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ISOMETRIES ON THE UNIT SPHERE OF THE ℓ^1-SUM OF STRICTLY CONVEX NORMED SPACES

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ABSTRACT. We study the extension property of isometries on the unit sphere of the ℓ^1-sum of strictly normed spaces, which is a special case of Tingley's isometric extension problem. In this paper, we will give some sufficient conditions such that such isometries can be extended to the whole space.

1. INTRODUCTION

Let E, F be real normed spaces. The classical Mazur–Ulam theorem states that every surjective isometry $T : E \to F$ must be affine. P. Mankiewicz [5] extended this result by showing that if $U \subset E$ and $V \subset F$ are either open connected or convex bodies and $V_0 : U \to V$ is a surjective isometry, then there exists a surjective affine isometry $V : E \to F$ such that $V|_U = V_0$. Motivated by these results, Tingley [9] proposed the following isometric extension problem. Suppose that E is a normed space, and let $S_1(E) = \{x \in E : \|x\| = 1\}$ be the unit sphere of E.

(IEP) Let E, F be real normed spaces. Suppose that $V_0 : S_1(E) \to S_1(F)$ is a surjective isometry. Is V_0 necessarily the restriction of a linear isometry on the whole space?

If this problem has a positive answer, then the local geometric property of a mapping on the unit sphere will determine the global property of the mapping

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on the whole space. So far, many papers have been devoted to the isometric extension problem. The isometric extension problem has been considered based on the representation of isometries and solved in the positive for some classical real Banach space (see, e.g., [1] and the references therein). However, the problem is still open, even if E, F are finite dimensional. Recently, V. Kadets and M. Martin [3] and R. Tanaka [7], [8] studied the isometric extension property on finite-dimensional spaces.

Wang and Orihara [10] gave the representation of surjective isometries between unit spheres of the ℓ^1-sum of strictly convex normed spaces and solved the isometric extension problem in this case. Let $E = (\oplus \sum_{\gamma\in\Gamma} E_\gamma)_{\ell^1}$ and $F = (\oplus \sum_{\delta\in\Delta} E_\delta)_{\ell^1}$, where Γ and Δ are index sets and E_γ, F_δ are strictly convex normed spaces for each $\gamma \in \Gamma$ and $\delta \in \Delta$. If V_0 is an surjective isometry from $S_1(E)$ onto $S_1(F)$, then Wang and Orihara [10] showed that there exists a bijection π from Γ onto Δ such that V_0 is of the form

$$V_0(x) = \sum_{\delta\in\Delta} \|x_{\pi^{-1}(\delta)}\| V_0\Big(\frac{x_{\pi^{-1}(\delta)}}{\|x_{\pi^{-1}(\delta)}\|}\Big), \quad \forall x = (x_\gamma)_{\gamma\in\Gamma} \in S_1(E).$$

Moreover, Liu [4] and Ding and Li [2] studied the surjective isometries between the unit spheres of the ℓ^β-sum $(0 < \beta < 1)$ (or ℓ^∞-sum) of strictly convex normed spaces and solved the isometric extension problem affirmatively in these cases.

These results consider the isometries between the unit spheres of spaces of the same type. In this paper, we will study the isometries on the unit sphere of the ℓ^1-sum of strictly convex normed spaces whose range is in a general normed space. We will give some sufficient conditions to solve the isometric extension problem in this case.

We will introduce some notation, which will be used in the rest of the paper. Let Γ be an index set containing at least two points, and let $\{G_\gamma : \gamma \in \Gamma\}$ be a family of strictly convex normed spaces. The ℓ^1-sum of $\{G_\gamma\}_{\gamma\in\Gamma}$ is defined by

$$\Big(\oplus \sum_{\gamma\in\Gamma} G_\gamma\Big)_{\ell^1} = \Big\{x = (x_\gamma)_{\gamma\in\Gamma} : x_\gamma \in G_\gamma(\gamma \in \Gamma), \sum_{\gamma\in\Gamma} \|x_\gamma\| < \infty\Big\},$$

and the norm is defined by

$$\|x\| = \sum_{\gamma\in\Gamma} \|x_\gamma\|, \quad \forall x = (x_\gamma)_{\gamma\in\Gamma} \in \Big(\oplus \sum_{\gamma\in\Gamma} G_\gamma\Big)_{\ell^1}.$$

