A NEW APPROACH TO THE NONSINGULAR CUBIC BINARY MOMENT PROBLEM

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ABSTRACT. We present an alternative solution to nonsingular cubic moment problems, using techniques that are expected to be useful for higher-degree truncated moment problems. In particular, we apply the theory of recursively determinate moment matrices to deal with a case of rank-increasing moment matrix extensions.

1. Introduction

Given a doubly indexed finite sequence of real numbers $\beta \equiv \beta^{(m)} = \{\beta_{00}, \beta_{10}, \beta_{01}, \ldots, \beta_{m,0}, \beta_{m-1,1}, \ldots, \beta_{1,m-1}, \beta_{0,m}\}$ with $\beta_{00} > 0$, the *truncated real moment problem* (TRMP) entails seeking necessary and sufficient conditions for the existence of a positive Borel measure μ supported in the real plane $\mathbb{R}^2$ such that

$$\beta_{ij} = \int x^i y^j \, d\mu \quad (i, j \in \mathbb{Z}_+, 0 \le i + j \le m).$$

When such a measure exists, we say that μ is a *representing measure* for β and that TRMP is *soluble*.

There is a parallel *truncated complex moment problem* (TCMP) for a finite sequence of complex numbers $\gamma \equiv \gamma^{(m)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,m}, \gamma_{1,m-1}, \ldots, \gamma_{m-1,1}, \gamma_{m,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$. Here TCMP consists of finding a positive Borel

measure μ supported in the complex plane $\mathbb{C}$ such that $\gamma_{ij} = \int \bar{z}^i z^j \, d\mu$ $(i, j \in \mathbb{Z}_+,$ $0 \leq i + j \leq m)$. It is well known that TRMP and TCMP are equivalent for an even integer m (see [4, Proposition 1.12]), and hence any techniques developed for TCMP are transferable to TRMP. Both problems are simply referred to as the *truncated moment problem* (TMP).

In a series of articles, for the case when $m = 2d$, the first named author and Fialkow found solutions for various truncated moment problems; for instance, we obtained complete solutions for $m = 2$ and $m = 4$ (see [1], [4], [9], [8]). Some solutions are based on matrix positivity and extension, combined with a so-called *functional calculus* (which is discussed in Section 2) for the columns of the associated moment matrix. This matrix is defined as follows. For a real moment sequence $\beta^{(2d)}$ of even degree, the *moment matrix* $\mathcal{M}(d) \equiv \mathcal{M}(d)(\beta^{(2d)})$ is given by

$$\mathcal{M}(d)(\beta^{(2d)}) := (\beta_{\mathbf{i+j}})_{\mathbf{i,j} \in \mathbb{Z}_+^2 : |\mathbf{i}|, |\mathbf{j}| \leq 2d}.$$

If we label the columns of $\mathcal{M}(d)$ with the degree lexicographical order, 1, X, Y, X^2, $XY, Y^2, \ldots, X^d, \ldots, Y^d$, we can then use the functional calculus for columns of $\mathcal{M}(d)$, introduced in [1]. The moment matrix $\mathcal{M}(d)$ is Hankel by rectangular blocks; for instance,

$$\mathcal{M}(2) \equiv \left(\begin{array}{c|cc|ccc} \beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} \\ \hline \beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} \\ \beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} \\ \hline \beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} \\ \beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} \\ \beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} \end{array} \right). \tag{1.1}$$

When $m = 2d + 1$, a general solution to partial cases of TMP can be found in [12] and [13] as well as a solution to the *truncated matrix moment problem*; a solution to the *cubic complex moment problem* (when $m = 3$) was given in [12]. Kimsey [10] obtained an existence proof of the cubic moment problem in the complex case; the proof entails a considerably more involved case analysis than what is presented below. At the same time, the analysis in [10] allows one to have some control over at least one point in the support of the representing measure. Related to this, in Remarks 3.2 and 3.4, we will establish a connection between the support of the representing measure and the column relations associated with the moment matrix extensions $\mathcal{M}(2)$ and $\mathcal{M}(3)$.

In [12], explicit examples of cubic bisequences $\beta^{(3)}$ were given such that $\mathcal{M}(1)$ is positive and invertible and yet $\beta^{(3)}$ does not have a 3-atomic representing measure. For the case of $\beta^{(3)}$ with $\mathcal{M}(1)$ positive semidefinite (possibly noninvertible), Kimsey [10, Theorems 3.1, 3.2] proved that $\beta^{(3)}$ always admits a representing measure with at most four atoms. Finally, we mention that a full abstract solution to the odd total degree moment problem in several variables was given by Kimsey in [11].

