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ON EXTREME CONTRACTIONS AND THE NORM ATTAINMENT SET OF A BOUNDED LINEAR OPERATOR

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In honor of T. K. Mukherjee

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ABSTRACT. In this paper we completely characterize the norm attainment set of a bounded linear operator between Hilbert spaces. In fact, we obtain two different characterizations of the norm attainment set of a bounded linear operator between Hilbert spaces. We further study the extreme contractions on various types of finite-dimensional Banach spaces, namely Euclidean spaces, and strictly convex spaces. In particular, we give an elementary alternative proof of the well-known characterization of extreme contractions on a Euclidean space, which works equally well for both the real and the complex case. As an application of our exploration, we prove that it is possible to characterize real Hilbert spaces among real Banach spaces, in terms of extreme contractions on their 2-dimensional subspaces.

1. Introduction and preliminaries

The purpose of the present article is to explore the norm attainment set of a bounded linear operator between Hilbert spaces. We also study the extreme contractions on a finite-dimensional Banach space, from the point of view of operator-norm attainment. Let us first establish the relevant notation and terminologies.

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Let $\mathbb{X}$, $\mathbb{Y}$ denote Banach spaces defined over $\mathbb{K}$, the field of scalars. In this article, unless otherwise stated, $\mathbb{K}$ can be either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Let $B_{\mathbb{X}}$ and $S_{\mathbb{X}}$ denote the unit ball and the unit sphere of $\mathbb{X}$, respectively (i.e., $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}$ and $S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$). We reserve the symbol $\mathbb{H}$ for Hilbert spaces. It is rather obvious that $\mathbb{H}$ is also a Banach space, with respect to the usual norm induced by the inner product $\langle , \rangle$ on $\mathbb{H}$. For any two elements x and y in $\mathbb{X}$, x is said to be orthogonal to y in the sense of Birkhoff and James in [1], [4], [5], written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all scalars λ. If the norm on $\mathbb{X}$ is induced by an inner product, then Birkhoff–James orthogonality coincides with the usual inner product orthogonality (i.e., $x \perp_B y$ if and only if $\langle x, y\rangle = 0$).

Let $\mathbb{B}(\mathbb{X}, \mathbb{Y})$ denote the Banach space of all bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$, endowed with the usual operator norm. We write $\mathbb{B}(\mathbb{X}, \mathbb{Y}) = \mathbb{B}(\mathbb{X})$ if $\mathbb{X} = \mathbb{Y}$. For $T \in \mathbb{B}(\mathbb{X}, \mathbb{Y})$, let M_T denote the set of unit vectors at which T attains norm, that is,

$$M_T = \big\{x \in S_{\mathbb{X}} : \|Tx\| = \|T\|\big\}.$$

In this paper, in the context of Hilbert spaces $\mathbb{H}_1$, $\mathbb{H}_2$, we completely characterize M_T for any $T \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$. In fact, we obtain two different characterizations, arising out of different motivations, of the norm attainment set of a bounded linear operator between Hilbert spaces. This answers a question raised very recently in [8], for the special case of Hilbert spaces.

It is easy to observe that for a nonzero $T \in \mathbb{B}(\mathbb{X}, \mathbb{Y})$, T is a scalar multiple of an isometry if and only if $M_T = S_{\mathbb{X}}$. It follows from the works of Koldobsky [7] and Blanco and Turnšek [2] that a nonzero $T \in \mathbb{B}(\mathbb{X}, \mathbb{Y})$ is a scalar multiple of an isometry if and only if T preserves Birkhoff–James orthogonality. In particular, it follows that a nonzero scalar multiple of an isometry always takes an orthogonal basis to an orthogonal basis (in the sense of Birkhof–James). In this paper, we observe an analogous result for any bounded linear operator defined on a finite-dimensional Hilbert space. We would like to add here that Wójcik made a detailed study of this remarkable property of bounded linear operators on a Hilbert space in [11] and [12]. Indeed, it follows from his works that if $\mathbb{H}$ is a finite-dimensional Hilbert space, then, given any $T \in \mathbb{B}(\mathbb{H})$, there exists an orthonormal basis $\mathbb{S}$ of $\mathbb{H}$ such that T preserves orthogonality on $\mathbb{S}$.