In the remainder of the paper, for any $\gamma \in \Gamma$ and $x_\gamma \in G_\gamma$, the element in G, whose support is the singleton $\{\gamma\}$ and the γth coordinate is x_γ, is also denoted by x_γ for convenience. Suppose that E is a normed space, and suppose that $V_0 : S_1(G) \to S_1(E)$ is an isometry from the unit sphere of G into the unit sphere of E. For any $x = (x_\gamma)_{\gamma\in\Gamma} \in (\oplus \sum_{\gamma\in\Gamma} G_\gamma)_{\ell^1}$, $\mathrm{supp}(x)$ is denoted by the support set of x; that is, $\mathrm{supp}(x) = \{\gamma \in \Gamma : x_\gamma \neq 0\}$. For any $x, y \in G$, we say that x, y are *orthogonal*, written by $x \perp y$, if $\mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset$.

2. The isometries on the unit sphere of ℓ^1-sum spaces

Lemma 2.1. *Suppose that* $x = (x_\gamma)_{\gamma\in\Gamma}, y = (y_\gamma)_{\gamma\in\Gamma} \in G$. *Then* $\|x + y\| = \|x\| + \|y\|$ *if and only if, for each* $\gamma \in \operatorname{supp}(x) \cap \operatorname{supp}(y)$, *there exists* $a_\gamma > 0$ *such that* $y_\gamma = a_\gamma x_\gamma$.

Proof. Since $\|x + y\| = \|x\| + \|y\|$ and $\|x_\gamma + y_\gamma\| \le \|x_\gamma\| + \|y_\gamma\|$ for every $\gamma \in \Gamma$, it follows that, for each $\gamma \in \Gamma$, $\|x_\gamma + y_\gamma\| = \|x_\gamma\| + \|y_\gamma\|$. Note that each G_γ is strictly convex; then one can derive that, for every $\gamma \in \operatorname{supp}(x) \cap \operatorname{supp}(y)$, there exists $a_\gamma > 0$ such that $y_\gamma = a_\gamma x_\gamma$. □

Lemma 2.2. *Suppose that* V_0 *satisfies that* $-V_0[S_1(G)] \subset V_0[S_1(G)]$. *Then, for any* $\gamma \in \Gamma$ *and* $x_\gamma \in G_\gamma$, *we have that* $V_0(-x_\gamma) = -V_0(x_\gamma)$.

Proof. For any $\gamma \in \Gamma$ and $x_\gamma \in G_\gamma$, there must exist an element $y \in S_1(G)$ such that $V_0(y) = -V_0(x_\gamma)$. Then, for each $\gamma_1 \in \Gamma$ with $\gamma_1 \neq \gamma$, we have that $x_\gamma \perp \frac{y_{\gamma_1}}{\|y_{\gamma_1}\|}$ whenever $y_{\gamma_1} \neq 0$.

Let $z = (z_\gamma)_{\gamma\in\Gamma}$, where $z_\gamma = \frac{1}{2}x_\gamma$, $z_{\gamma_1} = \frac{1}{2}\frac{y_{\gamma_1}}{\|y_{\gamma_1}\|}$ and $z_{\gamma'} = 0$ for all $\gamma' \neq \gamma, \gamma_1$. Evidently, z belongs to the unit sphere of G, which implies that there exists $u \in S_1(G)$ such that $V_0(u) = -V_0(z)$ and hence $\|u - z\| = 2$. By Lemma 2.1 we can derive that $u_\gamma = 0$ or $u_\gamma = -a_\gamma x_\gamma$ for some $a_\gamma > 0$. Therefore,

$$\big\|V_0(x_\gamma) - V_0(u)\big\| = \|u - x_\gamma\| = \|u\| + \|x_\gamma\| = 2,$$

and then

$$2 = \big\|V_0(z) - V_0(y)\big\| = \|z - y\|.$$

Thus, it follows from Lemma 2.1 that $-y_{\gamma_1} = a_{\gamma_1}\frac{y_{\gamma_1}}{\|y_{\gamma_1}\|}$ for some positive number a_{γ_1}. This is impossible since $y_{\gamma_1} \neq 0$. □

By Lemmas 2.1 and 2.2 we can derive that $V_0(-x) = -V_0(x)$ for all $x \in S_1(G)$, which is stated in the following theorem.