We know from [4, Proposition 1.12] that the complex and real truncated moment problems are equivalent, in the sense that there exists a bridging map that allows one to translate the hypotheses and conclusions for TCMP into similar hypotheses and conclusions for TRMP, and vice versa. Our approach here, however, focuses on the real cubic moment problem, and analyzes it in its own right.

In this article, we consider cubic *real* moment problems and present an alternative solution to the "nonsingular" case (i.e., $\mathcal{M}(1)$ invertible; see Section 3 for the formal definition). Our idea is to extend the initial data $\beta^{(3)}$ to an even-degree $\beta^{(4)}$, for which the associated moment matrix $\mathcal{M}(2)$ has rank 3 or 4. We then prove that $\mathcal{M}(2)$ (i) is a flat extension of $\mathcal{M}(1)$, that (ii) it is a flat extension of a 4×4 submatrix, or that (iii) it admits a flat extension $\mathcal{M}(3)$. In all three cases, we find a finitely atomic representing measure for $\beta^{(3)}$. In particular, we identify the support of the minimal representing measure as the intersection of three nondegenerate conics (in the rank 3 case) and of two nondegenerate conics (in the rank 4 case).

We anticipate that the present work will contribute to our understanding of higher-degree moment problems, beginning with the quintic moment problem. We also expect that solutions to odd-degree moment problems will be applied to solve the subnormal completion problem studied in [7].

2. Preliminaries

When we build a moment matrix $\mathcal{M}(2)$ out of a cubic finite sequence, the lower right-hand 3×3 block will include all quartic moments, which will need to remain undefined. To obtain our main results, we will choose appropriate quartic moments and show that $\mathcal{M}(2)$ has a representing measure. In order to describe this process in detail, we need to review basic TMP notation and results pertaining to the even-degree case.

Necessary conditions. In order to discuss basic necessary conditions for the existence of a measure, let μ be a representing measure of the even-degree moment sequence $\beta \equiv \beta^{(2d)}$. First, we recall that

$$0 \leq \int \left| p(x,y) \right|^2 d\mu = \sum_{i,j,k,l} a_{ij} a_{kl} \int x^{i+l} y^{j+k} \, d\mu = \sum_{i,j,k,l} a_{ij} a_{kl} \beta_{i+l} \beta_{j+k}$$

if and only if $\mathcal{M}(d) \geq 0$.

Let $\mathcal{P}_k$ denote the set of bivariate polynomials in $\mathbb{R}[x,y]$ whose degree is at most k, and let $\mathcal{C}_{\mathcal{M}(d)}$ denote the column space of $\mathcal{M}(d)$. For $k \leq d$, we now define an assignment from $\mathcal{P}_k$ to $\mathcal{C}_{\mathcal{M}(d)}$; given a polynomial $p(x,y) \equiv \sum_{ij} a_{ij} x^i y^j$, we let $p(X,Y) := \sum_{ij} a_{ij} X^i Y^j$ (so that $p(X,Y) \in \mathcal{C}_{\mathcal{M}(d)}$), which defines the above-mentioned functional calculus. We also let $\mathcal{Z}(p)$ denote the zero set of p, and we define the *algebraic variety* of β by

$$\mathcal{V} \equiv \mathcal{V}(\beta) := \bigcap_{p(x,y)=0, \deg p \leq d} \mathcal{Z}(p). \tag{2.1}$$

If $\widehat{p}$ denotes the column vector of coefficients of p, then we know that $p(X, Y) = \mathcal{M}(d)\widehat{p}$; as a consequence, $p(X, Y) = \mathbf{0}$ if and only if $\widehat{p} \in \ker \mathcal{M}(d)$. Another necessary condition we will use is $\operatorname{supp} \mu \subseteq \mathcal{V}(\beta)$ and $r := \operatorname{rank} \mathcal{M}(d) \leq \operatorname{card} \operatorname{supp} \mu \leq v := \operatorname{card} \mathcal{V}$; this condition is called the *variety condition* (see [1]). In addition, if p is any polynomial of degree at most $2d$ such that $p|_{\mathcal{V}} \equiv 0$, then the *Riesz functional* Λ must satisfy $\Lambda(p) := \int p \, d\mu = 0$, which is referred to as *Consistency* of the moment sequence. The main results in [6] state that the above-mentioned conditions together with Consistency are sufficient for solubility in the extremal case ($r = v$). Moreover, Curto, Fialkow, and Möller [6] showed that Consistency cannot be replaced by the weaker condition that $\mathcal{M}(d)$ is *recursively generated*, (RG); that is, if $p(X, Y) = \mathbf{0}$, then $(pq)(X, Y) = \mathbf{0}$ for each polynomial q with $\deg(pq) \leq d$.