We also explore extreme contractions on various types of finite-dimensional Banach spaces—namely, Euclidean spaces and strictly convex spaces. We consider $T \in \mathbb{B}(\mathbb{X}, \mathbb{Y})$ a contraction if $\|T\| \leq 1$. If, in addition, T is also an extreme point of the unit ball of $\mathbb{B}(\mathbb{X}, \mathbb{Y})$, then T is said to be an *extreme contraction* between $\mathbb{X}$ and $\mathbb{Y}$. Extreme contractions of $\mathbb{B}(\mathbb{H})$, where $\mathbb{H}$ is a complex Hilbert space, have been completely characterized by Kadison in [6] as isometries or coisometries. Interestingly enough, for real Hilbert spaces, the same characterization of extreme contractions was obtained much later on by Grząślewicz in [3]. In the present work, we give an elementary alternative proof of the same, for the case of finite-dimensional Hilbert spaces. We would like to emphasize that while our proof is essentially finite-dimensional, it works all the same for both real and complex Hilbert spaces.

A Banach space $\mathbb{X}$ is considered strictly convex if, for any two elements $x, y \in \mathbb{X}$, $\|x+y\| = \|x\|+\|y\|$ implies that $y = kx$ for some $k \geq 0$. Equivalently, $\mathbb{X}$ is strictly convex if and only if every point of $S_{\mathbb{X}}$ is an extreme point of the unit ball $B_{\mathbb{X}}$. We study extreme contractions on finite-dimensional strictly convex Banach spaces. We prove that if $\mathbb{X}$ is an n-dimensional Banach space and $\mathbb{Y}$ is any strictly convex Banach space, then $T \in \mathbb{B}(\mathbb{X}, \mathbb{Y})$ is an extreme contraction if $\|T\| = 1$ and M_T contains n linearly independent vectors. As an application of this result, we prove that it is possible to characterize real Hilbert spaces among real Banach spaces, in terms of extreme contractions on 2-dimensional subspaces of it.

2. Main results

As promised in the Introduction, we would like to begin this section with a complete characterization of the norm attainment set of a bounded linear operator between Hilbert spaces. We will see that preservation of Birkhoff–James orthogonality by a bounded linear operator at certain points plays a pivotal role in the whole scheme of things. Let us also observe that the following result answers an open question raised in [8] for the special case of Hilbert spaces.

Theorem 2.1. *Let $\mathbb{H}_1$, $\mathbb{H}_2$ be Hilbert spaces and let $T \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$. Given any $x \in S_{\mathbb{H}_1}$, $x \in M_T$ if and only if the following two conditions are satisfied:*

(i) $\langle x, y\rangle = 0$ *implies that* $\langle Tx, Ty\rangle = 0$,

(ii) $\sup\{\|Ty\| : \|y\| = 1, \langle x, y\rangle = 0\} \leq \|Tx\|$.

Proof. Let us first prove the necessary part of the theorem. Let $x \in M_T$. It follows from the definition of $\|T\|$ that (ii) holds true. We also note that every Hilbert space is smooth. Since Birkhoff–James orthogonality coincides with usual inner product orthogonality in a Hilbert space, it follows from Theorem 2.2 of [8] that (i) holds true. This completes the proof of the theorem in one direction.

Let us now prove the sufficient part. Let $x \in S_{\mathbb{H}_1}$ be such that (i) and (ii) are satisfied. Let $z \in S_{\mathbb{H}_1}$ be chosen arbitrarily. It is easy to see that z can be written as $z = \alpha x + h$, where $\langle x, h\rangle = 0$ and α is a scalar. If $h = 0$, then $1 = \|z\| = |\alpha|$. Therefore, $\|Tz\| = |\alpha|\|Tx\| = \|Tx\|$. Let $h \neq 0$. We have, $1 = \|z\|^2 = \langle \alpha x + h, \alpha x + h\rangle = |\alpha|^2 + \|h\|^2$, since $\langle x, h\rangle = 0$. Now, by virtue of (i) and (ii), we have,

$$\begin{aligned}
\|Tz\|^2 &= \langle \alpha Tx + Th, \alpha Tx + Th\rangle \\
&= |\alpha|^2\|Tx\|^2 + \|Th\|^2 \\
&= |\alpha|^2\|Tx\|^2 + \|h\|^2\Big\|T\Big(\frac{h}{\|h\|}\Big)\Big\|^2 \\
&\leq |\alpha|^2\|Tx\|^2 + \|h\|^2\|Tx\|^2 \\
&= \|Tx\|^2.
\end{aligned}$$

