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THE RATE OF ALMOST-EVERYWHERE CONVERGENCE OF BOCHNER–RIESZ MEANS ON SOBOLEV SPACES

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Abstract. We investigate the convergence rate of the generalized Bochner–Riesz means $S_R^{\delta,\gamma}$ on L^p-Sobolev spaces in the sharp range of δ and p ($p \geq 2$). We give the relation between the smoothness imposed on functions and the rate of almost-everywhere convergence of $S_R^{\delta,\gamma}$. As an application, the corresponding results can be extended to the n-torus $\mathbb{T}^n$ by using some transference theorems. Also, we consider the following two generalized Bochner–Riesz multipliers, $(1-|\xi|^{\gamma_1})_+^\delta$ and $(1-|\xi|^{\gamma_2})_+^\delta$, where γ_1, γ_2, δ are positive real numbers. We prove that, as the maximal operators of the multiplier operators with respect to the two functions, their $L^2(|x|^{-\beta})$-boundedness is equivalent for any γ_1, γ_2 and fixed δ.

1. Introduction

Let $R > 0$. We consider the generalized Bochner–Riesz means $S_R^{\delta,\gamma}$ on the Euclidean space $\mathbb{R}^n$ defined via the Fourier transform by

$$(S_R^{\delta,\gamma} f)^\wedge(\xi) = \Big(1 - \frac{|\xi|^\gamma}{R^\gamma}\Big)_+^\delta \widehat{f}(\xi), \tag{1.1}$$

where δ and γ are two real numbers satisfying $\delta > -1$ and $\gamma > 0$. Also, we may initially assume that the f's are functions in the Schwartz class $\mathscr{S}(\mathbb{R}^n)$. The function $(1 - |\xi|^\gamma)_+^\delta$ is called the *generalized Bochner–Riesz multiplier*, and we

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denote it by $m^{\delta,\gamma}$. The kernel $B_R^{\delta,\gamma}$ of $S_R^{\delta,\gamma}$ is defined by

$$B_R^{\delta,\gamma}(x) = \int_{\mathbb{R}^n} \Big(1 - \frac{|\xi|^\gamma}{R^\gamma}\Big)_+^\delta e^{2\pi i\xi\cdot x}\, d\xi = \big(((1-|\cdot|^\gamma)_+^\delta)^\vee\big)_{1/R}.$$

We may write $S_R^{\delta,\gamma}$ as a convolution operator

$$S_R^{\delta,\gamma} f = B_R^{\delta,\gamma} * f.$$

The associated maximal operator of $S_R^{\delta,\gamma}$ is defined by

$$(S_*^{\delta,\gamma} f)(x) = \sup_{R>0} \big|(S_R^{\delta,\gamma} f)(x)\big|.$$

When $\gamma = 2$, the classical Bochner–Riesz means that $S_R^{\delta,2}$ is one of the most significant and awesome operators in analysis since it is connected to a large number of important conjectures in harmonic analysis, such as the Bochner–Riesz conjecture, the disk conjecture, and the Kakeya conjecture, among many others. Hence, research on $S_R^{\delta,2}$ has attracted many authors (we direct the reader to [23], [20], [5], [3], [1], [26], [13], [6], [14], [15] and the references therein, among numerous research papers). In particular, the study of the convergence of $S_R^{\delta,2}$ is a long-standing subject in the classical theory of Fourier analysis. The number $\delta_0 = (n-1)/2$ is called the *critical index* of L^1. When $\delta > \delta_0$, $\lim_{R\to\infty} S_R^{\delta,2} f(x) = f(x)$ almost everywhere for any $f \in L^1(\mathbb{R}^n)$, while Stein [19] found an $L^1(\mathbb{R}^n)$-function f for which $\limsup_{R\to\infty} |S_R^{\delta_0,2} f(x)| = \infty$ almost everywhere. Finding a suitable subspace of $L^1(\mathbb{R}^n)$ related to almost-everywhere convergence of $S_R^{\delta_0,2}(f)(x)$ thus became an open problem, which was solved in 1982 by Lu, Taibleson, and Weiss [16] when they introduced the block space (see the related work of Stein [21] and Fefferman [9]). Also, for $\delta > \delta_0$, Stein, Taibleson, and Weiss [22] considered the weak-type boundedness for maximal operators of $S_R^{\delta_p,2}$ on the Hardy space H^p, and they proved that $\lim_{R\to\infty} S_R^{\delta_p,2} f(x) = f(x)$ almost everywhere for any $f \in H^p$, where $0 < p < 1$ and $\delta_p = n/p - (n+1)/2$. The most challenging research involves the case in which δ is below the critical index δ_0. In this case, we list two famous conjectures below.

Conjecture 1.1 (Bochner–Riesz Conjecture 1, (see [21, p. 390])). *For* $0 < \delta < (n-1)/2$,

$$\|S_R^{\delta,2} f\|_{L^p} \preceq \|f\|_{L^p}$$

if and only if

$$\frac{2n}{n+1+2\delta} < p < \frac{2n}{n-1-2\delta}.$$

Conjecture 1.2 (Bochner–Riesz Conjecture 2, (see [21, p. 390], [11, Section 5.5])). *If* $0 < \delta < (n-1)/2$, $f \in L^p$, *and*

$$\frac{2n}{n+1+2\delta} < p < \frac{2n}{n-1-2\delta},$$

then

$$\lim_{R\to\infty} S_R^{\delta,2} f(x) = f(x), \quad a.e.$$

The second conjecture was partially solved by Carbery, Rubio de Francia, and Vega [2] in the range $[2, \frac{2n}{n-1-2\delta})$ of index p. To reach the conclusion, they considered the boundedness of $S_*^{\delta,2}$ on the weighted space $L^2(\mathbb{R}^n, |x|^{-\beta}\,dx)$ and established the following Theorem A. Then invoking Theorem A, they obtained Theorem B below, which is the desired conclusion.

Theorem A ([2, p. 514]). *Let $\delta > 0$ and $0 \le \beta < 1 + 2\delta \le n$. Then there is a constant $C = C(\beta, \delta, n)$ such that*

$$\int_{\mathbb{R}^n} \big|(S_*^{\delta,2} f)(x)\big|^2 |x|^{-\beta}\,dx \le C^2 \int_{\mathbb{R}^n} \big|f(x)\big|^2 |x|^{-\beta}\,dx$$

for all functions $f \in \mathscr{S}(\mathbb{R}^n)$.

Theorem B ([2, p. 513]). *Let $\delta > 0$ and $n \ge 2$. Then for all f in $L^p(\mathbb{R}^n)$ with $2 \le p < \frac{2n}{n-1-2\delta}$, we have*

$$\lim_{R\to\infty} S_R^{\delta,2} f(x) = f(x)$$

for almost all $x \in \mathbb{R}^n$.

Besides the convergence problem, another interesting question is to study the convergence rate of $S_R^{\delta,\gamma} f(x)$ if f satisfies some smoothness condition. At the critical index δ_0, this problem was studied by Lu and Wang [17] and Fan and Zhao [7], who introduced the block Sobolev space $I_\lambda(B_q)(\mathbb{R}^n)$ to study the convergence speed of the generalized Bochner–Riesz means $S_R^{\delta_0,\gamma}$. Thus, the relation between the smoothness imposed on functions in the block space and the rate of almost-everywhere convergence of $S_R^{\delta_0,\gamma}$ is addressed. For $\delta > \delta_0$, Fan and Zhao [8] studied the convergence of $S_R^{\delta,\gamma}$ with $\delta = \delta_p$ on the Hardy–Sobolev spaces $I_\lambda(H^p)(\mathbb{R}^n)$ and revealed the relation between the smoothness imposed on the Hardy space $H^p(\mathbb{R}^n)$ and the rate of almost-everywhere convergence of the generalized Bochner–Riesz means. As the authors mentioned in [8], the study of generalized Bochner–Riesz means $S_R^{\delta,\gamma}$ on some corresponding Sobolev-type spaces is not merely a simple extension of using γ to replace 2, but rather it is naturally raised from the approximation theory in order to enhance the saturation of approximation.

