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THE PERTURBATION CLASS OF ALGEBRAIC OPERATORS AND APPLICATIONS

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Abstract. In this article, we completely describe the perturbation class, the commuting perturbation class, and the topological interior of the class of all bounded linear algebraic operators. As applications, we also focus on the stability of the essential ascent spectrum and the essential descent spectrum under finite-rank perturbations.

1. Introduction

Throughout this article, X denotes an infinite-dimensional complex Banach space, and $\mathcal{B}(X)$ denotes the algebra of all bounded linear operators on X.

Given a subset $\Lambda \subset \mathcal{B}(X)$, the *perturbation class* $\mathcal{P}(\Lambda)$ and the *commuting perturbation class* $\mathcal{P}_{\mathrm{c}}(\Lambda)$ of Λ are, respectively, defined by

$$\mathcal{P}(\Lambda) = \big\{S \in \mathcal{B}(X) : T + S \in \Lambda \text{ for every } T \in \Lambda\big\}$$

and

$$\mathcal{P}_{\mathrm{c}}(\Lambda) = \big\{S \in \mathcal{B}(X) : T + S \in \Lambda \text{ for every } T \in \Lambda \text{ commuting with } S\big\}.$$

Note that in order to check whether an operator S satisfies the definition of a perturbation class, we have to study the properties of $T + S$ for T in a family of operators, which can be tedious in the general case.

The concept of perturbation class has been considered in other situations. For example, it is well known that the perturbation class of all invertible elements in a

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Banach algebra is the radical of that algebra, and that its commuting perturbation class is the set of all quasinilpotent elements (see [9], [12]). For a detailed exposition on the perturbation classes problem, we direct the reader to [1], [6], [7], [9], and [12] and the references therein.

In this article, we consider the perturbation classes problem, as well as the commuting perturbation classes problem, for algebraic operators. Recall that an operator $T \in \mathcal{B}(X)$ is said to be *algebraic* if there exists a nonzero complex polynomial P such that $P(T) = 0$. In particular, when T is annihilated by a complex polynomial of degree at most 2, it is said to be *quadratic*.

The many applications of algebraic operators in applied linear algebra have been investigated by several mathematicians (see, e.g., [2], [5], [11]). Denote by $\mathcal{A}(X)$ the set of all algebraic operators in $\mathcal{B}(X)$, and denote by $\mathcal{F}(X)$ the set of all finite-rank operators in $\mathcal{B}(X)$. Note that one of the main results in [11, Proposition 2.4] states that the perturbation class of quadratic operators in $\mathcal{B}(X)$ is the 1-dimensional subspace $\mathbb{C}I$.

For an operator $T \in \mathcal{B}(X)$, write $\ker(T)$ for its kernel and $\operatorname{ran}(T)$ for its range. For two subspaces M and M' of X, we write $M \stackrel{e}{=} M'$ if there exist finite-dimensional subspaces L and L' such that $M \subseteq M' + L'$ and $M' \subseteq M + L$. The *essential ascent* $\mathrm{a_e}(T)$ and *essential descent* $\mathrm{d_e}(T)$ of $T \in \mathcal{B}(X)$ are, respectively, defined by

$$\mathrm{a_e}(T) = \inf\big\{n \geq 0 \colon \ker(T^n) \stackrel{e}{=} \ker(T^{n+1})\big\}$$

and

$$\mathrm{d_e}(T) = \inf\big\{n \geq 0 \colon \operatorname{ran}(T^n) \stackrel{e}{=} \operatorname{ran}(T^{n+1})\big\},$$

where the infimum over the empty set is taken to be infinite. Operators with finite essential ascent or descent seem to have been first studied in [8]. These operators play a significant role in more general studies in [10].

Let us consider the sets

$$\mathcal{A}_\mathrm{e}(X) = \big\{T \in \mathcal{B}(X) : \mathrm{a_e}(T) \text{ is finite and } \operatorname{ran}(T^{\mathrm{a_e}(T)+1}) \text{ is closed}\big\}$$

and

$$\mathcal{D}_\mathrm{e}(X) = \big\{T \in \mathcal{B}(X) : \mathrm{d_e}(T) \text{ is finite}\big\}.$$

