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A GROUP STRUCTURE ON $\mathbb{D}$ AND ITS APPLICATION FOR COMPOSITION OPERATORS

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Dedicated to Professor Anthony To-Ming Lau

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ABSTRACT. We present a group structure on $\mathbb{D}$ via the automorphisms which fix the point 1. Through the induced group action, each point of $\mathbb{D}$ produces an equivalence class that turns out to be a Blaschke sequence. We show that the corresponding Blaschke products are minimal/atomic solutions of the functional equation $\psi \circ \varphi = \lambda \psi$, where λ is a unimodular constant and φ is an automorphism of the unit disk. We also characterize all Blaschke products that satisfy this equation, and we study its application in the theory of composition operators on model spaces K_Θ.

1. INTRODUCTION

Given any analytic self-map of the open unit disk $\mathbb{D}$, that is, $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$, we define the composition mapping

$$\begin{aligned} C_\varphi : \mathrm{Hol}(\mathbb{D}) &\longrightarrow \mathrm{Hol}(\mathbb{D}), \\ f &\longmapsto f \circ \varphi. \end{aligned}$$

There are several natural questions about this mapping. The most studied question is the following. Given Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, which reside in $\mathrm{Hol}(\mathbb{D})$, under what conditions do we have $C_\varphi \mathcal{X} \subset \mathcal{Y}$? In particular, the special case $\mathcal{Y} = \mathcal{X}$ has been extensively studied for various classes of Banach spaces $\mathcal{X}$.

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The classical *subordination principle* of Littlewood can be rephrased in this language. In terms of composition operators, it says that the mapping $C_\varphi : H^p \longrightarrow H^p$ is a well-defined bounded operator on all Hardy spaces H^p, $0 < p \leq \infty$. This question has also been studied on numerous other function spaces such as Hardy spaces, Hardy–Orlics spaces or other weighted Hardy spaces, Dirichlet series, Bergman spaces, Besov spaces, Bloch spaces, and the Dirichlet space. However, the literature on this subject is simply out of control and it is rather impossible to take into account all contributions to this topic. In this paper, we present some facts about Blaschke products which are interesting in their own right, and, moreover, they help to characterize composition operators with inner symbol that map a model space into itself.

The closed invariant subspaces of the forward shift operator

$$\begin{aligned} S : H^2 &\longrightarrow H^2, \\ f &\longmapsto zf, \end{aligned}$$

were completely characterized by Beurling in his seminal work [2]. He showed that, apart from the trivial case $\{0\}$, they are precisely of the form ΘH^2, where Θ is an inner function. Hence, using a simple Hilbert space technic, the invariant subspaces of the backward shift S^* are precisely $(\Theta H^2)^\perp$. This collection is so important in mathematical analysis that it merits a specific name. Due to its application in the theory of contractions,

$$K_\Theta := (\Theta H^2)^\perp = H^2 \ominus \Theta H^2$$

is called a *model space*. The monograph [3] is a great source for model spaces and their operators.

Studying composition operators on model spaces is a new topic and there are still several open questions about them. In [8], the authors studied the compactness and membership in Schatten classes of the mapping $C_\varphi : K_\Theta \longrightarrow H^2$. In [10], a complete characterization of φ's for which C_φ leaves K_Θ invariant, when Θ is a finite Blaschke product, is given. The paper [11] is devoted to a comprehensive study of C_φ when φ is an inner function. In this situation, we are faced with the functional equation

$$\psi\big(\varphi(z)\big) \times \omega(z) = \psi(z) \quad (z \in \mathbb{D}), \tag{1.1}$$

where all three functions ψ, φ, and ω are inner. With an iteration technique and appealing to the structure of inner functions, (1.1) simplifies to

$$\psi\big(\varphi(z)\big) = \lambda\psi(z) \quad (\lambda \in \mathbb{T}, z \in \mathbb{D}), \tag{1.2}$$

where φ has its fixed (Denjoy–Wolff) point on $\mathbb{T}$. This equation is a special case of the celebrated *Schröder equation*, which has a long and rich history. As a matter of fact, its first treatment dates back to 1884 when Königs [7] classified the eigenvalues λ for mappings φ with fixed points inside $\mathbb{D}$. See [14, Chapter 6] for more details on Königs's solution. The composition of inner functions has also a very rich history (see [4]). In particular, with special attention to Blaschke products, it has been extensively studied in [6] and [12].

Despite the vast literature on Schröder's equation, not much is known when the Denjoy–Wolff point of φ is on $\mathbb{T}$ and it calls for further investigation. This paper is devoted to a complete characterization of the Blaschke products ψ which satisfy (1.2), with φ being an automorphism of $\mathbb{D}$. To do so, in Section 3 we define a noncommutative group structure on $\mathbb{D}$ which stems from automorphisms of $\mathbb{D}$ which fix the point 1. Essential properties of this group are discussed in this section. Then in Section 4, we introduce a family of abelian subgroups of $\mathbb{D}$. These subgroups will provide the main apparatus to spot all the minimal Blaschke sequences. To achieve this goal, we need an explicit formula for the nth iterate of an element in the group $\mathbb{D}$; this formula is obtained in Section 5. Eventually, in Sections 6 and 7, we study the orbits of the subgroup actions on $\mathbb{D}$ and show that these orbits are two-sided Blaschke sequences.

The main outcome is Theorem 7.1, in which we show that the corresponding Blaschke sequence is in fact a *minimal* or *atomic* solution of the functional equation. More precisely, this means that no proper subproduct satisfies the functional equation. Eventually, in Section 8, the stage is ready to characterize all Blaschke products that satisfy the functional equation. Theorem 8.1 is a complete characterization of Blaschke products that fulfill the functional equation (1.2). At the end, we discuss some applications of these objects (minimal Blaschke products) in the theory of composition operators on model spaces.

2. Automorphisms of $\mathbb{D}$

Let γ be an arbitrary unimodular constant, and let α be an arbitrary point in $\mathbb{D}$. The Möbius transformation

$$b(z) = \gamma\frac{\alpha - z}{1 - \bar{\alpha}z}$$

is an automorphism of the open unit disk with a simple zero at α. Conversely, any automorphism of the disk has the above form.

In order to use it in the formation of a Blaschke product, we define the Blaschke factor

$$b_\alpha(z) := \begin{cases} \frac{|\alpha|}{\alpha}\frac{\alpha - z}{1-\bar{\alpha}z} & \text{if } \alpha \neq 0, \\ z & \text{if } \alpha = 0. \end{cases}$$

But, some other variations of b are needed in our discussion.

Depending on the number of fixed points, apart from the identity, the Möbius transformations divide into two classes: either they have just one fixed point, or they have two distinct fixed points. Since an automorphism of the open unit disk maps bijectively $\mathbb{D}$ into itself, and also $\mathbb{T}$ into itself, there are certain restrictions on the location of these fixed points. See [1, Section 1.2] for more on this topic.

The point 1 is a fixed point of b if and only if

$$1 = \gamma\frac{\alpha - 1}{1 - \bar{\alpha}}.$$

Hence, b takes the form

$$\varphi_\alpha(z) := \frac{1-\bar{\alpha}}{1-\alpha}\frac{z-\alpha}{1-\bar{\alpha}z}, \tag{2.1}$$

where α is a parameter running through $\mathbb{D}$. A simple computation shows that the other fixed point of φ_α is

$$\kappa_\alpha := -\frac{\alpha(1-\bar{\alpha})}{\bar{\alpha}(1-\alpha)}. \tag{2.2}$$

In our calculation, we will also need the quantity

$$A_\alpha := \varphi'_\alpha(1) = \frac{1-|\alpha|^2}{|1-\alpha|^2}. \tag{2.3}$$

As a matter of fact, A_α is the angular derivative (in the sense of Carathéodory) of φ_α at the fixed point 1. Moreover, note that

$$A_\alpha = 1 \quad \Longleftrightarrow \quad \kappa_\alpha = 1.$$

Finally, given $z_0 \in \mathbb{D}$, we define the unimodular constant γ_{α,z_0} by

$$\gamma_{\alpha,z_0} := \begin{cases} \varphi_{z_0}(\kappa_\alpha) & \text{if } A_\alpha < 1, \\ 1 & \text{if } A_\alpha = 1, \\ \overline{\varphi_{z_0}(\kappa_\alpha)} & \text{if } A_\alpha > 1. \end{cases}$$

This constant will appear in several occasions below. For more information on the structure of automorphisms of $\mathbb{D}$ and Blaschke products, see [13, p. 176] or [9, p. 155].