This proves that, given any $z \in S_{\mathbb{H}_1}$, $\|Tz\| \leq \|Tx\|$. In other words, it must be true that $x \in M_T$. This completes the proof of the sufficient part and establishes the theorem. $\square$

While Theorem 2.1 completely characterizes M_T for $T \in \mathbb{L}(\mathbb{H}_1, \mathbb{H}_2)$, it does not say anything regarding a very significant aspect of operator-norm attainment between Hilbert spaces. It follows from Theorem 2.2 of [10] that, for $T \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$, M_T is always the unit sphere of some subspace of $\mathbb{H}_1$. Indeed, this observation is a characteristic property of Hilbert spaces and illustrates an important operator-theoretic difference between the geometries of Hilbert spaces and Banach spaces. It is therefore apparent that information regarding the dimension of the subspace, whose unit sphere is M_T, is extremely valuable in the study of the possible norm attainment set of a bounded linear operator between Hilbert spaces. Unfortunately, Theorem 2.1, at least in its explicit form, does not provide us with any clue in this regard. We next obtain another complete characterization of the norm attainment set of a bounded linear operator between Hilbert spaces, that effectively addresses this question.

Theorem 2.2. *Let $\mathbb{H}_1$, $\mathbb{H}_2$ be Hilbert spaces and let $T \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$. Given any $x \in S_{\mathbb{H}_1}$, $x \in M_T$ if and only if $\langle Tx, Ty \rangle = \|T\|^2 \langle x, y \rangle$ for every $y \in \mathbb{H}_1$. Consequently, the dimension of the subspace, whose unit sphere is M_T, is equal to the geometric multiplicity of the greatest eigen value (which is equal to $\|T\|^2$) of T^*T.*

Proof. The sufficient part of the theorem follows trivially, by choosing $y = x$. Let us now prove the necessary part of the theorem. Suppose that $x \in M_T$. We claim that $x \in M_{T^*T}$. Indeed, $\langle T^*Tx, x \rangle = \langle Tx, Tx \rangle = \|Tx\|^2 = \|T\|^2$. We also have, $\langle T^*Tx, x \rangle \leq \|T^*Tx\| \|x\| \leq \|T^*T\| = \|T\|^2$. Therefore, it follows that $\|T^*Tx\| = \|T^*T\|$ (i.e., $x \in M_{T^*T}$) and completes the proof of our claim. Since we have, $\langle T^*Tx, x \rangle = \|T^*Tx\| \|x\|$, it follows from the equality condition of the Cauchy–Schwarz inequality that $T^*Tx = \lambda x$, for some $\lambda \geq 0$. We now observe the chain of equalities $|\lambda| = \|\lambda x\| = \|T^*Tx\| = \|T^*T\| = \|T\|^2$, to conclude that $T^*Tx = \|T\|^2 x$. Therefore, given any $y \in \mathbb{H}_1$, we have

$$\langle Tx, Ty \rangle = \langle T^*Tx, y \rangle = \big\langle \|T\|^2 x, y \big\rangle = \|T\|^2 \langle x, y \rangle.$$

This completes the proof of the necessary part.

The last part of the theorem follows directly from our observation that $x \in M_T$ if and only if x is an eigen vector of the bounded linear operator T^*T, corresponding to its greatest eigen value $\|T\|^2$. This establishes the theorem in its entirety. □

Remark 2.3. In view of Theorem 2.1 and Theorem 2.2, we would like to remark that, for a bounded linear operator T between general Banach spaces $\mathbb{X}$, $\mathbb{Y}$, obtaining a complete characterization of M_T seems to be much more difficult. To the best of our knowledge, this problem remains open.

Next, following the study done by Wójcik in [11] and [12], we observe that, if $\mathbb{H}$ is a finite-dimensional Hilbert space, then given any $T \in \mathbb{B}(\mathbb{H})$, there exists an orthonormal basis $\mathbb{S}$ of $\mathbb{H}$ such that T preserves orthogonality on $\mathbb{S}$. The proof of the following result was given by Wójcik in Theorem 2.1 of [11] and Theorem 2.2 of [12] in two different ways.