Theorem 2.3. *Suppose that* V_0 *satisfies that* $-V_0[S_1(G)] \subset V_0[S_1(G)]$. *Then, for any* $x \in S_1(G)$, *we have that*

$$V_0(-x) = -V_0(x).$$

Proof. For any $x \in S_1(G)$, there exists $y \in S_1(G)$ such that $V_0(y) = -V_0(x)$. For any $\gamma \in \operatorname{supp}(y)$, it follows from Lemma 2.2 that

$$\Big\|y + \frac{y_\gamma}{\|y_\gamma\|}\Big\| = \Big\|V_0(y) - V_0\Big(-\frac{y_\gamma}{\|y_\gamma\|}\Big)\Big\| = \Big\|V_0\Big(\frac{y_\gamma}{\|y_\gamma\|}\Big) - V_0(x)\Big\| = \Big\|\frac{y_\gamma}{\|y_\gamma\|} - x\Big\|$$

and

$$\Big\|y - \frac{y_\gamma}{\|y_\gamma\|}\Big\| = \Big\|V_0(y) - V_0\Big(\frac{y_\gamma}{\|y_\gamma\|}\Big)\Big\| = \Big\|V_0(x) + V_0\Big(\frac{y_\gamma}{\|y_\gamma\|}\Big)\Big\| = \Big\|x + \frac{y_\gamma}{\|y_\gamma\|}\Big\|.$$

That is,

$$1 = 1 + \|y_\gamma\| - \|y_\gamma\| = \Big\|x_\gamma - \frac{y_\gamma}{\|y_\gamma\|}\Big\| - \|x_\gamma\|$$

and

$$1 - \|y_\gamma\| - \|y_\gamma\| = \Big\|x_\gamma + \frac{y_\gamma}{\|y_\gamma\|}\Big\| - \|x_\gamma\|.$$

The strict convexity of G_γ implies that there exists $a_\gamma > 0$ such that $x_\gamma = -a_\gamma y_\gamma$, and then

$$1 - 2\|y_\gamma\| + a_\gamma\|y_\gamma\| = \big|1 - a_\gamma\|y_\gamma\|\big|.$$

Since $a_\gamma\|y_\gamma\| \le \|x_\gamma\| \le 1$, one can derive that $a_\gamma = 1$. This implies that, for each $\gamma \in \operatorname{supp}(y)$, $y_\gamma = -x_\gamma$, and then $y = -x$. $\square$

In order to give the representation of V_0, we need the following result, which plays the important role in the proof of Corollary 2.6.

Lemma 2.4. *Suppose that V_0 satisfies that $-V_0[S_1(G)] \subset V_0[S_1(G)]$. Then, for any $k \in \mathbb{N}$, we have the following:*

(i) *For any $\{x_i\}_{1\le i\le k}$ with $x_i \perp x_j$ $(1 \le i \ne j \le k)$, we have that*

$$\Big\|\sum_{i=1}^{k} \lambda_i V_0(x_i)\Big\| = \sum_{i=1}^{k} |\lambda_i|, \quad \forall \lambda_1, \dots, \lambda_k \in \mathbb{R}.$$

(ii) *For any $\{\gamma_i\}_{1\le i\le k} \subset \Gamma$, $x_{\gamma_i} \in S_1(G_{\gamma_i})$ $(1 \le i \le k)$ and $\{\lambda_i\}_{1\le i\le k} \subset \mathbb{R}$, if there exists an element $y \in S_1(G)$ such that $V_0(y) = \sum_{i=1}^{k} \lambda_i V_0(x_{\gamma_i})$, then we have that $y = \sum_{i=1}^{k} \lambda_i x_{\gamma_i}$.*