The Sobolev-type spaces mentioned above, the block Sobolev space $I_\lambda(B_q)(\mathbb{R}^n)$, the Hardy–Sobolev spaces $H^p(\mathbb{R}^n)$, and the L^p-Sobolev spaces $I_\lambda(L^p)(\mathbb{R}^n)$ were introduced by Strichartz [24], [25] in a more general setting. Let I_λ denote the Riesz potential operators of order λ on $\mathbb{R}^n$ for $\lambda \in \mathbb{R}$, which may act on functions or tempered distributions. The Fourier transform of a Schwartz function (or even a tempered distribution f) satisfies $(I_\lambda f)^\wedge(\xi) = |\xi|^{-\lambda} \widehat{f}(\xi)$. If X is any function space or a space of tempered distributions, one can define Sobolev spaces based on X using $I_\lambda(X)$ to be the image of X under I_λ. Thus, by this definition in [25, p. 129], we know that $I_\lambda(L^p)(\mathbb{R}^n)$ are the classical homogeneous Sobolev spaces for $p \ge 1$ and that $I_\lambda(H^p)(\mathbb{R}^n)$ are the Hardy–Sobolev spaces for $0 < p \le 1$.

Now we are in a position to state our main results in this article. We will study the convergence rate of $S_R^{\delta,\gamma} f$ in the sharp range of p and δ as in Theorem B, when the index δ is below the critical index. Precisely, our first aim is to analyze the convergence rate of the Bochner–Riesz means $S_R^{\delta,\gamma} f$ on L^p-Sobolev spaces and to obtain the relation between the smoothness imposed on functions and the rate of almost-everywhere convergence. As an application, by using a transference theorem by Kenig and Tomas [12] for $1 < p \le \infty$, we will transfer our results to the n torus T^n.

Below we formulate our main theorems. They are new even when $\gamma = 2$.

Theorem 1.3. *Let $\delta > 0$, let $n \ge 2$, and let $2 \le p < \frac{2n}{n-1-2\delta}$. If f is in $I_\lambda(L^p)(\mathbb{R}^n)$, then for $0 \le \lambda < \gamma$,*

$$(S_R^{\delta,\gamma} f)(x) - f(x) = o(1/R^\lambda), \quad \text{a.e. as } R \to \infty,$$

and for $\lambda = \gamma$,

$$(S_R^{\delta,\gamma} f)(x) - f(x) = O(1/R^\gamma), \quad \text{a.e. as } R \to \infty.$$

Moreover, the rate $O(1/R^\gamma)$ is sharp at the endpoint $\lambda = \gamma$.

Clearly, Theorem 1.3 recovers Theorem B when $\lambda = 0$. We consider the maximal function

$$(M_\lambda^{\delta,\gamma} f)(x) = \sup_{R>0} \big| R^\lambda \big\{ (S_R^{\delta,\gamma} f)(x) - f(x) \big\} \big|.$$

Let $g := I_{-\lambda} f$. Define the operator $T_{R,\lambda}^{\delta,\gamma}$ by

$$(T_{R,\lambda}^{\delta,\gamma} g)(x) = \int_{\mathbb{R}^n} \widehat{g}(\xi) \frac{(1 - |\frac{\xi}{R}|^\gamma)_+^\delta - 1}{|\frac{\xi}{R}|^\lambda} e^{2\pi i x\cdot\xi}\, d\xi,$$

and $(M_\gamma g)(x) = \sup_{R>0} |(T_{R,\lambda}^{\delta,\gamma} g)(x)|$. We have $M_\lambda^{\delta,\gamma} f = M_\gamma g$.

To prove Theorem 1.3, we will need the following estimate on $M_\gamma g$.

Theorem 1.4. *Let $\delta > 0$ and $0 \le \beta < 1 + 2\delta \le n$. Then there is a constant $C = C(\beta, \delta, n)$ such that*

$$\int_{\mathbb{R}^n} \big|(M_\gamma g)(x)\big|^2 |x|^{-\beta}\, dx \le C^2 \int_{\mathbb{R}^n} \big|g(x)\big|^2 |x|^{-\beta}\, dx, \tag{1.2}$$

for all functions $g \in \mathscr{S}(\mathbb{R}^n)$.

Remark 1.5. Moreover, M_γ has a unique bounded sublinear extension $\overline{M_\gamma}$ on $L^2(\mathbb{R}^n, |x|^{-\beta}\, dx)$ which also satisfies (1.2). Also, for each $R > 0$, $T_{R,\lambda}^{\delta,\gamma}$ has a unique bounded linear extension $\overline{T_{R,\lambda}^{\delta,\gamma}}$ on $L^2(\mathbb{R}^n, |x|^{-\beta}\, dx)$ and the relationship $(\overline{M_\gamma} g)(x) = \sup_{R>0} |(\overline{T_{R,\lambda}^{\delta,\gamma}} g)(x)|$ holds for all $g \in L^2(\mathbb{R}^n, |x|^{-\beta}\, dx)$. In the following, for simplicity, we denote $\overline{T_{R,\lambda}^{\delta,\gamma}}$ by $T_{R,\lambda}^{\delta,\gamma}$ and we denote $\overline{M_\gamma}$ by M_γ.

Our idea is mainly inspired from Fan and Zhao's previous work (see [7], [8]). Precisely, observe that the Fourier transform of $R^\lambda \{ (S_R^{\delta,\gamma} f) - f \}$ is $\mu(\cdot/R)\, \widehat{g}$, where $g := I_{-\lambda} f$ and the multiplier μ is given by

$$\mu(\xi) = \frac{(1 - |\xi|^\gamma)_+^\delta - 1}{|\xi|^\lambda}, \quad \xi \ne 0 \qquad \text{and} \qquad \mu(0) = \lim_{t\to 0^+} \frac{(1 - t^\gamma)_+^\delta - 1}{t^\lambda}.$$

Hence we will decompose the multiplier μ as a sum of μ_0, μ_1, μ_∞, centralizing at 0, 1, and near ∞, respectively. Then each corresponding kernel will be carefully estimated.

Here we make some further comments about Theorem 1.4. In the proof of Theorem 1.4, we essentially prove that the $L^2(|x|^{-\beta})$-boundedness of $S_*^{\delta,\gamma}$ can be reduced to the $L^2(|x|^{-\beta})$-boundedness of $S_*^{\delta,2}$ so that Theorem A can be employed. Consider the following two generalized Bochner–Riesz multipliers, $m^{\delta,\gamma_1} = (1-|\xi|^{\gamma_1})_+^\delta$ and $m^{\delta,\gamma_2} = (1-|\xi|^{\gamma_2})_+^\delta$, where γ_1, γ_2, δ are positive real numbers. As the maximal operators of the multiplier operators with respect to m^{δ,γ_1} and m^{δ,γ_2}, does the $L^2(|x|^{-\beta})$-boundedness of S_*^{δ,γ_1} imply the $L^2(|x|^{-\beta})$-boundedness of S_*^{δ,γ_2}? Furthermore, is the $L^2(|x|^{-\beta})$-boundedness of S_*^{δ,γ_1} and S_*^{δ,γ_2} equivalent for any γ_1, γ_2, and fixed δ? Our second aim gives it a positive answer by proving the following.

