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COMMUTING CONTRACTIVE IDEMPOTENTS IN MEASURE ALGEBRAS

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Dedicated to Professor Anthony To-Ming Lau, in honor of his contributions to, and leadership in, the international abstract harmonic analysis community, the Canadian mathematical community, and my career

Communicated by K. F. Taylor

ABSTRACT. We determine when contractive idempotents in the measure algebra of a locally compact group commute. We consider a dynamical version of the same result. We also look at some properties of groups of measures whose identity is a contractive idempotent.

Let G be a locally compact group. When G is abelian, Cohen [1] characterized all of the idempotents in the measure algebra $\mathrm{M}(G)$. For nonabelian G, the idempotent probabilities were characterized by Kawada and Itô [3], while the contractive idempotents were characterized by Greenleaf [2]. We give an exact statement of their results in Theorem 0.1 below. For certain compact groups, the central idempotent measures were characterized by Rider [7], in a manner which is pleasingly reminiscent of Cohen's result on abelian groups. Rider points out a counterexample to his result when some assumptions are dropped. This has motivated our Example 1.3(i) below.

Discussion of contactive idempotents has been conducted in the setting of locally compact quantum groups by Neufang, Salmi, Skalski, and the present author [5].

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Under certain assumptions, results of Stromberg [10] and Muhkerjea [4] show that convolution powers of a probability measure converge either to an idempotent or to 0 (see Theorem 2.1 and Remark 2.5 below). We study limits of convolution powers of products of contractive idempotents whose supports generate a compact subgroup.

We close with a study of certain groups of measures identified by Greenleaf [2] and Stokke [9] whose identities are contractive idempotents.

0.1. **Notation and background.** We will always let G denote a locally compact group with measure algebra $\mathrm{M}(G)$. We let $\mathcal{K}(G)$ denote the collection of all compact subgroups of G. For K in $\mathcal{K}(G)$, we let m_K denote the normalized Haar measure on K as an element of $\mathrm{M}(G)$. We will identify the group algebra $\mathrm{L}^1(K)$ as a subalgebra of $\mathrm{M}(G)$ via the identification $f \mapsto f m_K$; that is, for $u \in \mathcal{C}_0(G)$, we define

$$\int_G u\, d(f m_K) = \int_K u(k) f(k)\, dk,$$

where $dk = dm_K(k)$. For K in $\mathcal{K}(G)$, we let $\widehat{K}_1$ denote the space of multiplicative characters on K. Hence $\widehat{K}_1$ is the dual group of $K/[K,K]$, where $[K,K]$ is the closed commutator subgroup. If K is abelian, we will write $\widehat{K}$ for $\widehat{K}_1$.

Let us recall what is known about contractive idempotents.

Theorem 0.1.

(i) (Kawada and Itô [3]) *If μ in $\mathrm{M}(G)$ is a probability with $\mu * \mu = \mu$, then there is K in $\mathcal{K}(G)$ with $\mu = m_K$.*

(ii) (Greenleaf [2]) *If μ in $\mathrm{M}(G)$ is nonzero and contractive, $\|\mu\| \leq 1$, and $\mu * \mu = \mu$, then there is K in $\mathcal{K}(G)$ and ρ in $\widehat{K}_1$ for which $\mu = \rho m_K$.*

Observe that all measures above are self-adjoint:

$$\int_G u\, d(\rho m_K)^* = \int_K u(k^{-1})\overline{\rho(k)}\, dk = \int_K u(k)\rho(k)\, dk = \int_G u\, d(\rho m_K)$$

thanks to unimodularity of the compact group K.

1. Main result

In order to proceed, let us consider some conditions under which products of groups are groups.

Lemma 1.1. *Let $K_1, K_2 \in \mathcal{K}(G)$. Then the following are equivalent:*

(i) *$K_1K_2 = \{k_1k_2 : k_1 \in K_1, k_2 \in K_2\} \in \mathcal{K}(G)$,*

(ii) *K_1K_2 is closed under inversion, and*

(iii) *$K_1K_2 = K_2K_1$.*

Proof. Note first that K_1K_2 is always a compact subset of G which contains the identity e. If (i) holds, then (ii) holds. We have that $(K_1K_2)^{-1} = K_2K_1$, which immediately shows the equivalence of (ii) and (iii). Finally, if (iii) holds, then it is clear that K_1K_2 is closed under multiplication. Thus, since (iii) implies (ii), we see that K_1K_2 is closed under multiplication and inversion, and hence we obtain (i). □

We observe that $K_1K_2 \in \mathcal{K}(G)$ in the following situations:

(i) $K_1 \subset K_2$, and

(ii) $K_1 \subset N_G(K_2) = \{s \in G : sK_2s^{-1} = K_2\}$.

If $K_1 \cap K_2 = \{e\}$ and $K_1K_2 \in \mathcal{K}(G)$, then (K_1, K_2) is referred to as a *matched pair* (see [12]), and K_1K_2 is a *Zappa–Szép product* (see [14], [11]). Indeed, we note that the representation k_1k_2 of an element of K_1K_2 is unique, for if $k_1k_2 = k_1'k_2'$, then $(k_1')^{-1}k_1 = k_2'k_2^{-1} = e$. Since, in general, we will not assume that $K_1 \cap K_2 = \{e\}$, or even that this intersection is normal in K_1K_2, when the latter is a group, our situation appears to generalize that of a matched pair.

Is there a "nice" characterization of when $K_1K_2 \in \mathcal{K}(G)$?

To proceed, we will use a nonnormal form of the Weyl integration formula. If H is a locally compact group and $L \in \mathcal{K}(H)$, then any continuous multiplicative function $\delta : L \to \mathbb{R}^{>0}$ is trivial. Thus the modular function Δ of H satisfies $\Delta|_L = 1$, which is the modular function of L. Hence the left homogeneous space H/L admits a left H-invariant Haar measure $m_{H/L}$. We have for u in $\mathcal{C}_c(H)$ that

$$\int_H u(h)\,dh = \int_{H/L}\int_L u(hl)\,dl\,d(hL), \tag{1.1}$$

where $d(hL) = dm_{H/L}(hL)$.