Proof. We will prove this lemma by induction. When $k = 2$, for any $x_1, x_2 \in S_1(G)$ with $x_1 \perp x_2$ and for any $\theta_1, \theta_2 = 1$ or -1, by Theorem 2.3 we have that

$$\big\|\theta_1 V_0(x_1) + \theta_2 V_0(x_2)\big\| = \|\theta_1 x_1 + \theta_2 x_2\| = \|\theta_1 x_1\| + \|\theta_2 x_2\| = 2.$$

Therefore, for any $\lambda_1, \lambda_2 \in \mathbb{R}$, we have that

$$\big\|\lambda_1 V_0(x_1) + \lambda_2 V_0(x_2)\big\| = |\lambda_1| + |\lambda_2|.$$

Suppose that $\gamma_1 \ne \gamma_2$, $x_{\gamma_i} \in S_1(G_{\gamma_i})$ $(i = 1, 2)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$; if there exists a $y \in S_1(G)$ such that $V_0(y) = \lambda_1 V_0(x_1) + \lambda_2 V_0(x_2)$, then for any $i = 1, 2$ we can derive that

$$\|y + x_{\gamma_i}\| = \big\|V_0(y) + V_0(x_{\gamma_i})\big\| = |1 + \lambda_i| + 1 - |\lambda_i|$$

and

$$\|y - x_{\gamma_i}\| = \big\|V_0(y) - V_0(x_{\gamma_i})\big\| = |1 - \lambda_i| + 1 - |\lambda_i|.$$

This implies that

$$\|y_{\gamma_i} + x_{\gamma_i}\| - \|y_{\gamma_i}\| = 1 + \lambda_i - |\lambda_i| \tag{2.1}$$

and

$$\|y_{\gamma_i} - x_{\gamma_i}\| - \|y_{\gamma_i}\| = 1 - \lambda_i - |\lambda_i|. \tag{2.2}$$

Since G_{γ_i} is strictly convex, there exists a real number a_{γ_i} such that $y_{\gamma_i} = a_{\gamma_i} x_{\gamma_i}$, where $|a_{\gamma_i}| = \|a_{\gamma_i} x_{\gamma_i}\| = \|y_{\gamma_i}\| \le 1$. It follows from (2.1) and (2.2) that $a_{\gamma_i} = \lambda_i$ for each $i = 1, 2$, which implies that $y = \lambda_1 x_{\gamma_1} + \lambda_2 x_{\gamma_2}$.

Suppose that the lemma is true for $k-1$; then for any $\{x_i\}_{1\le i\le k}\subset S_1(G)$ with $x_i\perp x_j$ for $1\le i\ne j\le k$, we have that, for any $\lambda_1,\dots,\lambda_k\in\mathbb{R}$,

$$\begin{aligned}
&\Big\|\sum_{i=1}^{k}\lambda_i V_0(x_i)\Big\| \\
&= \Big\|\Big\|\sum_{i=1}^{k-1}\lambda_i V_0(x_i)\Big\|\sum_{i=1}^{k-1}\frac{\lambda_i}{\|\sum_{i=1}^{k-1}\lambda_i V_0(x_i)\|}V_0(x_i)+\lambda_k V_0(x_k)\Big\| \\
&= \Big\|\Big\|\sum_{i=1}^{k-1}\lambda_i V_0(x_i)\Big\|V_0\Big(\sum_{i=1}^{k-1}\frac{\lambda_i x_i}{\|\sum_{i=1}^{k-1}\lambda_i V_0(x_i)\|}\Big)+\lambda_k V_0(x_k)\Big\| \\
&= \Big\|\sum_{i=1}^{k-1}\lambda_i V_0(x_i)\Big\|+|\lambda_k|=\sum_{i=1}^{k}|\lambda_i|.
\end{aligned}$$