Theorem 1.6. *Let $\delta > 0$ and let $0 \le \beta < 1 + 2\delta \le n$. Suppose that γ_1, γ_2, δ are positive real numbers. Then the $L^2(|x|^{-\beta})$-boundedness of S_*^{δ,γ_1} and S_*^{δ,γ_2} is equivalent for any γ_1, γ_2, and fixed δ.*

We were informed by the referee of the following result closely related to this article.

Theorem C ([4, p. 87])**.** *Let $\delta > 0$ and $n \ge 2$. If f is in $I_\lambda(L^2)(\mathbb{R}^n)$, then for $0 \le \lambda < \gamma$,*

$$(S_R^{\delta,\gamma} f)(x) - f(x) = o(1/R^\lambda), \quad \text{a.e. as } R \to \infty,$$

and for $\lambda = \gamma$,

$$(S_R^{\delta,\gamma} f)(x) - f(x) = O(1/R^\gamma), \quad \text{a.e. as } R \to \infty.$$

It is worth comparing our Theorem 1.3 to Theorem C. In fact, for a fixed $\delta > 0$ and $2 \le p < \frac{2n}{n-1-2\delta}$, in Theorem 1.3 we obtain the almost-everywhere convergence of $(S_*^{\delta,\gamma} f)(x)$ for all $f \in I_\lambda(L^p)(\mathbb{R}^n)$, while Theorem C only states the result for $f \in I_\lambda(L^2)(\mathbb{R}^n)$. Also, our proof for Theorem 1.3 is quite different from that of Theorem C.

This article is organized as follows. In Section 2, we define some necessary notation, definitions, and lemmas that will be used throughout this work. Since the proofs of Theorems 1.3 and 1.4 are based on Theorem 1.6, we first prove Theorem 1.6 in Section 3. Theorems 1.3 and 1.4 will be proved in Section 4. In Section 5, we extend Theorems 1.3 and 1.4 on $\mathbb{R}^n$ to the n-torus $\mathbb{T}^n$. We use the letter C throughout to denote positive constants that may vary at each occurrence but are independent of the essential variables. Also, the statement $A \sim B$ denotes that there exist positive constants C, C_0 such that $C_0 \le A/B \le C$, and the statement $f \preceq g$ denotes that there exists a positive constant C such that $f \le Cg$.

2. Some preliminaries

We begin with some preliminaries. A multi-index α is an ordered n-tuple of nonnegative integers. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\partial_x^\alpha f$ or $\partial^\alpha f$ denotes the derivative $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f$. Let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. The number $|\alpha|$ indicates the total order of differentiation of $\partial_x^\alpha f$. For $x \in \mathbb{R}^n$ and

$\alpha = (\alpha_1, \dots, \alpha_n)$ a multi-index, we set $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Let $\beta = (\beta_1, \dots, \beta_n)$ be a multi-index. The notation $\alpha \le \beta$ means that β ranges over all multi-indices satisfying $0 \le \alpha_j \le \beta_j$ for all $1 \le j \le n$ (for more details, see [10, p. 104]). Let $[r]$ be the greatest integer less than or equal to the real number r. Given a function φ on $\mathbb{R}^n$ and $\varepsilon > 0$, we denote by φ_ε the following function: $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$. For the sake of conciseness, we denote $L^2(|x|^{-\beta}) = L^2(\mathbb{R}^n, |x|^{-\beta}\,dx)$ and $L^p = L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Let $\phi_0, \phi_1, \phi_\infty \in C^\infty(\mathbb{R}^n)$ be radial functions satisfying the following conditions: (i) ϕ_0 is supported on the set $\{\xi \in \mathbb{R}^n : |\xi| < 1/2\}$, ϕ_1 is supported on the annulus $\{\xi \in \mathbb{R}^n : 1/4 < |\xi| < 2\}$, and ϕ_∞ is supported on the set $\{\xi \in \mathbb{R}^n : |\xi| > 3/2\}$; (ii) $\phi_0(\xi) + \phi_1(\xi) + \phi_\infty(\xi) = 1$, $0 \le \phi_j(\xi) \le 1$ for $j = 0, 1, \infty$, and

$$\phi_0(\xi) = \begin{cases} 1, & |\xi| \le 1/4, \\ 0, & |\xi| \ge 1/2, \end{cases} \qquad \phi_1(\xi) = \begin{cases} 1, & 1/2 \le |\xi| \le 3/2, \\ 0, & |\xi| \le 1/4 \text{ and } |\xi| \ge 2, \end{cases}$$
$$\phi_\infty(\xi) = \begin{cases} 1, & |\xi| \ge 2, \\ 0, & |\xi| \le 3/2. \end{cases}$$

Let $\Psi_\infty(\xi) = \phi_1(\xi) + \phi_\infty(\xi)$. Obviously, $\Psi_\infty \in C^\infty(\mathbb{R}^n)$ is supported on the set $\{\xi : |\xi| > 1/4\}$ satisfying $0 \le \Psi_\infty(\xi) \le 1$ and $\Psi_\infty(\xi) = 1$ on the set $\{\xi : |\xi| \ge 1/2\}$. Assume that the kernel $B_R^{\delta,\gamma}$ of $S_R^{\delta,\gamma}$ is defined by

$$B_R^{\delta,\gamma}(x) = \int_{\mathbb{R}^n} \Big(1 - \frac{|\xi|^\gamma}{R^\gamma}\Big)_+^\delta e^{2\pi i \xi\cdot x}\,d\xi.$$

We rewrite $S_R^{\delta,\gamma}$ as a convolution operator

$$S_R^{\delta,\gamma} f = B_R^{\delta,\gamma} * f.$$

Let $g := I_{-\lambda} f$, $T_{R,\lambda}^{\delta,\gamma} g := R^\lambda\{(S_R^{\delta,\gamma} f) - f\}$. Then the Fourier transform of the function $T_{R,\lambda}^{\delta,\gamma} g$ can be written as $m_{\lambda,0}^{\delta,\gamma}(\cdot/R)\widehat{g} + m_{\lambda,1}^{\delta,\gamma}(\cdot/R)\widehat{g} - m_{\lambda,\infty}^{\delta,\gamma}(\cdot/R)\widehat{g}$, where

$$m_{\lambda,0}^{\delta,\gamma}(\xi) = \frac{\phi_0(\xi)}{|\xi|^\lambda}\big((1 - |\xi|^\gamma)_+^\delta - 1\big), \qquad m_{\lambda,1}^{\delta,\gamma}(\xi) = \frac{\phi_1(\xi)}{|\xi|^\lambda}\big(1 - |\xi|^\gamma\big)_+^\delta,$$
$$m_{\lambda,\infty}^{\delta,\gamma}(\xi) = \frac{\Psi_\infty(\xi)}{|\xi|^\lambda}.$$

Here and in what follows, for simplicity, we define the operators

$$(T_{R,\lambda,j}^{\delta,\gamma} g)^\wedge(\xi) = m_{\lambda,j}^{\delta,\gamma}(\xi/R)\widehat{g}(\xi),$$

and we denote by $(K_{\lambda,j}^{\delta,\gamma})_{1/R}(x) = R^n K_{\lambda,j}^{\delta,\gamma}(Rx)$ the kernel of $T_{R,\lambda,j}^{\delta,\gamma}$, where

$$K_{\lambda,j}^{\delta,\gamma}(x) = {m_{\lambda,j}^{\delta,\gamma}}^\vee(x) = \int_{\mathbb{R}^n} m_{\lambda,j}^{\delta,\gamma}(\xi) e^{2\pi i \xi\cdot x}\,d\xi, \quad j = 0, 1, \infty.$$