Theorem 1.2. *Let $K_1, K_2 \in \mathcal{K}(G)$, $\rho_1 \in \widehat{(K_1)}_1$, and $\rho_2 \in \widehat{(K_2)}_1$. Then $\rho_1 m_{K_1}$ and $\rho_2 m_{K_2}$ commute if and only if one of the following cases holds for $K = K_1 \cap K_2$:*

(i) *$\rho_1|_K \neq \rho_2|_K$, in which case $(\rho_1 m_{K_1}) * (\rho_2 m_{K_2}) = 0$, or*

(ii) *$\rho_1|_K = \rho_2|_K$, $K_1K_2 \in \mathcal{K}(G)$, and the function*

$\rho : K_1K_2 \to \mathbb{C}$ given by $\rho(k_1k_2) = \rho_1(k_1)\rho_2(k_2)$ for k_1 in K_1 and k_2 in K_2

defines a character, in which case $(\rho_1 m_{K_1}) * (\rho_2 m_{K_2}) = \rho m_{K_1K_2}$.

In particular, the idempotent probabilities m_{K_1} and m_{K_2} commute if and only if $K_1K_2 \in \mathcal{K}(G)$, in which case we have $m_{K_1} * m_{K_2} = m_{K_1K_2}$.

Proof. We let $\nu = (\rho_1 m_{K_1}) * (\rho_2 m_{K_2})$. Notice that

$$\rho_1 m_{K_1} \text{ and } \rho_2 m_{K_2} \text{ commute if and only if } \nu^* = \nu. \tag{1.2}$$

For u in $\mathcal{C}_0(G)$ we have

$$\begin{aligned}\int_G u\,d\nu &= \int_{K_1}\int_{K_2} u(k_1k_2)\rho_1(k_1)\rho(k_2)\,dk_1\,dk_2 \\ &= \int_{K_1/K}\int_K\int_{K_2} u(k_1kk_2)\rho_1(k_1k)\rho(k_2)\,dk_2\,dk\,d(k_1K) \\ &= \int_{K_1/K}\int_{K_2} u(k_1k_2)\int_K \rho_1(k_1k)\rho_2(k^{-1}k_2)\,dk\,dk_2\,d(k_1K) \\ &= \int_{K_1/K}\int_{K_2}\Big[\int_K \rho_1(k)\overline{\rho_2(k)}\,dk\Big]u(k_1k_2)\rho_1(k_1)\rho_2(k_2)\,dk_2\,d(k_1K).\end{aligned} \tag{1.3}$$

The orthogonality of characters entails that the quantity $\int_K \rho_1(k)\overline{\rho_2(k)}\,dk$ is either 1 or 0, depending on whether $\rho_1|_K = \rho_2|_K$ or not. In the latter case, we see that

$\nu = 0$, and hence $(\rho_2 m_{K_2}) * (\rho_1 m_{K_1}) = \nu^* = 0 = \nu$, and we see that condition (i) holds.

Hence for the remainder of the proof, let us suppose that $\rho_1|_K = \rho_2|_K$. Then the function $\rho : K_1K_2 \to \mathbb{T}$ given as in (ii) is well defined. Indeed, if $k_1k_2 = k_1'k_2'$, then $(k_1')^{-1}k_1 = k_2'k_2^{-1} \in K$, and our assumption allows us to apply ρ_1 to the left, and ρ_2 to the right, to gain the same result. Furthermore, $(k_1, k_2) \mapsto \rho_1(k_1)\rho_2(k_2) = \rho(k_1k_2) : K_1 \times K_2 \to \mathbb{T}$ is continuous and hence factors continuously through the topological quotient space K_1K_2 of $K_1 \times K_2$.

We now wish to show that $\operatorname{supp}\nu = K_1K_2$. The inclusion $\operatorname{supp}\nu \subseteq K_1K_2$ is standard. Conversely, if k_1^o in K_1, k_2^o in K_2, and $\varepsilon > 0$ are given and we let $u, v \in \mathcal{C}_0(G)$ satisfy

$$u \geq 0 \qquad \text{and} \qquad u(k_1^o k_2^o) > \varepsilon > 0, \qquad \text{and} \qquad v|_{K_1K_2} = \bar{\rho},$$

then we may find open U_j in K_j containing k_j^o, for $j = 1, 2$, so that $U_1 \times U_2 \subseteq \{(k_1, k_2) \in K_1 \times K_2 : u(k_1k_2) > \varepsilon\}$, and our assumptions entail that

$$\begin{aligned}\int_G uv\, d\nu &= \int_{K_1}\int_{K_2} u(k_1k_2)\, dk_1\, dk_2 \\ &\geq \int_{U_1}\int_{U_2} u(k_1k_2)\, dk_1\, dk_2 \geq m_{K_1}(U_1)m_{K_2}(U_2)\varepsilon > 0.\end{aligned}$$

Hence $K_1K_2 \subseteq \operatorname{supp}\nu$. Notice that if it were the case that $\nu = 0$, this would contradict our present calculation of $\operatorname{supp}\nu$, and hence the assumption that $\rho_1|_K = \rho_2|_K$. Thus $\nu = 0$ only when $\rho_1|_K \neq \rho_2|_K$, showing that (i) fully characterizes this situation. We observe that

$$\operatorname{supp}\nu^* = (\operatorname{supp}\nu)^{-1} = K_2K_1. \tag{1.4}$$

Let us now assume that $\rho_1 m_{K_1}$ and $\rho_2 m_{K_2}$ commute. Then, by (1.2), $\nu = \nu^*$ and hence by (1.4) and Lemma 1.1, we have that $K_1K_2 \in \mathcal{K}(G)$. To complete the calculation we observe the following isomorphism of left K_1-spaces, generalizing the second isomorphism theorem of groups:

$$K_1K_2/K_2 \cong K_1/K, \quad kK_2 \mapsto kK. \tag{1.5}$$

Hence for $w \in \mathcal{C}(K_1K_2)$, which is constant of left cosets of K_2, we have $\int_{K_1K_2/K_2} w(k)\, d(kK_2) = \int_{K_1/K} w(k_1)\, d(k_1K)$ for the unique choices of left-invariant probability measures on the homogeneous spaces. We thus find that, for u in $\mathcal{C}_0(G)$,

$$\begin{aligned}\int_G u\, d\nu &= \int_{K_1/K}\int_{K_2} u(k_1k_2)\rho(k_1k_2)\, dk_2\, d(k_1K) \\ &= \int_{K_1K_2/K_2}\int_{K_2} u(k_1k_2)\rho(k_1k_2)\, dk_2\, d(kK_2) \\ &= \int_{K_1K_2} u(k)\rho(k)\, dk = \int_G u\, d(\rho m_{K_1K_2}),\end{aligned} \tag{1.6}$$

and so $\nu = \rho m_{K_1K_2}$. Since $\nu * \nu = \nu$, as $\rho_1 m_{K_1}$ and $\rho_2 m_{K_2}$ commute, and $m_{K_1K_2}$ is the normalized Haar measure of a compact subgroup, it follows that