Theorem 2.4. *Let $\mathbb{H}$ be a finite-dimensional Hilbert space and let $T \in \mathbb{B}(\mathbb{H})$. Then there exists an orthonormal basis $\mathbb{S} = \{x_1, x_2, \ldots, x_n\}$ of $\mathbb{H}$ such that T preserves orthogonality on $\mathbb{S}$ (i.e., $\langle Tx_i, Tx_j \rangle = 0$, whenever $i \neq j$).*

Remark 2.5. Although T preserves orthogonality on the orthonormal basis $\mathbb{S}$ of $\mathbb{H}$, it is not necessarily true that $T(\mathbb{S})$ is an orthonormal basis of $\mathbb{H}$. Indeed, $T(\mathbb{S})$ may not be a basis of $\mathbb{H}$ at all. However, if T is invertible then $T(\mathbb{S})$ is also an orthogonal basis of $\mathbb{H}$. If, in addition, T is an isometry then certainly $T(\mathbb{S})$ is an orthonormal basis of $\mathbb{H}$.

Our next goal is to study extreme contractions on Banach spaces and Hilbert spaces. Let us begin with an easy but useful proposition.

Proposition 2.6. *Let $\mathbb{X}$ be an n-dimensional Banach space and let $\mathbb{Y}$ be any Banach space. Let $T \in \mathbb{B}(\mathbb{X}, \mathbb{Y})$ be such that $\|T\| = 1$, T attains norm at n linearly independent unit vectors $x_1, x_2, \ldots, x_n$, and each Tx_i is an extreme point of $B_{\mathbb{Y}}$. Then T is an extreme contraction in $\mathbb{B}(\mathbb{X}, \mathbb{Y})$.*

Proof. If possible, suppose that T is not an extreme contraction. Then there exists $T_1, T_2 \in \mathbb{B}(\mathbb{X}, \mathbb{Y})$ such that $T_1, T_2 \neq T$, $\|T_1\| = \|T_2\| = 1$, and $T = tT_1 + (1-t)T_2$, for some $t \in (0,1)$. Therefore, for each $i \in \{1, 2, \ldots, n\}$, we have $Tx_i = tT_1x_i + (1-t)T_2x_i$. We also note that $T_1x_i, T_2x_i \in B_{\mathbb{Y}}$, as $\|T_1\| = \|T_2\| = 1$. Since Tx_i is an extreme point of $B_{\mathbb{Y}}$, it follows that $T_1x_i = T_2x_i = Tx_i$ for each $i \in \{1, 2, \ldots, n\}$. However, this implies that T_1, T_2 agree with T on a basis of $\mathbb{X}$, and therefore, $T_1 = T_2 = T$. This contradicts our initial assumption that $T_1, T_2 \neq T$ and completes the proof of the proposition. □

Since in a strictly convex space, every point of the unit sphere is an extreme point of the unit ball, the proof of the following proposition is now immediate.

Proposition 2.7. *Let $\mathbb{X}$ be an n-dimensional Banach space and let $\mathbb{Y}$ be any strictly convex Banach space. Let $T \in \mathbb{B}(\mathbb{X}, \mathbb{Y})$ be such that $\|T\| = 1$ and T attains norm at n linearly independent unit vectors $x_1, x_2, \ldots, x_n$. Then T is an extreme contraction in $\mathbb{B}(\mathbb{X}, \mathbb{Y})$.*

If $\mathbb{H}$ is a Hilbert space, then the extreme contractions in $\mathbb{B}(\mathbb{H})$ are precisely isometries and coisometries. We invite the reader to look through [6] for the complex case and [3] for the real case. Here we give an alternate elementary proof of the same result, when the Hilbert space is finite-dimensional. We would like to remark that our proof, besides being elementary, remains valid for both real and complex cases. In order to prove the desired result, we require the following fact from [8] (see [9] for the complex case).

Theorem 2.8. *Let $\mathbb{X}$, $\mathbb{Y}$ be smooth Banach spaces and let $T \in \mathbb{B}(\mathbb{X}, \mathbb{Y})$. If $x \in M_T$, then T preserves orthogonality at x (i.e., $x \perp_B y$ if and only if $Tx \perp_B Ty$).*

Now, we have the promised characterization of extreme contractions on finite-dimensional Hilbert spaces, as follows.

Theorem 2.9. *Let $\mathbb{H}$ be a finite-dimensional Hilbert space. Then $T \in \mathbb{B}(\mathbb{H})$ is an extreme contraction if and only if T is an isometry.*

Proof. Let us first prove the easier sufficient part of the theorem. Let $\dim \mathbb{H} = n$. Let $T \in \mathbb{B}(\mathbb{H})$ be an isometry. It is easy to see that $\|T\| = 1$ and that $M_T = S_{\mathbb{H}}$. Therefore, there exists n linearly independent unit vectors at which T attains norm. Since every Hilbert space is strictly convex, it now follows from Proposition 2.7 that T is an extreme contraction in $\mathbb{B}(\mathbb{H})$. This completes the proof of the sufficient part.

