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OPERATOR APPROXIMATE BIPROJECTIVITY OF LOCALLY COMPACT QUANTUM GROUPS

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ABSTRACT. We initiate a study of operator approximate biprojectivity for quantum group algebra $L^1(\mathbb{G})$, where $\mathbb{G}$ is a locally compact quantum group. We show that if $L^1(\mathbb{G})$ is operator approximately biprojective, then $\mathbb{G}$ is compact. We prove that if $\mathbb{G}$ is a compact quantum group and $\mathbb{H}$ is a non-Kac-type compact quantum group such that both $L^1(\mathbb{G})$ and $L^1(\mathbb{H})$ are operator approximately biprojective, then $L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{H})$ is operator approximately biprojective, but not operator biprojective.

1. Introduction and preliminaries

The concept of biprojectivity in the theory of homological Banach algebras was first introduced by Helemskiĭ [6] and later developed systematically in [7]. For example, the group algebra $L^1(G)$ of a locally compact group G is biprojective if and only if G is compact. But when we work with a completely contractive Banach algebra—in particular, the Fourier algebra $A(G)$—then evidence suggests that it is best to work with operator homology.

The class of locally compact quantum groups was first introduced and studied by Kustermans and Vaes [8], [9]. Recall that a (*von Neumann algebraic*) *locally compact quantum group* is a quadruple $\mathbb{G} = (L^\infty(\mathbb{G}), \Delta, \phi, \psi)$, where $L^\infty(\mathbb{G})$ is a von Neumann algebra with identity element 1 and a comultiplication

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$\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\mathbb{G})$. Moreover, ϕ and ψ are normal faithful semifinite left and right Haar weights on $L^\infty(\mathbb{G})$, respectively. Here $\bar{\otimes}$ denotes the von Neumann algebra tensor product.

The Gelfand–Naimark–Segal construction applied to the left Haar weight ϕ of every locally compact quantum group $\mathbb{G}$ gives a Hilbert space $L^2(\mathbb{G})$. There exists a left fundamental unitary operator W on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$ implementing the comultiplication Δ via

$$\Delta(x) = W^*(1 \otimes x)W \quad \big(x \in L^\infty(\mathbb{G})\big).$$

(For more details see [8], [9].)

The predual of $L^\infty(\mathbb{G})$ is denoted by $L^1(\mathbb{G})$. Then the preadjoint of the comultiplication Δ induces on $L^1(\mathbb{G})$ an associative completely contractive multiplication $\Delta_* : L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{G}) \to L^1(\mathbb{G})$, where $\widehat{\otimes}$ is the projective tensor product of operator spaces. Therefore, $L^1(\mathbb{G})$ is a Banach algebra under the product $*$ given by $f * g := \Delta_*(f \otimes g) \in L^1(\mathbb{G})$ for all $f, g \in L^1(\mathbb{G})$. Moreover, the module actions of $L^1(\mathbb{G})$ on $L^\infty(\mathbb{G})$ are given by

$$f \cdot x := (\mathrm{id} \otimes f)\big(\Delta(x)\big) \qquad \text{and} \qquad x \cdot f := (f \otimes \mathrm{id})\big(\Delta(x)\big)$$

for all $f \in L^1(\mathbb{G})$ and $x \in L^\infty(\mathbb{G})$.

The *reduced quantum group C^*-algebra* of $L^\infty(\mathbb{G})$ is defined as

$$C_0(\mathbb{G}) := \overline{\big\{(\mathrm{id} \otimes \omega)(W); \omega \in B\big(L^2(\mathbb{G})\big)_*\big\}}^{\|\cdot\|}.$$

We say that $\mathbb{G}$ is *compact* if $C_0(\mathbb{G})$ is a unital C^*-algebra. For compact quantum groups, it follows that ϕ is finite and that $\phi = \psi$. Moreover, there is a unique *Haar state* on $L^\infty(\mathbb{G})$, that we denote by $h_{\mathbb{G}}$, such that

$$h_{\mathbb{G}}(x \cdot f) = h_{\mathbb{G}}(f \cdot x) = h_{\mathbb{G}}(x)f(1) \quad \big(x \in L^\infty(\mathbb{G}), f \in L^1(\mathbb{G})\big).$$

In this case, we denote the compact quantum group $\mathbb{G}$ by $(L^\infty(\mathbb{G}), \Delta^{\mathbb{G}})$.

Recall that a locally compact quantum group $\mathbb{G}$ is called *amenable* if there exists a functional $m \in L^\infty(\mathbb{G})^*$ such that $\|m\| = m(1) = 1$ and $m(x \cdot f) = m(x)f(1)$ for all $x \in L^\infty(\mathbb{G})$ and $f \in L^1(\mathbb{G})$. In this case, m is called a *left-invariant mean* on $L^\infty(\mathbb{G})$. Moreover, $\mathbb{G}$ is called *coamenable* if $L^1(\mathbb{G})$ has a bounded approximate identity.

The noncommutative operator biprojectivity of quantum group algebra $L^1(\mathbb{G})$ of a locally compact quantum group $\mathbb{G}$ has been studied by several authors (see [2], [4], [5]). It was shown by Aristov [2, Theorem 4.7] that if $L^1(\mathbb{G})$ is operator biprojective, then $\mathbb{G}$ must be compact. Conversely, if $\mathbb{G}$ is a compact Kac algebra, then $\mathbb{G}$ is operator biprojective (see [2, Theorem 4.12]). This leads him to the following question.