$(\rho m_{K_1K_2}) * (\rho m_{K_1K_2}) = (\rho * \rho)m_{K_1K_2}$, whence $\rho = \rho * \rho$. We could simply appeal to Theorem 0.1(ii), to see that, since $\|\nu\| \leq \|\rho_1 m_{K_1}\|\|\rho_2 m_{K_2}\| = 1$, $\rho \in (\widehat{K_1K_2})_1$. However, let us give a direct verification, using only the present tools. We may interchange the roles of K_1 and K_2 above, and define $\tilde{\rho} : K_2K_1 \to \mathbb{T}$ by $\tilde{\rho}(k_2k_1) = \rho_2(k_2)\rho_1(k_1)$, which, like ρ, is well defined and continuous. We also see, by the computation (1.6), that $\nu = \tilde{\rho}m_{K_2K_1} = \tilde{\rho}m_{K_1K_2}$. Hence $\tilde{\rho} = \rho$ on K_1K_2. But it then follows that ρ is a homomorphism: if $k = k_1k_2$, $l = l_1l_2$, $k_1, l_1 \in K_1$, $k_2, l_2 \in K_2$, then we have $k_2l_1 = l_1'k_2'$ for some l_1' in K_1 and k_2' in K_2, and hence

$$\begin{aligned}\rho(k_1k_2l_1l_2) &= \rho_1(k_1l_1')\rho_2(k_2'l_2) = \rho_1(k_1)\rho(l_1'k_2')\rho_2(l_2) \\ &= \rho_1(k_1)\tilde{\rho}(k_2l_1)\rho_2(l_2) = \rho_1(k_1)\rho_2(k_2)\rho_1(l_1)\rho_2(l_2) = \rho(k_1k_2)\rho(l_1l_2).\end{aligned}$$

Conversely, if the conditions of (ii) are assumed, then computations (1.3) and (1.6) show that $(\rho_1 m_{K_1}) * (\rho_2 m_{K_2}) = \rho m_{K_1K_2}$ and show the same with the roles of $\rho_1 m_{K_1}$ and $\rho_2 m_{K_2}$, reversed. $\square$

Example 1.3. (i) Let $G = K \rtimes A$, where A is a compact group acting as continuous automorphisms on the group K, so we obtain group law $(k, \alpha)(k', \beta) = (k\alpha(k'), \alpha\beta)$. We identify K and A with their cannonical copies in G. Suppose there is ρ in $\widehat{K}_1$ for which $\rho \circ \alpha \neq \rho$ for some α in A, and hence for α on an open subset of A. (A specific example would be to take $K = \mathbb{T}$, $A = \{\mathrm{id}, \sigma\}$ where $\sigma(t) = t^{-1}$, and $\rho(t) = t^n$ where $n \in \mathbb{Z} \setminus \{0\}$.) Then for $u \in \mathcal{C}(G)$ we obtain for ρ as above,

$$\int_G u\, d\big[(\rho m_K) * m_A\big] = \int_K \int_A u(k, \alpha)\rho(k)\, d\alpha\, dk,$$

while, since the modular function on the compact group A qua automorphisms on K is 1, we have

$$\begin{aligned}\int_G u\, d\big[m_A * (\rho m_K)\big] &= \int_A \int_K u\big(\alpha(k), \alpha\big)\rho(k)\, dk\, d\alpha \\ &= \int_A \int_K u(k, \alpha)\rho \circ \alpha^{-1}(k)\, dk\, d\alpha.\end{aligned}$$

Thus ρm_K and m_A do not commute. The only assumption missing from Theorem 1.2 is that $(k, \alpha) \mapsto \rho(k)$ is not a character on G.

(ii) Let $n \geq 5$, and let S_n denote the symmetric group on a set of n elements, let S_{n-1} denote the stabilizer subgroup of any fixed element, and let C be the cyclic subgroup generated by any full n-cycle. Then $S_n = S_{n-1}C$, as may be easily checked, and $\{S_{n-1}, C\}$ is a "nontrivial" matched pair in the sense that neither subgroup is normal in G.

We note that the only nontrivial coabelian normal subgroup of S_n is $A_n = \ker \mathrm{sgn}$, as A_n is simple and of index 2; hence $(\widehat{S_n})_1 = \{1, \mathrm{sgn}\}$. Hence if ρ_2 in $\widehat{C} \setminus \{1\}$ satisfies $\rho_2 \neq \mathrm{sgn}|_C$, then, for any ρ_1 in $(\widehat{S_{n-1}})_1$, it follows from Theorem 1.2 that $(\rho_1 m_{S_{n-1}}) * (\rho_2 m_C) \neq (\rho_2 m_C) * (\rho_1 m_{S_{n-1}})$.

2. Dynamical considerations

If S is a subset of G, let $\langle S\rangle$ denote the smallest closed subgroup containing S.

Theorem 2.1 (Stromberg [10])**.** *If μ is a probability in $\mathrm{M}(G)$, for which $K = \langle \operatorname{supp}\mu\rangle \in \mathcal{K}(G)$, then the weak* limit, $\lim_{n\to\infty}\mu^{*n}$, exists if and only if $\operatorname{supp}\mu$ is contained in no coset of a closed proper normal subgroup of K. Moreover, this limit equals the Haar measure m_K.*

We observe that $\operatorname{supp}\mu^* = (\operatorname{supp}\mu)^{-1}$, and hence in the assumptions above we have $\lim_{n\to\infty}(\mu^*)^{*n} = m_K$, too.

Since $\operatorname{supp}(m_K * m_L) = KL$, as was checked in the proof of Theorem 1.2, it follows that for K, L in $\mathcal{K}(G)$ for which $\langle KL\rangle$ is compact, we have $\lim_{n\to\infty}(m_K * m_L)^{*n} = m_{\langle KL\rangle} = \lim_{n\to\infty}(m_L * m_K)^{*n}$. For example, in $S = \mathrm{SU}(2)$, any two distinct (maximal) tori T_1 and T_2 generate S. Indeed, the only subgroups of S with nontrivial connected components are tori, or S itself. Hence $m_S = \lim_{n\to\infty}(m_{T_1} * m_{T_2})^{*n}$.

Furthermore, we can deduce from the observation above that m_L and m_K commute if and only if $KL = \langle KL\rangle$, giving the special case of Theorem 1.2.

Motivated by the above considerations, we consider the following dynamical result.