On the other hand, for any $\{\gamma_i\}_{1\le i\le k}\subset\Gamma, x_{\gamma_i}\in S_1(G_{\gamma_i})$ $(1\le i\le k)$ and $\{\lambda_i\}_{1\le i\le k}\subset\mathbb{R}$, if there exists $y\in S_1(G)$ such that $V_0(y)=\sum_{i=1}^{k}\lambda_i V_0(x_{\gamma_i})$, then for each $i=1,2,\dots,k$ we have that

$$\|y+x_{\gamma_i}\|=\big\|V_0(y)+V_0(x_{\gamma_i})\big\|=|1+\lambda_i|+1-|\lambda_i|$$

and

$$\|y-x_{\gamma_i}\|=\big\|V_0(y)-V_0(x_{\gamma_i})\big\|=|1-\lambda_i|+1-|\lambda_i|.$$

Since G_{γ_i} is strictly convex, there exists a $a_{\gamma_i}\in\mathbb{R}$ with $|a_{\gamma_i}|\le 1$ such that $y_{\gamma_i}=a_{\gamma_i}x_{\gamma_i}$. Moreover, we can derive that $a_{\gamma_i}=\lambda_i$ for every $i=1,2,\dots,k$, and hence $y=\sum_{i=1}^{k}\lambda_i x_{\gamma_i}$. □

Corollary 2.5. *If V_0 satisfies the following conditions,*

(Δ_1) $-V_0[S_1(G)]\subset V_0[S_1(G)]$,

(Δ_2) *for any $\gamma\in\Gamma, x_\gamma\in S_1(G_\gamma)$, and $y\in S_1(G)$ with $x_\gamma\perp y$, we have that $\lambda_1V_0(x_\gamma)+\lambda_2V_0(y)\in V_0[S_1(G)]$ for all $\lambda_1,\lambda_2\in\mathbb{R}$ with $|\lambda_1|+|\lambda_2|=1$,*

then, for any $k\in\mathbb{N}, x_{\gamma_i}\in S_1(G_{\gamma_i})$ $(i=1,2,\dots,k)$, and $\lambda_1,\dots,\lambda_k\in\mathbb{R}$ with $\sum_{i=1}^{k}|\lambda_i|=1$, we have that

$$V_0\Big(\sum_{i=1}^{k}\lambda_i x_{\gamma_i}\Big)=\sum_{i=1}^{k}\lambda_i V_0(x_{\gamma_i}).$$

Therefore, we can get the representation of V_0.

Corollary 2.6. *Suppose that V_0 satisfies conditions (Δ_1) and (Δ_2); then for any $x=(x_\gamma)_{\gamma\in\Gamma}\in S_1(G)$ we have that*

$$V_0(x)=\sum_{x_\gamma\ne 0}\|x_\gamma\|V_0\Big(\frac{x_\gamma}{\|x_\gamma\|}\Big).$$

Proof. By condition (Δ_2) and Corollary 2.5, for any $k\in\mathbb{N}$ and $\{\gamma_i\}_{1\le i\le k}\subset\operatorname{supp}(x)$, we have that

$$\sum_{i=1}^{k}\frac{\|x_{\gamma_i}\|}{\sum_{i=1}^{k}\|x_{\gamma_i}\|}V_0\Big(\frac{x_{\gamma_i}}{\|x_{\gamma_i}\|}\Big)\in V_0\big[S_1(G)\big]$$

and

$$V_0\Big(\sum_{i=1}^{k}\frac{x_{\gamma_i}}{\sum_{i=1}^{k}\|x_{\gamma_i}\|}\Big)=\sum_{i=1}^{k}\frac{x_{\gamma_i}}{\sum_{i=1}^{k}\|x_{\gamma_i}\|}V_0\Big(\frac{x_{\gamma_i}}{\|x_{\gamma_i}\|}\Big).$$

Then one can derive that

$$V_0(x)=V_0\Big(\sum_{x_\gamma\neq 0}x_\gamma\Big)=\sum_{x_\gamma\neq 0}\|x_\gamma\|V_0\Big(\frac{x_\gamma}{\|x_\gamma\|}\Big).$$

□

By the above lemmas, we can obtain the main result of this paper.