3. The group $(\mathbb{D}, *)$

In a rather surprising way, the open unit disk $\mathbb{D}$ becomes a group. The law of composition is defined by

$$\alpha * \beta := \frac{\beta(1-\bar{\beta}) + \alpha(1-\beta)}{(1-\bar{\beta}) + \alpha\bar{\beta}(1-\beta)} \quad (\alpha, \beta \in \mathbb{D}). \tag{3.1}$$

This algebraic structure is rewarding and has numerous interesting properties. The rather strange law of composition comes from the composition of some judiciously chosen automorphisms of the disk. This is clarified below.

Theorem 3.1. $(\mathbb{D}, *)$ *is a (nonabelian) group. The identity element is* 0*, and the inverse of* α *is* $-\alpha\frac{1-\bar{\alpha}}{1-\alpha}$.

Proof. To reveal the mystery behind the complicated law $*$, consider the collection

$$\mathcal{G} = \{\varphi_\alpha : \alpha \in \mathbb{D}\}.$$

As we observed in Section 2, the set $\mathcal{G}$ precisely consists of *all* automorphisms of the open unit disk with a *fixed point* at 1. This fact is essential. Since the automorphisms $\varphi_\alpha \circ \varphi_\beta$ and φ_α^{-1} fix the point 1, we deduce that $\varphi_\alpha \circ \varphi_\beta \in \mathcal{G}$ and $\varphi_\alpha^{-1} \in \mathcal{G}$. Hence, equipped with the law of composition of functions, $\mathcal{G}$ is a group.

Now, we use the parameterization of $\mathcal{G}$ by $\mathbb{D}$ and transfer the algebraic structure of $\mathcal{G}$ to $\mathbb{D}$. Since $\mathcal{G}$ is a group, given $\alpha, \beta \in \mathbb{D}$, there is a *unique* $\gamma \in \mathbb{D}$ such that $\varphi_\alpha \circ \varphi_\beta = \varphi_\gamma$. We define the isomorphism such that $\gamma = \alpha * \beta$; that is,

$$\varphi_\alpha \circ \varphi_\beta = \varphi_{\alpha * \beta} \quad (\alpha, \beta \in \mathbb{D}). \tag{3.2}$$

We proceed to find an explicit formula for γ. In fact, we have

$$\begin{aligned}
\varphi_{\alpha*\beta}(z) &= (\varphi_\alpha \circ \varphi_\beta)(z) \\
&= \frac{1-\bar{\alpha}}{1-\alpha}\,\frac{\frac{1-\bar{\beta}}{1-\beta}\frac{z-\beta}{1-\bar{\beta}z}-\alpha}{1-\bar{\alpha}\frac{1-\bar{\beta}}{1-\beta}\frac{z-\beta}{1-\bar{\beta}z}} \\
&= \frac{1-\bar{\alpha}}{1-\alpha}\,\frac{(1-\bar{\beta})(z-\beta)-\alpha(1-\beta)(1-\bar{\beta}z)}{(1-\beta)(1-\bar{\beta}z)-\bar{\alpha}(1-\bar{\beta})(z-\beta)} \\
&= \frac{1-\bar{\alpha}}{1-\alpha}\,\frac{((1-\bar{\beta})+\alpha\bar{\beta}(1-\beta))z-(\beta(1-\bar{\beta})+\alpha(1-\beta))}{((1-\beta)+\bar{\alpha}\beta(1-\bar{\beta}))-(\bar{\beta}(1-\beta)+\bar{\alpha}(1-\bar{\beta}))z}.
\end{aligned}$$

Looking at the zero of the last quotient shows that (3.1) holds.

We constructed the group $(\mathbb{D}, *)$ such that it is an isomorphic copy of $(\mathcal{G}, \circ)$. As the first consequence, since $\varphi_0 = id$, the point 0 is the identity element of $(\mathbb{D}, *)$. Using (3.1), it is also easy to see that

$$\alpha * 0 = 0 * \alpha = \alpha \quad (\alpha \in \mathbb{D}).$$

Similarly, the expression

$$\varphi_\alpha^{-1}(z) = \frac{1-\bar{\alpha}}{1-\alpha}\,\frac{z+\alpha\frac{1-\bar{\alpha}}{1-\alpha}}{1+\bar{\alpha}\frac{1-\alpha}{1-\bar{\alpha}}z} = \varphi_{-\alpha\frac{1-\bar{\alpha}}{1-\alpha}}(z)$$

gives the formula for the inverse of α, something that can also be directly verified via (3.1); that is,

$$\alpha * \left(-\alpha\frac{1-\bar{\alpha}}{1-\alpha}\right) = \left(-\alpha\frac{1-\bar{\alpha}}{1-\alpha}\right) * \alpha = 0 \quad (\alpha \in \mathbb{D}).$$ □

Fix $\alpha \in \mathbb{D}$. To avoid the confusion with the law of multiplication in the complex plane, for $n \geq 1$, we write

$$\alpha_n := \alpha * \alpha * \cdots * \alpha \quad (n \text{ times}, n \geq 1)$$

and, appealing to the formula for the inverse of α in $\mathbb{D}$ given in Theorem 3.1, we define

$$\alpha_{-n} := \left(-\alpha\frac{1-\bar{\alpha}}{1-\alpha}\right)_n \quad (n \geq 1).$$

Since 0 is the identity element in $\mathbb{D}$, we put $\alpha_0 := 0$. Hence, each $\alpha \in \mathbb{D}$ gives birth to a two-sided sequence $(\alpha_n)_{n\in\mathbb{Z}}$, and with this notation, we have the crucial identity

$$\varphi_\alpha^{[n]} = \varphi_{\alpha_n} \quad (n \in \mathbb{Z}). \tag{3.3}$$

The notation $f^{[k]}$ means $f \circ \cdots \circ f$, k times. This observation immediately implies

$$\varphi_{\alpha_m} \circ \varphi_{\alpha_n} = \varphi_{\alpha_{m+n}} \quad (m, n \in \mathbb{Z}). \tag{3.4}$$

This identity will be used frequently. Theorem 3.1 and (2.1) also reveal that

$$\varphi_\alpha(0) = -\alpha\frac{1-\bar{\alpha}}{1-\alpha} = \alpha_{-1} \quad (\alpha \in \mathbb{D}). \tag{3.5}$$

To obtain another useful formula, note that $\varphi_\beta \circ \varphi_{\alpha_{-1}}$ and $\varphi_{\varphi_\alpha(\beta)}$ both belong to $\mathcal{G}$ and vanish at $\varphi_\alpha(\beta)$. Hence,

$$\varphi_\beta \circ \varphi_{\alpha_{-1}} = \varphi_{\varphi_\alpha(\beta)} \quad (\alpha, \beta \in \mathbb{D}).$$

As a special case, we have

$$\varphi_{z_0} \circ \varphi_{\alpha_{-n}} = \varphi_{w_n} \quad (\alpha, z_0 \in \mathbb{D}, n \in \mathbb{Z}), \tag{3.6}$$

where $w_n = \varphi_{\alpha_n}(z_0)$. The importance of this formula will be revealed below.