Theorem 2.2. *Let $K_j \in \mathcal{K}(G)$ and $\rho_j \in \widehat{(K_j)}_1$ for $j = 1,\dots,m$ for which $L = \langle K_1\cdots K_m\rangle \in \mathcal{K}(G)$. Then the weak* limit*

$$\lim_{n\to\infty}\left[(\rho_1 m_{K_1}) * \cdots * (\rho_m m_{K_m})\right]^{*n}$$

always exists. It is ρm_L, provided there is a ρ in $\widehat{L}_1$ for which $\rho|_{K_j} = \rho_j$ for each j, and 0 otherwise.

Proof. We let $\nu = (\rho_1 m_{K_1})*\cdots*(\rho_m m_{K_m})$. Then each ν^{*n}, being a product of contractive elements, satisfies $\|\nu^{*n}\| \le 1$. The Peter–Weyl theorem tells us that the algebra $\mathrm{Trig}(L)$ consiting of matrix coefficients of finite-dimensional unitary representations, is uniformly dense in in $\mathcal{C}(L)$. Hence, since $\operatorname{supp}\nu \subseteq L$ and $\|\nu\| \le 1$, and $\|\nu^{*n}\| \le 1$ for each n, it suffices to determine, for any finite-dimensional unitary representation $\pi : L \to \mathrm{U}(d)$, the nature of the limit

$$\lim_{n\to\infty}\pi(\nu^{*n}) = \lim_{n\to\infty}\int_L \pi(l)\, d\nu^{*n}(l) \quad \text{in } M_d(\mathbb{C}). \tag{2.1}$$

It is well known, and simple to compute, that each

$$\pi(\nu^{*n}) = \pi(\nu)^n = \left[\pi(\rho_1 m_{K_1})\cdots\pi(\rho_1 m_{K_1})\right]^n.$$

For each $j = 1,\dots,m$, the Schur orthogonality relation tell us that

$$\pi(\rho_j m_{K_j}) = \int_{K_j}\rho_j(k)\pi(k)\, dk = p_j,$$

where p_j is the orthogonal projection onto the space of vectors ξ for which $\pi(k)\xi = \overline{\rho_j(k)}\xi$ for each k in K_j. Hence it follows that

$$\pi(\nu) = p_1\cdots p_m \qquad \text{and} \qquad \pi(\nu^{*n}) = (p_1\cdots p_n)^n.$$

Since each p_j is contractive, the eigenvalues of $\pi(\nu)$ are of modulus not exceeding 1. Furthermore, if $\|\pi(\nu)\xi\|_2 = \|\xi\|_2$ (Hilbertian norm), then we find that

$$\|\xi\|_2 = \|p_1 \cdots p_m \xi\|_2 \leq \|p_2 \cdots p_m \xi\|_2 \leq \cdots \leq \|p_m \xi\|_2 \leq \|\xi\|_2,$$

and so equality holds at each place. But we see then that ξ is in the range of p_m, hence of p_{j-1} if it is in the range of p_j, and thus in the mutual range R_π of each of $p_1, \ldots, p_m$. In particular, each eigenvalue of $\pi(\nu)$ is either 1 or has modulus strictly less than 1. Hence, if we consider the Jordan form of $\pi(\nu) = p_1 \cdots p_m$, we see that $\lim_{n\to\infty} \pi(\nu)^n = q$, where q is the necessarily contractive, hence orthogonal, range projection onto R_π. But then, for ξ in R_π and k_j in K_j, $j = 1, \ldots, m$, we have

$$\pi(k_1 \cdots k_n)\xi = \pi(k_1) \cdots \pi(k_m)\xi = \overline{\rho_1(k_1)} \cdots \overline{\rho_n(k_n)}\xi.$$

If we have $\xi \neq 0$, then $\mathbb{C}\xi$ is $\pi(K_1 \cdots K_m)$-invariant, and hence π-invariant as $L = \langle K_1 \cdots K_m \rangle$. Moreover, there is, then, ρ in $\widehat{L}_1$ for which $\pi(l)\xi = \rho(l)\xi$, and it follows that $\rho|_{K_j} = \rho_j$. Notice that this ρ is determined independently of the choice of ξ, and hence even the choice of π. In particular, if no such ρ exists (i.e., for every finite-dimensional unitary representation, $R_\pi = \{0\}$), then we have $\lim_{n\to\infty} \nu^{*n} = 0$, in the weak* sense. When this ρ does exists, we see for u in $\mathcal{C}_0(G)$ that each $\int_G u\, d(\nu^{*n})$ is given by the nm-fold iterated integral

$$\begin{aligned}
&\int_{K_1} \cdots \int_{K_m} \cdots \int_{K_1} \cdots \int_{K_m} u(k_{11} \cdots k_{1m} \cdots k_{n1} \cdots k_{nm}) \\
&\qquad \rho_1(k_{11}) \cdots \rho_m(k_{1m}) \cdots \rho_1(k_{n1}) \cdots \rho_m(k_{nm})\, dk_{nm} \cdots dk_{n1} \cdots dk_{1m} \cdots dk_{11} \\
&\quad = \int_{K_1} \cdots \int_{K_m} \cdots\cdots \int_{K_1} \cdots \int_{K_m} u(k_{11} \cdots k_{nm}) \rho(k_{11} \cdots k_{nm})\, dk_{nm} \cdots dk_{11} \\
&\quad = \int_G u\rho\, d\big([m_{K_1} * \cdots * m_{K_m}]^{*n}\big).
\end{aligned} \tag{2.2}$$

It is easy to verify, as in the proof of Theorem 1.2, that $\sup(m_{K_1} * \cdots * m_{K_m}) = K_1 \cdots K_m$. Hence, by Theorem 2.1, we obtain the weak* limit

$$\lim_{n\to\infty} \nu^{*n} = \rho m_L$$

as desired. □

In fact, the above result generalizes the necessity direction of Theorem 1.2.

Corollary 2.3. *Let K_j and ρ_j, $j = 1, \ldots, m$, be as in Theorem 2.2 above, and let $L = K_1 \cdots K_m$. If $\nu = (\rho_1 m_{K_1}) * \cdots * (\rho_m m_{K_m})$ is idempotent, then either $\nu = 0$ or $L = \langle L \rangle \in \mathcal{K}(G)$ and there is ρ in $\widehat{L}_1$ with $\rho|_{K_j} = \rho_j$ for each j.*

Proof. Suppose $\nu \neq 0$. By a similar method as in the proof of Theorem 1.2, we see that $\operatorname{supp} \nu = L$. Moreover, if ν is idempotent, then $\lim_{n\to\infty} \nu^{*n} = \nu$. Hence we obtain that $L = \langle L \rangle$, and there exists a multiplicative character ρ on L, as promised, thanks to Theorem 2.2. □

Though Corollary 2.3 generalizes the necessity direction of Theorem 1.2, the proof of the earlier result is more self-contained, and does not rely on Stromberg's result. Furthermore, the sufficiency direction of Theorem 1.2 cannot be generalized so easily, even with probability idempotent measures.