Is $L^1(\mathbb{G})$ operator biprojective for any compact quantum group $\mathbb{G}$?

Recently, in [4], Caspers, Lee, and Ricard gave a complete answer to the above open problem. Indeed, they characterized operator biprojectivity of $L^1(\mathbb{G})$ as follows. Let $\mathbb{G}$ be a locally compact quantum group. Then $L^1(\mathbb{G})$ is operator biprojective if and only if $\mathbb{G}$ is compact and of Kac type.

Furthermore, operator approximate homological notions like operator approximate biprojectivity and operator approximate biflatness of completely contractive Banach algebras were introduced and studied by several authors (see, e.g., [12], [11], and also [1]). Therefore, it is a natural question whether operator approximate biprojectivity of $L^1(\mathbb{G})$ is equivalent to $\mathbb{G}$ being compact and of Kac type?

In this paper, we prove that when $L^1(\mathbb{G})$ is *operator approximately biprojective*, then $\mathbb{G}$ is compact. Furthermore, we show that if $\mathbb{G}$ is compact and $L^1(\mathbb{G})$ is operator approximately biprojective with special type of approximate splitting morphism, then $\mathbb{G}$ is of Kac type.

Moreover, it is shown that, for a coamenable compact quantum group $\mathbb{G}$ and a compact quantum group $\mathbb{H}$, if $L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{H})$ is operator approximately biprojective, then $L^1(\mathbb{G})$ is also operator approximately biprojective. Finally, we intend to prove that if $\mathbb{G}$ is a compact quantum group and $\mathbb{H}$ is a non-Kac-type compact quantum group such that $L^1(\mathbb{G})$ and $L^1(\mathbb{H})$ are operator approximately biprojective, then $L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{H})$ is also operator approximately biprojective, but clearly not operator biprojective.

2. Operator approximate biprojectivity

Recall from [12] that a completely contractive Banach algebra $\mathcal{A}$ is called *operator approximately biprojective* if there is a net (ρ_γ) of completely bounded $\mathcal{A}$-bimodule morphisms from $\mathcal{A}$ into $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\pi\circ\rho_\gamma(a)\to a$ for all $a\in\mathcal{A}$, where $\pi:\mathcal{A}\widehat{\otimes}\mathcal{A}\to\mathcal{A}$ is the product morphism. Also, from [11], $\mathcal{A}$ is called *operator approximately biflat* if there is a net $\theta_\gamma:(\mathcal{A}\widehat{\otimes}\mathcal{A})^*\to\mathcal{A}^*$ of completely bounded $\mathcal{A}$-bimodule morphisms such that $\mathrm{W^*OT}\text{-}\lim_\gamma\theta_\gamma\circ\pi^*=\mathrm{id}_{\mathcal{A}^*}$.

It is known that $L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{G})$ can be regarded as an operator $L^1(\mathbb{G})$-bimodule via the following module structure:

$$f\cdot(h\otimes k)\cdot g:=f*h\otimes k*g\quad\big(f,g,h,k\in L^1(\mathbb{G})\big).$$

Therefore, $L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\mathbb{G})$ becomes an operator $L^1(\mathbb{G})$-bimodule via the following structure:

$$f\cdot(x\otimes y)\cdot g:=x\cdot g\otimes f\cdot y\quad\big(f,g\in L^1(\mathbb{G}),x,y\in L^\infty(\mathbb{G})\big).$$

Now we have the following definition.

Definition 2.1. Let $\mathbb{G}$ be a locally compact quantum group. Then

(i) $L^1(\mathbb{G})$ is *operator approximately biprojective* if there is a net $\rho_\gamma:L^1(\mathbb{G})\to L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{G})$ of completely bounded $L^1(\mathbb{G})$-bimodule morphisms such that $\lim_\gamma\Delta_*\circ\rho_\gamma(f)=f$ for all $f\in L^1(\mathbb{G})$ (we call $(\rho_\gamma)_\gamma$ an *approximate splitting morphism*);

(ii) $L^1(\mathbb{G})$ is *operator approximately biflat* if there is a net $\theta_\gamma:L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\mathbb{G})\to L^\infty(\mathbb{G})$ of completely bounded $L^1(\mathbb{G})$-bimodule morphisms such that $\mathrm{W^*OT}\text{-}\lim_\gamma\theta_\gamma\circ\Delta=\mathrm{id}_{L^\infty(\mathbb{G})}$.

The following lemma is a restatement from Bédos and Tuset. We omit the proof because it can easily be adapted from [3, Proof of Proposition 3.1].

Lemma 2.2 ([3, Proposition 3.1]). *Let $\mathbb{G}$ be a locally compact quantum group such that there exists a left- (or right)-invariant functional $f \in L^1(\mathbb{G})$ such that $f|_{C_0(\mathbb{G})} \neq 0$. Then $\mathbb{G}$ is compact.*

Theorem 2.3. *Let $\mathbb{G}$ be a locally compact quantum group. If $L^1(\mathbb{G})$ is operator approximately biprojective, then $\mathbb{G}$ is compact.*

Proof. Let $L^1(\mathbb{G})$ be operator approximately biprojective with splitting morphism $(\rho_\gamma)_\gamma$. We put,

$$T = \mathrm{id}_{L^1(\mathbb{G})} \otimes 1 : L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{G}) \to L^1(\mathbb{G}) : f \otimes g \mapsto g(1)f.$$