Then

$$T_{R,\lambda}^{\delta,\gamma} g = T_{R,\lambda,0}^{\delta,\gamma} g + T_{R,\lambda,1}^{\delta,\gamma} g - T_{R,\lambda,\infty}^{\delta,\gamma} g$$

and each operator $T^{\delta,\gamma}_{R,\lambda,j}g$ can be written as a convolution operator

$$T^{\delta,\gamma}_{R,\lambda,j}g(x) = (K^{\delta,\gamma}_{\lambda,j})_{1/R} * g(x),$$

for $j = 0, 1, \infty$. Let $M_\gamma g$ be the maximal function defined by

$$(M_\gamma g)(x) = \sup_{R>0}\Big|\int_{\mathbb{R}^n} \widehat{g}(\xi)\frac{(1-|\frac{\xi}{R}|^\gamma)^\delta_+ - 1}{|\frac{\xi}{R}|^\lambda}e^{2\pi i\xi\cdot x}\,d\xi\Big|.$$

Then we have

$$\begin{aligned}(M^{\delta,\gamma}_\lambda f)(x) &= \sup_{R>0}\big|R^\lambda\big\{(S^{\delta,\gamma}_R f)(x) - f(x)\big\}\big| \\ &= \sup_{R>0}\big|(T^{\delta,\gamma}_{R,\lambda}g)(x)\big| \\ &\le \sum_{j=0,1,\infty}(M_{\gamma,j}g)(x),\end{aligned}$$

where $M_{\gamma,j}g$ are the associated maximal operators

$$(M_{\gamma,j}g)(x) = \sup_{R>0}\big|(T^{\delta,\gamma}_{R,\lambda,j}g)(x)\big|, \quad j = 0, 1, \infty.$$

We next introduce the following two useful lemmas, both of which can be found in [7]; they play a crucial role in the proof of Theorem 1.4.

Lemma 2.1 ([7, Lemma 2]). *For $0 \le \lambda < \gamma$ and $\delta > -1$, we have*

$$\big|K^{\delta,\gamma}_{\lambda,0}(x)\big| \preceq \frac{1}{(1+|x|)^{n+\gamma-\lambda}},$$

and for $\lambda = \gamma$, we have

$$\big|K^{\delta,\gamma}_{\gamma,0}(x)\big| \preceq \frac{1}{(1+|x|)^L} \quad \text{for any } L > 0.$$

Lemma 2.2 ([7, Lemma 3]). *For $\lambda > 0$ and $\delta > -1$, we have*

$$\big|K^{\delta,\gamma}_{\lambda,\infty}(x)\big| \preceq \frac{1}{|x|^{n-\lambda}} \quad \text{if } |x| < 1,$$

and if $|x| \ge 1$, then we have

$$\big|K^{\delta,\gamma}_{\lambda,\infty}(x)\big| \preceq \frac{1}{|x|^L} \quad \text{for any } L > n.$$

We also need the following lemma which can be found in [10].

Lemma 2.3 ([10, p. 92]). *If a function φ has an integrable radially decreasing majorant Φ, then the estimate*

$$\sup_{t>0}\big|(f * \varphi_t)(x)\big| \le \|\Phi\|_{L^1} M(f)(x)$$

is valid for all locally integrable functions f on $\mathbb{R}^n$, where M is the Hardy–Littlewood maximal operator.

3. Proof of Theorem 1.6

The main purpose of this section is to establish Theorem 1.6. To do this, we first state an auxiliary result in [18]. Our idea is also partially motivated from [18].

Lemma 3.1 ([18, Lemma 7]). *For any $\rho > 0$, $\delta > 0$, we have $[(1+|\cdot|^{\rho})^{-\delta}]^{\vee} \in L^1(\mathbb{R}^n)$.*

Proof of Theorem 1.6. By symmetry, we just need to give the implication relationship on one side. Without loss of generality, we suppose that S_*^{δ,γ_1} is bounded on $L^2(|x|^{-\beta})$. Its proof can be completed by an iteration argument. Pick a positive integer k such that $2^k\gamma_2 > \max\{n+1, \gamma_1\}$. Recall that $\phi_1 \in \mathscr{S}(\mathbb{R}^n)$ is supported on the annulus $\{\xi \in \mathbb{R}^n : 1/4 < |\xi| < 2\}$ and that it satisfies

$$\phi_1(\xi) = \begin{cases} 1, & 1/2 \leq |\xi| \leq 3/2, \\ 0, & |\xi| \leq 1/4 \text{ and } |\xi| \geq 2. \end{cases}$$

We can decompose the generalized Bochner–Riesz multiplier $m^{\delta,2^k\gamma_2}$ as

$$\begin{aligned} m^{\delta,2^k\gamma_2} &= \big(1-|\xi|^{2^k\gamma_2}\big)_+^{\delta} \\ &\simeq \Big(\frac{1-|\xi|^{2^k\gamma_2}}{1-|\xi|^{\gamma_1}}\Big)^{\delta}\phi_1(\xi)\big(1-|\xi|^{\gamma_1}\big)_+^{\delta} + \big(1-|\xi|^{2^k\gamma_2}\big)_+^{\delta}\big(1-\phi_1(\xi)\big) \\ &=: m_I + m_{II}. \end{aligned} \tag{3.1}$$

For m_{II}, from the assumption $2^k\gamma_2 > n+1$ and the definition of ϕ_1, we obtain that m_{II} is supported on the disk $\{\xi \in \mathbb{R}^n : 0 \leq |\xi| \leq 1/2\}$ and $m_{II} \in C_c^{n+1}(\mathbb{R}^n)$. Integrating by parts $n+1$ times, we see that $[m_{II}(\cdot)]^{\vee} \in L^1(\mathbb{R}^n)$. Hence, Lemma 2.3 and Young's inequality yield that

$$\sup_{R>0}\big|\big([m_{II}(\cdot)]^{\vee}\big)_{1/R} * g(x)\big| \preceq M(g)(x).$$

Invoking the $L^2(|x|^{-\beta})$-boundedness of M when $-n < \beta < n$, we obtain that

$$\int_{\mathbb{R}^n} \big|\sup_{R>0}\big|\big([m_{II}(\cdot)]^{\vee}\big)_{1/R} * g(x)\big|\big|^2 |x|^{-\beta}\,dx \preceq \int_{\mathbb{R}^n} \big|g(x)\big|^2 |x|^{-\beta}\,dx. \tag{3.2}$$

For the first term m_I, we have

$$\begin{aligned} &\sup_{R>0}\big|\big([m_I(\cdot)]^{\vee}\big)_{1/R} * g(x)\big| \\ &\quad \preceq \sup_{R>0}\Big|\Big\{\Big(\Big(\frac{1-|\cdot|^{2^k\gamma_2}}{1-|\cdot|^{\gamma_1}}\Big)^{\delta}\phi_1(\cdot)\Big)^{\vee}\Big\}_{1/R}\Big| * (S_*^{\delta,\gamma_1}g)(x), \end{aligned} \tag{3.3}$$

where

$$(S_*^{\delta,\gamma_1}g)(x) = \sup_{R>0}\big|(S_R^{\delta,\gamma_1}g)(x)\big| = \sup_{R>0}\big|(B_R^{\delta,\gamma_1} * g)(x)\big|$$

and

$$B_R^{\delta,\gamma_1} = \big(((1-|\cdot|^{\gamma_1})_+^{\delta})^{\vee}\big)_{1/R}.$$