Example 2.4. The special orthogonal group $S = \mathrm{SO}(3)$ admits the well-known Euler angle decomposition: $S = T_1T_2T_1$, where

$$T_1 = \left\{ k_1(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix} : 0 \leq t \leq 2\pi \right\} \qquad \text{and}$$

$$T_2 = \left\{ k_2(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} : 0 \leq t \leq 2\pi \right\}.$$

We note that the multiplication $T_1 \times (T_2/\{I, k_2(\pi)\}) \times T_1 \to S$ is a diffeomorphism. For u in $\mathcal{C}(S)$, we have

$$\int_{T_1}\int_{T_2}\int_{T_1} u\, d(m_{T_1} * m_{T_2} * m_{T_1}) = \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} u\big(k_1(t_1)k_2(t_2)k_1(t_3)\big)\frac{dt_3\, dt_2\, dt_1}{8\pi^3},$$

whereas the Haar measure m_S gives the integral

$$\int_S u\, dm_S = \int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} u\big(k_1(t_1)k_2(t_2)k_1(t_3)\big)\sin t_2 \frac{dt_3\, dt_2\, dt_1}{8\pi^2}.$$

Hence, considering T_1-spherical functions (i.e., u in $\mathcal{C}(T_1\backslash S/T_1)$), we see that

$$m_{T_1T_2T_1} = m_S \neq m_{T_1} * m_{T_2} * m_{T_1}.$$

Remark 2.5. We note the following result, shown (implicitly) by Muhkerjea [4, Theorem 2]. *If μ is a probability in $\mathrm{M}(G)$, for which $\langle \operatorname{supp}\mu\rangle \notin \mathcal{K}(G)$, then the weak* limit satisfies* $\lim_{n\to\infty} \mu^{*n} = 0$.

Hence if $K_1, \ldots, K_m$ in $\mathcal{K}(G)$ have $\langle K_1 \cdots K_m\rangle \notin \mathcal{K}(G)$, we see that $\lim_{n\to\infty}(m_{K_1} * \cdots * m_{K_m})^{*n} = 0$, which is rather antithetical to having $m_{K_1} * \cdots * m_{K_m}$ be an idempotent.

As a simple example, consider any two nontrivial finite subgroups K and L of discrete groups Γ and Λ, and consider each as a subgroup of the free product $\Gamma * \Lambda$. For a Lie theoretic example, consider the Iwasawa decomposition KAN of $S = \mathrm{SL}_2(\mathbb{R})$. Compute that if $a \in A \setminus \{I\}$, then $aKa^{-1} \neq K$. Since K is maximal compact, we see that $\langle KaKa^{-1}\rangle \notin \mathcal{K}(S)$.

It is the case that if for $K_1, \ldots, K_m$ in $\mathcal{K}(G)$ we have $H = \langle K_1 \cdots K_m\rangle \notin \mathcal{K}(G)$, then for ρ_j in $\widehat{(K_j)}_1$, $j = 1, \ldots, m$, the weak* limit satisfies

$$\lim_{n\to\infty}\big[(\rho_1 m_{K_1}) * \cdots * (\rho_m m_{K_m})\big]^{*n} = 0. \tag{2.3}$$

In the case that $\rho_i|_{K_i\cap K_j} \neq \rho_j|_{K_i\cap K_j}$ for some $i \neq j$, we have $(\rho_1 m_{K_1}) * \cdots * (\rho_m m_{K_m}) = 0$, as may be computed, by a straightforward adaptation of (1.3). If there is a continuous multiplicative character $\rho : H \to \mathbb{T}$ such that $\rho|_{K_j} = \rho_j$ for each j, then the computation (2.2) and Mukherjea's theorem give the result. In presence or absence of these assumptions, (2.3) follows from a result which should appear in a future work of Neufang, Salmi, Skalski, and the present author. In fact, the same result implies Theorem 2.2. However, the proof given in the present note uses simpler methods.

3. On groups of measures

Greenleaf's motivation for studying idempotent measures was their use in the study of contractive homomorphisms $\mathrm{L}^1(H) \to \mathrm{M}(G)$. In doing so, he required a description of certain groups of measures, given in Theorem 3.1 below. We are interested in determining how these groups interact under convolution product with each other. Stokke [9] conducted a study of Greenleaf's groups, and also devised a more general class of groups (see (3.1)). We show that the latter class is indeed more general.

For any subgroup H of G, let

$$N_G(H) = \{g \in G : gHg^{-1} = H\} \qquad \text{and}$$
$$Z_G(H) = \{g \in G : gh = hg \text{ for all } h \text{ in } H\}$$

denote its normalizer and centralizer, respectively. Notice that, for another subgroup L, we have $L \subseteq Z_G(H)$ if and only if $H \subseteq Z_G(L)$. Notice too that for the topological closure $\overline{H}$, we have $N_G(\overline{H}) = N_G(H)$ and $Z_G(\overline{H}) = Z_G(H)$, and hence these subgroups are closed.

Given K in $\mathcal{K}(G)$ and ρ in $\widehat{K}_1$, we let

$$N_{K,\rho} = N_G(K) \cap N_G(\ker\rho)$$

and then let $q : N_{K,\rho} \to N_{K,\rho}/\ker\rho$ be the quotient map. We let

$$G_{K,\rho} = q^{-1}\big(Z_{N_{K,\rho}/\ker\rho}(K/\ker\rho)\big).$$

Hence g in $G_{K,\rho}$ normalizes both K and $\ker\rho$ and commutes with elements of K modulo $\ker\rho$. We then consider, in $\mathrm{M}(G)$, the subgroup

$$\Gamma_{\rho m_K} = \big\{z\delta_g * (\rho m_K) : z \in \mathbb{T} \text{ and } g \in G_{K,\rho}\big\}.$$

We remark that $G_{K,\rho} = \{g \in G : \delta_g * (\rho m_K) = (\rho m_K) * \delta_g\}$, and $\Gamma_{\rho m_K}$ is a topological group with the weak*-topology on $\mathrm{M}(G)$ and multiplication $z\delta_g * (\rho m_K) * z'\delta_{g'} * (\rho m_K) = zz'\delta_{gg'} * (\rho m_K)$.

Theorem 3.1.