In summary, positive semidefiniteness alone is sufficient to solve the quadratic moment problem ($d = 1$) (see [1]). However, for $d > 1$, the solubility of TMP requires more. For instance, the solution of the quartic moment problem ($d = 2$) requires positive semidefiniteness, the variety condition, and the (RG) property (this last property requires that the moment matrix be recursively generated; see [1], [4], [9]).

Flat extensions. We recall that $\mathcal{M}(d)$ is said to be *flat* if $\operatorname{rank} \mathcal{M}(d) = \operatorname{rank} \mathcal{M}(d-1)$; that is, $\mathcal{M}(d)$ is a rank-preserving positive extension of $\mathcal{M}(d-1)$. In this case, $\mathcal{M}(d)$ has a unique rank $\mathcal{M}(d)$-atomic measure. Furthermore, it is known that if $\mathcal{M}(d)$ has a positive extension $\mathcal{M}(d + k)$ for some $k \in \mathbb{Z}_+$, which in turn admits a flat extension $\mathcal{M}(d+k+1)$, then β has a rank $\mathcal{M}(d+k)$-atomic measure (see [2, Theorem 1.5]). This result is referred to as the *flat extension theorem*; it is probably the most efficient, concrete solution to TMP, even though the construction of an extension is usually difficult for a high-degree TMP.

We will use the flat extension theorem in the proof of our main results; thus, we need to briefly describe the process of building a flat extension. Since a moment matrix extension $\mathcal{M}(d+1)$ of $\mathcal{M}(d)$ can be written as $\mathcal{M}(d+1) = \begin{pmatrix} \mathcal{M}(d) & B(d+1) \\ B(d+1)^* & C(d+1) \end{pmatrix}$, for some rectangular matrices $B(d + 1)$ and $C(d + 1)$, we can adapt a classical result given by Šmul'jan in the search for a positive $\mathcal{M}(d + 1)$.

Theorem 2.1 (Šmul'jan's theorem [14, main theorem])**.** *Let A, B, C be matrices of complex numbers, with A and C square matrices. Then*

$$\tilde{A} := \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \iff \begin{cases} A \geq 0, \\ B = AW \quad (\text{for some } W), \\ C \geq W^*AW. \end{cases}$$

Moreover, $\operatorname{rank} \tilde{A} = \operatorname{rank} A \iff C = W^*AW$.

Remark 2.2. When the extension $\tilde{A}$ in Theorem 2.1 has the same rank as A, we say that $\tilde{A}$ is a *flat extension* of A. Besides satisfying this theorem, an extension $\mathcal{M}(d+1)$ must maintain the moment matrix structure; that is, the C-block must be Hankel. This condition makes generating flat extensions quite difficult in many instances.

We now discuss how we can find an explicit formula for a representing measure. Suppose that $\mathcal{M}(d)$ admits a positive extension $\mathcal{M}(d+k)$ for some $k \in \mathbb{Z}_+$ that has a flat extension $\mathcal{M}(d+k+1)$. Thus, β has a rank $\mathcal{M}(d+k)$-atomic measure μ; also, let $r := \text{rank}\,\mathcal{M}(d+k)$. The flat extension theorem says that the algebraic variety $\mathcal{V}$ of $\mathcal{M}(d+k+1)$ consists of exactly r points, and we may write $\mathcal{V} = \{(x_1, y_1), \ldots, (x_r, y_r)\}$. Denote the Vandermonde matrix V as

$$V = \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 & \cdots & x_1^{d+k} & \cdots & y_1^{d+k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_r & y_r & x_r^2 & x_r y_r & y_r^2 & \cdots & x_r^{d+k} & \cdots & y_r^{d+k} \end{pmatrix}. \tag{2.2}$$

If $\mathcal{B} := \{\mathbf{t}_1, \ldots, \mathbf{t}_r\}$ is the basis for the column space of $\mathcal{M}(d+k)$ and if $V_{\mathcal{B}}$ is the submatrix of V with columns labeled as in $\mathcal{B}$, then we can find the densities by solving

$$V_{\mathcal{B}}^T \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_r \end{pmatrix}^T = \begin{pmatrix} \Lambda(\mathbf{t}_1) & \Lambda(\mathbf{t}_2) & \cdots & \Lambda(\mathbf{t}_r) \end{pmatrix}^T. \tag{2.3}$$

Finally, we have $\mu = \sum_{k=1}^{r} \rho_k \delta_{(x_k, y_k)}$.