First, define the function F by

$$F(t)=\begin{cases}\frac{1-t^{\nu}}{1-t}, & t\neq 1,\\ \nu, & t=1.\end{cases}$$

Then it is easy to see that $F\in C^{\infty}(\mathbb{R})$ and that it is infinitely differentiable at the point $t=1$. Pick $\nu=\frac{2^k\gamma_2}{\gamma_1}$, $t=|\xi|^{\gamma_1}$. Invoking the assumption $2^k\gamma_2>\gamma_1$ and the definition of $\phi_1(\xi)$, we have $(F(|\xi|^{\gamma_1}))^{\delta}\phi_1(\xi)\in C_c^{\infty}(\mathbb{R}^n)\subseteq\mathscr{S}(\mathbb{R}^n)$; that is, $\left(\frac{1-|\xi|^{2^k\gamma_2}}{1-|\xi|^{\gamma_1}}\right)^{\delta}\phi_1(\xi)\in\mathscr{S}(\mathbb{R}^n)$. Therefore, we obtain that

$$\left(\left(\frac{1-|\cdot|^{2^k\gamma_2}}{1-|\cdot|^{\gamma_1}}\right)^{\delta}\phi_1(\cdot)\right)^{\vee}\in\mathscr{S}(\mathbb{R}^n)\subseteq L^1(\mathbb{R}^n). \tag{3.4}$$

From (3.4), using Lemma 2.3 combined with (3.3), we can reach the conclusion that

$$\sup_{R>0}\left|\left(\left[m_I(\cdot)\right]^{\vee}\right)_{1/R}*g(x)\right|\preceq M(S_*^{\delta,\gamma_1}g)(x).$$

Again, invoking the $L^2(|x|^{-\beta})$-boundedness of M when $-n<\beta<n$, and by the hypothesis of Theorem 1.6 that S_*^{δ,γ_1} is bounded on $L^2(|x|^{-\beta})$, we obtain that for $\delta>0$ and $0\leq\beta<1+2\delta\leq n$,

$$\int_{\mathbb{R}^n}\left|\sup_{R>0}\left|\left(\left[m_I(\cdot)\right]^{\vee}\right)_{1/R}*g(x)\right|\right|^2|x|^{-\beta}\,dx\preceq\int_{\mathbb{R}^n}\left|g(x)\right|^2|x|^{-\beta}\,dx. \tag{3.5}$$

Combining (3.1), (3.2), and (3.5), we obtain that

$$\int_{\mathbb{R}^n}\left|\sup_{R>0}\left|\left(\left[m^{\delta,2^k\gamma_2}(\cdot)\right]^{\vee}\right)_{1/R}*g(x)\right|\right|^2|x|^{-\beta}\,dx\preceq\int_{\mathbb{R}^n}\left|g(x)\right|^2|x|^{-\beta}\,dx,$$

which yields the inequality

$$\int_{\mathbb{R}^n}\left|(S_*^{\delta,2^k\gamma_2}g)(x)\right|^2|x|^{-\beta}\,dx\preceq\int_{\mathbb{R}^n}\left|g(x)\right|^2|x|^{-\beta}\,dx. \tag{3.6}$$

Decompose the multiplier $m^{\delta,2^{k-1}\gamma_2}$ as

$$\left(1-|\xi|^{2^{k-1}\gamma_2}\right)_+^{\delta}=\left(1+|\xi|^{2^{k-1}\gamma_2}\right)^{-\delta}\left(1-|\xi|^{2^k\gamma_2}\right)_+^{\delta};$$

that is, $m^{\delta,2^{k-1}\gamma_2}=(1+|\xi|^{2^{k-1}\gamma_2})^{-\delta}m^{\delta,2^k\gamma_2}$. Using Lemmas 3.1 and 2.3, we have

$$S_*^{\delta,2^{k-1}\gamma_2}g\preceq M(S_*^{\delta,2^k\gamma_2}g)(x).$$

Recalling the $L^2(|x|^{-\beta})$-boundedness of M when $-n<\beta<n$ and (3.6), we further obtain that

$$\int_{\mathbb{R}^n}\left|(S_*^{\delta,2^{k-1}\gamma_2}g)(x)\right|^2|x|^{-\beta}\,dx\preceq\int_{\mathbb{R}^n}\left|g(x)\right|^2|x|^{-\beta}\,dx. \tag{3.7}$$

Write $m^{\delta,2^{k-2}\gamma_2}=(1+|\xi|^{2^{k-2}\gamma_2})^{-\delta}m^{\delta,2^{k-1}\gamma_2}$. From (3.7), we have

$$\int_{\mathbb{R}^n}\left|(S_*^{\delta,2^{k-2}\gamma_2}g)(x)\right|^2|x|^{-\beta}\,dx\preceq\int_{\mathbb{R}^n}\left|g(x)\right|^2|x|^{-\beta}\,dx.$$

Repeating the process of the above argument $k-2$ times, we finally obtain that

$$\int_{\mathbb{R}^n} \left|(S_*^{\delta,\gamma_2} g)(x)\right|^2 |x|^{-\beta}\, dx \preceq \int_{\mathbb{R}^n} \left|g(x)\right|^2 |x|^{-\beta}\, dx,$$

which concludes the proof of Theorem 1.6. □

4. Proofs of Theorem 1.3 and Theorem 1.4

We postpone the proof of Theorem 1.4 until the end of this section. First, let us describe how we can complete the proof of Theorem 1.3 by virtue of Theorem 1.4. We need the following lemma.

Lemma 4.1 ([11, p. 410]). *Let $0 < r < p < \infty$ and let $n(1-\frac{r}{p}) < \beta < n$. Then $L^p(\mathbb{R}^n)$ is contained in $L^r(\mathbb{R}^n) + L^r(\mathbb{R}^n, |x|^{-\beta})$.*

Proof. For any $g \in L^p(\mathbb{R}^n)$, write $g = g_1 + g_2$, where $g_1 = g\chi_{|\cdot|\le 1}$, $g_2 = g\chi_{|\cdot|>1}$. Noting that $0 < r < p < \infty$, $n(1-\frac{r}{p}) < \beta < n$, and combining with Hölder's inequality, it is easy to check that $g_1 \in L^r(\mathbb{R}^n)$ and $g_2 \in L^r(\mathbb{R}^n, |x|^{-\beta})$. □

Now we explain how to deduce Theorem 1.3 from Lemma 4.1 and Theorem 1.4.

Proof of Theorem 1.3. For any given function $f \in I_\lambda(L^p)(\mathbb{R}^n)$, let $g := I_{-\lambda} f$. Then $g \in L^p(\mathbb{R}^n)$. For $2 \le p < \frac{2n}{n-1-2\delta}$, from Lemma 4.1 we can choose β satisfying $0 \le n(1-\frac{2}{p}) < \beta < 1+2\delta < n$, for which $L^p \subseteq L^2 + L^2(|x|^{-\beta)})$. We have the decomposition formula

$$g = g_1 + g_2,$$

where $g_1 \in L^2(\mathbb{R}^n)$ and $g_2 \in L^2(\mathbb{R}^n, |x|^{-\beta})$. Accordingly,

$$f = f_1 + f_2, \tag{4.1}$$

where $f_1 = I_\lambda g_1 \in I_\lambda(L^2(\mathbb{R}^n))$ and $f_2 = I_\lambda g_2 \in I_\lambda(L^2(\mathbb{R}^n, |x|^{-\beta}))$. Since $\mathscr{S}(\mathbb{R}^n) \cap I_\lambda(L^2)(\mathbb{R}^n)$ is dense in $I_\lambda(L^2)(\mathbb{R}^n)$, for any $\varepsilon > 0$, we choose an $l_1 \in \mathscr{S}(\mathbb{R}^n) \cap I_\lambda(L^2)(\mathbb{R}^n)$ such that $h_1 = f_1 - l_1$ and

$$\|h_1\|_{I_\lambda(L^2)(\mathbb{R}^n)} < \varepsilon.$$

Since $l_1 \in \mathscr{S}(\mathbb{R}^n)$, for any fixed $s > 0$, we conclude that

$$\left|\left\{x \in \mathbb{R}^n : \varlimsup_{R\to\infty} \left|R^\lambda \left((S_R^{\delta,\gamma} l_1)(x) - l_1(x)\right)\right| > s/2\right\}\right| = 0.$$