4. The subgroup $\mathbb{D}_\kappa$

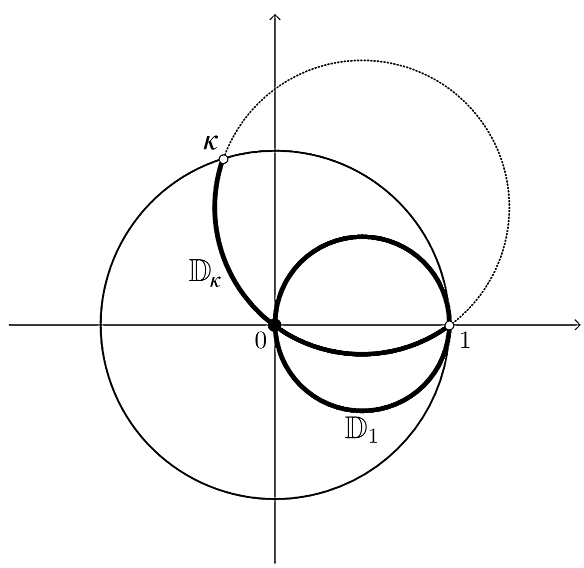
The fixed points of φ_α are 1 and κ_α, where the latter is given by (2.2). Hence, the study of $\mathbb{D}$ naturally bifurcates into two cases: $\kappa_\alpha = 1$ and $\kappa_\alpha \neq 1$. Note that as α runs through $\mathbb{D}$, the fixed point κ runs through all of $\mathbb{T}$.

For a fixed $\kappa \in \mathbb{T}$, we also define

$$\mathbb{D}_\kappa := \{\alpha \in \mathbb{D} : \kappa_\alpha = \kappa\} = \big\{\alpha \in \mathbb{D} : \alpha + \kappa\bar{\alpha} = (1+\kappa)|\alpha|^2\big\}. \tag{4.1}$$

The last expression shows that the points of $\mathbb{D}_\kappa$ are part of the circle passing through the points 1, 0, and κ that are inside $\mathbb{D}$. Also note that, for $\kappa = -1$, we have the degenerate case

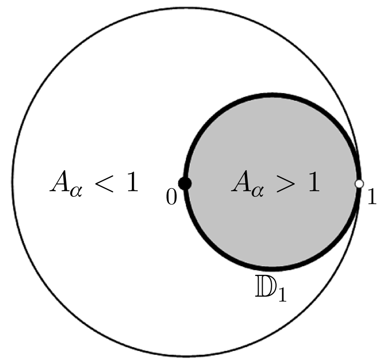
$$\mathbb{D}_{-1} = \big\{\alpha \in \mathbb{D} : \alpha(1-\bar{\alpha}) = \bar{\alpha}(1-\alpha)\big\} = (-1, 1).$$



One special case is of particular interest. If $\kappa = 1$, then

$$\begin{aligned} \mathbb{D}_1 &= \big\{\alpha \in \mathbb{D} : \alpha(1-\bar{\alpha}) = -\bar{\alpha}(1-\alpha)\big\} \\ &= \{\alpha \in \mathbb{D} : \alpha + \bar{\alpha} = 2|\alpha|^2\} \\ &= \big\{x + iy : (x-1/2)^2 + y^2 = 1/4\big\} \setminus \{1\}. \end{aligned} \tag{4.2}$$

Hence, $\mathbb{D}_1$ is precisely the circle of radius 1 inside $\mathbb{D}$ that is tangent to point 1, of course without counting the boundary point 1. At such points, φ_α has just one fixed point (i.e., the point 1), and this makes the difference in the following. The subgroup $\mathbb{D}_1$ is also the borderline for the values of A_α. On $\mathbb{D}_1$, we precisely have $A_\alpha = 1$, while inside it $A_\alpha > 1$, and in between $\mathbb{D}$ and $\mathbb{D}_1$ we have $A_\alpha < 1$. This is important when we study the iterates of an element in an equivalence class (Section 6).



Theorem 4.1. *Let $\kappa \in \mathbb{T}$. Then $\mathbb{D}_\kappa$ is an abelian subgroup of $\mathbb{D}$. Moreover, on $\mathbb{D}_\kappa$, the law of composition simplifies to*

$$\alpha * \beta = \frac{\alpha + \beta - (1+\bar{\kappa})\alpha\beta}{1 - \bar{\kappa}\alpha\beta} \quad (\alpha, \beta \in \mathbb{D}_\kappa).$$

Proof. Direct verification of this fact is possible. However, it is easier to just note that if κ is the fixed point of φ_α and φ_β, then it also stays fixed under $\varphi_\alpha \circ \varphi_\beta^{-1} = \varphi_{\alpha * \beta_{-1}}$. Hence, for each $\alpha, \beta \in \mathbb{D}_\kappa$, we have $\alpha * \beta_{-1} \in \mathbb{D}_\kappa$. Clearly, $0 \in \mathbb{D}_\kappa$. Thus, $\mathbb{D}_\kappa$ is a subgroup of $\mathbb{D}$.

To obtain a simpler formula for $*$ in $\mathbb{D}_\kappa$, note that, by (3.1) and (4.1), we have

$$\begin{aligned}
\alpha * \beta &= \frac{\beta(1-\bar{\beta}) + \alpha(1-\beta)}{(1-\bar{\beta}) + \alpha\bar{\beta}(1-\beta)} \\
&= \frac{-\kappa\bar{\beta}(1-\beta) + \alpha(1-\beta)}{(1-\bar{\beta}) - \alpha\bar{\kappa}\beta(1-\bar{\beta})} \\
&= \frac{(1-\beta)(-\kappa\bar{\beta} + \alpha)}{(1-\bar{\beta})(1-\bar{\kappa}\alpha\beta)} \\
&= \frac{\alpha + \beta - (1+\bar{\kappa})\alpha\beta}{1 - \bar{\kappa}\alpha\beta} \quad (\alpha, \beta \in \mathbb{D}_\kappa).
\end{aligned}$$

This formula also reveals that $\mathbb{D}_\kappa$ is abelian. $\square$

5. A formula for the iterates of α

Using an interesting technique of complex analysis, we now obtain an explicit formula for α_n.

Theorem 5.1. *Let $\alpha \in \mathbb{D}$. Then we have*

$$\alpha_n = \begin{cases} \frac{\kappa_\alpha(1-A_\alpha^n)}{1-\kappa_\alpha A_\alpha^n} & \text{if } \kappa_\alpha \neq 1, \\ \frac{n\alpha}{1+(n-1)\alpha} & \text{if } \kappa_\alpha = 1 \end{cases} \quad (n \in \mathbb{Z}).$$

In particular, except the identity element 0, no other element of $\mathbb{D}$ is of finite order.

Proof. Direct verification of the above formula is feasible, but it is not a pleasant task. We present another more interesting approach. Given $\kappa \in \mathbb{T}$, define

$$\phi_\kappa(z) := \begin{cases} \frac{z-\kappa}{z-1} & \text{if } \kappa \neq 1, \\ \frac{z}{z-1} & \text{if } \kappa = 1. \end{cases}$$

This function satisfies $\phi_\kappa \circ \phi_\kappa = id$. Now, we need to consider two cases.

Case I: $\kappa_\alpha \neq 1$. We have

$$(\phi_{\kappa_\alpha} \circ \varphi_\alpha \circ \phi_{\kappa_\alpha})(z) = \frac{z}{A_\alpha},$$

and thus we deduce that

$$(\phi_{\kappa_\alpha} \circ \varphi_\alpha^{[n]} \circ \phi_{\kappa_\alpha})(z) = \frac{z}{A_\alpha^n} \quad (n \in \mathbb{Z}).$$

Therefore,

$$\varphi_\alpha^{[n]}(z) = \phi_{\kappa_\alpha}\Big(\frac{\phi_{\kappa_\alpha}(z)}{A_\alpha^n}\Big) \quad (n \in \mathbb{Z}), \tag{5.1}$$

which simplifies to

$$\varphi_\alpha^{[n]}(z) = \frac{(1-\kappa_\alpha A_\alpha^n)z - \kappa_\alpha(1-A_\alpha^n)}{(1-A_\alpha^n)z + (A_\alpha^n - \kappa_\alpha)} \quad (n \in \mathbb{Z}). \tag{5.2}$$

Now, according to (3.3), $\varphi_\alpha^{[n]} = \varphi_{\alpha_n}$, and by considering the zero of $\varphi_\alpha^{[n]}$ we obtain the required above formula.

