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INJECTION THEOREM FOR LOCAL DITKIN SETS

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Abstract. For Figà-Talamanca–Herz algebras $A_p(G)$, $1 < p < \infty$, of a locally compact group G and a closed subgroup H of G, we prove an injection theorem for local Ditkin sets.

1. Introduction and preliminaries

Let G be a locally compact abelian group, let H be a closed subgroup, and let F be a closed subset of H. The following result is well known and classical (see [8, Theorem 7.4.13]): F is a Ditkin set in G if and only if F is a Ditkin set in the subgroup H. In this article we extend this statement to a class of noncommutative groups, including the amenable groups (Theorem 4.1). In this generalization we replace $L^1(\widehat{G})$ by the Fourier algebra $A_2(G)$ or, more generally, by the Figà-Talamanca–Herz algebra $A_p(G)$. Partial results (for $p = 2$ or for every $1 < p < \infty$) are already known. For normal subgroups see [2, Théorème 12], and for neutral subgroups see [1, Corollary 7]). For other directly related works, see also E. Kaniuth and A. Lau's [5, Theorem 3.4] and K. Parthasarathy and N. S. Kumar's [7, Theorem 3.5].

We use a natural action (denoted $u \cdot T$) of $A_p(G)$ on the Banach space $\mathcal{L}$ of all bounded operators of $L^p(G)$. Our proof requires a new characterization of the notion of locally p-Ditkin sets involving not only convolution operators but also general bounded operators of L^p. A closed subset F of G is *locally p-Ditkin* if and

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only if whenever $u \in A_p(G)$ vanishes on F and $Y \in CV_p(G)$ has support in F, then the equation $u \cdot X = Y$ has no solution $X \in \mathcal{L}$ unless $Y = 0$ (Propositions 3.4 and 3.5).

2. An action of $A_p(G)$ on $\mathcal{L}$

Let $\mathcal{A}_p(G)$ be the set of all pairs $((r_n), (s_n))$ where (r_n) is a sequence of $\mathcal{L}^p(G)$ and (s_n) is a sequence of $\mathcal{L}^{p'}(G)$ such that $\sum N_p(r_n) N_{p'}(s_n)$ converges. For $((r_n), (s_n)) \in \mathcal{A}_p(G)$ we denote by $T_{((r_n),(s_n))}$ the trace class operator

$$\langle T_{((r_n),(s_n))} f, g \rangle = \sum \big\langle [r_n], g \big\rangle \big\langle f, [\overline{s_n}] \big\rangle$$

(we write $T_{((r_n),(s_n))} \in \mathcal{T}$). Putting

$$F_{((r_n),(s_n))}(x, y) = \sum r_n(x) s_n(y)$$

if

$$\sum r_n(x) s_n(y)$$

converges and 0 otherwise, we get

$$\langle T_{((r_n),(s_n))} f, g \rangle = \int_{G \times G} F_{((r_n),(s_n))}(x, y) \overline{g(x)} f(y)\, dx\, dy.$$

The integral formula for $T_{((r_n),(s_n))}$ permits to associate in a bilinear way to $\varphi \in C^b(G \times G)$ and $S \in \mathcal{T}$ an operator φS of $\mathcal{L}$ with $\|\varphi S\| \le \|\varphi\|_\infty \|S\|_{\mathcal{T}}$. Setting for $\psi : G \times G \to \mathbb{C}$ $(\Xi\psi)(x, y) = \psi(y, x)$ and for $\varphi : G \to \mathbb{C}$ $(M_G\varphi)(x, y) = \varphi(yx^{-1})$, we get for $u \in A_p(G)$ and $S \in \mathcal{T}$ $\Xi M_G u S \in \mathcal{T}$ and

$$\|\Xi M_G u S\|_{\mathcal{T}} \le \|u\|_{A_p(G)} \|S\|_{\mathcal{T}}.$$

Via the pairing of $\mathcal{L}$ with $\mathcal{T}$, we obtain therefore an action of $A_p(G)$ on $\mathcal{L}$: for $u \in A_p(G)$ and for $U \in \mathcal{L}$ the operator $M_G u U$ is defined by

$$\langle M_G u U, S \rangle_{\mathcal{L},\mathcal{T}} = \langle U, \Xi M_G u S \rangle_{\mathcal{L},\mathcal{T}}$$

for every $S \in \mathcal{T}$.