Let $\widetilde{h_1} = I_{-\lambda} h_1$. Then

$$\|\widetilde{h_1}\|_{L^2(\mathbb{R}^n)} = \|I_{-\lambda} h_1\|_{L^2(\mathbb{R}^n)} = \|h_1\|_{I_\lambda(L^2(\mathbb{R}^n))}.$$

Making use of the sublinearity of the maximal function, together with Theorem 1.4, it follows that for any $s > 0$,

$$\begin{aligned}
&\left|\left\{x \in \mathbb{R}^n : (M_\lambda^{\delta,\gamma} f_1)(x) > s\right\}\right| \\
&\quad = \left|\left\{x \in \mathbb{R}^n : \varlimsup_{R\to\infty} \left|R^\lambda \left((S_R^{\delta,\gamma} f_1)(x) - f_1(x)\right)\right| > s\right\}\right| \\
&\quad \le \left|\left\{x \in \mathbb{R}^n : (M_\lambda^{\delta,\gamma} h_1)(x) > s/2\right\}\right| \\
&\quad = \left|\left\{x \in \mathbb{R}^n : (M_\gamma \widetilde{h_1})(x) > s/2\right\}\right|
\end{aligned}$$

$$\preceq \Big(\frac{\|\widetilde{h_1}\|_{L^2(\mathbb{R}^n)}}{s}\Big)^2 = \Big(\frac{\|h_1\|_{I_\lambda(L^2(\mathbb{R}^n))}}{s}\Big)^2 < \Big(\frac{\varepsilon}{s}\Big)^2.$$

Letting $\varepsilon \to 0$, we deduce that

$$\big|\{x \in \mathbb{R}^n : (M_\lambda^{\delta,\gamma} f_1)(x) > s\}\big| = 0,$$

which concludes that

$$(S_R^{\delta,\gamma} f_1)(x) - f_1(x) = o(1/R^\lambda), \quad \text{a.e. } x \in \mathbb{R}^n \text{ as } R \to \infty.$$

Since the measure $|x|^{-\beta}\,dx$ is absolutely continuous with respect to the Lebesgue measure, we have

$$(S_R^{\delta,\gamma} f_2)(x) - f_2(x) = o(1/R^\lambda), \quad \text{a.e. } x \in \mathbb{R}^n \text{ as } R \to \infty.$$

Thus, recalling (4.1), we can obtain that for any given function $f \in I_\lambda(L^p)(\mathbb{R}^n)$, $0 \le \lambda \le \gamma$, we have

$$(S_R^{\delta,\gamma} f)(x) - f(x) = o(1/R^\lambda), \quad \text{a.e. } x \in \mathbb{R}^n \text{ as } R \to \infty.$$

Let us now consider the sharpness of the convergence rate $O(1/R^\gamma)$ in the case $\lambda = \gamma$. For any fixed $\delta > 0$, we pick an $f \in \mathscr{S}(\mathbb{R}^n) \cap I_\lambda(L^2)(\mathbb{R}^n)$ and $f \not\equiv 0$. There exists a point x_0 satisfying $(|\cdot|^\gamma \widehat{f})^\vee(x_0) \neq 0$; that is,

$$\int_{\mathbb{R}^n} \widehat{f}(\xi)|\xi|^\gamma e^{2\pi i\xi\cdot x_0}\,d\xi \neq 0.$$

Using the continuity of the Fourier transform, we have

$$\int_{\mathbb{R}^n} \widehat{f}(\xi)|\xi|^\gamma e^{2\pi i\xi\cdot x}\,d\xi \neq 0 \tag{4.2}$$

in a neighborhood of x_0. Write

$$\begin{aligned} R^\gamma\big\{(S_R^{\delta,\gamma} f)(x) - f(x)\big\} &= R^\gamma \int_{\mathbb{R}^n} \Big(\Big(1 - \Big|\frac{\xi}{R}\Big|^\gamma\Big)_+^\delta - 1\Big)\widehat{f}(\xi) e^{2\pi i x\cdot\xi}\,d\xi \\ &= R^\gamma \int_{|\xi|<R} \Big(\Big(1 - \Big|\frac{\xi}{R}\Big|^\gamma\Big)_+^\delta - 1\Big)\widehat{f}(\xi) e^{2\pi i x\cdot\xi}\,d\xi \\ &\quad + R^\gamma \int_{|\xi|\ge R} \Big(\Big(1 - \Big|\frac{\xi}{R}\Big|^\gamma\Big)_+^\delta - 1\Big)\widehat{f}(\xi) e^{2\pi i x\cdot\xi}\,d\xi \\ &=: I_R(x) + II_R(x). \end{aligned}$$

Since $f \in \mathscr{S}(\mathbb{R}^n)$, we have $|II_R(x)| \preceq R^\gamma \int_{|\xi|\ge R} |\widehat{f}(\xi)|\,d\xi \preceq R^{-L}$, for any $L > 0$. By Taylor's expansion, we notice that

$$\big(1 - |\xi|^\gamma\big)_+^\delta - 1 = \delta|\xi|^\gamma + O\big(|\xi|^{2\gamma}\big),$$

where $\gamma > 0$ and $|\xi| \le 1$. Then for $|\xi| \le R$,

$$\Big(1 - \Big|\frac{\xi}{R}\Big|^{\gamma}\Big)_{+}^{\delta} - 1 = \delta\Big|\frac{\xi}{R}\Big|^{\gamma} + O\Big(\Big|\frac{\xi}{R}\Big|^{2\gamma}\Big).$$

So we have

$$\begin{aligned} I_R(x) &= -\delta \int_{|\xi|<R} |\xi|^{\gamma} \widehat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi + O\Big(\frac{1}{R^{\gamma}} \cdot \int_{|\xi|<R} |\xi|^{2\gamma} \widehat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi\Big) \\ &= -\delta \int_{|\xi|<R} |\xi|^{\gamma} \widehat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi + O\Big(\frac{1}{R^{\gamma}}\Big). \end{aligned}$$

Thus,

$$\big|R^{\gamma}\big\{(S_R^{\delta,\gamma} f)(x) - f(x)\big\}\big| = \Big|\delta \int_{|\xi|<R} |\xi|^{\gamma} \widehat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi\Big| + O\Big(\frac{1}{R^{\gamma}}\Big) + R^{-L},$$

for any $L > 0$. From (4.2), we obtain that

$$\lim_{R\to\infty} R^{\gamma}\big((S_R^{\delta_p,\gamma} f)(x) - f(x)\big) \ne 0$$

in a set of positive measure. This concludes the proof of Theorem 1.3. □

Now we return to prove Theorem 1.4.