The corresponding *essential ascent spectrum* $\sigma^\mathrm{e}_\mathrm{asc}(T)$ and *essential descent spectrum* $\sigma^\mathrm{e}_\mathrm{des}(T)$ of $T \in \mathcal{B}(X)$ are, respectively, defined by

$$\sigma^\mathrm{e}_\mathrm{asc}(T) = \big\{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{A}_\mathrm{e}(X)\big\}$$

and

$$\sigma^\mathrm{e}_\mathrm{des}(T) = \big\{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{D}_\mathrm{e}(X)\big\}.$$

In [4], the authors studied the essential ascent spectrum, showing that an operator $F \in \mathcal{B}(X)$ has some finite-rank power if and only if $\sigma^\mathrm{e}_\mathrm{asc}(T + F) = \sigma^\mathrm{e}_\mathrm{asc}(T)$ for every $T \in \mathcal{B}(X)$ commuting with F. Similar results for the essential descent spectrum were established in [3]. These characterizations—of commuting perturbations leaving invariant the essential ascent spectrum and the essential descent spectrum of the operators in their commutants—can be considered as characterizations of the commuting perturbation class of the sets $\mathcal{A}_{\mathrm{e}}(X)$ and $\mathcal{D}_{\mathrm{e}}(X)$. Indeed, one can easily get

$$\mathcal{P}_{\mathrm{c}}\big(\mathcal{A}_{\mathrm{e}}(X)\big) = \mathcal{P}_{\mathrm{c}}\big(\mathcal{D}_{\mathrm{e}}(X)\big) = \big\{F \in \mathcal{B}(X) : F^n \text{ has finite rank for some } n \geq 1\big\}.$$

As applications of the perturbation class of algebraic operators, we also consider here the perturbation classes problem for the essential ascent spectrum and the essential descent spectrum.

This article is organized as follows. In the Section 2, we establish that the perturbation class of $\mathcal{A}(H)$, where H is an infinite-dimensional complex Hilbert space, is the subspace $\mathbb{C}I + \mathcal{F}(H)$. We also look at the stability of the essential ascent spectrum and the essential descent spectrum under finite-rank perturbations. Section 3 is devoted to proving that the commuting perturbation class of $\mathcal{A}(X)$ is itself. In the final section, we show that the topological interior of $\mathcal{A}(X)$ is empty.

2. Perturbation class of $\mathcal{A}(H)$, $\mathcal{A}_{\mathrm{e}}(H)$, and $\mathcal{D}_{\mathrm{e}}(H)$

It is well known that the set $\mathcal{A}(X)$ is stable under finite-rank perturbations, and hence its perturbation class contains the subspace $\mathbb{C}I + \mathcal{F}(X)$. The main result of this section is the following theorem, which states that the perturbation class of algebraic operators consists exactly of finite-rank operators plus scalar multiples of the identity.

Theorem 2.1. *Let H be an infinite-dimensional complex Hilbert space. Then*

$$\mathcal{P}\big(\mathcal{A}(H)\big) = \mathbb{C}I + \mathcal{F}(H).$$

Before proving this theorem, we need to establish the following auxiliary results. Let $z \in X$ and $f \in X^*$ be nonzero, where X^* denotes the topological dual space. As usual, we will denote by $z \otimes f$ the rank 1 operator given by $(z \otimes f)(x) = f(x)z$ for all $x \in X$. Note that every rank 1 operator in $\mathcal{B}(X)$ can be written in this form.

Lemma 2.2. *Let $N \in \mathcal{B}(X)$ be a nilpotent operator of index n, and let $x \in X$ be such that $N^{n-1}x \neq 0$. Then there exists a closed N-invariant subspace Y such that $X = \operatorname{Span}\{N^k x : 0 \leq k \leq n-1\} \oplus Y$.*

Proof. Since N is nilpotent, the vectors $N^k x$, $0 \leq k \leq n-1$, are linearly independent. Let $f \in X^*$ be a linear form satisfying

$$f(N^k x) = \delta_{k,n-1} \quad \text{for } 0 \leq k \leq n-1.$$

Consider also the operator $P \in \mathcal{B}(X)$ defined by

$$P = x \otimes f N^{n-1} + Nx \otimes f N^{n-2} + \cdots + N^{n-1}x \otimes f.$$