Then it is easily verified that

$$T(f \cdot \omega) = f * T(\omega), \qquad T(\omega \cdot f) = f(1)T(\omega)$$

and $T(\omega)(1) = \Delta_*(\omega)(1)$ for all $f \in L^1(\mathbb{G})$ and $\omega \in L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{G})$. Define the net

$$\xi_\gamma : L^1(\mathbb{G}) \to L^1(\mathbb{G})$$

of completely bounded left $L^1(\mathbb{G})$-module morphisms by $\xi_\gamma := T \circ \rho_\gamma$. Moreover, for each γ, we have

$$\xi_\gamma(f * g) = g(1)\xi_\gamma(f)$$

for all $f, g \in L^1(\mathbb{G})$. This implies that, for each $f \in L^1(\mathbb{G})$ and $g \in I_0(\mathbb{G})$, we have

$$\xi_\gamma(f * g) = \xi_\gamma(f)g(1) = 0,$$

where $I_0(\mathbb{G}) := \{g \in L^1(\mathbb{G}) : g(1) = 0\}$. Since the linear span of $L^1(\mathbb{G}) * I_0(\mathbb{G})$ is dense in $I_0(\mathbb{G})$ (see [2, Theorem 4.4]), we conclude that $\xi_\gamma(g) = 0$ for all $g \in I_0(\mathbb{G})$. Fix $f_0 \in L^1(\mathbb{G})$ with $f_0(1) = 1$. Since $L^1(\mathbb{G})$ is operator approximately biprojective, $\lim_\gamma \Delta_*(\rho_\gamma(f_0)) = f_0$. Therefore, we can find γ_0 such that $\Delta_*(\rho_{\gamma_0}(f_0))(1) \neq 0$. Hence, if we set $g_0 = \xi_{\gamma_0}(f_0) \in L^1(\mathbb{G})$, then we have

$$\begin{aligned} g_0(1) &= \xi_{\gamma_0}(f_0)(1) = T\big(\rho_{\gamma_0}(f_0)\big)(1) \\ &= \Delta_*\big(\rho_{\gamma_0}(f_0)\big)(1) \neq 0. \end{aligned}$$

This shows that $g_0|_{C_0(\mathbb{G})} \neq 0$. Moreover, for each $f \in L^1(\mathbb{G})$, we have $f * f_0 - f_0 * f \in I_0(\mathbb{G})$, which implies that

$$\begin{aligned} f * g_0 &= f * \xi_{\gamma_0}(f_0) = \xi_{\gamma_0}(f * f_0) \\ &= \xi_{\gamma_0}(f_0 * f) = \xi_{\gamma_0}(f_0)f(1) \\ &= f(1)g_0. \end{aligned}$$

Therefore, g_0 is a left-invariant functional in $L^1(\mathbb{G})$ such that $g_0|_{C_0(\mathbb{G})} \neq 0$, whence $\mathbb{G}$ is compact by Lemma 2.2. □

The proof of the following result is a modification of the argument used in the proof of Theorem 2.3.

Theorem 2.4. *Let $\mathbb{G}$ be a locally compact quantum group. If $L^1(\mathbb{G})$ is operator approximately biflat, then $\mathbb{G}$ is amenable.*

Proof. Let $L^1(\mathbb{G})$ be operator approximately biflat. In that case, then, there is a net $\theta_\gamma : L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ of completely bounded $L^1(\mathbb{G})$-bimodule morphisms such that W*-$\lim_\gamma \theta_\gamma(\Delta(1)) = 1$, where 1 is the identity element of $L^\infty(\mathbb{G})$. Fix $f_0 \in L^1(\mathbb{G})$ with $f_0(1) = 1$. Then we may find γ_0 such that

$$\theta_{\gamma_0}\big(\Delta(1)\big)(f_0) \neq 0.$$

Now, let us consider the completely bounded, left $L^1(\mathbb{G})$-module morphism here, $\Gamma : L^\infty(\mathbb{G})^* \to L^\infty(\mathbb{G})^*$, defined by $\Gamma := T^{**} \circ \theta^*_{\gamma_0}$, where T is defined as in the proof of Theorem 2.3. Moreover, for each $g \in L^1(\mathbb{G})$, each $x \in L^\infty(\mathbb{G})$, and each $\omega \in L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{G})$, we have

$$\big\langle g \cdot T^*(x), \omega\big\rangle = \big\langle T^*(x), \omega \cdot g\big\rangle = \big\langle x, T(\omega \cdot g)\big\rangle = g(1)\big\langle T^*(x), \omega\big\rangle.$$

This shows that $g \cdot T^*(x) = g(1)T^*(x)$. Therefore, for each $f, g \in L^1(\mathbb{G})$ and $x \in L^\infty(\mathbb{G})$, we obtain

$$\begin{aligned}
\big\langle \Gamma(f * g), x\big\rangle &= \big\langle \theta^*_{\gamma_0}(f * g), T^*(x)\big\rangle \\
&= \big\langle \theta^*_{\gamma_0}(f) \cdot g, T^*(x)\big\rangle \\
&= \big\langle \theta^*_{\gamma_0}(f), g \cdot T^*(x)\big\rangle \\
&= g(1)\big\langle \theta^*_{\gamma_0}(f), T^*(x)\big\rangle \\
&= g(1)\big\langle \Gamma(f), x\big\rangle,
\end{aligned}$$

where, in the second equality, we used the fact that θ_{γ_0}, and so it holds that $\theta^*_{\gamma_0}$ is a $L^1(\mathbb{G})$-bimodule morphism. Hence,