Proof of Theorem 1.4. The sublinearity of the maximal function $M_{\gamma}g$ leads to

$$(M_{\gamma}g)(x) \le \sum_{j=0,1,\infty} (M_{\gamma,j}g)(x), \tag{4.3}$$

where $M_{\gamma,j}g$ is the associated maximal operator

$$(M_{\gamma,j}g)(x) = \sup_{R>0}\big|(T_{R,\lambda,j}^{\delta,\gamma} g)(x)\big|.$$

Invoking Lemmas 2.1 and 2.2, we know that for any $0 \le \lambda \le \gamma$, $K_{\lambda,j} \in L^1(\mathbb{R}^n)$ for $j = 0, \infty$. Combining with Lemma 2.3, we have

$$M_{\gamma,j}g(x) \preceq M(g)(x), \quad j = 0, \infty,$$

which says that $M_{\gamma,j}g$ ($j = 0, \infty$) can be dominated by the Hardy–Littlewood maximal function. Taking note of the fact that when $-n < \beta < n$, the Hardy–Littlewood maximal function operator is bounded on $L^2(\mathbb{R}^n, |x|^{-\beta}\, dx)$ (see [10, Theorem 7.1.9, Example 7.1.7]), we know that

$$\|M_{\gamma,j}g\|_{L^2(|x|^{-\beta})} \preceq \|g\|_{L^2(|x|^{-\beta})} \quad \text{for } j = 0, \infty. \tag{4.4}$$

Invoking (4.3) and (4.4), to prove Theorem 1.4 it suffices to show that

$$\|M_{\gamma,1}g\|_{L^2(|x|^{-\beta})} \preceq \|g\|_{L^2(|x|^{-\beta})}. \tag{4.5}$$

Thus, we mainly pay attention to the estimate of $M_{\gamma,1}g$. Let

$$\Phi(x) = \Big(\frac{\phi_1(\cdot)}{|\cdot|^{\lambda}}\Big)^{\vee},$$

and let $\Phi_{1/R}(x) = R^n\Phi(Rx)$ for $R > 0$. Then

$$(T^{\delta,\gamma}_{R,\lambda,1} g)(x) = \Phi_{1/R} * B^{\delta,\gamma}_R * g(x)$$

and

$$\begin{aligned}(M_{\gamma,1} g)(x) &= \sup_{R>0} \left|(T^{\delta,\gamma}_{R,\lambda,1} g)(x)\right| \\ &\le \sup_{R>0} |\Phi_{1/R}| * (S^{\delta,\gamma}_* g)(x) \\ &\preceq M(S^{\delta,\gamma}_* g)(x),\end{aligned}$$

where M is the Hardy–Littlewood maximal operator, since Φ is a Schwartz function by its choice. Invoking the $L^2(|x|^{-\beta})$-boundedness of M when $-n < \beta < n$, we have

$$\begin{aligned}\|M_{\gamma,1} g\|_{L^2(|x|^{-\beta})} &\preceq \left\|M(S^{\delta,\gamma}_* g)\right\|_{L^2(|x|^{-\beta})} \\ &\preceq \|S^{\delta,\gamma}_* g\|_{L^2(|x|^{-\beta})}.\end{aligned}$$

Thus, in order to prove (4.5), we only need to show that

$$\|S^{\delta,\gamma}_* g\|_{L^2(|x|^{-\beta})} \preceq \|g\|_{L^2(|x|^{-\beta})}. \tag{4.6}$$

From Theorem 1.6, we know that the $L^2(|x|^{-\beta})$-boundedness of $S^{\delta,2}_*$ implies the $L^2(|x|^{-\beta})$-boundedness of $S^{\delta,\gamma}_*$. Recalling Theorem A, we deduce the $L^2(|x|^{-\beta})$-boundedness of $S^{\delta,\gamma}_*$ in (4.6). This finishes the proof of Theorem 1.4. □

5. Generalized Bochner–Riesz means on the torus

In this section, we study the generalized Bochner-Riesz means operators on the torus $\mathbb{T}^n$. Let $m = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$, and let $|m| = (m_1^2 + m_2^2 + \cdots + m_n^2)^{1/2}$. Functions on $\mathbb{T}^n$ are functions $\widetilde{f}$ on $\mathbb{R}^n$ that satisfy $\widetilde{f}(x) = \widetilde{f}(x+m)$ for all $x \in \mathbb{R}^n$ and $m \in \mathbb{Z}^n$. For a complex-valued function $\widetilde{f}$ in $L^1(\mathbb{T}^n)$ and $m \in \mathbb{Z}^n$, we define the mth Fourier coefficient of $\widetilde{f}$ by $c_m = \int_{\mathbb{T}^n} \widetilde{f}(x) e^{-2\pi i m\cdot x}\, dx$. The Fourier series of $\widetilde{f}$ at $x \in \mathbb{T}^n$ is the series $\widetilde{f}(x) \sim \sum_{m\in\mathbb{Z}^n} c_m e^{2\pi i m\cdot x}$. The Riesz potential $\widetilde{I}_{-\lambda}$ of order $\lambda \ge 0$ is defined as $\widetilde{I}_{-\lambda}(\widetilde{f})(x) \sim \sum_{m\in\mathbb{Z}^n} |m|^\lambda c_m e^{-2\pi i m\cdot x}$, where $\{c_m\}$ is the set of Fourier coefficients of $\widetilde{f}$. For the sake of simplicity, without loss of generality, we initially consider functions $\widetilde{f} \in C^\infty(\mathbb{T}^n)$, that is, $\widetilde{f}$ is a C^∞-function satisfying

$$\widetilde{f}(x) = \widetilde{f}(x+m)$$

for any $x \in \mathbb{R}^n$ and $m \in \mathbb{Z}^n$. Similar to the definition established in (1.1) on $\mathbb{R}^n$, the generalized Bochner–Riesz means $S^{\delta,\gamma}_R$ of order δ on the torus $\mathbb{T}^n$ is the operator

$$(\widetilde{S^{\delta,\gamma}_R}\widetilde{f})(x) = \sum_{\substack{m\in\mathbb{Z}^n \\ |m|\le R}} \left(1 - \frac{|m|^\gamma}{R^\gamma}\right)^\delta c_m e^{2\pi i m\cdot x}$$

defined on integrable functions $\widetilde{f}$ on $\mathbb{T}^n$, where δ and γ are two real numbers satisfying $\delta > -1$ and $\gamma > 0$. Denote $(m^{\delta,\gamma})^\vee(y)$ by $L^{\delta,\gamma}(y)$, where $y \in \mathbb{R}^n$. Then we have

$$(\widetilde{S_R^{\delta,\gamma}}\widetilde{f})(x) = \sum_{\substack{l\in\mathbb{Z}^n \\ |l|\le R}} m^{\delta,\gamma}\Big(\frac{l}{R}\Big)c_l e^{2\pi i l\cdot x} = \widetilde{f} * \widetilde{B_R^{\delta,\gamma}}(x),$$

for any integrable function $\widetilde{f}$ on $\mathbb{T}^n$ and $x \in \mathbb{T}^n$, where $\widetilde{B_R^{\delta,\gamma}}$ is a function whose sequence of Fourier coefficients is $\{m^{\delta,\gamma}(\frac{l}{R})\}_{l\in\mathbb{Z}^n}$. By the Poisson summation formula (see [10, Theorem 3.2.8]), we have

$$\widetilde{B_R^{\delta,\gamma}}(x) = \sum_{k\in\mathbb{Z}^n} m^{\delta,\gamma}\Big(\frac{k}{R}\Big)e^{2\pi i k\cdot x} = R^n \sum_{l\in\mathbb{Z}^n} L^{\delta,\gamma}\big((x+l)R\big).$$