Case II: $\kappa_\alpha = 1$. The proof has the same spirit, except that we use ϕ_1. In this case, we have

$$(\phi_1 \circ \varphi_\alpha \circ \phi_1)(z) = z + \frac{\alpha}{1-\alpha},$$

and thus

$$(\phi_1 \circ \varphi_\alpha^{[n]} \circ \phi_1)(z) = z + \frac{n\alpha}{1-\alpha} \quad (n \in \mathbb{Z}).$$

Therefore,

$$\varphi_\alpha^{[n]}(z) = \phi_1\Big(\phi_1(z) + \frac{n\alpha}{1-\alpha}\Big) \quad (n \in \mathbb{Z}), \tag{5.3}$$

which simplifies to

$$\varphi_\alpha^{[n]}(z) = \frac{(1-\alpha+n\alpha)z - n\alpha}{n\alpha z + 1 - \alpha - n\alpha} \quad (n \in \mathbb{Z}). \tag{5.4}$$

The result now follows. □

With similar techniques, one can also show that

$$\alpha_n = \frac{A_\alpha(1 + A_\alpha + \cdots + A_\alpha^{n-1})B}{1 + A_\alpha(1 + A_\alpha + \cdots + A_\alpha^{n-1})B} \quad (n \in \mathbb{Z}),$$

where

$$B = \frac{\alpha - |\alpha|^2}{1 - |\alpha|^2}.$$

But, we do not need this representation in the following.

Note that if $\alpha \in \mathbb{D}_\kappa$, then its iterates form a subgroup in $\mathbb{D}_\kappa$. In particular,

$$(\alpha_n)_{n\in\mathbb{Z}} \subset \mathbb{D}_\kappa.$$

This observation is exploited in the next section.

6. An equivalence relation

The operation

$$\begin{aligned} \diamond : (\mathbb{D}, *) \times \mathbb{D} &\longrightarrow \mathbb{D}, \\ (\alpha, z) &\longmapsto \varphi_\alpha(z) \end{aligned}$$

defines a group action on the set $\mathbb{D}$. The required condition $\alpha \diamond (\beta \diamond z) = (\alpha * \beta) \diamond z$ is precisely a reformulation of (3.2). Since $(\varphi_{w_{-1}} \circ \varphi_z)(z) = w$, this action is transitive and thus it creates just one orbit on $\mathbb{D}$. Hence, we restrict ourselves to some subgroups of $(\mathbb{D}, *)$ to obtain better equivalence classes.

Fix $\alpha \in \mathbb{D}$. Then the subgroup it creates in $(\mathbb{D}, *)$ is precisely $(\alpha_n)_{n\in\mathbb{Z}}$. The orbits, or equivalent classes, created by this subgroup are as follows. Two points z_1 and z_2 are in the same orbit, and we write $z_1 \sim_\alpha z_2$, if and only if there is an integer $n \in \mathbb{Z}$ such that

$$\varphi_\alpha^{[n]}(z_1) = \varphi_{\alpha_n}(z_1) = z_2.$$

Since φ_α is an automorphism, it maps $\mathbb{D}$ and $\mathbb{T}$, respectively, to themselves bijectively. Hence, the equivalence class generated by a point $z \in \mathbb{D}$ is a sequence that is entirely in $\mathbb{D}$. A similar statement holds for the points of $\mathbb{T}$. More information on the equivalence classes are gathered below. Since $\alpha = 0$ corresponds to the identity mapping on $\mathbb{D}$, the following result (while properly modified) becomes trivial in this case. Thus, we assume that $\alpha \neq 0$.

Theorem 6.1. *Let $\alpha \in \mathbb{D}$, $\alpha \neq 0$. Then the following assertions hold.*

(i) *The equivalence class generated by $z_0 \in \mathbb{D}$ is precisely $(\varphi_{\alpha_n}(z_0))_{n\in\mathbb{Z}}$, which consists of distinct points of $\mathbb{D}$. In particular, the equivalence class generated by 0 is the sequence $(\alpha_n)_{n\in\mathbb{Z}}$.*

(ii) *We have*

$$\lim_{n\to\pm\infty} \varphi_{\alpha_n}(z_0) = 1 \quad (\textit{if } A_\alpha = 1)$$

and

$$\lim_{n\to+\infty} \varphi_{\alpha_n}(z_0) = \kappa_\alpha \qquad \textit{while} \qquad \lim_{n\to-\infty} \varphi_{\alpha_n}(z_0) = 1 \quad (\textit{if } A_\alpha > 1)$$

and

$$\lim_{n\to+\infty} \varphi_{\alpha_n}(z_0) = 1 \qquad \textit{while} \qquad \lim_{n\to-\infty} \varphi_{\alpha_n}(z_0) = \kappa_\alpha \quad (\textit{if } A_\alpha < 1).$$

Proof. (i) That the equivalence class generated by $z_0 \in \mathbb{D}$ is precisely $(\varphi_{\alpha_n}(z_0))_{n\in\mathbb{Z}}$ is rather trivial. This fact says that the equivalence class generated by z_0 consists of the past, present, and future of z_0 under the transformation φ_α. See formulas (5.2) and (5.4). For any $\alpha \in \mathbb{D}$, the automorphism φ_α has no fixed point inside $\mathbb{D}$. Hence, the class $(\varphi_{\alpha_n}(z_0))_{n\in\mathbb{Z}}$ consists of distinct points. To find the equivalence class of 0, apply (2.1) to get

$$\varphi_{\alpha_n}(0) = -\alpha_n \frac{1 - \overline{\alpha_n}}{1 - \alpha_n} \quad (n \in \mathbb{Z}).$$

But, by Theorem 3.1 and (3.4),

$$-\alpha_n \frac{1-\overline{\alpha_n}}{1-\alpha_n} = \text{ inverse of } \alpha_n \text{ in } (\mathbb{D},*) = \alpha_{-n} \quad (n \in \mathbb{Z}).$$

Thus, by part (i),

$$\big(\varphi_{\alpha_n}(0)\big)_{n\in\mathbb{Z}} = (\alpha_{-n})_{n\in\mathbb{Z}} = (\alpha_n)_{n\in\mathbb{Z}}.$$

(ii) If $A_\alpha = 1$, then we rewrite (5.4) as

$$\varphi_\alpha^{[n]}(z) = \frac{(1-\alpha)z + n\alpha(z-1)}{(1-\alpha)+n\alpha(z-1)}.$$

This representation shows that

$$\lim_{n\to\pm\infty} \varphi_{\alpha_n}(z_0) = 1.$$

Note that $A_\alpha = 1$ happens precisely on $\mathbb{D}_1$. But, if $A_\alpha \neq 1$, then we rewrite (5.2) as

$$\varphi_\alpha^{[n]}(z) = \frac{-A_\alpha^n \kappa_\alpha(z-1) + (z-\kappa_\alpha)}{-A_\alpha^n(z-1)+(z-\kappa_\alpha)}.$$

Now, there are two possibilities. If $A_\alpha > 1$, which corresponds to the points α inside the disk surrounded by $\mathbb{D}_1$, then

$$\lim_{n\to+\infty} \varphi_{\alpha_n}(z_0) = \kappa_\alpha \qquad \text{while} \qquad \lim_{n\to-\infty} \varphi_{\alpha_n}(z_0) = 1.$$

But, if $A_\alpha < 1$, which corresponds to the points $\alpha \in \mathbb{D}$, but outside the disk surrounded by $\mathbb{D}_1$, then

$$\lim_{n\to+\infty} \varphi_{\alpha_n}(z_0) = 1 \qquad \text{while} \qquad \lim_{n\to-\infty} \varphi_{\alpha_n}(z_0) = \kappa_\alpha. \qquad \square$$

We can also provide a geometric interpretation of the equivalence classes. Chapter 3 of [13] contains a comprehensive study of the geometric behavior of Möbius transformation. A very short glimpse of this visual interpretation is provided below.