Degree 1 *transformations.* We briefly review a tool that will allow us to convert the given moment problem into a simpler one; this tool is known as the *invariance of moment problems under degree* 1 *transformations*. The complex version is provided in [4]; we adapt the notation in [4] to obtain a real version.

For $a, b, c, d, e, f \in \mathbb{R}$ with $bf \neq ce$, let $\Psi(x, y) \equiv (\Psi_1(x, y), \Psi_2(x, y)) := (a + bx + cy, d + ex + fy)$ for $x, y \in \mathbb{R}$. If Λ_β denotes the Riesz functional associated with β, then given $\beta \equiv \beta^{(2d)}$ we build a new (equivalent) moment sequence $\tilde{\beta} \equiv \tilde{\beta}^{(2d)} \equiv \{\tilde{\beta}_{ij}\}$ given by $\tilde{\beta}_{ij} := \Lambda_\beta(\Psi_1^i \Psi_2^j)$ $(0 \leq i + j \leq 2d)$. We immediately check that $\Lambda_{\tilde{\beta}}(p) = \Lambda_\beta(p \circ \Psi)$ for every $p \in \mathcal{P}_d$.

Proposition 2.3 (Invariance under degree 1 transformations; [4, Proposition 1.7]). *Let $\mathcal{M}(d)$ and $\tilde{\mathcal{M}}(d)$ be the moment matrices associated with β and $\tilde{\beta}$, respectively, and let $J\hat{p} := \widehat{p \circ \Psi}$ $(p \in \mathcal{P}_d)$. The following statements hold:*

(i) $\tilde{\mathcal{M}}(d) = J^T \mathcal{M}(d) J$;

(ii) *J is invertible;*

(iii) $\tilde{\mathcal{M}}(d) \geq 0 \Leftrightarrow \mathcal{M}(d) \geq 0$;

(iv) $\text{rank}\,\tilde{\mathcal{M}}(d) = \text{rank}\,\mathcal{M}(d)$;

(v) *the formula $\mu = \tilde{\mu} \circ \Psi$ establishes a one-to-one correspondence between the sets of representing measures for β and $\tilde{\beta}$, which preserves measure class and cardinality of the support; moreover, $\varphi(\text{supp}\,\mu) = \text{supp}\,\tilde{\mu}$;*

(vi) *$\mathcal{M}(d)$ admits a flat extension if and only if $\tilde{\mathcal{M}}(d)$ admits a flat extension.*

We will now apply Proposition 2.3 to a cubic real binary moment sequence $\beta \equiv \beta^{(3)} : \{\beta_{00}, \beta_{10}, \beta_{01}, \beta_{20}, \beta_{11}, \beta_{02}, \beta_{30}, \beta_{21}, \beta_{12}, \beta_{03}\}$ with $\beta_{00} > 0$. Our strategy is to enlarge $\beta^{(3)}$ to $\beta^{(4)}$ by adding new undetermined moments of degree 4; this extended finite sequence has an associated moment matrix $\mathcal{M}(2)$. As we will see in the remainder of this article, it is enough to consider the case when $\mathcal{M}(2)$ is "normalized"; that is, $\mathcal{M}(1)$ is the identity matrix. The case when $\mathcal{M}(1)$ is singular can be dealt with easily using the results in [1] and [3]. We thus assume

that $\beta_{00} = 1$ and that the principal 2×2 and 3×3 minors of $\mathcal{M}(1)$, d_2 and d_3, respectively, are strictly positive. A calculation using *Mathematica* [15] reveals that

$$d_2 = -\beta_{10}^2 + \beta_{20},$$
$$d_3 = -\beta_{02}\beta_{10}^2 + 2\beta_{01}\beta_{10}\beta_{11} - \beta_{11}^2 - \beta_{01}^2\beta_{20} + \beta_{02}\beta_{20}.$$

Consider now the degree 1 transformation

$$\Psi(x,y) \equiv (a + bx + cy, d + ex + fy),$$

where $a := \frac{\beta_{01}\beta_{20} - \beta_{10}\beta_{11}}{\sqrt{d_2 d_3}}$, $b := \frac{\beta_{11} - \beta_{01}\beta_{10}}{\sqrt{d_2 d_3}}$, $c := -\sqrt{\frac{d_2}{d_3}}$, $d := -\frac{\beta_{10}}{\sqrt{d_2}}$, $e := \frac{1}{\sqrt{d_2}}$, and $f := 0$. Observe that

$$bf - ce = -\sqrt{\frac{1}{d_3}} \neq 0.$$

Through this transformation, and using [4, Proposition 1.7], any positive semidefinite $\mathcal{M}(2)$ with a nonsingular $\mathcal{M}(1)$ can be translated to