Let us now prove the comparatively trickier necessary part. Let $T \in \mathbb{B}(\mathbb{H})$ be an extreme contraction. It is easy to observe that $\|T\| = 1$. Since $\mathbb{H}$ is finite-dimensional, by the standard compactness argument, there exists a unit vector $x_1 \in M_T$. Applying Theorem 2.4, let us construct an orthonormal basis $\mathbb{S} = \{x_1, x_2, \ldots, x_n\}$ of $\mathbb{H}$ such that T preserves orthogonality on $\mathbb{S}$ (i.e., $\langle Tx_i, Tx_j\rangle = 0$, whenever $i \neq j$). Note that without loss of generality, we may and do choose that $\|Tx_i\| \geq \|Tx_j\|$ if $i < j$. If $x_i \in M_T$ for each $i \in \{1, 2, \ldots, n\}$, then applying Theorem 2.2 of [10], it is easy to see that $M_T = S_{\mathbb{H}}$. Since $\|T\| = 1$, it follows that T is an isometry and we having nothing more to prove. Let us assume that $x_1, \ldots, x_k \in M_T$ and $x_{k+1}, \ldots, x_n \notin M_T$. Let us now consider the following two cases and reach a contradiction in each of the cases to complete the proof of the theorem.

Case I: $\|Tx_{k+1}\| > 0$. Let us choose $\epsilon > 0$ such that $(1+\epsilon)^2\|Tx_{k+1}\|^2 < 1$. We would like to remark that since $x_{k+1} \notin M_T$, such a choice of ϵ is always possible. Define a linear operator $T_1 \in \mathbb{B}(\mathbb{H})$ in the following way:

$$\begin{aligned} T_1 x_i &= T x_i \quad \text{for each } i \in \{1, \ldots, k\}, \\ T_1 x_{k+i} &= (1+\epsilon) T x_{k+i} \quad \text{for each } i \in \{1, \ldots, n-k\}. \end{aligned}$$

Note that $T_1 \neq T$. We claim that $\|T_1\| = 1$. Let $z = \sum_{i=1}^n \alpha_i x_i \in S_{\mathbb{H}}$, for some scalars α_i. We have, $\sum_{i=1}^n |\alpha_i|^2 = 1$.

If $\alpha_{k+1} = \cdots = \alpha_n = 0$ then $\|T_1 z\| = \|Tz\|$. On the other hand, if $\alpha_1 = \cdots = \alpha_k = 0$, then $T_1 z = \sum_{i=1}^{n-k} \alpha_{k+i}(1+\epsilon)Tx_{k+i}$. Therefore,

$$\begin{aligned} \|T_1 z\|^2 &= \sum_{i=1}^{n-k} (1+\epsilon)^2 |\alpha_{k+i}|^2 \|Tx_{k+i}\|^2 \\ &\leq \sum_{i=1}^{n-k} (1+\epsilon)^2 |\alpha_{k+i}|^2 \|Tx_{k+1}\|^2 \\ &= (1+\epsilon)^2 \|Tx_{k+1}\|^2 < \|T\|^2. \end{aligned}$$

Let us assume that at least one of $\alpha_1, \ldots, \alpha_k$ (say α_1) is nonzero and at least one of $\alpha_{k+1}, \ldots, \alpha_n$ (say α_{k+1}) is nonzero. Choosing $w = -\overline{\alpha_{k+1}} x_1 + \overline{\alpha_1} x_{k+1}$, it is easy to see that $\langle z, w\rangle = 0$. However, an easy computation reveals that $\langle T_1 z, T_1 w\rangle = -\alpha_1\alpha_{k+1}(1 - (1+\epsilon)^2\|Tx_{k+1}\|^2) \neq 0$. This proves that T_1 does not preserve orthogonality at such a z, and therefore T_1 cannot attain norm at such a z. Since T_1 must attain norm at some point of $S_{\mathbb{H}}$ and $\|T_1 x_1\| = \|Tx_1\| = \|T\| = 1$, we conclude that $\|T_1\| = 1$.

