu}})_{2n} = (T_{\boldsymbol{\mu}})_n \circ E_{12}$, we have

$$\begin{aligned}
\big\|(T_{\boldsymbol{\mu}})_n\big\| &= \sup\big\{\big\|(T_{\boldsymbol{\mu}})_n(\mathbf{v})\big\|_{M_n(\mathbb{M}_p)} : \mathbf{v} \in \mathsf{ball}\big(M_n(\mathsf{V})\big)\big\} \\
&= \sup\big\{\big\|E_{12}(T_{\boldsymbol{\mu}})_{2n}(U_{\mathbf{v}})\big\|_{M_n(\mathbb{M}_p)} : \mathbf{v} \in \mathsf{ball}\big(M_n(\mathsf{V})\big)\big\} \\
&\le \sup\big\{\big\|(T_{\boldsymbol{\mu}})_{2n}(U_{\mathbf{v}})\big\|_{M_{2n}(\mathbb{M}_p)} : \mathbf{v} \in \mathsf{ball}\big(M_n(\mathsf{V})\big)\big\} \\
&\le \sup\big\{\big\|(T_{\boldsymbol{\mu}})_{2n}(\mathbf{u})\big\|_{M_{2n}(\mathbb{M}_p)} : \mathbf{u} \in \mathcal{U}\big(M_{2n}(\mathsf{V})\big)\big\} \\
&\le \sup_{r\in\mathbb{N}} \sup\big\{\big\|(T_{\boldsymbol{\mu}})_r(\mathbf{u})\big\|_{M_r(\mathbb{M}_p)} : \mathbf{u} \in \mathcal{U}\big(M_r(\mathsf{V})\big)\big\} \le \|T_{\boldsymbol{\mu}}\|_{\mathrm{cb}}.
\end{aligned}$$

Taking the supremum over all n on the left-hand side, the result follows. □

Now we specialize to the case $\mathsf{V} = C(\Omega)$. We can identify $M_n(C(\Omega))$ (completely isometrically) with $C(\Omega; \mathbb{M}_n)$, and under this identification $\mathcal{U}(M_n(C(\Omega)))$ is identified with $C(\Omega; \mathbb{U}_n)$. To ease notation, we denote this unitary group by Γ_n. Then, given $\mathbf{u} = [u_{ij}] \in \Gamma_n$, define $J_{n,\mathbf{u}} : M(\Omega) \to \mathbb{M}_n$ by

$$J_{n,\mathbf{u}}(\mu) := \big[\mu(u_{ij})\big]. \tag{4}$$

If $\boldsymbol{\mu} = [\mu_{st}] \in M_p(M(\Omega))$, then the *canonical shuffle* $M_p(\mathbb{M}_n) \cong M_n(\mathbb{M}_p)$ maps $(J_{n,\mathbf{u}})_p(\boldsymbol{\mu})$ to $(T_{\boldsymbol{\mu}})_n(\mathbf{u})$, and so by Lemma 5.1 we get

$$\sup_n \sup_{\mathbf{u}\in\Gamma_n} \big\|(J_{n,\mathbf{u}})_p(\boldsymbol{\mu})\big\| = \sup_n \sup_{\mathbf{u}\in\Gamma_n} \big\|(T_{\boldsymbol{\mu}})_n(\mathbf{u})\big\| = \|\boldsymbol{\mu}\|_{(\mathrm{SD})}. \tag{5}$$

Let $\mathcal{M} := \prod^{\mathsf{B}}_{n\in\mathbb{N}} \prod^{\mathsf{B}}_{\mathbf{u}\in\Gamma_n} \mathbb{M}_n$, and define $J : M(\Omega) \longrightarrow \mathcal{M}$ to be the direct product of the maps $J_{n,\mathbf{u}}$. Equation (5) implies that, for each $p \in \mathbb{N}$, we have

$$\big\|J_p(\boldsymbol{\mu})\big\| = \|\boldsymbol{\mu}\|_{(\mathrm{SD})} \quad \text{for all } \boldsymbol{\mu} \in M_p\big(M(\Omega)\big),$$

so that $J : M(\Omega) \to \mathcal{M}$ is a complete isometry. Note that for each $\omega \in \Omega$, $J_{\mathbf{u}}(\delta_\omega) = \mathbf{u}(\omega) \in \mathbb{U}_n$, and hence $J(\delta_\omega) \in \mathbf{U}(\mathcal{M})$. So, to complete the proof of Proposition 4.4, there is only one thing left to check.

Lemma 5.2. *J is* w*–w* *continuous.*

The analogous statement for the *canonical embedding* of a dual operator space can be found in [3, Proposition 2.1], and a description of the pre-adjoint is given without proof after [3, Proposition 3.1]. For sake of completeness, we give the details.