(i) (Greenleaf [2]) *Any closed group of contractive measures has identity of the form ρm_K of Theorem 0.1(ii) and is a subgroup of $\Gamma_{\rho m_K}$.*

(ii) (Stokke [9], after [2]) *The map*

$$(z, g) \mapsto z\delta_g * (\rho m_K) : \mathbb{T} \times G_{K,\rho} \to \Gamma_{\rho m_K}$$

is continuous and open and with compact kernel $\{(\rho(k), k) : k \in K\} \cong K$.

Remark 3.2. We give a mild simplification of Stokke's argument, which will help us below.

(i) *Let*

$$\Omega_{K,\rho} = (\mathbb{T} \times G_{K,\rho})/\big\{\big(\rho(k), k\big) : k \in K\big\}.$$

Then the one-point compactification $\Omega_{K,\rho} \sqcup \{\infty\}$ (resp., topological coproduct, if $G_{K,\rho}$ is compact) is homeomorphic to $\Gamma_{\rho m_K} \cup \{0\}$.

Indeed, consider the semigroup homomorphism on $(\mathbb{T} \times G_{K,\rho}) \sqcup \{\infty\}$ given by $(z, g) \mapsto z\delta_g * (\rho m_K)$, $\infty \mapsto 0$, which has kernel $\{(\rho(k), k) : k \in K\}$ at the identity—a fact which we will take for granted, thanks to arguments in [9] and [2]. It suffices to verify that this semigroup homorphism is continuous and that $\Gamma_{\rho m_K} \cup \{0\}$ is weak*-compact. Let (z_i, g_i) be a net in $\mathbb{T} \times G_{K,\rho}$ such that $z_i\delta_{g_i} * (\rho m_K) \to \mu$ in i. If (g_i) is unbounded in $G_{K,\rho}$, we may pass to subnet and assume $g_i \to \infty$. But then, for u in $\mathcal{C}_0(G)$, $(u(g_i\cdot))$ converges to zero uniformly on compact sets, thanks to uniform continuity of u. It follows that $\mu = 0$. Otherwise, (g_i) is bounded in $G_{K,\rho}$, and by passing to subnet, we may assume that $(z_i, g_i) \to (z, g)$ in $\mathbb{T} \times G_{K,\rho}$. But then, for u in $\mathcal{C}_0(G)$, $(u(g_i\cdot))$ converges to $u(g\cdot)$ uniformly on compact sets, and it follows that $\mu = z\delta_g * (\rho m_K)$. Notice that any limit point of a net in $\Gamma_{\rho m_K}$ is in $\Gamma_{\rho m_K} \cup \{0\}$, so the latter set is weak*-closed, and hence weak*-compact, as it is a subset of the weak*-compact unit ball of $\mathrm{M}(G)$.

(ii) *If H is any closed subgroup of $G_{K,\rho}$, then*

$$\big((\mathbb{T} \times H)/\{\big(\rho(k), k\big) : k \in K \cap H\}\big) \sqcup \{\infty\}$$

is homeomorphic to $\{z\delta_g * (\rho m_K) : z \in \mathbb{T} \text{ and } g \in H\} \cup \{0\}$. Moreover, the latter set is weak-compact.* These facts are immediate from (i) above.

For a set Σ of contractive measures, let $[\Sigma]$ denote the smallest weak*-closed semigroup containing Σ.

Proposition 3.3. *Suppose that K_1, K_2, ρ_1, and ρ_2 satisfy the conditions of Theorem 1.2(ii), and let ρ be as given there. Then*

$$[\Gamma_{\rho_1 m_{K_1}} * \Gamma_{\rho_2 m_{K_2}}] \cap \Gamma_{\rho m_{K_1K_2}} = \big\{z\delta_g * (\rho m_{K_1K_2}) : z \in \mathbb{T}, g \in \langle H_1H_2\rangle\big\},$$

where $H_1 = G_{K_1,\rho_1} \cap G_{K_1K_2,\rho}$ and $H_2 = G_{K_2,\rho_2} \cap G_{K_1K_2,\rho}$.

Proof. Let us record some observations about contractive idempotents. First we have that $\mathrm{supp}(\rho m_K) = K$. If g in G and z in $\mathbb{T}$ are such that $z\delta_g*(\rho m_K) = \rho' m_{K'}$, then $gK = \mathrm{supp}(z\delta_g * (\rho m_K)) = \mathrm{supp}(\rho' m_{K'}) = K'$, so $K = K'$ and $g \in K$.

To see the inclusion of the first set into the second, let $g_1 \in G_{K_1,\rho_1}$, $g_2 \in G_{K_2,\rho_2}$. Then we compute

$$\delta_{g_1} * (\rho_1 m_{K_1}) * \delta_{g_2} * (\rho_2 m_{K_2}) = \delta_{g_1} * (\rho m_{K_1K_2}) * \delta_{g_2} = \delta_{g_1g_2} * \delta_{g_2^{-1}} * (\rho m_{K_1K_2}) * \delta_{g_2},$$

where $\delta_{g_2^{-1}} * (\rho m_{K_1K_2}) * \delta_{g_2}$ is a contractive idempotent. If we assume that there is g in $G_{K_1K_2,\rho}$ and z in $\mathbb{T}$, for which

$$\delta_{g_1g_2} * \delta_{g_2^{-1}} * (\rho m_{K_1K_2}) * \delta_{g_2} = z\delta_g * (\rho m_{K_1K_2}),$$

then it follows from the argument in the paragraph above that $g^{-1}g_1g_2 \in K_1K_2$. Hence $g^{-1}g_1 \in K_1K_2 \subseteq G_{K_1K_2,\rho}$ so $g_1 \in H_1$. Also, as $g \in N_G(K_1K_2)$, we have $g_2 \in K_1K_2g \subseteq G_{K_1K_2,\rho}$, and we obtain that $g_2 \in H_2$. By Remark 3.2(ii), any nonzero limit of products of elements of $\{z\delta_g * (\rho m_{K_1K_2}) : z \in \mathbb{T}, g \in \langle H_1H_2\rangle\}$ remains in that set.

To see the reverse inclusion, we let $g_1 \in H_1$ and $g_2 \in H_2$ and we observe that

$$\delta_{g_1 g_2} * (\rho m_{K_1 K_2}) = \delta_{g_1} * (\rho m_{K_1 K_2}) * \delta_{g_2} = \delta_{g_1} * (\rho_1 m_{K_1}) * \delta_{g_2} * (\rho_2 m_{K_2}).$$

We use Remark 3.2(i), to see that nonzero limits of products of such elements remain in $\Gamma_{\rho m_{K_1 K_2}}$.