$$\Gamma(f * g) = g(1)\Gamma(f) \quad \text{for all} f, g \in L^1(\mathbb{G}).$$

As the linear span of $L^1(\mathbb{G}) * I_0(\mathbb{G})$ is dense in $I_0(\mathbb{G})$, we obtain that $\Gamma|_{I_0(\mathbb{G})} = 0$. Define the functional $m \in L^\infty(\mathbb{G})^*$ by $m := \Gamma(f_0)$. Since $\Delta(1) = T^*(1)$, it follows that

$$m(1) = \theta_{\gamma_0}\big(\Delta(1)\big)(f_0) \neq 0.$$

Moreover, for each $f \in L^1(\mathbb{G})$ and $x \in L^\infty(\mathbb{G})$,we have

$$m(x \cdot f) - f(1)m(x) = \Gamma\big(f * f_0 - f(1)f_0\big)(x) = 0.$$

This shows that m is a left-invariant functional on $L^\infty(\mathbb{G})$ such that $m(1) \neq 0$. Therefore, the last part of the proof of [10, Theorem 2.1] shows that there exists a left-invariant mean on $L^\infty(\mathbb{G})$, which implies that $\mathbb{G}$ is amenable. □

A *finite-dimensional corepresentation* of a compact quantum group $(L^\infty(\mathbb{G}), \Delta^{\mathbb{G}})$ is a matrix $v = (v_{ij}) \in M_n(L^\infty(\mathbb{G}))$, where n is called the *dimension* of v such that

$$\Delta^{\mathbb{G}}(v_{ij}) = \sum_{k=1}^{n} v_{ik} \otimes v_{kj} \quad (1 \leq i, j \leq n).$$

Moreover, v is called *unitary* if v is a unitary matrix, and it is called *irreducible* if we have $\{X \in M_n(\mathbb{C}) : Xv = vX\} = \mathbb{C}I_n$, where it holds that n is the dimension of v. Recall that for any compact quantum group $\mathbb{G}$ there exists a maximal family

$\{v^\alpha : \alpha \in \mathbb{I}\}$ of finite-dimensional irreducible unitary corepresentation of $\mathbb{G}$. For each $\alpha \in \mathbb{I}$, we denote by n_α the dimension of v^α.

By [5, Proposition 2.1], for each $\alpha \in \mathbb{I}$ there exists a unique positive invertible matrix $F^\alpha \in M_n(\mathbb{C})$ with $m_\alpha := \operatorname{Tr} F^\alpha = \operatorname{Tr}(F^\alpha)^{-1}$ such that for each α, we can assume that $F^\alpha = \operatorname{diag}(\lambda_1^\alpha, \dots, \lambda_{n_\alpha}^\alpha)$. Moreover, for each $\alpha, \beta \in \mathbb{I}$, $1 \le i, j \le n_\alpha$, and $1 \le k, l \le n_\beta$, we have

$$h_{\mathbb{G}}\big((v_{kl}^\beta)^* v_{ij}^\alpha\big) = \delta_{\alpha\beta}\delta_{jl}\delta_{ki}\frac{1}{m_\alpha \lambda_i^\alpha}, \qquad h_{\mathbb{G}}\big(v_{kl}^\beta (v_{ij}^\alpha)^*\big) = \delta_{\alpha\beta}\delta_{ik}\delta_{jl}\frac{\lambda_j^\alpha}{m_\alpha}.$$

If $F^\alpha = I_{n_\alpha}$, where I_{n_α} is the identity matrix, for all $\alpha \in \mathbb{I}$, then $\mathbb{G}$ is of Kac type or is a Kac algebra.

Theorem 2.5. *Let $\mathbb{G}$ be a compact quantum group. Consider the following statements.*

1. *$L^1(\mathbb{G})$ is operator approximately biprojective with approximate splitting morphism (ρ_γ).*
2. *There exists a normal completely bounded net $\theta_\gamma : L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ such that*

$$W^*OT\text{-}\lim_\gamma \theta_\gamma\Delta = \mathrm{id}_{L^\infty(\mathbb{G})}, \qquad \Delta\theta_\gamma = (\theta_\gamma \otimes \mathrm{id})(\mathrm{id} \otimes \Delta) = (\mathrm{id} \otimes \theta_\gamma)(\Delta \otimes \mathrm{id}).$$

3. *There exists a normal completely bounded net $\theta_\gamma : L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ such that for each θ_γ there exists a family $\{X^{\alpha,\gamma} \in M_{n_\alpha} : \alpha \in \mathbb{I}\}$ such that, for $\alpha, \beta \in \mathbb{I}$, $1 \le i, j \le n_\alpha$, and $1 \le k, l \le n_\beta$,*

$$\theta_\gamma(v_{ij}^\alpha \otimes v_{kl}^\beta) = \delta_{\alpha\beta} X_{jk}^{\alpha,\gamma} v_{il}^\alpha, \qquad \lim_\gamma \sum_{r=1}^{n_\alpha} X_{rr}^{\alpha,\gamma} = 1.$$