Definition 2.1. Let G be a locally compact group and $1 < p < \infty$. For $u \in A_p(G)$ and $T \in \mathcal{L}(L^p(G))$ we put

$$u \cdot T = \overline{\tau_p\big(M_G(u) \tau_p \overline{T} \tau_p\big)\tau_p},$$

where $\tau_p(f)(x) = f(x^{-1}) \Delta_G(x^{-1})^{1/p}$ for $f : G \to \mathbb{C}$.

Proposition 2.2. *Let G be a locally compact group and $1 < p < \infty$. Then:*

(1) *for $u \in A_p(G)$, $T \in \mathcal{L}$ and $\alpha \in \mathbb{C}$ we have $\alpha(u \cdot T) = (\overline{\alpha} u) \cdot T = u \cdot (\alpha T)$;*
(2) *$(uv) \cdot T = u \cdot (v \cdot T)$ for $u, v \in A_p(G)$ and $T \in \mathcal{L}$;*
(3) *$(u + v) \cdot T = u \cdot T + v \cdot T$ for $u, v \in A_p(G)$ and $T \in \mathcal{L}$;*
(4) *$u \cdot (S + T) = u \cdot S + u \cdot T$ for $u \in A_p(G)$ and $S, T \in \mathcal{L}$;*
(5) *$\|u \cdot T\| \le \|u\|_{A_p} \|T\|$ for $u \in A_p(G)$ and $T \in \mathcal{L}$;*
(6) *for $u \in A_p(G)$ and $T \in CV_p(G)$ we have $u \cdot T = uT$.*

Proof. We verify (6) at first for $u = \overline{k} * \check{l}$ with $k, l \in C_{00}(G)$. For every $S \in \mathcal{T}$ we get

$$\langle \overline{k} * \check{l} \cdot T, S\rangle_{\mathcal{L},\mathcal{T}} = \int_G \langle M_{_{t^{-1}}(\check{\bar{l}})} T M_{_{t^{-1}}(\check{k})}, S\rangle_{\mathcal{L},\mathcal{T}}\, dt.$$

Putting for $\varphi, \psi \in C_{00}(G)$ $S = T_{(\varphi),(\overline{\psi})}$ we have

$$\langle \overline{k} * \check{l} \cdot T, S\rangle_{\mathcal{L},\mathcal{T}} = \int_G \big\langle T M_{_{t^{-1}}(\check{k})}[\varphi], M_{_{t^{-1}}(\check{l})}[\psi]\big\rangle_{L^p, L^{p'}}\, dt,$$

and, consequently,

$$\big\langle \overline{k} * \check{l} \cdot T[\varphi], [\psi]\big\rangle_{L^p, L^{p'}} = \int_G \big\langle T\big[{}_{t^{-1}}(\check{k})\varphi\big], \big[{}_{t^{-1}}(\check{l})\psi\big]\big\rangle_{L^p, L^{p'}}\, dt.$$

For an arbitrary $u \in A_p(G)$ we choose for $\varepsilon > 0$ two sequences (k_n), (l_n) of $C_{00}(G)$ such that $\sum N_p(k_n) N_{p'}(l_n) < \infty$ and such that

$$u = \sum \overline{k_n} * \check{l}_n.$$

Let N be a positive integer with

$$\sum_{N+1}^{\infty} N_p(k_n) N_{p'}(l_n) < \varepsilon\big(2\|T\|\big)^{-1}.$$

It suffices then to put

$$v = \sum_{n=N+1}^{\infty} \overline{k_n} * \check{l}_n$$

to get

$$\|u \cdot T - v \cdot T\| < \frac{\varepsilon}{2},$$

and, consequently,

$$\|u \cdot T - uT\| \le \|u \cdot T - v \cdot T\| + \|v \cdot T - vT\| + \|vT - uT\| < \varepsilon. \qquad \square$$