Either argument above can be easily redone, multiplied by elements of $\mathbb{T}$. □

Example 3.4. (i) In the notation above, suppose that $G_{K_1,\rho_1} = G$. This happens, for example, if K_1 is in the center of G. Indeed, then $\ker \rho_1$ is in the center of G, and $K_1/\ker \rho_1$ is in the center of $G/\ker \rho_1$. Then, in the assumption of Proposition 3.3, we have $G_{K_1,\rho_1} \cap G_{K_1K_2,\rho} = G_{K_1K_2,\rho}$, and hence

$$[\Gamma_{\rho_1 m_{K_1}} * \Gamma_{\rho_2 m_{K_2}}] \cap \Gamma_{\rho m_{K_1 K_2}} = \Gamma_{\rho m_{K_1 K_2}}.$$

(ii) In the notation above, we always have that $K_1 \subseteq G_{K_1,\rho_1}$ and $K_2 \subseteq G_{K_2,\rho_2}$. Hence, if $G = K_1 K_2$, then by Proposition 3.3, we have

$$[\Gamma_{\rho|_{K_1} m_{K_1}} * \Gamma_{\rho|_{K_2} m_{K_2}}] \cap \Gamma_{\rho m_G} = \Gamma_{\rho m_G}$$

for any $\rho \in \widehat{G}_1$. This works even for "nontrivial" matched pairs in the sense of Example 1.3(ii).

(iii) Let T be any nontrivial compact abelian group, let σ be given on $T \times T$ by $\sigma(t_1,t_2) = (t_2,t_1)$, and let $G = (T \times T) \rtimes \{\mathrm{id},\sigma\}$. Let $\rho_1, \rho_2 \in \widehat{T}$ (dual group of T), so $\rho_1 \times \rho_2 \in \widehat{T \times T}$. Then $N_G(T \times \{e\}) = T \times T$ is abelian, and hence it is easy to follow the definition to see $G_{T\times\{e\},\rho_1} = T \times T$. By symmetry, $G_{\{e\}\times T,\rho_2} = T \times T$, as well.

On the other hand, $N_G(T \times T) = G$, and $\sigma(\ker \rho_1 \times \rho_2) = \ker \rho_1 \times \rho_2$, so $N_G(\ker \rho_1 \times \rho_2) = G$. Also,

$$G/\ker \rho_1 \times \rho_2 = \big[(T \times T)/\ker \rho_1 \times \rho_2\big] \rtimes \{\mathrm{id},\sigma\} \cong \rho_1 \times \rho_2(T \times T) \times \{\mathrm{id},\sigma\}$$

is abelian; that is, σ acts trivially on the image $\rho_1\times\rho_2(T\times T) \cong (T\times T)/\ker \rho_1 \times \rho_2$. Hence $G_{T\times T,\rho_1\times\rho_2} = G$. Thus, by Proposition 3.3, we have

$$[\Gamma_{\rho_1 m_{T\times\{e\}}} * \Gamma_{\rho_2 m_{\{e\}\times T}}] \cap \Gamma_{(\rho_1\times\rho_2) m_{T\times T}} \subsetneq \Gamma_{(\rho_1\times\rho_2) m_{T\times T}}.$$

We now consider some groups of measures considered in [9]. For K in $\mathcal{K}(G)$ and ρ in $\widehat{K}_1$, let

$$\mathcal{M}_{\rho m_K} = \big\{\nu \in \mathrm{M}(G) : \nu^* * \nu = \rho m_K = \nu * \nu^*\big\}. \tag{3.1}$$

Notice that, if $\nu \in \mathcal{M}_{\rho m_K}$, then the operator $\xi \mapsto \nu * \xi$ on $\mathrm{L}^2(G)$ is a partial isometry with support and range projection $\xi \mapsto (\rho m_K) * \xi$. Since the injection $\nu \mapsto (\xi \mapsto \nu * \xi)$ from $\mathrm{M}(G)$ into bounded operators on $\mathrm{L}^2(G)$ is injective, it follows that, for ν in $\mathcal{M}_{\rho m_K}$, $\nu * (\rho m_K) = \nu = (\rho m_K) * \nu$. We call $\mathcal{M}_{\rho m_K}$ the *intrinsic unitary group* at ρm_K. It is clear that $\Gamma_{\rho m_K} \subseteq \mathcal{M}_{\rho m_K}$.

Our goal is to make a modest determination of the scope of $\mathcal{M}_{m_K}$ for an idempotent probability measure. We begin with an analogue of a well-known characterization of the structure of the connected component of the invertible group of a Banach algebra. This lemma plays more of a role in motivating the methods below than in producing a result that we shall use directly.

Lemma 3.5. *Let H be a locally compact group. Then the connected component of the identity of $\mathcal{M}_{\delta_e}$ in $\mathrm{M}(H)$ is the group*

$$\mathcal{M}_{\delta_e,0} = \big\{\exp\lambda_1 \cdots \exp\lambda_n : \lambda_1,\dots,\lambda_n \in \mathrm{M}(H)_{\mathrm{ska}}, n \in \mathbb{N}\big\},$$

where $\mathrm{M}(H)_{\mathrm{ska}} = \{\lambda \in \mathrm{M}(H) : \lambda^* = -\lambda\}$*, the real linear space of skew-adjoint measures.*

Proof. There exists norm-open neighborhoods B of 0 and U of δ_e, in $\mathrm{M}(H)$, on which $\exp : B \to U$ is a homeomorphism. There is a logarithm defined on a neighborhood of δ_e, and analytic functional calculus shows that these are mutually inverse. We may suppose that B is symmetric and closed under the adjoint.

If $\nu \in U \cap \mathcal{M}_{\delta_e}$, then there is some λ in B for which $\nu = \exp\lambda$, and we have $\exp(\lambda^*) = \exp(\lambda)^* = \nu^* = \nu^{-1} = \exp(-\lambda)$, and hence $\lambda^* = -\lambda$, by assumption on B. If $\nu = \exp\lambda_1 \cdots \exp\lambda_n$, with $\lambda_1,\dots,\lambda_n \in \mathrm{M}(H)_{\mathrm{ska}}$, and ν' in $\mathcal{M}_{\delta_e}$ is so close to ν that $\nu^* * \nu' \in U$, then $\nu^* * \nu' = \exp\lambda_{n+1}$ for some λ_{n+1} in $\mathrm{M}(H)_{\mathrm{ska}}$. The subgroup of all such products is hence open in $\mathcal{M}_{\delta_e}$ and clearly connected, and thus the connected component of δ_e. □

We say that a locally compact group H is *Hermitian* if each self-adjoint element of $\mathrm{L}^1(H)$ has a real spectrum. See [6] for notes on the class of Hermitian groups.