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1)$\Rightarrow$(2). Take $\theta_\gamma := \rho_\gamma^*$. Then, for each $x, y \in L^\infty(\mathbb{G})$ and $f, g \in L^1(\mathbb{G})$, we have

$$\begin{aligned}\big\langle \Delta\theta_\gamma(x \otimes y), f \otimes g\big\rangle &= \big\langle x \otimes y, \rho_\gamma\Delta_*(f \otimes g)\big\rangle \\ &= \big\langle x \otimes y, f \cdot \rho_\gamma(g)\big\rangle \\ &= \big\langle x \otimes y, (\Delta \otimes \mathrm{id})_*\big(f \otimes \rho_\gamma(g)\big)\big\rangle \\ &= \big\langle (\mathrm{id} \otimes \theta_\gamma)(\Delta \otimes \mathrm{id})(x \otimes y), f \otimes g\big\rangle.\end{aligned}$$

Thus, $\Delta\theta_\gamma = (\mathrm{id} \otimes \theta_\gamma)(\Delta \otimes \mathrm{id})$. Similarly, $\Delta\theta_\gamma = (\theta_\gamma \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)$. Moreover, it is easy to see that W*OT-$\lim_\gamma \theta_\gamma\Delta = \mathrm{id}_{L^\infty(\mathbb{G})}$.

(2)$\Rightarrow$(3). By an argument similar to [5, Proof of Proposition 3.2], for $1 \le i, j \le n_\alpha$, and $1 \le k, l \le n_\beta$, we have

$$\theta_\gamma(v_{ij}^\alpha \otimes v_{kl}^\beta) \in \operatorname{span}\{v_{i1}^\alpha, \dots, v_{in_\alpha}^\alpha\} \cap \operatorname{span}\{v_{1l}^\beta, \dots, v_{n_\beta l}^\beta\}.$$

By linear independence, we see that $\theta_\gamma(v_{ij}^\alpha \otimes v_{kl}^\beta) = 0$ if $\alpha \ne \beta$. If $\alpha = \beta$, then we see immediately that there is a scalar $X_{jk}^{\alpha,\gamma}$ such that

$$\theta_\gamma(v_{ij}^\alpha \otimes v_{kl}^\alpha) = X_{jk}^{\alpha,\gamma} v_{il}^\alpha.$$

Now, by assumption, we have

$$\begin{aligned} v_{ij}^{\alpha} &= W^* - \lim_{\gamma} \theta_{\gamma} \Delta(v_{ij}^{\alpha}) \\ &= W^* - \lim_{\gamma} \sum_{k=1}^{n_\alpha} \theta_{\gamma}(v_{ik}^{\alpha} \otimes v_{kj}^{\alpha}) \\ &= W^* - \lim_{\gamma} \sum_{k=1}^{n_\alpha} X_{kk}^{\alpha,\gamma} v_{ij}^{\alpha}. \end{aligned}$$

Therefore, $\lim_{\gamma} \sum_{k=1}^{n_\alpha} X_{kk}^{\alpha,\gamma} = 1$. □

We now end this section with a result which shows that, for a compact quantum group $\mathbb{G}$, operator approximate biprojectivity of $L^1(\mathbb{G})$ with special type of approximate splitting morphism implies that $\mathbb{G}$ is of Kac type. The proof of the following proposition is similar to that of [5, Theorem 3.3]; we provide the details for the convenience of the reader.

Proposition 2.6. *Let $\mathbb{G}$ be a compact quantum group such that $L^1(\mathbb{G})$ is operator approximately biprojective with approximate splitting morphism $\rho_\gamma : L^1(\mathbb{G}) \to L^1(\mathbb{G}) \hat{\otimes} L^1(\mathbb{G})$, and suppose further that every $\theta_\gamma = \rho_\gamma^*$ is an $L^\infty(\mathbb{G})$-bimodule map, in the sense that $\theta_\gamma(\Delta(a)x\Delta(b)) = a\theta_\gamma(x)b$ for all $x \in L^\infty(\mathbb{G}) \bar{\otimes} L^\infty(\mathbb{G})$ and $a, b \in L^\infty(\mathbb{G})$. Then the Haar state $h_{\mathbb{G}}$ is tracial, so $\mathbb{G}$ is a Kac algebra.*

Proof. Let $\alpha \in \mathbb{I}$ and let $\alpha_0 \in \mathbb{I}$ be such that $v^{\alpha_0} = 1$, the trivial corepresentation. The linear span $\mathcal{A} := \operatorname{span}\{v_{ij}^{\alpha} : \alpha \in \mathbb{I}, 1 \le i, j \le n_\alpha\}$ forms a unital Hopf $*$-algebra which is dense in $C_0(\mathbb{G})$ and $\{v_{ij}^{\alpha} : \alpha \in \mathbb{I}, 1 \le i, j \le n_\alpha\}$ forms a basis for $\mathcal{A}$. As $h_{\mathbb{G}}(v_{ij}^{\alpha}) = \delta_{\alpha\alpha_0}$ and $h_{\mathbb{G}}(v_{ij}^{\alpha}(v_{il}^{\alpha})^*) = \delta_{jl}\frac{\lambda_j^\alpha}{m_\alpha}$, for $1 \le i, j, k \le n_\alpha$, we obtain that

$$v_{ij}^{\alpha}(v_{il}^{\alpha})^* \otimes v_{kj}^{\alpha}(v_{lj}^{\alpha})^* = \delta_{jl}\frac{\lambda_j^\alpha}{m_\alpha} 1 \otimes \delta_{kl}\frac{\lambda_j^\alpha}{m_\alpha} 1 + R,$$