A simple computation shows that $P^2 = P$, and therefore $X = \ker(I-P) \oplus \ker(P)$. Since N commutes with P, we infer that $\ker(P)$ is a closed N-invariant subspace. Furthermore, we have $PN^k x = N^k x$ for every $0 \leq k \leq n-1$, and hence

$$\operatorname{Span}\{N^k x : 0 \leq k \leq n-1\} \subseteq \ker(I-P) = \operatorname{ran}(P) \subseteq \operatorname{Span}\{N^k x : 0 \leq k \leq n-1\},$$

so that $\ker(I-P)=\operatorname{Span}\{N^k x : 0\le k\le n-1\}$. Consequently, $\ker(P)$ is the desired subspace. ☐

Lemma 2.3. *Let $S\in\mathcal{B}(X)$ be a nilpotent operator having an infinite-dimensional range. Then there exists $T\in\mathcal{A}(X)$ such that $T+S\notin\mathcal{A}(X)$.*

Proof. Let n_0 be the nilpotence index of S, and let $x_0\in X$ be such that $S^{n_0-1}x_0\neq 0$. Lemma 2.2 ensures the existence of an S-invariant complement X_1 of $\operatorname{Span}\{S^k x_0 : 0\le k\le n_0-1\}$. Choose a linear form $f_0\in X^*$ satisfying $f_0\equiv 0$ on X_1 and

$$f_0(S^k x_0)=\delta_{k,n_0-1}\quad \text{for } 0\le k\le n_0-1.$$

Since $\operatorname{ran}(S)$ is infinite-dimensional, then $S_{|X_1}$ is nilpotent of index $n_1\ge 2$. Take $x_1\in X_1$ with $S^{n_1-1}x_1\neq 0$, and we denote by X_2 an S-invariant subspace such that $X_1=\operatorname{Span}\{S^k x_1 : 0\le k\le n_1-1\}\oplus X_2$. Consider a linear form $f_1\in X^*$ satisfying $f_1\equiv 0$ on $\operatorname{Span}\{S^k x_0 : 0\le k\le n_0-1\}\oplus X_2$ and

$$f_1(S^k x_1)=\delta_{k,n_1-1}\quad \text{for } 0\le k\le n_1-1.$$

Repeating the same argument, we get linearly independent sets

$$\{S^k x_m : 0\le k\le n_m-1 \text{ and } m\ge 0\}\subset X \quad\text{and}\quad \{f_m : m\ge 0\}\subset X^*$$

such that

$$S^{n_m-1}x_m\neq 0,\qquad S^{n_m}x_m=0\quad\text{and}\quad f_m(S^k x_p)=\delta_{k,n_m-1}\delta_{p,m}\quad \text{for } m,p,k\ge 0.$$

Now, consider the operator $T\in\mathcal{B}(X)$ given by

$$T=\sum_{k\ge 0}\alpha_k S^{n_{k+1}-2}x_{k+1}\otimes f_k,$$

where the α_k's are nonzero complex numbers for which $\sum_{k\ge 0}|\alpha_k|\|S^{n_{k+1}-2}x_{k+1}\otimes f_k\|$ is finite. Since $f_i(S^{n_{j+1}-2}x_j)=0$ for every $i,j\ge 0$, then

$$\Big(\sum_{k=0}^{m}\alpha_k S^{n_{k+1}-2}x_{k+1}\otimes f_k\Big)^2=0\quad \text{for every } m\ge 0.$$

Thus, $T^2=0$ and $T\in\mathcal{A}(X)$. Let us show that $T+S\notin\mathcal{A}(X)$. Put $u=S^{n_0-2}x_0$. Then, for every integer $p\ge 1$, we get

$$\begin{aligned}
(T+S)u&=S^{n_0-1}x_0,\\
(T+S)^2u&=\alpha_0 S^{n_1-2}x_1,\\
(T+S)^3u&=\alpha_0 S^{n_1-1}x_1,\\
(T+S)^4u&=\alpha_0\alpha_1 S^{n_2-2}x_2,\\
&\ \vdots\\
(T+S)^{2p}u&=\alpha_0\cdots\alpha_{p-1}S^{n_p-2}x_p,\\
(T+S)^{2p+1}u&=\alpha_0\cdots\alpha_{p-1}S^{n_p-1}x_p,
\end{aligned}$$

which means that $\{u,(T+S)u,\dots,(T+S)^n u\}$ is a linearly independent set for every integer $n\ge 0$, and hence $T+S$ is nonalgebraic by [2, Theorem 4.2.7]. This completes the proof. ☐

In the following, we derive an analogous result for idempotent operators.