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & \tilde{\beta}_{30} & \tilde{\beta}_{21} & \tilde{\beta}_{12} \\ 0 & 0 & 1 & \tilde{\beta}_{21} & \tilde{\beta}_{12} & \tilde{\beta}_{03} \\ 1 & \tilde{\beta}_{30} & \tilde{\beta}_{21} & \tilde{\beta}_{40} & \tilde{\beta}_{31} & \tilde{\beta}_{22} \\ 0 & \tilde{\beta}_{21} & \tilde{\beta}_{12} & \tilde{\beta}_{31} & \tilde{\beta}_{22} & \tilde{\beta}_{13} \\ 1 & \tilde{\beta}_{12} & \tilde{\beta}_{03} & \tilde{\beta}_{22} & \tilde{\beta}_{13} & \tilde{\beta}_{04} \end{pmatrix} =: \mathcal{M}[a_0, a_1, a_2, a_3], \qquad (2.4)$$

where $a_i := \tilde{\beta}_{3-i,i}$.

Recursively determinate moment problems. Our approach to the nonsingular cubic moment problem will require a key result from the theory of recursively determinate moment matrices (see [5, Theorems 2.3, 2.5, Corollary 2.4]), which we now briefly describe. We first recall that a moment matrix $\mathcal{M}(d)$ is *recursively determinate* if there are column dependence relations in $\mathcal{M}(d)$ of the form

$$X^n = p(X,Y) \quad (p \in \mathcal{P}_{n-1}); \qquad (2.5)$$
$$Y^m = q(X,Y) \quad (q \in \mathcal{P}_m, m \leq n, \text{ and } q \text{ has no } y^m \text{ term}). \qquad (2.6)$$

One of the main results in [5] follows.

Lemma 2.4 ([5, Corollary 2.4], with $d = n = m = 2$, so that $d = n + m - 2$). *Assume that $\mathcal{M}(2)$ is positive semidefinite which admits column relations of the form (2.5) and (2.6), with $n = m = 2$. Then $\mathcal{M}(2)$ admits a flat extension $\mathcal{M}(3)$.*

We recall that, in general, the solubility of a quartic moment problem requires the variety condition. However, Lemma 2.4 says that the variety condition is superfluous if a positive semidefinite $\mathcal{M}(2)$ with invertible $\mathcal{M}(1)$ has only two column relations $X^2 = p(X,Y)$ and $Y^2 = q(X,Y)$, where p and q are linear polynomials. In such a case, $\mathcal{M}(2)$ has a flat extension $\mathcal{M}(3)$, and therefore a 4-atomic representing measure. It follows that the pair of equations $x^2 = p(x,y)$ and $y^2 = q(x,y)$ has exactly four common real roots.

3. Cubic binary moment problems

As we have indicated before, the nontrivial cases of the cubic binary moment problem arise when the submatrix $\mathcal{M}(1)$ of $\beta^{(3)}$ is nonsingular. Moreover, as noted in [10], the positive semidefiniteness of $\mathcal{M}(1)$ is always a necessary condition for the existence of a representing measure. Thus, in the remainder of this article, we focus on cubic binary moment problems with $\mathcal{M}(1)$ positive definite. When this happens, we say that $\beta^{(3)}$ is a *nonsingular* cubic binary moment sequence.

Main results. Using the degree 1 transformation introduced in Section 2, if $\beta^{(3)}$ is a nonsingular cubic binary moment sequence, then we may always assume, without loss of generality, that $\beta^{(3)} : \{1, 0, 0, 1, 0, 1, a_0, a_1, a_2, a_3\}$ and we may write

$$
\mathcal{M}(2) := \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & a_0 & a_1 & a_2 \\
0 & 0 & 1 & a_1 & a_2 & a_3 \\
1 & a_0 & a_1 & \beta_{40} & \beta_{31} & \beta_{22} \\
0 & a_1 & a_2 & \beta_{31} & \beta_{22} & \beta_{13} \\
1 & a_2 & a_3 & \beta_{22} & \beta_{13} & \beta_{04}
\end{pmatrix}, \tag{3.1}
$$

where β_{40}, β_{31}, β_{22}, β_{13}, and β_{04} are undetermined new moments. We will prove that the extended $\beta^{(4)}$ obtained from $\beta^{(3)}$ by adding the quartic moments β_{40}, β_{31}, β_{22}, β_{13}, and β_{04} admits a representing measure, for appropriate choices of the new moments; as a result, $\beta^{(3)}$ also admits a representing measure μ. The smallest cardinality of $\operatorname{supp} \mu$ will be 3 in some cases, and 4 in others.