where $R \in \ker h_{\mathbb{G}} \otimes \ker h_{\mathbb{G}}$. Moreover, by Theorem 2.5, $\theta_\gamma(v_{ij}^{\alpha} \otimes v_{kl}^{\beta}) = \delta_{\alpha\beta} X_{jk}^{\alpha,\gamma} v_{il}^{\alpha}$ for all γ. It follows that $h_{\mathbb{G}} \circ \theta_\gamma = h_{\mathbb{G}} \otimes h_{\mathbb{G}}$. Therefore, we see that

$$\begin{aligned} X_{jk}^{\alpha,\gamma} \frac{\lambda_j^\alpha}{m_\alpha} &= h_{\mathbb{G}}\big(X_{jk}^{\alpha,\gamma} v_{ij}^{\alpha}(v_{ij}^{\alpha})^*\big) \\ &= h_{\mathbb{G}}\big(\theta_\gamma(v_{ij}^{\alpha} \otimes v_{kj}^{\alpha})(v_{ij}^{\alpha})^*\big) \\ &= h_{\mathbb{G}} \circ \theta_\gamma\big((v_{ij}^{\alpha} \otimes v_{kj}^{\alpha})\Delta(v_{ij}^{\alpha})^*\big) \\ &= \sum_{l=1}^{n_\alpha} h_{\mathbb{G}} \circ \theta_\gamma\big(v_{ij}^{\alpha}(v_{il}^{\alpha})^* \otimes v_{kj}^{\alpha}(v_{lj}^{\alpha})^*\big) \\ &= \sum_{l=1}^{n_\alpha} h_{\mathbb{G}}\Big(\delta_{jl}\frac{\lambda_j^\alpha}{m_\alpha} 1\Big) h_{\mathbb{G}}\Big(\delta_{kl}\frac{\lambda_j^\alpha}{m_\alpha} 1\Big) \\ &= \delta_{jk}\Big(\frac{\lambda_j^\alpha}{m_\alpha}\Big)^2, \end{aligned}$$

where in the third equality we use the assumption that $\theta_\gamma(x\Delta(b)) = \theta_\gamma(x)b$ for all $x \in L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\mathbb{G})$ and $b \in L^\infty(\mathbb{G})$. Thus, $X^{\alpha,\gamma}_{jk} = \delta_{jk}\frac{\lambda^\alpha_j}{m_\alpha}$.

Since $h_{\mathbb{G}}((v^\alpha_{is})^* v^\alpha_{ij}) = \delta_{sj}\frac{1}{\lambda^\alpha_i m_\alpha}$ and $h_{\mathbb{G}}((v^\alpha_{sj})^* v^\alpha_{kj}) = \delta_{sk}\frac{1}{\lambda^\alpha_k m_\alpha}$, it follows that

$$(v^\alpha_{is})^* v^\alpha_{ij} \otimes (v^\alpha_{sj})^* v^\alpha_{kj} = \delta_{sj}\frac{1}{\lambda^\alpha_i m_\alpha} \otimes \delta_{sk}\frac{1}{\lambda^\alpha_k m_\alpha} + r,$$

where $r \in \ker h_{\mathbb{G}} \otimes \ker h_{\mathbb{G}}$. Furthermore, we have

$$X^{\alpha,\gamma}_{jk}(v^\alpha_{ij})^* v^\alpha_{ij} = (v^\alpha_{ij})^*\theta_\gamma(v^\alpha_{ij} \otimes v^\alpha_{kj}) = \sum_{s=1}^{n_\alpha} \theta_\gamma\big((v^\alpha_{is})^* v^\alpha_{ij} \otimes (v^\alpha_{sj})^* v^\alpha_{kj}\big).$$

Again, by applying $h_{\mathbb{G}}$, we see that

$$X^{\alpha,\gamma}_{jk}\frac{1}{\lambda^\alpha_i m_\alpha} = \sum_{s=1}^{n_\alpha} \delta_{sj}\frac{1}{\lambda^\alpha_i m_\alpha}\delta_{sk}\frac{1}{\lambda^\alpha_k m_\alpha} = \delta_{jk}\frac{1}{\lambda^\alpha_i \lambda^\alpha_k m_\alpha^2}.$$

Therefore, $\delta_{jk}\frac{\lambda^\alpha_j}{m_\alpha} = \delta_{jk}\frac{1}{\lambda^\alpha_k m_\alpha}$. Thus, $\lambda^\alpha_k = 1$ for all α and $1 \le k \le n_\alpha$, and so $h_{\mathbb{G}}$ is tracial, which implies that $\mathbb{G}$ is a Kac algebra. □

3. Tensor product and operator approximate biprojectivity

Let $(\mathbb{G}, \Delta^{\mathbb{G}})$ and $(\mathbb{H}, \Delta^{\mathbb{H}})$ be two compact quantum groups. Then the tensor product of $\mathbb{G}$ and $\mathbb{H}$ is the compact quantum group $\mathbb{G}\otimes\mathbb{H} := (L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\mathbb{H}), \Delta^{\mathbb{G}\otimes\mathbb{H}})$ with comultiplication defined by

$$\Delta^{\mathbb{G}\otimes\mathbb{H}} := (\mathrm{id}_{L^\infty(\mathbb{G})} \otimes \Sigma \otimes \mathrm{id}_{L^\infty(\mathbb{H})})(\Delta^{\mathbb{G}} \otimes \Delta^{\mathbb{H}}) : \mathbb{M} \to \mathbb{M}\bar{\otimes}\mathbb{M},$$

where Σ is the flip map from $L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\mathbb{H})$ to $L^\infty(\mathbb{H})\bar{\otimes}L^\infty(\mathbb{G})$ and $\mathbb{M} = L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\mathbb{H})$. The Haar integral of $\mathbb{G}\otimes\mathbb{H}$ is denoted by $h_{\mathbb{G}\otimes\mathbb{H}}$ and is defined by $h_{\mathbb{G}} \otimes h_{\mathbb{H}}$.