Remark 2.3. As a consequence of (6), for $u \in A_p(G)$ and for $T \in PM_p(G)$ we have $\langle v, u \cdot T\rangle = \langle uv, T\rangle$ for every $v \in A_p(G)$. In particular, for G abelian, $u \in A_2(G)$, and $T \in CV_p(G)$, we have

$$\widehat{u \cdot T} = \Phi_{\widehat{G}}^{-1}(\overline{u}) * \widehat{T},$$

where $\Phi_{\widehat{G}}(f)(x) = \int_{\widehat{G}} f(\chi)\chi(x)\, d\chi$ for $f \in L^1(\widehat{G})$ and $x \in G$.

For H an arbitrary closed subgroup of G we denote by i the canonical extension to $CV_p(H)$ of the inclusion of H into G.

The proof of the injection theorem for locally Ditkin sets strongly depends on the following theorem (see [3, Theorem 5]).

Theorem 2.4. *Let G be a locally compact group, let H be a closed subgroup, and let $1 < p < \infty$. Then there is a linear map $\mathcal{P}$ of $\mathcal{L}(L^p(G))$ into $\mathcal{L}(L^p(H))$ such that*

(1) $\|\mathcal{P}(T)\| \le \|T\|$ *for every* $T \in \mathcal{L}(L^p(G))$;

(2) $\mathcal{P}(i(S)) = S$ *for every* $S \in CV_p(H)$*;*
(3) $\mathcal{P}(u \cdot T) = \mathrm{Res}_H\, u \cdot \mathcal{P}(T)$ *for every* $u \in A_p(G)$ *and every* $T \in \mathcal{L}(L^p(G))$.

3. A characterization of Ditkin sets involving bounded operators

Using the generalization of Wiener's theorem to $CV_p(G)$ twice, we first obtain the following lemma.

Lemma 3.1. *Let G be a locally compact group, let $1 < p < \infty$, and let $T \in CV_p(G)$. If $k * l \cdot T = 0$ for every $k, l \in C_{00}(G)$, we have necessarily $T = 0$.*

Proof. Suppose that $T \neq 0$. By the generalization of Wiener's theorem (see [4]) there is $x \in \operatorname{supp} T$. Choose $k, l \in C_{00}(G)$ such that $k * l(x) \neq 0$; this implies (see [4, p. 119]) $x \in \operatorname{supp}(k * l) \cdot T$ and therefore $k * l \cdot T \neq 0$. □

Definition 3.2. Let G be a locally compact group, let $1 < p < \infty$, and let F be a closed subset of G. We say that F is a *locally p-Ditkin* subset of G if for every $u \in A_p(G) \cap C_{00}(G)$ vanishing on F and for every $\varepsilon > 0$ there is $v \in A_p(G) \cap C_{00}(G)$ with $\operatorname{supp} v \cap F = \emptyset$ and $\|u - uv\| < \varepsilon$. The set is said to be a *p-Ditkin* set of G if for every $u \in A_p(G)$ vanishing on F and for every $\varepsilon > 0$ there is $v \in A_p(G) \cap C_{00}(G)$ with $\operatorname{supp} v \cap F = \emptyset$ and $\|u - uv\| < \varepsilon$.

Remark 3.3. If G is amenable, then every locally p-Ditkin set is indeed p-Ditkin.

The following characterization of locally p-Ditkin set will be useful. For G abelian and $p = 2$, see [8, Theorem 7.4.17].

Proposition 3.4. *Let G be a locally compact group, let F be a closed subset of G, and let $1 < p < \infty$. Then the following statements are equivalent:*

(1) *the set F is locally p-Ditkin;*
(2) *for every $T \in PM_p(G)$ and for every $u \in A_p(G)$ with compact support, with* $\operatorname{supp} u \cdot T \subset F$ *and* $\mathrm{Res}_F\, u = 0$ *we have* $\langle u, T \rangle = 0$*;*
(3) *for every $T \in CV_p(G)$ with compact support and for every $u \in A_p(G)$ with compact support, with* $\operatorname{supp} u \cdot T \subset F$ *and* $\mathrm{Res}_F\, u = 0$ *we have* $\langle u, T \rangle = 0$*;*
(4) *for every $T \in CV_p(G)$ and for every $u \in A_p(G)$ with* $\operatorname{supp} u \cdot T \subset F$ *and* $\mathrm{Res}_F\, u = 0$ *we have* $u \cdot T = 0$.