By Fubini's theorem, it is easy to check that $R^\lambda\{(\widetilde{S_R^{\delta,\gamma}}\widetilde{f}) - \widetilde{f}\}$ can be written as the Fourier series

$$R^\lambda\big\{(\widetilde{S_R^{\delta,\gamma}}\widetilde{f}) - \widetilde{f}\big\}(x) = \sum_{l\in\mathbb{Z}^n} |l|^\lambda c_l \mu\Big(\frac{l}{R}\Big)e^{2\pi i l\cdot x},$$

where $\{c_l\}$ is the set of Fourier coefficients of $\widetilde{f}$ and μ is the same as in Section 1. Precisely, the multiplier μ is given by

$$\mu(\xi) = \frac{(1-|\xi|^\gamma)_+^\delta - 1}{|\xi|^\lambda}, \quad \xi \neq 0 \qquad \text{and} \qquad \mu(0) = \lim_{t\to 0^+} \frac{(1-t^\gamma)_+^\delta - 1}{t^\lambda}.$$

Denote by $K^{\delta,\gamma,\lambda}$ the inverse Fourier transform of μ, that is, $K^{\delta,\gamma,\lambda}(y) = \mu^\vee(y) = \int_{\mathbb{R}^n} \mu(\xi)e^{2\pi i\xi\cdot y}\,d\xi$, where $y \in \mathbb{R}^n$. Let $\widetilde{g} := \widetilde{I}_{-\lambda}\widetilde{f}$ and $\widetilde{T_{R,\lambda}^{\delta,\gamma}}\widetilde{g} := R^\lambda\{(\widetilde{S_R^{\delta,\gamma}}\widetilde{f}) - \widetilde{f}\}$, and let the associated maximal operator of $\widetilde{T_{R,\lambda}^{\delta,\gamma}}\widetilde{g}$ be defined by $(\widetilde{M_\gamma}\widetilde{g})(x) = \sup_{R>0} |(\widetilde{T_{R,\lambda}^{\delta,\gamma}}\widetilde{g})(x)|$. We can write

$$\widetilde{T_{R,\lambda}^{\delta,\gamma}}\widetilde{g}(x) = R^\lambda\big\{(\widetilde{S_R^{\delta,\gamma}}\widetilde{f}) - \widetilde{f}\big\}(x) = \widetilde{K_R^{\delta,\gamma,\lambda}} * \widetilde{g}(x),$$

where

$$\widetilde{K_R^{\delta,\gamma,\lambda}}(x) \sim \sum_{l\in\mathbb{Z}^n} \mu\Big(\frac{l}{R}\Big)e^{2\pi i l\cdot x} = R^n \sum_{l\in\mathbb{Z}^n} K^{\delta,\gamma,\lambda}\big((x+l)R\big).$$

For the convolution operator $\varphi_t * f(x)$, one can define its periodic version via the Fourier series by

$$\widetilde{\varphi}_t * \widetilde{f}(x) \sim \sum_{k\in\mathbb{Z}^n} c_k \widehat{\varphi}(tk)e^{-2\pi i k\cdot x},$$

where c_k is the mth Fourier coefficient of $\widetilde{f}$. We recall the following result on the transference of maximal multipliers.

Theorem D ([12, Theorem 1])**.** *Let $1 < p \leq \infty$. Suppose that $\widehat{\varphi}$ is a bounded and continuous function on $\mathbb{R}^n$. If*

$$\Big\| \sup_{0<t\leq 1} |\varphi_t * g| \Big\|_{L^p(\mathbb{R}^n)} \preceq \|g\|_{L^p(\mathbb{R}^n)}$$

for all $g \in L^p(\mathbb{R}^n) \cap S(\mathbb{R}^n)$, then we have

$$\Big\| \sup_{0<t\leq 1} |\widetilde{\varphi}_t * \widetilde{g}| \Big\|_{L^p(\mathbb{T}^n)} \preceq \|\widetilde{g}\|_{L^p(\mathbb{T}^n)}$$

for all $\widetilde{g} \in L^p(\mathbb{T}^n)$.

Using Theorem D, we have the following theorem.

Theorem 5.1. *Let $\delta > 0$, $p \geq 2$. If $\widetilde{f}$ is in $\widetilde{I}_\lambda(L^p)(\mathbb{T}^n)$, then for $0 \leq \lambda < \gamma$,*

$$(\widetilde{S_R^{\delta,\gamma}}\widetilde{f})(x) - \widetilde{f}(x) = o(1/R^\lambda), \quad \text{a.e. as } R \to \infty,$$

and for $\lambda = \gamma$,

$$(\widetilde{S_R^{\delta,\gamma}}\widetilde{f})(x) - \widetilde{f}(x) = O(1/R^\gamma), \quad \text{a.e. as } R \to \infty.$$

Moreover, the rate $O(1/R^\gamma)$ is sharp at the endpoint $\lambda = \gamma$ in the sense that $O(1/R^\gamma)$ cannot be replaced by $o(1/R^\gamma)$.

The sharpness of the theorem easily follows the same argument as in the proof of Theorem 1.3. For the proof of convergence, by checking the proof for Theorem 1.3, it suffices to prove the following result.

Theorem 5.2. *Let $\delta > 0$. If $\widetilde{f}$ is in $\widetilde{I}_\lambda(L^2)(\mathbb{T}^n)$, then*

$$\|\widetilde{M_\gamma}\widetilde{g}\|_{L^2(\mathbb{T}^n)} \preceq \|\widetilde{g}\|_{L^2(\mathbb{T}^n)}. \tag{5.1}$$

Proof. From Theorem 1.4, taking $\beta = 0$, we obtain that M_γ is bounded on $L^2(\mathbb{R}^n)$. To prove Theorem 5.2, it is easy to see that μ is a bounded and continuous function on $\mathbb{R}^n$. Then by Theorem C we obtain the transference result (5.1). □

Now we return to explain how to deduce Theorem 5.1 from Theorem 5.2.

Proof of Theorem 5.1. By standard arguments as in the proof of Theorem 1.3, Theorem 5.2 implies that

$$(\widetilde{S_R^{\delta,\gamma}}\widetilde{f})(x) - \widetilde{f}(x) = o(1/R^\lambda), \quad \text{a.e. as } R \to \infty,$$

for $\widetilde{f}$ in $\widetilde{I}_\lambda(L^2)(\mathbb{T}^n)$. For $p \geq 2$, we note that $\widetilde{I}_\lambda(L^p)(\mathbb{T}^n) \subseteq \widetilde{I}_\lambda(L^2)(\mathbb{T}^n)$ because $\mathbb{T}^n$ is compact. In fact, if $\widetilde{f} \in \widetilde{I}_\lambda(L^p)(\mathbb{T}^n)$, then by Hölder's inequality we have

$$\|\widetilde{f}\|_{\widetilde{I}_\lambda(L^2)(\mathbb{T}^n)} = \|\widetilde{I}_\lambda\widetilde{f}\|_{L^2(\mathbb{T}^n)} \preceq \|\widetilde{I}_\lambda\widetilde{f}\|_{L^p(\mathbb{T}^n)} = \|\widetilde{f}\|_{\widetilde{I}_\lambda(L^p)(\mathbb{T}^n)}.$$

Since $\widetilde{I}_\lambda(L^p)(\mathbb{T}^n) \subseteq \widetilde{I}_\lambda(L^2)(\mathbb{T}^n)$ for any $p \geq 2$, we have

$$(\widetilde{S_R^{\delta,\gamma}}\widetilde{f})(x) - \widetilde{f}(x) = o(1/R^\lambda), \quad \text{a.e. as } R \to \infty,$$

for any $\widetilde{f} \in \widetilde{I}_\lambda(L^p)(\mathbb{T}^n)$, $p \geq 2$. □

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