Theorem 2.7. *Suppose that V_0 satisfies (Δ_2) and the following condition:*

(Δ_3) *There exists a perturbation π of Γ such that, for any $\gamma\in\Gamma, x_\gamma, y_\gamma\in S_1(G_\gamma)$, and $\lambda_1,\lambda_2\in\mathbb{R}$, if $\|\lambda_1V_0(x_\gamma)+\lambda_2V_0(y_\gamma)\|=1$, then $\lambda_1V_0(x_\gamma)+\lambda_2V_0(y_\gamma)\in V_0[S_1(G_{\pi(\gamma)})]$.*

Then V_0 can be extended to be an isometric mapping V from G to E.

Proof. Define a mapping V from G into E by

$$V_0(x)=\sum_{\gamma\in\Gamma}y_\gamma,\quad \forall x=(x_\gamma)_{\gamma\in\Gamma}\in G,$$

where

$$y_\gamma=\begin{cases}\|x_\gamma\|V_0(\frac{x_\gamma}{\|x_\gamma\|}) & \text{if } x_\gamma\neq 0,\\ 0 & \text{if } x_\gamma=0.\end{cases}$$

By Corollary 2.6, we have that V is an extension of V_0; that is, $V|_{S_1(G)}=V_0$ and V is a positively homogeneous operator satisfying $\|V(x)\|=\|x\|$ for all $x\in G$.

For any $\tilde{x}_\gamma,\tilde{y}_\gamma\in G_\gamma$ with $\|\tilde{x}_\gamma\|,\|\tilde{y}_\gamma\|\le 1$ and $e\in S_1(G_{\gamma_1})$, where $\gamma_1\neq\gamma$, we can construct two elements $u=(u_\sigma)_{\sigma\in\Gamma}$ and $v=(v_\sigma)_{\sigma\in\Gamma}$ in G as follows:

$$u_\sigma=\begin{cases}\tilde{x}_\gamma & \text{if } \sigma=\gamma,\\ (1-\|\tilde{x}_\gamma\|)e & \text{if } \sigma=\gamma_1,\\ 0 & \text{if } \sigma\neq\gamma,\gamma_1\end{cases}$$

and

$$v_\sigma=\begin{cases}\tilde{y}_\gamma & \text{if } \sigma=\gamma,\\ (1-\|\tilde{y}_\gamma\|)e & \text{if } \sigma=\gamma_1,\\ 0 & \text{if } \sigma\neq\gamma,\gamma_1.\end{cases}$$

It is easy to check that $\|u\|=\|v\|=1$. It follows from the construction of V that

$$\begin{aligned}\big\|V(u)-V(v)\big\|&=\sum_{\gamma\in\Gamma}\big\|V(u_\sigma)-V(v_\sigma)\big\|\\&=\big\|V(\tilde{x}_\gamma)-V(\tilde{y}_\gamma)\big\|+\big\|\big(1-\|\tilde{x}_\gamma\|\big)V(e)-\big(1-\|\tilde{y}_\gamma\|\big)V(e)\big\|\\&=\big\|V(\tilde{x}_\gamma)-V(\tilde{y}_\gamma)\big\|+\big|\|\tilde{x}_\gamma\|-\|\tilde{y}_\gamma\|\big|\end{aligned}$$

and

$$\begin{aligned}\|u-v\| &= \sum_{\sigma\in\Gamma}\|u_\sigma - v_\sigma\| \\ &= \|\tilde{x}_\gamma - \tilde{y}_\gamma\| + \big\|\big(1-\|\tilde{x}_\gamma\|\big)e - \big(1-\|\tilde{y}_\gamma\|\big)e\big\| \\ &= \|\tilde{x}_\gamma - \tilde{y}_\gamma\| + \big|\|\tilde{x}_\gamma\| - \|\tilde{y}_\gamma\|\big|.\end{aligned}$$

Since $\|V(u)-V(v)\| = \|V_0(u)-V_0(v)\| = \|u-v\|$, one can derive that

$$\big\|V(\tilde{x}_\gamma) - V(\tilde{y}_\gamma)\big\| = \|\tilde{x}_\gamma - \tilde{y}_\gamma\|. \tag{2.3}$$