Theorem 6.1 shows that the points $(\varphi_{\alpha_n}(z_0))_{n\in\mathbb{Z}}$ reside on some curves passing through 1, κ_α, and z_0, and tend to the boundary points 1 and κ_α as $n \to \pm\infty$.

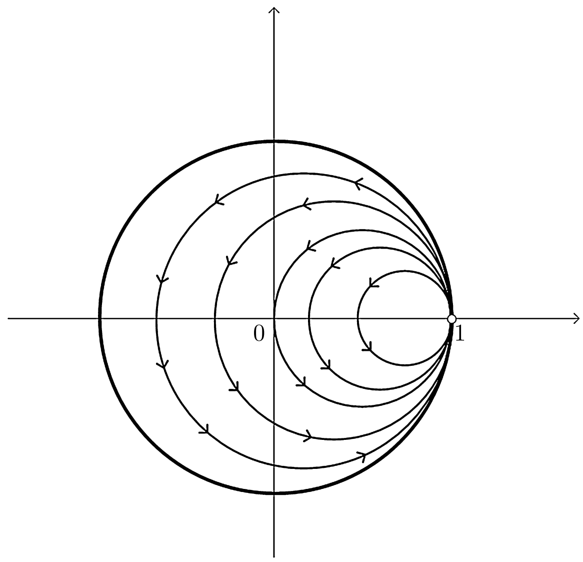
Parabolic case: $\kappa_\alpha = 1$. The relation (5.3) reveals that the equivalence class $(\varphi_{\alpha_n}(z_0))_{n\in\mathbb{Z}}$ is on the image of the line

$$t \longmapsto \phi_1(z_0) + \frac{\alpha}{1-\alpha} t \quad (t \in \mathbb{R}),$$

under the mapping ϕ_1. Since $\phi_1(\infty) = 1$ and $\phi_1(\phi_1(z_0)) = z_0$, the image is a circle passing through the points 1 and z_0. Different values of $\phi_1(z_0)$ correspond to different parallel lines. Hence, their images are circles which are tangent at 1. One particular circle corresponds to the line passing through $\phi_1(z_0) = 1/2$. In this case, we have

$$\phi_1\Big(\frac{1}{2} + \frac{\alpha}{1-\alpha} t\Big) = \frac{t + \frac{1-\alpha}{2\alpha}}{t + \frac{1-\bar\alpha}{2\bar\alpha}} \in \mathbb{T}.$$

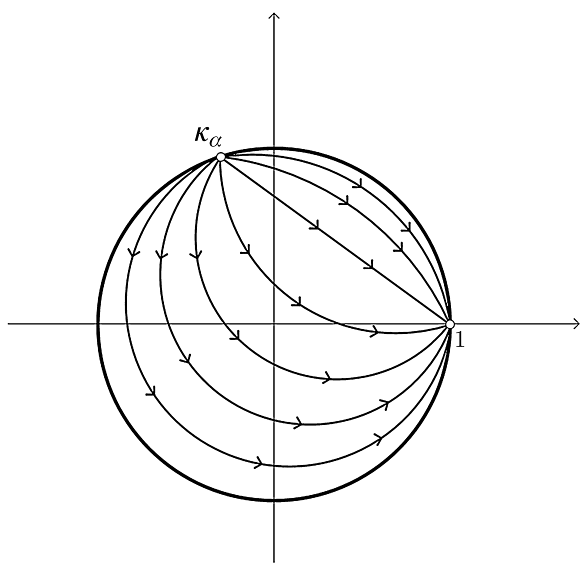
Hence, the image of this last line is the unit circle $\mathbb{T}$. In other words, the iterates of boundary points stay on $\mathbb{T}$ and (except 1) they form a two-sided sequence which converges to 1 from both sides.



Hyperbolic case: $\kappa_\alpha \neq 1$. By (5.1), we see that the equivalence class $(\varphi_{\alpha_n}(z_0))_{n\in\mathbb{Z}}$ is on the image of

$$t \longmapsto \frac{\phi_{\kappa_\alpha}(z_0)}{A_\alpha^t} \quad (t \in \mathbb{R}).$$

Since $A_\alpha \in (0,\infty)\setminus\{1\}$, the image is a line passing through 0 and $\phi_{\kappa_\alpha}(z_0)$. Since $\phi_{\kappa_\alpha}(\infty) = 1$, $\phi_{\kappa_\alpha}(0) = \kappa_\alpha$, and $\phi_{\kappa_\alpha}(\phi_{\kappa_\alpha}(z_0)) = z_0$, the image is a circle passing through the points 1, κ_α, and z_0. The following figure shows the paths when $A_\alpha < 1$.



For $A_\alpha > 1$, we just need to reverse the directions.

7. Minimal Blaschke products

In this section, we take the first step in finding the solutions of the equation $\psi \circ \varphi_\alpha = \lambda\psi$ by showing that each equivalence class of $\sim_\alpha$ in $\mathbb{D}$ produces a Blaschke product that is a *minimal solution* of the equation. Therefore, having the freedom to choose $\alpha \in \mathbb{D}$ and any of the equivalence classes it generates, the following result provides a vast variety of solutions of the functional equation. In fact, we can go even further and extract all Blaschke products that satisfy the equation.

Theorem 7.1. *Fix $\alpha \in \mathbb{D}$, $\alpha \neq 0$. Let $z_0 \in \mathbb{D}$, and let $(z_n)_{n\in\mathbb{Z}} \subset \mathbb{D}$ be the corresponding equivalence class generated by $\sim_\alpha$. Then $(z_n)_{n\in\mathbb{Z}}$ is a two-sided infinite Blaschke sequence and the corresponding Blaschke product*

$$B_{\alpha,z_0} = \prod_{n=-\infty}^{\infty} b_{z_n}$$

satisfies the functional equation

$$B_{\alpha,z_0} \circ \varphi_\alpha = \gamma_{\alpha,z_0} B_{\alpha,z_0}.$$

Moreover, no proper divisor ψ of B_{α,z_0} satisfies any functional equations of the form $\psi \circ \varphi_\alpha = \lambda\psi$, $\lambda \in \mathbb{T}$.

Proof. According to Theorem 6.1(i), without loss of generality, we can assume

$$z_n = \varphi_{\alpha_n}(z_0) \quad (n \in \mathbb{Z}).$$

Hence,

$$\begin{aligned} 1 - |z_n|^2 &= 1 - \left|\varphi_{\alpha_n}(z_0)\right|^2 \\ &= \frac{(1-|\alpha_n|^2)(1-|z_0|^2)}{|1-\overline{\alpha_n} z_0|^2} \\ &\leq \frac{1+|z_0|}{1-|z_0|}\big(1-|\alpha_n|^2\big). \end{aligned}$$

Therefore, to deal with $(1-|\alpha_n|^2)$, in light of Theorem 5.1, we consider two cases.

Parabolic case: $\kappa_\alpha = 1$. Using (4.2), we have

$$\begin{aligned} 1 - |z_n|^2 &\leq \frac{1+|z_0|}{1-|z_0|}\left(1 - \left|\frac{n\alpha}{1+(n-1)\alpha}\right|^2\right) \\ &= \frac{1+|z_0|}{1-|z_0|}\,\frac{1+(n-1)(\alpha+\bar\alpha)-(2n-1)|\alpha|^2}{|1+(n-1)\alpha|^2} \\ &\leq \frac{1+|z_0|}{1-|z_0|}\,\frac{1-|\alpha|^2}{|1+(n-1)\alpha|^2} = O(1/n^2) \quad (n \longrightarrow \pm\infty). \end{aligned}$$

Hence, $\mathcal{C}$ is a double-sided Blaschke sequence.

Hyperbolic case: $\kappa_\alpha \neq 1$. We have

$$\begin{aligned} 1 - |z_n|^2 &\leq \frac{1+|z_0|}{1-|z_0|}\left(1 - \left|\frac{\kappa_\alpha(1-A_\alpha^n)}{1-\kappa_\alpha A_\alpha^n}\right|^2\right) \\ &= \frac{1+|z_0|}{1-|z_0|}\,\frac{(2-\kappa_\alpha-\bar\kappa_\alpha)A_\alpha^n}{|1-\kappa_\alpha A_\alpha^n|^2} = O(q^{|n|}) \quad (n \longrightarrow \pm\infty), \end{aligned}$$

where $q := \min\{A_\alpha, 1/A_\alpha\} < 1$. Hence, again $\mathcal{C}$ is a double-sided Blaschke sequence (indeed, with a geometric rate of convergence).