Using Theorem 2.1, we first determine under what conditions the extended matrix $\mathcal{M}(2)$ will be a flat extension of $\mathcal{M}(1)$. First, to ensure the positive semidefiniteness of $\mathcal{M}(2)$, and if we let $W := B(2)$ (the upper right-hand 3×3 block of $\mathcal{M}(2)$), we see that $C(2)$ (the lower right-hand 3×3 block of $\mathcal{M}(2)$) must satisfy the inequality $C(2) \geq W^T \mathcal{M}(1) W$, with equality characterizing flatness. Now,

$$
W^T \big(\mathcal{M}(1) \big)^{-1} W = W^T W
$$
$$
= \begin{pmatrix}
1 + a_0^2 + a_1^2 & a_0 a_1 + a_1 a_2 & 1 + a_0 a_2 + a_1 a_3 \\
a_0 a_1 + a_1 a_2 & a_1^2 + a_2^2 & a_1 a_2 + a_2 a_3 \\
1 + a_0 a_2 + a_1 a_3 & a_1 a_2 + a_2 a_3 & 1 + a_2^2 + a_3^2
\end{pmatrix}. \tag{3.2}
$$

Consequently, $\mathcal{M}(2)$ is a flat extension of $\mathcal{M}(1)$ if and only if

$$
\beta_{40} = 1 + a_0^2 + a_1^2, \tag{3.3}
$$
$$
\beta_{31} = a_0 a_1 + a_1 a_2, \tag{3.4}
$$
$$
\beta_{13} = a_1 a_2 + a_2 a_3, \tag{3.5}
$$
$$
\beta_{04} = 1 + a_2^2 + a_3^2, \quad \text{and} \tag{3.6}
$$
$$
k := (1 + a_0 a_2 + a_1 a_3) - (a_1^2 + a_2^2) = 0. \tag{3.7}
$$

Condition (3.7) is equivalent to the commutativity of the matrices defined in [10]. We are now ready to prove our first result.

Theorem 3.1. *Let $\beta^{(3)}$ be a nonsingular cubic binary moment sequence, let k be as in (3.7), and assume that $k = 0$. Then $\beta^{(3)}$ admits a 3-atomic representing measure.*

Proof. From the discussion preceding the statement of Theorem 3.1, the new quartic moments β_{40}, β_{31}, β_{13}, and β_{04} must be defined using (3.3)–(3.7), respectively. As for β_{22}, we must use $1 + a_0 a_2 + a_1 a_3$, which in this case equals $a_1^2 + a_2^2$, because $k = 0$. With these definitions, we easily conclude that $\mathcal{M}(2)$ is a flat moment matrix extension of $\mathcal{M}(1)$, which gives the desired result.   □

Remark 3.2. Observe that the proof of Theorem 3.1 shows that the three column relations in $\mathcal{M}(2)$ are

$$X^2 = 1 + a_0 X + a_1 Y,$$
$$XY = a_1 X + a_2 Y,$$
$$Y^2 = 1 + a_2 X + a_3 Y.$$

From this it follows that the support of the unique representing measure is the 3-point intersection of a vertical parabola, a nondegenerate hyperbola with a horizontal asymptote and a vertical asymptote, and a horizontal parabola; all three conics are completely determined by the initial data.

When $k \neq 0$, it is not possible to select new quartic moments so that $\mathcal{M}(2)$ is a flat extension of $\mathcal{M}(1)$. Therefore, any positive semidefinite moment matrix extension $\mathcal{M}(2)$ will satisfy $\operatorname{rank} \mathcal{M}(2) \geq 4$. Nevertheless, the following theorem shows that it is always possible to choose a set of quartic moments such that $\operatorname{rank} \mathcal{M}(2) = 4$. Once those moments have been appropriately chosen, the extended moment matrix $\mathcal{M}(2)$ will admit a flat extension $\mathcal{M}(3)$, and therefore a 4-atomic representing measure for $\beta^{(4)}$, which is also a representing measure for the initial data sequence $\beta^{(3)}$.