Theorem 3.1. *Let $\mathbb{G}$ be a coamenable compact quantum group, and let $\mathbb{H}$ be a compact quantum group. If $L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{H})$ is operator approximately biprojective, then $L^1(\mathbb{G})$ is operator approximately biprojective.*

Proof. Let $\mathbb{A} := L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{H})$ be operator approximately biprojective with approximate splitting morphism $\rho_\gamma : \mathbb{A} \to \mathbb{A}\widehat{\otimes}\mathbb{A}$. Coamenability of $\mathbb{G}$ implies that $L^1(\mathbb{G})$ has a bounded approximate identity, say $(e_\alpha)_\alpha$. It is known that for compact quantum group $\mathbb{H}$, the Haar state $h_{\mathbb{H}}$ is a normal invariant mean on $L^\infty(\mathbb{H})$ and therefore $h_{\mathbb{H}} * h_{\mathbb{H}} = h_{\mathbb{H}}$. Then for every $f_1, f_2 \in L^1(\mathbb{G})$, we have

$$\begin{aligned}
\rho_\gamma(f_1 * f_2 \otimes h_{\mathbb{H}}) &= \rho_\gamma\big((f_1 \otimes h_{\mathbb{H}})(f_2 \otimes h_{\mathbb{H}})\big)\\
&= (f_1 \otimes h_{\mathbb{H}}) \cdot \rho_\gamma(f_2 \otimes h_{\mathbb{H}})\\
&= \lim_\alpha (f_1 * e_\alpha \otimes h_{\mathbb{H}}) \cdot \rho_\gamma(f_2 \otimes h_{\mathbb{H}})\\
&= \lim_\alpha f_1 \cdot (e_\alpha \otimes h_{\mathbb{H}}) \cdot \rho_\gamma(f_2 \otimes h_{\mathbb{H}})\\
&= f_1 \cdot \lim_\alpha \rho_\gamma(e_\alpha * f_2 \otimes h_{\mathbb{H}})\\
&= f_1 \cdot \rho_\gamma(f_2 \otimes h_{\mathbb{H}}).
\end{aligned}$$

Similarly, we can show a right-module version of this equation. Therefore, for each $f_1, f_2 \in L^1(\mathbb{G})$, we have

$$\rho_\gamma(f_1 * f_2 \otimes h_{\mathbb{H}}) = f_1 \cdot \rho_\gamma(f_2 \otimes h_{\mathbb{H}}) = \rho_\gamma(f_1 \otimes h_{\mathbb{H}}) \cdot f_2. \tag{3.1}$$

Now, we define $\theta : \mathbb{A}\widehat{\otimes}\mathbb{A} \to L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{G})$ by

$$\theta(f_1 \otimes h_1 \otimes f_2 \otimes h_2) = (h_1 * h_2)(1) f_1 \otimes f_2 \quad \big(f_1, f_2 \in L^1(\mathbb{G}), h_1, h_2 \in L^1(\mathbb{H})\big).$$

It is easy to see that θ is a completely contractive $L^1(\mathbb{G})$-bimodule morphism. We now define $\tilde{\rho}_\gamma : L^1(\mathbb{G}) \to L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{G})$ by

$$\tilde{\rho}_\gamma(f) = \theta \circ \rho_\gamma(f \otimes h_{\mathbb{H}}) \quad \big(f \in L^1(\mathbb{G})\big).$$

From (3.1), $\tilde{\rho}_\gamma$ is a completely contractive $L^1(\mathbb{G})$-bimodule morphism. Therefore, for each $f_1, f_2 \in L^1(\mathbb{G})$ and $h_1, h_2 \in L^1(\mathbb{H})$, we have

$$\begin{aligned}
(\mathrm{id}_{L^1(\mathbb{G})} \otimes 1)\Delta_*^{\mathbb{G}\otimes\mathbb{H}}(f_1 \otimes h_1 \otimes f_2 \otimes h_2) &= (\mathrm{id}_{L^1(\mathbb{G})} \otimes 1)(f_1 * f_2 \otimes h_1 * h_2) \\
&= (h_1 * h_2)(1) f_1 * f_2 \\
&= (h_1 * h_2)(1)\Delta_*^{\mathbb{G}}(f_1 \otimes f_2) \\
&= \Delta_*^{\mathbb{G}} \circ \theta(f_1 \otimes h_1 \otimes f_2 \otimes h_2).
\end{aligned}$$

Thus, $\Delta_*^{\mathbb{G}} \circ \theta = (\mathrm{id}_{L^1(\mathbb{G})} \otimes 1) \circ \Delta_*^{\mathbb{G}\otimes\mathbb{H}}$. It follows that for every $f \in L^1(\mathbb{G})$, we have

$$\begin{aligned}
\lim_\gamma \Delta_*^{\mathbb{G}} \circ \tilde{\rho}_\gamma(f) &= \lim_\gamma \Delta_*^{\mathbb{G}} \circ \theta \circ \rho_\gamma(f \otimes h_{\mathbb{H}}) \\
&= \lim_\gamma (\mathrm{id}_{L^1(\mathbb{G})} \otimes 1) \circ \Delta_*^{\mathbb{G}\otimes\mathbb{H}} \circ \rho_\gamma(f \otimes h_{\mathbb{H}}) \\
&= (\mathrm{id}_{L^1(\mathbb{G})} \otimes 1)(f \otimes h_{\mathbb{H}}) = f.
\end{aligned}$$

This shows that $L^1(\mathbb{G})$ is operator approximately biprojective. □

It is clear that when $\mathbb{G}$ is a compact quantum group and $\mathbb{H}$ is a non-Kac-type compact quantum group, then $\mathbb{G} \otimes \mathbb{H}$ is of non-Kac type. Therefore, the following proposition follows immediately from [4, Corollary 1.3].