For any $x=(x_\gamma)_{\gamma\in\Gamma}, y=(y_\gamma)_{\gamma\in\Gamma}\in G$ and $\gamma\in\Gamma$, if $V_0(x_\gamma)-V_0(y_\gamma)\neq 0$, then by (Δ_3) one can choose an element $z_{\pi(\gamma)}\in S_1(G_{\pi(\gamma)})$ such that

$$\begin{aligned}&V(x_\gamma)-V(y_\gamma) \\ &\quad= \big\|V(x_\gamma)-V(y_\gamma)\big\|\Big[\frac{1}{\|V(x_\gamma)-V(y_\gamma)\|}\Big(\|x_\gamma\|V_0\Big(\frac{x_\gamma}{\|x_\gamma\|}\Big) - \|y_\gamma\|V_0\Big(\frac{y_\gamma}{\|y_\gamma\|}\Big)\Big)\Big] \\ &\quad= \big\|V(x_\gamma)-V(y_\gamma)\big\|V_0(z_{\pi(\gamma)}).\end{aligned}$$

On the other hand, there exists a real number $M>0$ such that $x_\gamma/M, y_\gamma/M$ have norm less than 1 for all $\gamma\in\Gamma$, which implies that, by (2.3),

$$\begin{aligned}\big\|V(x_\gamma)-V(y_\gamma)\big\| &= M\Big\|V\Big(\frac{x_\gamma}{M}\Big) - V\Big(\frac{y_\gamma}{M}\Big)\Big\| = M\Big\|\frac{x_\gamma}{M} - \frac{y_\gamma}{M}\Big\| \\ &= \|x_\gamma - y_\gamma\| \quad \text{for all} \gamma\in\Gamma.\end{aligned}$$

Then, by Lemma 2.4, one can derive that

$$\begin{aligned}\big\|V(x)-V(y)\big\| &= \Big\|\sum_{x_\gamma\neq 0}\|x_\gamma\|V_0\Big(\frac{x_\gamma}{\|x_\gamma\|}\Big) - \sum_{y_\gamma\neq 0}\|y_\gamma\|V_0\Big(\frac{y_\gamma}{\|y_\gamma\|}\Big)\Big\| \\ &= \Big\|\sum_{\gamma\in\mathrm{supp}(x)\cap\mathrm{supp}(y)}\big(V(x_\gamma)-V(y_\gamma)\big)\Big\| \\ &\quad + \Big\|\sum_{x_\gamma\neq 0, y_\gamma=0}\|x_\gamma\|V_0\Big(\frac{x_\gamma}{\|x_\gamma\|}\Big)\Big\| + \Big\|\sum_{y_\gamma\neq 0, x_\gamma=0}\|y_\gamma\|V_0\Big(\frac{y_\gamma}{\|y_\gamma\|}\Big)\Big\| \\ &= \Big\|\sum_{\gamma\in\mathrm{supp}(x)\cap\mathrm{supp}(y)}\big\|V(x_\gamma)-V(y_\gamma)\big\|V_0(z_{\pi(\gamma)})\Big\| \\ &\quad + \sum_{x_\gamma\neq 0, y_\gamma=0}\|x_\gamma\| + \sum_{y_\gamma\neq 0, x_\gamma=0}\|y_\gamma\| \\ &= \sum_{\gamma\in\mathrm{supp}(x)\cap\mathrm{supp}(y)}\big\|V(x_\gamma)-V(y_\gamma)\big\| \\ &\quad + \sum_{x_\gamma\neq 0, y_\gamma=0}\|x_\gamma\| + \sum_{y_\gamma\neq 0, x_\gamma=0}\|y_\gamma\| \\ &= \sum_{\gamma\in\Gamma}\|x_\gamma - y_\gamma\| = \|x-y\|,\end{aligned}$$

and this shows that V is an isometry from G into E. □

Remark 2.8. We can obtain condition (Δ_1) from conditions (Δ_2) and (Δ_3). By Lemma 2.4, we can see that the condition (Δ_2) can be replaced by the following condition (Δ_2'):

(Δ_2') For any $x_\gamma \in S_1(G_\gamma)$ and $y \in S_1(G)$ with $x_\gamma \perp y$, if $\lambda_1, \lambda_2 \in \mathbb{R}$ with $|\lambda_1| + |\lambda_2| = 1$, we have that $\lambda_1 V_0(x_\gamma) + \lambda_2 V_0(y) \in V_0[S_1(G)]$.