Proposition 3.6. *Let $K \in \mathcal{K}(G)$.*

(i) *If $N_G(K) \supsetneq K$, then $\Gamma_{m_K} \subsetneq \mathcal{M}_{m_K}$.*
(ii) *If $N_G(K)/K$ contains either a nondiscrete closed abelian subgroup or a closed non-Hermitian subgroup, then the connected component of the identity $\mathcal{M}_{m_K,0}$ is unbounded.*

Proof. We let $H = N_G(K)/K$. We notice, in passing, that $N_G(K) = G_{K,1}$. Consider the map $\varphi : \mathrm{M}(H) \to \mathrm{M}(G)$ given for u in $\mathcal{C}_0(G)$ by

$$\int_G u\, d\varphi(\nu) = \int_H \int_K u(gk)\, dk\, dgK,$$

where we remark, in passing, that this quantity is equal to $\int_{N_G(K)} u(g)\, dg$, thanks to the Weyl integration formula. Since arbitrary elements of $\mathcal{C}_0(H)$ may be represented as $gK \mapsto \int_K u(gk)\, dk$, as above, we see that φ is injective, even isometric. In particular, using the definitions of the groups, it is easy to see that

$$\mathcal{M}_{m_K} \supseteq \varphi(\mathcal{M}_{\delta_{e_H}}) \qquad \text{and} \qquad \Gamma_{m_K} = \varphi(\Gamma_{\delta_{e_H}}) = \mathbb{T}\varphi(\delta_H),$$

where $\delta_H = \{\delta_h : h \in H\}$.

(i) To see that the inclusion $\Gamma_{m_K} \subseteq \mathcal{M}_{m_K}$ is proper, it suffices to see that $\Gamma_{\delta_{e_H}}$ is a proper subgroup of $\mathcal{M}_{\delta_{e_H}}$. Since H contains at least two elements, the real dimension of $\mathrm{M}(H)_{\mathrm{ska}}$ is at least 2. Since exp is analytic and a homeomorphism on a neighborhood $\widetilde{B}$ of 0 in $\mathrm{M}(H)_{\mathrm{ska}}$, $\mathcal{M}_{\delta_{e_H}}$ contains a manifold of real dimension at least 2. But since δ_H is norm discrete, we can pick $\widetilde{B}$ small enough so that $\exp(\widetilde{B}) \cap \Gamma_{\delta_{e_H}} \subset \mathbb{T}\delta_{e_H}$. Hence $\exp(\widetilde{B}) \not\subset \Gamma_{\delta_{e_H}}$.

(ii) If there exists $\nu = \nu^*$ in $\mathrm{M}(H)$ with a nonreal spectrum, then the one-parameter subgroup $\{\exp(it\nu)\}_{t\in\mathbb{R}}$ is unbounded and a subgroup of $\mathcal{M}_{\delta_{e_H}}$. The

Wiener–Pitt phenomenon shows that if H contains a closed nondiscrete abelian subgroup A, then such a ν exists. Indeed, if $\nu = \nu^*$ in $\mathrm{M}(A) \subseteq \mathrm{M}(H)$, then the Fourier–Steiltjes transform satisfies $\hat{\nu} = \widehat{\nu^*} = \overline{\hat{\nu}}$, and we appeal to Section 6.4 in [8]. If H contains a closed non-Hermitian subgroup, then we can choose ν to be absolutely continuous with respect to the Haar measure of that subgroup. □

It is not clear whether or not $\mathcal{M}_{m_K}$ is always locally compact with respect to the weak* topology.

Remark 3.7. (i) The proof of (i) above tells us that if $N_G(K)/K$ is infinite, then $\mathcal{M}_{m_K}$ contains manifolds of arbitrarily high dimension. Thus we see that $\mathcal{M}_{m_K}$ is not Lie, in this case.

(ii) If $N_G(K)$ is compact, and hence so too is $H = N_G(K)/K$ with dual object $\widehat{H}$, then $\mathcal{M}_{m_K} \cong \mathcal{M}_{\delta_{e_H}}$ is isomorphic to a subgroup of the product of unitary groups $\prod_{\pi\in\widehat{H}} \mathrm{U}(d_\pi)$, containing the dense restricted product subgroup, consisting of all elements which are I_{d_π} for all but finitely many indices π. Indeed, the Fourier–Steiltjes transform $\nu \mapsto (\pi(\nu))_{\pi\in\widehat{H}} : \mathrm{M}(H) \to \ell^\infty\text{-}\bigoplus_{\pi\in\widehat{H}} M_{d_\pi}(\mathbb{C})$ (notation as in (2.1)) injects $\mathcal{M}_{\delta_{e_H}}$ into the product group. Furthermore, consider u in $\prod_{\pi\in\widehat{H}} \mathrm{U}(d_\pi)$, where $u_\pi = I_{d_\pi}$ for all but $\pi_1, \dots, \pi_n$ in $\widehat{H}$, and $u_{\pi_k} = [u_{ij,k}]$ in $\mathrm{U}(d_{\pi_k})$ for $k = 1, \dots, n$. Consider the element of $\mathrm{M}(H)$, given by

$$\nu_u = \delta_e + \sum_{k=1}^{n} d_{\pi_k}\Big(\sum_{i,j=1}^{d_{\pi_k}} u_{ij,k}\pi_{k,ij} - \sum_{j=1}^{d_{\pi_k}} \pi_{k,jj}\Big) m_H,$$

where each set $\{\pi_{k,ij}\}_{i,j=1}^{d_{\pi_k}}$ is comprised of matrix coefficients of π_k with respect to an orthonormal basis for the space on which it acts. This element satisfies $\pi(\nu_u) = u_\pi$ for all π. Notice that $\nu_u * \nu_{u'} = \nu_{uu'}$ and $\nu_u^* = \nu_{u^*}$.

Notice that if we have an isomorphism $\mathcal{M}_{\delta_{e_H}}$ with $\prod_{\pi\in\widehat{H}} \mathrm{U}(d_\pi)$, then $\mathrm{M}(H)$ will be isomorphic to a C*-algebra (every matrix is a linear combination of four unitaries, and we appeal to the open mapping theorem to see that the Fourier–Steiltjes transform is surjective), and hence Arens-regular. It then follows from the hereditary properties of Arens regularity, and the main result of [13], that H is finite. In particular, since the Fourier–Steiltjes transform is weak*–weak* continuous, we conclude for infinite compact H that $\mathcal{M}_{\delta_{e_H}}$ is not weak*-compact.

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