Lemma 2.4. *Let H be an infinite-dimensional complex Hilbert space, and let $P \in \mathcal{B}(H)$ be an idempotent operator such that* $\dim \operatorname{ran}(P) = \dim \ker(P) = \infty$. *Then there exists $T \in \mathcal{A}(H)$ such that $T + P \notin \mathcal{A}(H)$.*

Proof. Note that $H = \ker(P) \oplus \operatorname{ran}(P)$ and, for every finite-codimensional subspace Y, we have that $Y \cap \operatorname{ran}(P)$ and $Y \cap \ker(P)$ are infinite-dimensional. Let $x_0 \in \ker(P)$ be such that $\|x_0\| = 1$, and choose $y_0 \in \operatorname{Span}\{x_0\}^{\perp} \cap \operatorname{ran}(P)$ satisfying $\|y_0\| = 1$. Now, take $x_1 \in \operatorname{Span}\{x_0, y_0\}^{\perp} \cap \ker(P)$ with $\|x_1\| = 1$, and let $y_1 \in \operatorname{Span}\{x_0, x_1, y_0\}^{\perp} \cap \operatorname{ran}(P)$ be such that $\|y_0\| = 1$. Repeating the same argument, we get an orthonormal system $\{x_n, y_n : n \geq 0\}$ such that

$$x_n \in \ker(P) \quad \text{and} \quad y_n \in \operatorname{ran}(P) \quad \text{for } n \geq 0.$$

One can easily verify that $\{2^{-1}(y_n + x_n), 2^{-1}(y_n - x_n) : n \geq 0\}$ is an orthonormal system. Put $M = \operatorname{Span}\{2^{-1}(y_n + x_n), 2^{-1}(y_n - x_n) : n \geq 0\}$, and define $S \in \mathcal{B}(H)$ by $S_{|M^{\perp}} = 0$. Then

$$S(y_n + x_n) = 0 \quad \text{and} \quad S(y_n - x_n) = -(y_n + x_n) + y_{n+1} + x_{n+1} \quad \text{for } n \geq 0.$$

Clearly, $S^2 = 0$. Moreover, for $u = 2^{-1}(y_0 + x_0)$, we have

$$\begin{aligned}
(S + 2P - I)u &= 2^{-1}(y_0 - x_0), \\
(S + 2P - I)^2 u &= 2^{-1}(y_1 + x_1), \\
(S + 2P - I)^3 u &= 2^{-1}(y_1 - x_1), \\
&\vdots \\
(S + 2P - I)^{2n} u &= 2^{-1}(y_n + x_n), \\
(S + 2P - I)^{2n+1} u &= 2^{-1}(y_n - x_n).
\end{aligned}$$

Thus $\{(S + 2P - I)^k u : 0 \leq k \leq n\}$ is a linearly independent set for every $n \geq 0$, and hence $S + 2P - I$ is nonalgebraic, so that $2^{-1}S + P \notin \mathcal{A}(H)$. To conclude, it suffices to take $T = 2^{-1}S$. $\square$

With these results in hand, we are ready to prove our main theorem.

Proof of Theorem 2.1. We have only to prove that $\mathcal{P}(\mathcal{A}(H)) \subseteq \mathbb{C}I + \mathcal{F}(H)$. For this, let $S \in \mathcal{P}(\mathcal{A}(H))$, and suppose to the contrary that $S \notin \mathbb{C}I + \mathcal{F}(H)$. Note that S is an algebraic operator, and hence $H = H_1 \oplus \cdots \oplus H_n$ where $H_k = \ker(S - \lambda_k)^{m_k}$ for $1 \leq k \leq n$, and the scalars λ_k are distinct. Clearly, the operators $N_k = (S - \lambda_k)_{|H_k}$, $1 \leq k \leq n$, are nilpotent and we have

$$S = (N_1 + \lambda_1) \oplus \cdots \oplus (N_n + \lambda_n).$$

We discuss two cases.

Case 1. There exists $j \in \{1, \ldots, n\}$ such that $\dim \operatorname{ran}(N_j) = \infty$. Then it follows from Lemma 2.3 that there exists a bounded algebraic operator T_j on H_j such that $T_j + N_j$ is nonalgebraic. Thus, if we consider the operator $T \in \mathcal{B}(H)$ given by $T_{|H_j} = T_j$ and $T_{|H_k} = 0$ for $k \neq j$, we get that $T \in \mathcal{A}(H)$ and $T + S \notin \mathcal{A}(H)$, which is a contradiction.