Proof. 1. (1) implies (2).

Let $T \in PM_p(G)$ and $u \in A_p(G) \cap C_{00}(G)$ with $\mathrm{Res}_H\, u = 0$ and $\operatorname{supp} u \cdot T \subset F$. Let ε be a positive real number. There is $v \in A_p(G) \cap C_{00}(G)$ with $\operatorname{supp} v \cap F = \emptyset$ and

$$\|u - uv\|_{A_p} < \big(1 + \|T\|\big)^{-1} \varepsilon.$$

From $\operatorname{supp}(uv) \cdot T = \emptyset$ it follows that $(uv) \cdot T = 0$. Choose now $w \in A_p(G) \cap C_{00}(G)$ such that $w(x) = 1$ for every $x \in \operatorname{supp} v$. We get

$$\langle uv, T \rangle = \langle v, u \cdot T \rangle = \langle vw, u \cdot T \rangle = \big\langle w, (uv) \cdot T \big\rangle = 0,$$

and finally

$$\big|\langle u, T \rangle\big| \leq \big|\langle u, T \rangle - \langle uv, T \rangle\big| + \big|\langle uv, T \rangle\big| < \varepsilon.$$

2. (2) implies (3).

The operator T having compact support belongs to $PM_p(G)$.

3. (3) implies (4).

Let $T \in CV_p(G)$, and let $u \in A_p(G)$ with $\operatorname{supp} u \cdot T \subset F$ and with $\operatorname{Res}_F u = 0$. Consider four arbitrary functions $\varphi, \psi, k, l \in C_{00}(G)$. The convolution operator $(k * l) \cdot T$ has compact support; the support of $(u\varphi * \psi) \cdot ((k * l) \cdot T)$ is contained in F; and $\operatorname{Res}_F u\varphi * \psi = 0$, and, consequently, $\langle u\varphi * \psi, (k * l) \cdot T\rangle = 0$. This implies $\langle k * l\varphi * \psi, u \cdot T\rangle = 0$, and, applying Lemma 3.1 twice, we conclude that $u \cdot T = 0$.

4. (4) implies (1).

Suppose that there exists a function $u \in A_p(G) \cap C_{00}(G)$ with $\operatorname{Res}_H u = 0$ which is not in the norm closure in $A_p(G)$ of

$$\big\{uv \;\big|\; v \in A_p(G) \cap C_{00}(G), \operatorname{supp} v \cap F = \emptyset\big\}.$$

There is $T \in PM_p(G)$ with $\langle u, T\rangle \neq 0$ and $\langle uv, T\rangle = 0$ for every $v \in A_p(G) \cap C_{00}(G)$ with $\operatorname{supp} v \cap F = \emptyset$.

Let $x \in G \setminus F$. We claim that $x \notin \operatorname{supp} u \cdot T$. Suppose that $x \in \operatorname{supp} u \cdot T$. Choose a compact neighborhood V of x such that $V \cap F = \emptyset$. There is $v \in A_p(G)$ with $\operatorname{supp} v \subset V$ and $\langle v, u \cdot T\rangle \neq 0$. But from $v \in C_{00}(G)$ and $\operatorname{supp} v \cap F = \emptyset$ we get $\langle uv, T\rangle = 0$. This implies $x \notin \operatorname{supp} u \cdot T$.

We have proved that $\operatorname{supp} u \cdot T \subset F$ and, consequently, that $u \cdot T = 0$. It suffices finally to choose $w \in A_p(G) \cap C_{00}(G)$ with $w(x) = 1$ on $\operatorname{supp} u$ to get $\langle u, T\rangle = \langle uw, T\rangle = \langle w, u \cdot T\rangle = 0$, which is a contradiction. □

To obtain our main result we need the following improvement of Proposition 3.4(1) ⇒ (4).