Let us now define another linear operator $T_2 \in \mathbb{B}(\mathbb{H})$ in the following way:

$$T_2 x_i = T x_i \quad \text{for each } i \in \{1, \ldots, k\},$$
$$T_2 x_{k+i} = (1-\epsilon) T x_{k+i} \quad \text{for each } i \in \{1, \ldots, n-k\}.$$

Clearly, $T_2 \neq T$. As in the case of T_1, it is easy to prove that $\|T_2\| = 1$. Therefore, we have proved the following facts:

(i) $\|T_1\| = \|T_2\| = \|T\| = 1$,
(ii) $T = \frac{1}{2}T_1 + \frac{1}{2}T_1$,
(iii) $T_1,\ T_2 \neq T$. However, this contradicts our initial assumption that $T \in \mathbb{B}(\mathbb{H})$ is an extreme contraction.

Case II: $\|Tx_{k+1}\| = 0$. It follows that $\|Tx_{k+i}\| = 0$ for each $i \in \{1, \ldots, n-k\}$. We observe that for each $i \in \{1, \ldots, k\}$, $(Tx_i)^{\perp} = \{y \in \mathbb{H} : \langle Tx_i, y \rangle = 0\}$ is a subspace of codimension 1 in $\mathbb{H}$. Therefore, it is easy to deduce that $\bigcap_{i=1}^{k}(Tx_i)^{\perp} \neq \emptyset$. Choose a fixed vector $w \in \bigcap_{i=1}^{k}(Tx_i)^{\perp} \cap S_{\mathbb{H}}$. Define a linear operator $T_1 \in \mathbb{B}(\mathbb{H})$ in the following way:

$$T_1 x_i = T x_i \quad \text{for each } i \in \{1, \ldots, n\} \setminus \{k+1\}, \qquad T_1 x_{k+1} = \frac{1}{2} w.$$

Clearly, $T_1 \neq T$. We claim that $\|T_1\| = 1$. Let $z = \sum_{i=1}^{n} \alpha_i x_i \in S_{\mathbb{H}}$, for some scalars α_i. We have $\sum_{i=1}^{n} |\alpha_i|^2 = 1$. We also have $\|T_1 z\|^2 = \sum_{i=1}^{k} |\alpha_i|^2 + \frac{1}{4}|\alpha_{k+1}|^2 \leq \sum_{i=1}^{n} |\alpha_i|^2 = 1$. Since $\|T_1 x_1\| = \|T x_1\| = 1$, we must have $\|T_1\| = 1$. Define another linear operator $T_2 \in \mathbb{B}(\mathbb{H})$ in the following way:

$$T_2 x_i = T x_i \quad \text{for each } i \in \{1, \ldots, n\} \setminus \{k+1\}, \qquad T_2 x_{k+1} = -\frac{1}{2} w.$$

As in the case of T_1, it is easy to observe that $T_2 \neq T$ and that $\|T_2\| = 1$. Therefore, we have proved the following facts:

(i) $\|T_1\| = \|T_2\| = \|T\| = 1$,
(ii) $T = \frac{1}{2}T_1 + \frac{1}{2}T_1$,
(iii) $T_1,\ T_2 \neq T$. However, this contradicts our initial assumption that $T \in \mathbb{B}(\mathbb{H})$ is an extreme contraction. This establishes the theorem. □

As an application of the results we obtained, we now obtain a characterization of real Hilbert spaces among all real Banach spaces in terms of extreme contractions. First, let us prove the following lemma in order to obtain the desired characterization,

Lemma 2.10. *Let $\mathbb{X}$ be a 2-dimensional real Banach space which is not strictly convex. Then there exists a linear operator $T \in \mathbb{B}(\mathbb{X})$ such that T is an extreme contraction in $\mathbb{B}(\mathbb{X})$ but T is not an isometry.*

Proof. Since $\mathbb{X}$ is not strictly convex, the unit sphere $S_{\mathbb{X}}$ contains a closed straight line segment $\mathbb{I}$. We note that $\mathbb{I}$ can be written as $\mathbb{I} = \{x + \lambda y : \lambda_1 \leq \lambda \leq \lambda_2\}$, where x is a fixed interior point of $\mathbb{I}$, y is a fixed point on $S_{\mathbb{X}}$ such that the straight line joining θ and y is parallel to $\mathbb{I}$ and λ_1, λ_2 are two fixed real numbers, one positive and the other negative. Moreover, assume that the segment $\mathbb{I}$ has maximal length (i.e., $x + \lambda_1 y := v_1$ and $x + \lambda_2 y := v_2$ are extreme points of

$B_{\mathbb{X}}$). From the description of y, it is quite clear that $x \perp_B y$. Let w be any fixed extreme point of $B_{\mathbb{X}}$. Let us define a linear operator $T \in \mathbb{B}(\mathbb{X})$ in the following way: $Tx = w$, $Ty = 0$.