To show that

$$B_{\alpha,z_0} = \prod_{n\in\mathbb{Z}} b_{z_n}$$

satisfies the functional equation $B_{\alpha,z_0} \circ \varphi_\alpha = B_{\alpha,z_0}$, we rewrite B_{α,z_0} in the form

$$B_{\alpha,z_0} = \prod_{n\in\mathbb{Z}} \gamma_n \varphi_{z_n},$$

where γ_n are appropriate constants such that $b_{z_n} = \gamma_n \varphi_{z_n}$; that is,

$$\gamma_n = -\frac{|z_n|}{z_n} \cdot \frac{1 - z_n}{1 - \bar{w}_n} \quad (n \in \mathbb{Z}).$$

Now, by (3.4) and (3.6),

$$\varphi_{z_n} \circ \varphi_\alpha = \varphi_{z_0} \circ \varphi_{\alpha_{-n}} \circ \varphi_\alpha = \varphi_{z_0} \circ \varphi_{\alpha_{-n+1}} = \varphi_{z_{n-1}}. \tag{7.1}$$

Therefore,

$$B_{\alpha,z_0} \circ \varphi_\alpha = \prod_{n\in\mathbb{Z}} \gamma_n \varphi_{z_n} \circ \varphi_\alpha = \prod_{n\in\mathbb{Z}} \gamma_n \varphi_{z_{n-1}} = \Big(\prod_{n\in\mathbb{Z}} \frac{\gamma_n}{\gamma_{n-1}}\Big) B_{\alpha,z_0}.$$

In the first place, even though it can be directly verified, the above calculation shows that this last product has to be convergent. Second, we have

$$\prod_{n\in\mathbb{Z}} \frac{\gamma_n}{\gamma_{n-1}} = \lim_{N\to+\infty} \prod_{n=-N+1}^{N} \frac{\gamma_n}{\gamma_{n-1}} = \lim_{N\to+\infty} \frac{\gamma_N}{\gamma_{-N}} = \frac{\lim_{N\to+\infty} \gamma_N}{\lim_{N\to-\infty} \gamma_N}.$$

Using Theorem 6.1(ii), we can compute both limits. In fact, the formula

$$z_n = \varphi_{\alpha_n}(z_0) = \frac{1 - \bar{\alpha}_n}{1 - \alpha_n} \frac{z_0 - \alpha_n}{1 - \bar{\alpha}_n z_0} \quad (n \in \mathbb{Z})$$

implies

$$\frac{1 - z_n}{1 - \bar{w}_n} = \frac{1 - z_0}{1 - \bar{z}_0} \frac{1 - \bar{\alpha}_n}{1 - \alpha_n} \frac{1 - \alpha_n \bar{z}_0}{1 - \bar{\alpha}_n z_0} \quad (n \in \mathbb{Z}).$$

Hence,

$$\gamma_n = \frac{1 - z_0}{1 - \bar{z}_0} \frac{1 - \alpha_n \bar{z}_0}{|1 - \alpha_n \bar{z}_0|} \frac{|\alpha_n - z_0|}{\alpha_n - z_0} \quad (n \in \mathbb{Z}),$$

and thus

$$\alpha_n \to 1 \quad \Longrightarrow \quad \gamma_n \to 1,$$

while

$$\alpha_n \to \kappa_\alpha \quad \Longrightarrow \quad \gamma_n \to \frac{1 - z_0}{1 - \bar{z}_0} \frac{1 - \kappa_\alpha \bar{z}_0}{\kappa_\alpha - z_0}.$$

Therefore, by Theorem 6.1(ii),

$$\prod_{n\in\mathbb{Z}} \frac{\gamma_n}{\gamma_{n-1}} = \begin{cases} \frac{1-z_0}{1-\bar{z}_0} \frac{1-\kappa_\alpha \bar{z}_0}{\kappa_\alpha - z_0} & \text{if } A_\alpha > 1, \\ 1 & \text{if } A_\alpha = 1, \\ \frac{1-\bar{z}_0}{1-z_0} \frac{\kappa_\alpha - z_0}{1-\kappa_\alpha \bar{z}_0} & \text{if } A_\alpha < 1. \end{cases}$$

In fact, the above calculation shows the motivation for the definition of γ_{α,z_0}. It is defined such that $\prod_{n\in\mathbb{Z}} \frac{\gamma_n}{\gamma_{n-1}} = \gamma_{\alpha,z_0}$. Thus, B_{α,z_0} satisfies the functional equation $B_{\alpha,z_0} \circ \varphi_\alpha = \gamma_{\alpha,z_0} B_{\alpha,z_0}$.

Finally, the identity (7.1) reveals that no proper divisor of B_{α,z_0} satisfies a functional equation of the form $\psi \circ \varphi_\alpha = \lambda\psi$. $\square$

By Theorem 6.1(i), the equivalence class generated by 0 is $(\alpha_n)_{n\in\mathbb{Z}}$ and, in this case, $\alpha_{-n} = \bar{\alpha}_n$. Hence, the corresponding minimal Blaschke product is

$$B(z) = z \prod_{n=1}^{+\infty} \frac{(\alpha_n - z)(\bar{\alpha}_n - z)}{(1-\alpha_n z)(1-\bar{\alpha}_n z)}.$$

By Theorem 7.1, this is the minimal Blaschke product that satisfies the equation $B \circ \varphi_\alpha = B$ and, moreover, $B(0) = 0$.

8. The general solution

Let ψ be an inner function satisfying $\psi \circ \varphi_\alpha = \lambda\psi$, and denote its zero set on $\mathbb{D}$ by $\mathcal{Z}(\psi)$. Then the equation $\psi \circ \varphi_\alpha = \lambda\psi$ implies $\psi \circ \varphi_{\alpha_{-1}} = \bar{\lambda}\psi$, and by induction we obtain

$$\psi \circ \varphi_{\alpha_n} = \lambda^n \psi \quad (n \in \mathbb{Z}).$$

This identity reveals that, if z_1 is a zero of ψ, then in fact the whole equivalence class $[z_n]_{n\in\mathbb{Z}}$, generated by $\sim_\alpha$, is among $\mathcal{Z}(\psi)$. Hence, we can write

$$\mathcal{Z}(\psi) = \bigcup_m \mathcal{C}_m,$$

where $(\mathcal{C}_m)_m$ is a (finite or infinite, and repetition allowed) collection of equivalence classes of $\sim_\alpha$ in $\mathbb{D}$. Note that since ψ is a nonconstant inner function, we must have

$$\sum_m \sum_{z_{mn}\in\mathcal{C}_m} \big(1 - |z_{mn}|\big) < \infty. \tag{8.1}$$

Thus,

$$B_{\alpha,(\mathcal{C}_m)_m} = \prod_m B_{\alpha,\mathcal{C}_m} \tag{8.2}$$

is a well-defined Blaschke product and, by Theorem 7.1, $B_{\alpha,(\mathcal{C}_m)_m}$ satisfies the functional equation

$$B_{\alpha,(\mathcal{C}_m)_m} \circ \varphi_\alpha = \lambda' B_{\alpha,(\mathcal{C}_m)_m},$$

where λ' is an appropriate unimodular constant. These types of Blaschke products form the main building blocks for a description of solutions of the equation $\psi \circ \varphi_\alpha = \lambda\psi$, $\lambda \in \mathbb{T}$.

Again thanks to Theorem 7.1, it is rather trivial that if we have a sequence which can be decomposed as above, then the corresponding Blaschke product is in fact a solution of the functional equation.