Proposition 3.5. *Let G be a locally compact group, let F be a closed subset of G, let $1 < p < \infty$, $u \in A_p(G)$, and let $T \in \mathcal{L}(L^p(G))$. Suppose that F is locally p-Ditkin, that $u \cdot T \in CV_p(G)$, that* $\operatorname{Res}_F u = 0$, *and that* $\operatorname{supp} u \cdot T \subset F$*; we have then $u \cdot T = 0$.*

Proof. Let $v \in A_p(G) \cap C_{00}(G)$, and let $\varepsilon > 0$. There is $w \in A_p(G) \cap C_{00}(G)$ with $\operatorname{supp} w \cap F = \emptyset$ and

$$\|vu - vuw\|_{A_p} < \big(1 + \|T\|\big)^{-1}\varepsilon.$$

We have $v \cdot (u \cdot T) \in CV_p(G)$ and $\operatorname{supp} v \cdot (u \cdot T) \subset F$, and therefore $(wuv) \cdot T = 0$. On the other hand,

$$\big\|(uv) \cdot T - (uvw) \cdot T\big\| \le \|uv - uvw\|_{A_p}\|T\| \le \frac{\varepsilon\|T\|}{(1 + \|T\|)} < \varepsilon.$$

This implies $(uv) \cdot T = 0$, and consequently $v \cdot (u \cdot T) = 0$, and finally $u \cdot T = 0$. □

Remark 3.6. This result is new even for $G = \mathbb{R}$ and $p = 2$.

4. Injection theorem for Ditkin sets

Theorem 4.1. *Let G be a locally compact group, let H be an arbitrary closed subgroup of G, $1 < p < \infty$, and let F be a closed subset of H. Then the following statements are equivalent:*

(1) *the set F is locally p-Ditkin in G;*
(2) *the set F is locally p-Ditkin in H.*

Proof. 1. (1) implies (2).

Let $u \in A_p(H) \cap C_{00}(H)$ with $\operatorname{Res}_F u = 0$ and $\varepsilon > 0$. According to C. Herz (see [4]) there is $v' \in A_p(G)$ with $\operatorname{Res}_H v' = u$. Choose $v'' \in A_p(G) \cap C_{00}(G)$ with $v''(x) = 1$ for every $x \in \operatorname{supp} u$. Putting $v = v'v''$ we get $v \in A_p(G) \cap C_{00}(G)$ and $\operatorname{Res}_H v = u$. There is, consequently, $w \in A_p(G) \cap C_{00}(G)$ with $\operatorname{supp} w \cap F = \emptyset$ and $\|v - vw\|_{A_p} < \varepsilon$. It follows that $\|u - u \operatorname{Res}_H w\|_{A_p} < \varepsilon$ with $\operatorname{supp} \operatorname{Res}_H w \cap F = \emptyset$.

2. (2) implies (1).

It suffices to show that, for $u \in A_p(G)$ and for $T \in CV_p(G)$ with $\operatorname{Res}_H u = 0$ and $\operatorname{supp} u \cdot T \subset F$, we have $u \cdot T = 0$.

By Lohoué's theorem (see [6, Théorème 5]) there is $S \in CV_p(H)$ such that $i(S) = u \cdot T$. Let $\mathcal{P}$ be the linear map of $\mathcal{L}(L^p(G))$ into $\mathcal{L}(L^p(H))$ of Theorem 2.4; then, applying (2) and (3) of Theorem 2.4, we get

$$S = \mathcal{P}\big(i(S)\big) = \mathcal{P}(u \cdot T) = \operatorname{Res}_H u \cdot \mathcal{P}(T).$$

Taking into account that $\operatorname{supp} i(S) = \operatorname{supp} S$ we obtain that $\operatorname{Res}_H u \cdot \mathcal{P}(T) \in CV_p(H)$ and that $\operatorname{supp} \operatorname{Res}_H u \cdot \mathcal{P}(T) \subset F$. We cannot assert that the operator $\mathcal{P}(T)$ belongs to $CV_p(H)$, but Proposition 3.5 implies $\operatorname{Res}_H u \cdot \mathcal{P}(T) = 0$; hence $S = 0$, and finally $u \cdot T = 0$. $\square$

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