Theorem 3.3. *Let $\beta^{(3)}$ be a nonsingular cubic binary moment sequence, let k be as in (3.7), and assume that $k \neq 0$. Then $\beta^{(3)}$ admits a 4-atomic representing measure.*

Proof. We will divide the proof into two cases: $k > 0$ and $k < 0$.

Case 1: ($k > 0$). As in the proof of Theorem 3.1, let β_{40}, β_{31}, β_{13}, and β_{04} be given by (3.3)–(3.7), respectively. Since $k > 0$, the positivity of $\mathcal{M}(2)$ will be preserved if we let $\beta_{22} := 1 + a_1 a_3 + a_2 a_4$. With this choice of β_{22}, the proposed extended matrix $\mathcal{M}(2)$ will be a positive semidefinite moment matrix, and such that the block $C(2)$ differs from $W^T (\mathcal{M}(1))^{-1} W$ in just the $(2, 2)$-entry. As a result, $\operatorname{rank} \mathcal{M}(2) = 4$. A simple calculation now reveals that

$$X^2 = 1 + a_0 X + a_1 Y,$$
$$Y^2 = 1 + a_2 X + a_3 Y.$$

We now know that $\mathcal{M}(2)$ is positive semidefinite and recursively determinate, and by Lemma 2.4, $\mathcal{M}(2)$ admits a 4-atomic representing measure; it follows that $\beta^{(3)}$ also admits a 4-atomic representing measure.

Case 2: ($k < 0$). Here our strategy is to allow the rank to increase as we transition from $\mathcal{M}(1)$ to the compression of $\mathcal{M}(2)$ to the first four rows and columns. This requires making the column X^2 *linearly independent* of the columns 1, X, and Y in $\mathcal{M}(2)$. It is straightforward to observe that this can be easily accomplished by letting

$$\beta_{40} := 2 + a_1^2 + a_2^2.$$

With this definition in hand, we now postulate that $\mathcal{M}(2)$ is a flat extension of its compression to the first four rows and columns. A calculation using *Mathematica* reveals that one can accomplish this by defining three of the remaining quartic moments as follows:

$$\beta_{31} := a_1 a_2 + a_2 a_3,$$
$$\beta_{22} := a_2^2 + a_3^2,$$
$$\beta_{13} := a_2 a_3 + a_3 a_4.$$

Having chosen these moments, we now use Theorem 2.1 to determine the remaining quartic moment, β_{04}. Since we wish to make $\mathcal{M}(2)$ a flat extension of its above-mentioned compression, a calculation using *Mathematica* immediately yields

$$\begin{aligned}
\beta_{04} = {} & 2 + a_1^4 + 2a_0 a_2 + a_0^2 a_2^2 + 2a_1^2 a_2^2 + a_2^4 + 2a_1 a_3 \\
& + 2a_0 a_1 a_2 a_3 + a_3^2 + a_1^2 a_3^2 \\
& - 2a_1^2 - 2a_0 a_1^2 a_2 - a_2^2 - 2a_0 a_2^3 - 2a_1^3 a_3 - 2a_1 a_2^2 a_3.
\end{aligned} \tag{3.8}$$

As a result, in $\mathcal{M}(2)$ we now have

$$XY = a_1 X + a_2 Y \tag{3.9}$$

and

$$Y^2 = p_1 1 + p_2 X + p_3 Y + p_4 X^2 \tag{3.10}$$

for suitable real scalars p_1, p_2, p_3, p_4 (which depend upon a_0, a_1, a_2, a_3); moreover, $p_4 = -k$. We will now build a flat moment matrix extension $\mathcal{M}(3)$ of $\mathcal{M}(2)$. This will prove that $\mathcal{M}(2)$ admits a 4-atomic representing measure; a fortiori, $\beta^{(3)}$ also admits a 4-atomic representing measure, just as in Case 1 above.