Proof of Lemma 5.2. Let $\mathbf{T}_n$ denote the space of trace class operators on $\mathbb{C}^n$, equipped with its canonical norm. Then

$$\mathcal{M}_* = \ell_1\text{-}\bigoplus_n \bigoplus_{\mathbf{u}\in C(\Omega;\mathbb{U}_n)} \mathbf{T}_n,$$

and for any $\mathbf{S} = (S_{n,\mathbf{u}}) \in \mathcal{M}_*$ and $\mu \in M(\Omega)$ we have

$$\big\langle J^*(\mathbf{S}), \mu \big\rangle_{M(\Omega)^* - M(\Omega)} = \sum_{n\in\mathbb{N}} \sum_{\mathbf{u}\in\Gamma_n} \operatorname{Tr}\big(S_{n,\mathbf{u}} J_{\mathbf{u}}(\mu)\big). \tag{6}$$

Define $h \in C(\Omega)$ by

$$h(\omega) = \sum_{n\in\mathbb{N}} \sum_{\mathbf{u}\in\Gamma_n} \operatorname{Tr}\big[S_{n,\mathbf{u}} \mathbf{u}(\omega)\big] \quad (\omega \in \Omega).$$

The sum is absolutely convergent, uniformly in ω, since $\sum_{n\in\mathbb{N}} \sum_{\mathbf{u}\in\Gamma_n} \|S_{n,\mathbf{u}}\|_1 < \infty$ by our assumption on $\mathbf{S}$. Therefore, we have

$$\mu(h) = \sum_{n\in\mathbb{N}} \sum_{\mathbf{u}\in\Gamma_n} \mu\big(\operatorname{Tr}\big[S_{n,\mathbf{u}} \mathbf{u}(\cdot)\big]\big) = \sum_{n\in\mathbb{N}} \sum_{\mathbf{u}\in\Gamma_n} \operatorname{Tr}\big(S_{n,\mathbf{u}} J_{\mathbf{u}}(\mu)\big), \tag{7}$$

and combining (6) and (7) gives $\langle J^*(\mathbf{S}), \mu \rangle = \mu(h)$. Thus $J^*(\mathcal{M}_*) \subseteq C(\Omega) = M(\Omega)_*$ as required. (In fact, our argument constructs the pre-adjoint J_* explicitly, as $J_*(\mathbf{S}) := h$.) ☐

Appendix: Products of WOT-continuous representations

The arguments that follow are surely not new. We include them because they highlight that local compactness plays no role in Remark 2.1.

Throughout, $\mathbb{I}$ is a fixed indexing set, not necessarily countable. Let $(\mathcal{H}_i)_{i\in\mathbb{I}}$ be a family of Hilbert spaces, and let $\mathcal{M} := \prod_i^{\mathsf{B}} \mathcal{B}(\mathcal{H}_i)$. Of course, $\mathsf{ball}(\mathcal{M}) = \prod_i \mathsf{ball}(\mathcal{B}(\mathcal{H}_i))$ as sets. Form the ℓ^2-direct sum $\mathcal{H} = \ell^2\text{-}\bigoplus_{i\in\mathbb{I}} \mathcal{H}_i$. If we regard $\mathcal{M}$ as a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ in the natural way, via block-diagonal embedding, then we may equip $\mathsf{ball}(\mathcal{M})$ with the relative WOT inherited from $\mathcal{B}(\mathcal{H})$, which will be denoted by τ. On the other hand, if we let τ_i be the relative WOT on $\mathsf{ball}(\mathcal{B}(\mathcal{H}_i))$ for each $i \in \mathbb{I}$, this gives us a product topology on $\prod_{i\in\mathbb{I}} \mathsf{ball}(\mathcal{H}_i) = \mathsf{ball}(\mathcal{M})$.

Lemma A.1 (Products of unit balls in the WOT)**.** *The identity map* $\mathsf{id} : (\mathsf{ball}(\mathcal{M}), \tau) \to \prod_{i\in\mathbb{I}} (\mathsf{ball}(\mathcal{B}(\mathcal{H}_i)), \tau_i)$ *is a homeomorphism.*

Proof. We start by showing that id is continuous. By the universal property of product topologies, it suffices to show that for each $k \in \mathbb{I}$ the coordinate projection $P_k : \mathsf{ball}(\mathcal{M}) \to \mathsf{ball}(\mathcal{B}(\mathcal{H}_k))$ is continuous for the respective weak operator topologies. But this follows immediately from the identity $P_k(T)\xi_k = T\imath_k(\xi_k)$, where $\imath_k : \mathcal{H}_k \hookrightarrow \mathcal{H}$ is the natural embedding of Hilbert spaces.

Note that $\mathsf{ball}(\mathcal{B}(\mathcal{H}))$ is compact and Hausdorff in the relative WOT; and since $\mathcal{M}$ is WOT-closed in $\mathcal{B}(\mathcal{H})$, it is easily checked that $\mathsf{ball}(\mathcal{M})$ is WOT-closed in $\mathsf{ball}(\mathcal{B}(\mathcal{H}))$. Hence $(\mathsf{ball}(\mathcal{M}), \tau)$ is compact and Hausdorff, so that id, being a continuous surjection from a compact space to a Hausdorff space, is an open mapping. ☐

Now let G be a topological group and, for each $i \in \mathbb{I}$, let $\sigma_i : G_i \to \mathcal{U}(\mathcal{H}_i)$ be a WOT-continuous unitary representation. Let $\sigma = \prod_i \sigma_i$ be the direct product representation $G \to \mathcal{U}(\mathcal{H})$. By the universal property of product topologies, $\sigma : G \to \prod_i(\mathsf{ball}(\mathcal{B}(\mathcal{H}_i)), \tau_i)$ is continuous. Applying Lemma A.1, we conclude that σ is WOT-continuous as required.

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