Proposition 3.2. *Let $\mathbb{G}$ be a compact quantum group, and let $\mathbb{H}$ be a non-Kac-type compact quantum group. Then $L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{H})$ is not operator biprojective.*

In the following theorem, we show that $L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{H})$ is operator approximately biprojective, when both $L^1(\mathbb{G})$ and $L^1(\mathbb{H})$ are operator approximately biprojective. However, when $\mathbb{H}$ is of non-Kac type, then Proposition 3.2 implies that $L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{H})$ is not operator biprojective.

Theorem 3.3. *Let $\mathbb{G}$ and $\mathbb{H}$ be compact quantum groups. If $L^1(\mathbb{G})$ and $L^1(\mathbb{H})$ are operator approximately biprojective, then $L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{H})$ is operator approximately biprojective.*

Proof. If we set $\mathbb{A} := L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{G})$, $\mathbb{B} := L^1(\mathbb{H})\widehat{\otimes}L^1(\mathbb{H})$, and $\mathbb{E} := L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{H})$, then there are approximate splitting morphisms $\rho_\gamma : L^1(\mathbb{G}) \to \mathbb{A}$ and ρ_η :

$L^1(\mathbb{H}) \to \mathbb{B}$. Let $\mathfrak{F}$ be the set of all (γ, η). Then $\mathfrak{F}$ is a directed set by setting $(\gamma_1, \eta_1) \preceq (\gamma_2, \eta_2)$ if and only if $\gamma_1 \preceq \gamma_2$ and $\eta_1 \preceq \eta_2$. Now, consider the net $\rho_{(\gamma,\eta)} : \mathbb{E} \to \mathbb{E}\widehat{\otimes}\mathbb{E}$ defined by

$$\rho_{(\gamma,\eta)} := (\mathrm{id}_{L^1(\mathbb{G})} \otimes \sigma \otimes \mathrm{id}_{L^1(\mathbb{H})}) \circ (\rho_\gamma \otimes \rho_\eta),$$

where σ is the flip map on $L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{H})$. It is not hard to check that any $\rho_{(\gamma,\eta)}$ is a completely contractive $\mathbb{E}$-bimodule morphism and that

$$\big((\mathrm{id}_{L^\infty(\mathbb{G})} \otimes \Sigma \otimes \mathrm{id}_{L^\infty(\mathbb{H})}) \circ (\Delta^{\mathbb{G}} \otimes \Delta^{\mathbb{H}})\big)_* = (\Delta^{\mathbb{G}} \otimes \Delta^{\mathbb{H}})_* \circ (\mathrm{id}_{L^\infty(\mathbb{G})} \otimes \Sigma \otimes \mathrm{id}_{L^\infty(\mathbb{H})})_*,$$

where Σ is the flip map on $L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\mathbb{H})$. Moreover, as $\Sigma_* \circ \sigma = \mathrm{id}_{\mathbb{E}}$, we obtain that

$$(\mathrm{id}_{L^\infty(\mathbb{G})} \otimes \Sigma \otimes \mathrm{id}_{L^\infty(\mathbb{H})})_* \circ (\mathrm{id}_{L^1(\mathbb{G})} \otimes \sigma \otimes \mathrm{id}_{L^1(\mathbb{H})}) = \mathrm{id}_{\mathbb{A}\widehat{\otimes}\mathbb{B}}.$$

Therefore, for each $f \in L^1(\mathbb{G})$ and $g \in L^1(\mathbb{G})$, we have

$$\begin{aligned}
\lim_{(\gamma,\eta)} \big\|(\Delta^{\mathbb{G}\otimes\mathbb{H}})_* \rho_{(\gamma,\eta)}(f \otimes g) - f \otimes g\big\| &= \lim_{(\gamma,\eta)} \big\|(\Delta^{\mathbb{G}} \otimes \Delta^{\mathbb{H}})_*(\rho_\gamma \otimes \rho_\eta)(f \otimes g) - f \otimes g\big\| \\
&= \lim_{(\gamma,\eta)} \big\|\Delta^{\mathbb{G}}_* \rho_\gamma(f) \otimes \Delta^{\mathbb{H}}_* \rho_\eta(g) - f \otimes g\big\| \\
&\le \lim_{(\gamma,\eta)} \big(\big\|\Delta^{\mathbb{G}}_* \rho_\gamma(f)\big\| \big\|\Delta^{\mathbb{H}}_* \rho_\eta(g) - g\big\| \\
&\qquad + \|g\| \big\|\Delta^{\mathbb{G}}_* \rho_\gamma(f) - f\big\|\big) = 0.
\end{aligned}$$

This shows that, $(\rho_{(\gamma,\eta))})$ is an approximate splitting morphism for $L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{H})$, which implies that it is operator approximately biprojective. □

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