Put $S = \psi / B_{\alpha,(\mathcal{C}_m)_m}$. The discussion above shows that S is a zero-free inner function (i.e., a singular inner function) that satisfies an equation of the form $S \circ \varphi_\alpha = \lambda'' S$, $\lambda'' \in \mathbb{T}$. The classification of such a function is still an open question. However, at last we deduce the following result.

Theorem 8.1. *Fix $\alpha \in \mathbb{D}$, $\alpha \neq 0$. If a Blaschke product B satisfies the functional equation $B \circ \varphi_\alpha = \lambda B$, then its zero set is a union of equivalence classes generated by $\sim_\alpha$. Reciprocally, if a sequence $(z_n)_n \subset \mathbb{D}$ is such that*

(i) *as in (8.2), it can be decomposed as a union of equivalence classes generated by* $\sim_\alpha$,
(ii) *and satisfies (8.1),*

then the corresponding Blaschke product B *is a solution of the functional equation* $B \circ \varphi_\alpha = \lambda B$, *with some unimodular constant* λ. *In particular, if* $\alpha \in \mathbb{D}_1$, *then* $\lambda = 1$.

If ψ_0 satisfies the equation $\psi \circ \varphi_\alpha = \psi$, and ω is any arbitrary inner function, then we also have

$$(\omega \circ \psi_0) \circ \varphi_\alpha = (\omega \circ \psi_0).$$

Hence, $\psi = \omega \circ \psi_0$ is also a solution of the equation $\psi \circ \varphi_\alpha = \psi$. For example, if B is any of the Blaschke products (8.2) for which $\gamma = 1$, then $\omega \circ B$ is a solution. What is rather surprising is that all solutions are obtained in this manner.

Theorem 8.2. *Let* $\alpha \in \mathbb{D}$, $\alpha \neq 0$. *Then the inner function* ψ *is a solution of the equation* $\psi \circ \varphi_\alpha = \psi$ *if and only if there is an inner function* ω *and a Blaschke product* B *of type (8.2) such that*

$$\psi = \omega \circ B.$$

Proof. Without loss of generality, assume that ψ is nonconstant. Then, by a celebrated result of Frostman [5], there is a $\beta \in \mathbb{D}$ such that $\widetilde{\psi} = b_\beta \circ \psi$ is a Blaschke product with simple zeros. As a matter of fact, in a sense (logarithmic capacity), there are many such β's. But, just one choice is enough for us.

Surely, $\widetilde{\psi}$ satisfies $\widetilde{\psi} \circ \varphi_\alpha = \widetilde{\psi}$. By induction, we get

$$\widetilde{\psi} \circ \varphi_{\alpha_n} = \widetilde{\psi} \quad (n \in \mathbb{Z}).$$

If z_0 is a zero of $\widetilde{\psi}$, then the above identity shows that $\varphi_{\alpha_n}(z_0)$ is also a zero of $\widetilde{\psi}$. Hence, we can classify the zeros of $\widetilde{\psi}$ as a union of equivalence classes of $\sim_\alpha$, for example, $(\mathcal{C}_m)_m$. This observation immediately reveals that, up to a unimodular constant, $\widetilde{\psi}$ is precisely a Blaschke product of type (8.2). Since $\psi = b_\beta^{-1} \circ \widetilde{\psi}$, the proof is complete. □

It is important to keep in mind that the representation $\psi = \omega \circ B$, given in Theorem 8.2, is far away from being unique. For example, in the proof of the theorem, we picked one of the Frostman shifts and then constructed B. Different shifts give different sets of zeros and thus different Blaschke products.

9. Application in model spaces

The functional equation (1.1), and its simplified form (1.2), stem from studies on composition operators on model spaces K_Θ. The following question is still wide open.

Open Question. For which symbols φ, does the composition operator C_φ map K_Θ into itself?

This question was initiated in [10] and then was followed in [11] for composition operators with an inner symbol. We state the following result from [11] with a new proof.

Theorem 9.1. *Let φ and θ be inner functions, and let*

$$\eta(z) = \begin{cases} (\theta \circ \varphi)(z) & \text{if } \theta(0) \neq 0 \text{ and } \varphi(0) = 0, \\ z(\theta \circ \varphi)(z) & \text{if } \theta(0) \neq 0 \text{ and } \varphi(0) \neq 0, \\ z\frac{\theta(\varphi(z))}{\varphi(z)} & \text{if } \theta(0) = 0. \end{cases}$$

Then the mapping

$$C_\varphi : K_\theta \longrightarrow K_\eta$$

is well defined and bounded. Moreover,

$$\langle C_\varphi K_\theta \rangle = K_\eta;$$

that is, K_η is the smallest closed S^*-invariant subspace of H^2 that contains the image of K_θ under C_φ.

Proof. We just treat the first case. Other cases are similar. We remember that the scalar product in K_θ is the L^2-scalar product. Write S for the forward shift operator. For each $h \in H^2$ and $n \geq 1$, it is well known that $S^{*n}\theta \in K_\theta$, $n \geq 1$, and, moreover, the sequence $(S^{*n}\theta)_{n\geq 1}$ is dense in K_θ. Indeed, the verification of these facts is straightforward. We have

$$\langle S^{*n}\theta, \theta h\rangle = \langle \theta, S^n \theta h\rangle = \langle \theta, z^n \theta h\rangle = \langle 1, z^n h\rangle = 0.$$

Hence, $S^{*n}\theta \in (\theta H^2)^\perp = K_\theta$, $n \geq 1$. Moreover, assume that $f \in K_\theta$ is such that $f \perp S^{*n}\theta$, $n \geq 1$. Then

$$0 = \langle f, S^{*n}\theta\rangle = \langle z^n f, \theta\rangle = \langle z^n, \theta \bar{f}\rangle \quad (n \geq 1),$$

which implies $\theta \bar{f} \in \overline{H^2}$, or, equivalently, $f \in \theta H^2$. Thus, $f = 0$; that is, the sequence $(S^{*n}\theta)_{n\geq 1}$ is dense in K_θ. Therefore, to show that C_φ maps K_θ into K_η, it is enough to verify that

$$C_\varphi(S^{*n}\theta) \in K_\eta \quad (n \geq 1).$$

For this, we show that $C_\varphi(S^{*n}\theta) \perp \eta H^2$.

Write $\theta(z) = \sum_{k=0}^{\infty} a_k z^k$. Then

$$C_\varphi(S^{*n}\theta) = \sum_{k=n}^{\infty} a_k \varphi^{k-n} = \Big(\theta \circ \varphi - \sum_{k=0}^{n-1} a_k \varphi^k\Big)\varphi^{-n}.$$

Since φ and η are inner, for each $h \in H^2$ and $n \geq 1$, we have

$$\begin{aligned} \big\langle C_\varphi(S^{*n}\theta), \theta h\big\rangle &= \Big\langle \Big(\theta \circ \varphi - \sum_{k=0}^{n-1} a_k \varphi^k\Big)\varphi^{-n}, \eta h\Big\rangle \\ &= \Big\langle \Big(\eta - \sum_{k=0}^{n-1} a_k \varphi^k\Big), \varphi^n \eta h\Big\rangle \\ &= \langle \eta, \varphi^n \eta h\rangle - \sum_{k=0}^{n-1} a_k \langle \varphi^k, \varphi^n \eta h\rangle \\ &= \langle 1, \varphi^n h\rangle - \sum_{k=0}^{n-1} a_k \langle 1, \varphi^{n-k} \eta h\rangle = 0. \end{aligned}$$

Remember $\varphi(0) = 0$, which was exploited in the last line. In short, we showed that $C_\varphi K_\theta \subset K_\eta$, which immediately also implies $\langle C_\varphi K_\theta \rangle \subset K_\eta$.