To define $\mathcal{M}(3)$, we aim to preserve the (RG) property. First, we observe that the columns 1, X, and Y in $\mathcal{M}(3)$ are obtained from those columns in $\mathcal{M}(2)$ by adding suitable cubic and quartic moments. Moreover, the columns XY and Y^2 are defined using (3.9) and (3.10), while the columns $X^2 Y$, XY^2, and Y^3 are obtained, via the functional calculus, from (3.9) and (3.10). For instance,

$$X^2 Y := a_1 X^2 + a_2 XY = a_1 X^2 + a_2(a_1 X + a_2 Y) = a_2 a_1 X + a_2^2 Y + a_1 X^2$$

and

$$\begin{aligned}
Y^3 := {} & p_1 Y + p_2 XY + p_3 Y^2 + p_4 X^2 Y \\
= {} & p_1 Y + p_2(a_1 X + a_2 Y) + p_3(p_1 1 + p_2 X + p_3 Y + p_4 X^2) \\
& + p_4(a_2 a_1 X + a_2^2 Y + a_1 X^2).
\end{aligned}$$

It follows that both $X^2 Y$ and Y^3 are linear combinations of the columns 1, X, Y, and X^2. Now, to define XY^2 one can use either (3.9) or (3.10). However, the (RG)

property requires that both definitions of XY^2 be compatible. In other words, the expressions

$$XY^2 \equiv a_1 XY + a_2 Y^2 = a_2 p_1 1 + (a_1^2 + a_2 p_2)X + (a_1 a_2 + a_2 p_3)Y + a_2 p_4 X^2$$
$$\big(\text{obtained using } (3.9)\big)$$

and

$$XY^2 \equiv p_1 X + p_2 X^2 + p_3 XY + p_4 X^3 = (p_1 + a_1 p_3)X + a_2 p_3 Y + p_2 X^2 + p_4 X^3$$
$$\big(\text{obtained using } (3.10)\big)$$

must be identical. Since $p_4 = -k \neq 0$, we immediately get

$$X^3 = \frac{1}{p_4}\big[a_2 p_1 1 + (a_1^2 + a_2 p_2 - p_1 - a_1 p_3)X + a_1 a_2 Y + (a_2 p_4 - p_2)X^2\big], \quad (3.11)$$

which we can then use to define the column X^3. Close examination of (3.11) at the level of the fourth row in $\mathcal{M}(3)$ leads to a formula for the quintic moment β_{50}. This value must then be inserted in the seventh row of X^2 to complete the definition of X^2 in $\mathcal{M}(3)$. As a result, in the new moment matrix $\mathcal{M}(3)$ we have exhibited each cubic column as a linear combination of columns associated with monomials of degree at most 2. This means that $\mathcal{M}(3)$ is a flat extension of $\mathcal{M}(2)$, as desired. The proof is now complete. $\qquad\square$

Remark 3.4. Observe that the proof of Theorem 3.3 shows that the two column relations in $\mathcal{M}(2)$ are as follows. In Case 1, we have

$$X^2 = 1 + a_0 X + a_1 Y,$$
$$Y^2 = 1 + a_2 X + a_3 Y.$$

From this we conclude that the support of the minimal representing measure is the 4-point intersection of a vertical parabola and a horizontal parabola. In Case 2, we have

$$XY = a_1 X + a_2 Y,$$
$$Y^2 = p_1 1 + p_2 X + p_3 Y + p_4 X^2.$$

Keeping in mind that $p_4 = -k > 0$, it follows that the support of the minimal representing measure is the 4-point intersection of two nondegenerate hyperbolas, one with horizontal and vertical asymptotes, and the other with oblique asymptotes.

As in Remark 3.2, both conics in Case 1 and both conics in Case 2 are completely determined by the initial data.

Remark 3.5. The quartic moment β_{04} defined by (3.8) is nonnegative, being a diagonal entry of the positive semidefinite matrix $\mathcal{M}(2)$. One can say more, however, by appealing to the theory of semidefinite programming. As is well known, a polynomial $f \in \mathcal{P}_{2d}$ is a sum of squares if and only if $f = \mathbf{z}^T Q \mathbf{z}$ for some square matrix $Q \geq 0$, where $\mathbf{z}$ is the vector of monomials of degree less than or equal to d. If we let $f \equiv f(a_0, a_1, a_2, a_3) := \beta_{04} - 1$ and $\mathbf{y} := (1, a_2, a_3, a_1^2, a_2^2, a_0 a_2, a_1 a_3)$,

a calculation using *Mathematica* reveals that $f \geq 0$ if and only if $\mathbf{y}^T R \mathbf{y} \geq 0$, where

$$
R := \begin{pmatrix}
1 & 0 & 0 & -1 & -1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 1 & -1 & -1 \\
-1 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & 0 & 0 & -1 & -1 & 1 & 1 \\
1 & 0 & 0 & -1 & -1 & 1 & 1
\end{pmatrix}. \tag{3.12}
$$

Since R is a flat extension of its 3×3 compression to the first three rows and columns, it is clear that $R \geq 0$. It follows that $\beta_{04} \equiv f + 1 \geq 1 > 0$.

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