Remark 2.9. If, in addition, V_0 is surjective, then it follows from the Mazur–Ulam theorem that the extension V of V_0 is a linear isometry.

Corollary 2.10. *Suppose that E is a normed space, and suppose that V_0 is an isometric mapping from $S_1[\ell^1(\Gamma)]$ into $S_1(E)$ satisfying the following condition:*

(Δ_2'') *If $\gamma_1, \gamma_2 \in \Gamma$ and $\gamma_1 \neq \gamma_2$, then for any $\lambda_1, \lambda_2 \in \mathbb{R}$ with $|\lambda_1| + |\lambda_2| = 1$ we have that $\lambda_1 V_0(e_{\gamma_1}) + \lambda_2 V_0(e_{\gamma_2}) \in V_0[S_1(\ell^1(\Gamma))]$.*

Then V_0 can be extended to be an operator $V : \ell^1(\Gamma) \to E$, defined on the whole space $\ell^1(\Gamma)$. Furthermore, if V_0 is a surjective operator, then V_0 can be extended to be a linear isometry from $\ell^1(\Gamma)$ to E.

Remark 2.11. Let $\ell^1_{(n)}$ be the n-dimensional Banach spaces with the ℓ^1-norm. Define an isometric mapping V_0 from $S_1(\ell^1_{(2)})$ into $S_1(\ell^1_{(3)})$ by

$$V_0(\xi_1, \xi_2) = \begin{cases} (0, \xi_1, \xi_2) & \text{if } \xi_1 < 0, \\ (\xi_1, 0, \xi_2) & \text{if } \xi_1 \geq 0, \end{cases} \quad \forall (\xi_1, \xi_2) \in S_1(\ell^1_{(2)}).$$

Then V_0 can be extended to be an isometric mapping defined on $\ell^1_{(2)}$ in the canonical way, but it is not linear.

In particular, if E is the ℓ^1-sum of strictly convex normed spaces, then, by Theorem 2.7, we can conclude the main result (Theorem 6) of [10].

Corollary 2.12. *Suppose that $(E_\gamma)_{\gamma\in\Gamma}$ and $(F_\delta)_{\delta\in\Delta}$ are sets of strictly convex normed spaces. Let $E = (\sum_{\gamma\in\Gamma} E_\gamma)_{\ell^1}$ and $F = (\sum_{\delta\in\Delta} F_\delta)_{\ell^1}$. If V_0 is an isometry from $S_1(E)$ into $S_1(F)$ satisfying $-V_0[S_1(E)] \subset V_0[S_1(E)]$ and, for any $\gamma \in \Gamma$, $x_\gamma \in S_1(E_\gamma)$ and $y \in S_1(E)$ with $x_\gamma \perp y$, then we have that $\lambda_1 V_0(x_\gamma) + \lambda_2 V_0(y) \in V_0[S_1(E)]$, where $\lambda_1, \lambda_2 \in \mathbb{R}$ with $|\lambda_1| + |\lambda_2| = 1$. Then V_0 can be extended to an isometry defined on the whole space E. In particular, if V_0 is a surjective isometry, then it must be extended to be a linear surjective isometry from E onto F.*

For any index set Γ and any Banach space G, it is well known that $\ell^1(\Gamma, G) = \ell^1(\Gamma) \otimes_\pi G$ (see [6, p. 20]). So we can derive a more general case of Theorem 2.7.

Corollary 2.13. *Suppose that Γ is an index set and that G is a strictly convex Banach space. Let V_0 be an isometric mapping from the unit sphere of $\ell^1(\Gamma) \otimes_\pi G$ into the unit sphere of another Banach space E. If V_0 satisfies (Δ_2) and (Δ_3), then it can be extended to be an isometry defined on $\ell^1(\Gamma) \otimes_\pi G$.*

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