We claim that $\|T\| = 1$. Clearly, $\|T\| \geq \|Tx\| = \|w\| = 1$. On the other hand, let $z = \alpha x + \beta y \in S_{\mathbb{X}}$, where α, β are scalars. We have $1 = \|\alpha x + \beta y\| \geq |\alpha|$, since $x \perp_B y$. Therefore, $\|Tz\| = \|\alpha w\| = |\alpha| \leq 1$. This proves that $\|T\| = 1$. It is now easy to observe that $v_1, v_2 \in M_T$. We also note that v_1, v_2 must be linearly independent and $Tv_1 = Tv_2 = w$. As w is an extreme point of $B_{\mathbb{X}}$, applying Proposition 2.6 we obtain that T is an extreme contraction in $\mathbb{B}(\mathbb{X})$. However, T cannot be an isometry, since $\|y\| = 1 > 0 = \|Ty\|$. This completes the proof of the lemma. □

Let us now proceed to establish the promised characterization of real Hilbert spaces.

Theorem 2.11. *A real Banach space $\mathbb{X}$ is a Hilbert space if and only if for every 2-dimensional subspace $\mathbb{Y}$ of $\mathbb{X}$, isometries are the only extreme contractions in $\mathbb{B}(\mathbb{Y})$.*

Proof. Let us first prove the necessary part of the theorem. If $\mathbb{X}$ is a Hilbert space then every 2-dimensional subspace $\mathbb{Y}$ of $\mathbb{X}$ is also a Hilbert space. Therefore, it follows from Theorem 2.9 that isometries are the only extreme contractions in $\mathbb{B}(\mathbb{Y})$.

Let us now prove the sufficient part. If possible, suppose that $\mathbb{X}$ is not a Hilbert space. Then there exists a 2-dimensional subspace $\mathbb{Y}$ of $\mathbb{X}$ such that $\mathbb{Y}$ is not a Hilbert space. Let us first assume that $\mathbb{Y}$ is strictly convex. As $\mathbb{Y}$ is not a Hilbert space, it follows from Theorem 2.2 of [10] that there exists a linear operator $T \in \mathbb{B}(\mathbb{Y})$ and two unit vectors $e_1, e_2 \in S_{\mathbb{Y}}$ such that T attains norm at $e_1, e_2 \in S_{\mathbb{Y}}$ but T does not attain norm at every point of $\mathrm{span}\{\mathrm{e_1}, \mathrm{e_2}\} \cap \mathrm{S}_{\mathbb{Y}}$. It is immediate that T must be nonzero. We also note that $M_T = M_{\frac{T}{\|T\|}}$. Therefore, without loss of generality, we may and do assume that $\|T\| = 1$. Since T does not attain norm at every point of $\mathrm{span}\{\mathrm{e_1}, \mathrm{e_2}\} \cap \mathrm{S}_{\mathbb{Y}}$, it is easy to deduce that e_1, e_2 must be linearly independent and that T cannot be an isometry. Since $\mathbb{Y}$ is strictly convex and e_1, e_2 are linearly independent, it follows from Proposition 2.7 that T is an extreme contraction in $\mathbb{B}(\mathbb{Y})$. However, this contradicts our hypothesis as T is not an isometry.

Next, let us assume that $\mathbb{Y}$ is not strictly convex. Lemma 2.10 ensures that there exists a linear operator $T \in \mathbb{B}(\mathbb{Y})$ such that T is an extreme contraction in $\mathbb{B}(\mathbb{Y})$ but T is not an isometry. This, once again, is a contradiction to our initial hypothesis. This establishes the theorem completely. □

As a concluding remark, we would like to add that while extreme contractions between Hilbert spaces are well understood, the scenario is far from complete in the more general setting of Banach spaces. In this paper, we have tried to illustrate the pivotal role played by the norm attainment set of a bounded linear operator in studying extreme contractions between Banach spaces. It is expected that the study will be further continued, in order to have a better understanding of extreme contractions and the norm attainment set of a bounded linear operator in the context of general Banach spaces.

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