It remains to show that the smallest closed subspace of H^2 which contains the range of C_φ (i.e., $C_\varphi K_\theta$) is precisely K_η. To this end, note that

$$k_\lambda^\theta(z) = \frac{1 - \overline{\theta(\lambda)}\theta(z)}{1 - \bar{\lambda} z}$$

is the reproducing kernel of the evaluation functional at the point $\lambda \in \mathbb{D}$. In particular, $k_\lambda^\theta \in K_\theta$ for any values of $\lambda \in \mathbb{D}$. Since $k_0^\theta = 1 - \overline{\theta(0)}\theta \in K_\theta$, we get $C_\varphi k_0^\theta = 1 - \overline{\theta(0)}\eta \in C_\varphi K_\theta$. Thus, remembering the assumption $\theta(0) \neq 0$, we deduce

$$S^* C_\varphi k_0^\theta = S^* \eta \in S^* C_\varphi K_\theta \subset \langle C_\varphi K_\theta \rangle.$$

But, we saw above that $S^*\eta$ is a generator of K_η. Hence, $K_\eta \subset \langle C_\varphi K_\theta \rangle$. □

The above theorem opens the gate to study composition operators C_φ, with φ inner, which map model spaces into themselves. According to Theorem 9.1, the mapping $C_\varphi : K_\theta \longrightarrow K_\theta$ is well defined if and only if $K_\eta \subset K_\theta$. Considering the hierarchy of inner functions, the inclusion $K_\eta \subset K_\theta$ happens if and only if η divides θ; that is,

$$\eta(z)\theta_1(z) = \theta(z) \quad (z \in \mathbb{D}),$$

where θ_1 is an inner function. Therefore, we are forced to consider three cases corresponding to the different definitions of η that were given in Theorem 9.1.

(i) If $\theta(0) \neq 0$ and $\varphi(0) = 0$, then $\eta = \theta \circ \varphi$ and we must have

$$\theta\big(\varphi(z)\big)\theta_1(z) = \theta(z) \quad (z \in \mathbb{D}).$$

Hence, there is an integer $n \geq 1$ and an inner function ϑ, with $\vartheta(0) \neq 0$, such that $\theta(z) = \vartheta(z^n)$ and $\varphi = \rho_{e^{i2k\pi/n}}$ for $1 \leq k \leq n$ (see [11, Theorem 4.1]).

(ii) If $\theta(0) \neq 0$ and $\varphi(0) \neq 0$, then $\eta(z) = z\theta(\varphi(z))$ and we must have

$$z\theta\big(\varphi(z)\big)\theta_1(z) = \theta(z) \quad (z \in \mathbb{D}).$$

The above equation does not hold for $z = 0$. Hence, there is no solution in this case.

(iii) Assume $\theta(0) = 0$; then $\eta(z) = z\theta(\varphi(z))/\varphi(z)$. This is the most interesting case, which leads us to unknown territories. We must have

$$\frac{\theta(\varphi(z))}{\varphi(z)}\theta_1(z) = \frac{\theta(z)}{z} \quad (z \in \mathbb{D}).$$

Put $\theta_2(z) = \theta(z)/z$. Hence, the above becomes

$$\theta_2\big(\varphi(z)\big)\theta_1(z) = \theta_2(z) \quad (z \in \mathbb{D}).$$

According to the Grand Iteration Theorem, φ has a fixed point p in $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$. If $p \in \mathbb{D}$, as it is discussed in detail in [11, Theorem 4.1], the possible solutions are

$$\theta(z) = \gamma z\big(\tau_p(z)\big)^m \quad (\gamma \in \mathbb{T}, m \geq 1)$$

with

$$\varphi = \tau_p \circ \rho_\lambda \circ \tau_p \quad (\lambda \in \mathbb{T})$$

and

$$\theta(z)=\gamma z\big(\tau_p(z)\big)^m\vartheta\big(\big(\tau_p(z)\big)^n\big)\quad(\gamma\in\mathbb{T},m\geq 1,n>1)$$

with

$$\varphi=\tau_p\circ\rho_{e^{i2k\pi/n}}\circ\tau_p\quad(1\leq k\leq n).$$

But if $p\in\mathbb{T}$, the situation is dramatically more complicated. Here, θ is of the form

$$\theta(z)=\gamma z\theta_2(z)\prod_{n=0}^{\infty}\theta_1\big(\varphi^{[n]}(z)\big),\tag{9.1}$$

where θ_1 is such that the product is convergent and θ_2 fulfills

$$\theta_2\big(\varphi(z)\big)=\lambda\theta_2(z)\quad(z\in\mathbb{D},\lambda\in\mathbb{T}).$$

That is why the classification obtained in Theorem 8.2 now becomes useful in shedding more light on this final case whenever θ is a Blaschke product. The general situation is still an open problem.

Theorem 9.2. *Let θ be a Blaschke product with $\theta(0)=0$. Assume that there is $\alpha\in\mathbb{D}$, $\alpha\neq 0$, such that φ_α maps K_θ into itself. Then the following hold:*

(a) *θ is of the form (9.1).*
(b) *The zeros of the Blaschke product $B(z)=\theta(z)/z$ can be decomposed as a union of equivalence classes generated by $\sim_\alpha$.*
(c) *If $\theta_1=1$, then the composition mapping C_{φ_α} is actually an isomorphism from K_θ into itself.*

Proof. Most of the proof is done in the above discussions. In particular, we saw that θ must have the form (9.1). We know that the Denjoy–Wolff point of φ_α is either 1 or κ_α. This is because φ_α has just two fixed points on $\overline{\mathbb{D}}$, and one of them has to be the Denjoy–Wolff fixed point. Therefore, by Theorem 8.1, the zeros of the Blaschke product θ_2 are decomposed as a union of equivalence classes generated by $\sim_\alpha$ and, by case (iii), the operator C_{φ_α} maps K_θ into itself. To show that C_φ is surjective whenever $\theta_1=1$, note that

$$K_{zB}=\mathbb{C}\oplus\operatorname{Span}\big\{k_{z_j}:B(z_j)=0\big\},$$

where k_{z_j} is the Cauchy reproducing kernel

$$k_{z_j}(z)=\frac{1}{1-\bar{z}_jz}.$$

We have $C_{\varphi_\alpha}1=1$ and, by Theorem 3.1,

$$C_{\varphi_\alpha}k_{z_j}(z)=\frac{1}{1-\bar{z}_j\varphi_\alpha(z)}=\frac{A+Bz}{1-\overline{\varphi_{\alpha_{-1}}(z_j)}z},$$

where A and B are some constants. Hence, $k_{\varphi_{\alpha_{-1}}(z_j)}$ belongs to the image of C_{φ_α}. We assumed that the zeros of B can be decomposed as a union of equivalence classes generated by $\sim_\alpha$. Therefore, by Theorem 6.1(i), the image contains all Cauchy kernels k_{z_j}, where z_j runs through the zeros of B. In short, this means that the mapping is surjective. □

We can also interpret Theorem 7.1 in the following way to state some facts about the point spectrum of C_{φ_α}. Writing the functional equation as $C_{\varphi_\alpha} B_{\alpha,z_0} = \gamma_{\alpha,z_0} B_{\alpha,z_0}$, it says that B_{α,z_0} is an eigenvector of C_{φ_α} corresponding to the eigenvalue γ_{α,z_0}. As usual, there are two cases to consider.

(a) If $\alpha \in \mathbb{D}_1$, then for any choice of z_0, we have $\gamma_{\alpha,z_0} = 1$. Hence, there are infinitely many Blaschke products that satisfy $C_{\varphi_\alpha} B_{\alpha,z_0} = B_{\alpha,z_0}$. In the first place, the mere existence of such eigenfunctions was an open question. Second, it is still unknown if C_{φ_α} can have other eigenvalues.

(b) If $\alpha \in \mathbb{D} \setminus \mathbb{D}_1$, then $\gamma_{\alpha,z_0} = \varphi_{z_0}(\kappa_\alpha)$ (or its conjugate) and as z_0 ranges over $\mathbb{D}$, the values of $\varphi_{z_0}(\kappa_\alpha)$ cover all of $\mathbb{T} \setminus \{1\}$. Hence, $\sigma_p(C_{\varphi_\alpha}) = \mathbb{T} \setminus \{1\}$ and each eigenvalue has infinitely many Blaschke products as its eigenvectors. (That the eigenvalues of C_{φ_α} must stay on $\mathbb{T}$ is rather elementary to verify.)

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