Case 2. For every $k \in \{1, \ldots, n\}$, $\dim \operatorname{ran}(N_k)$ is finite. Then, the operator $N = N_1 \oplus \cdots \oplus N_n$ has finite rank and

$$S = (\lambda_1 \oplus \cdots \oplus \lambda_n) + N.$$

Since $S \notin \mathbb{C}I + \mathcal{F}(H)$, we infer that $n \geq 2$ and there exist $r, t \in \{1, \ldots, n\}$, $r \neq t$, such that $\dim H_r = \dim H_t = \infty$. Without loss of generality, we may assume that $r = 0$ and $t = 1$. Consider the idempotent operator $P \in \mathcal{B}(H_1 \oplus H_2)$ defined by $P = 0 \oplus I$. It follows from Lemma 2.4 that there exists a bounded algebraic operator R on $H_1 \oplus H_2$ such that $R + P$ is nonalgebraic. To conclude, consider the operator $T \in \mathcal{B}(H)$ given by

$$T_{|H_1 \oplus H_2} = (\lambda_2 - \lambda_1)R + \lambda_1 \quad \text{and} \quad T_{|H_3 \oplus \cdots \oplus H_n} = 0.$$

Clearly, we have $T \in \mathcal{A}(H)$. Since $N_{|H_1 \oplus H_2}$ is finite rank and

$$S_{|H_1 \oplus H_2} = (\lambda_1 \oplus \lambda_2) + N_{|H_1 \oplus H_2} = (\lambda_2 - \lambda_1)P + \lambda_1 + N_{|H_1 \oplus H_2},$$

we assert that $(T + S)_{|H_1 \oplus H_2}$ is nonalgebraic, and hence $T + S \notin \mathcal{A}(H)$. This contradiction finishes the proof. $\square$

We continue by stating a question which arises in a natural way from our result.

Question 2.5. Does Theorem 2.1 remain true in the context of Banach spaces?

Now, as applications of Theorem 2.1, we characterize finite-rank operators as the class of operators leaving invariant the sets $\mathcal{A}_{\mathrm{e}}(H)$ and $\mathcal{D}_{\mathrm{e}}(H)$.

Theorem 2.6. *Let H be an infinite-dimensional complex Hilbert space. Then*

$$\mathcal{P}\big(\mathcal{A}_{\mathrm{e}}(H)\big) = \mathcal{P}\big(\mathcal{D}_{\mathrm{e}}(H)\big) = \mathcal{F}(H).$$

Proof. Since $\mathcal{A}_{\mathrm{e}}(H)$ and $\mathcal{D}_{\mathrm{e}}(H)$ are stable under finite-rank perturbations (see [10]), we have $\mathcal{F}(H) \subseteq \mathcal{P}(\mathcal{A}_{\mathrm{e}}(H))$ and $\mathcal{F}(H) \subseteq \mathcal{P}(\mathcal{D}_{\mathrm{e}}(H))$.

Let us show that $\mathcal{P}(\mathcal{A}_{\mathrm{e}}(H)) \subseteq \mathcal{F}(H)$. Let $F \in \mathcal{P}(\mathcal{A}_{\mathrm{e}}(H))$. It follows that

$$\sigma^{\mathrm{e}}_{\mathrm{asc}}(T + F) = \sigma^{\mathrm{e}}_{\mathrm{asc}}(T) \quad \text{for every } T \in \mathcal{B}(H).$$

In particular, this equality holds for every $T \in \mathcal{B}(H)$ commuting with F. Thus, it follows from [4, Theorem 3.2] that F^n is finite rank for some integer $n \geq 1$. Now, for every $T \in \mathcal{B}(H)$, we get by [4, Theorem 2.7] that

$$T \text{ is algebraic} \Leftrightarrow \sigma^{\mathrm{e}}_{\mathrm{asc}}(T) = \emptyset \Leftrightarrow \sigma^{\mathrm{e}}_{\mathrm{asc}}(T + F) = \emptyset \Leftrightarrow T + F \text{ is algebraic}.$$

Thus, Theorem 2.1 infers that $F \in \mathbb{C}I + \mathcal{F}(H)$. Taking into account that $F^n \in \mathcal{F}(H)$, we conclude that $F \in \mathcal{F}(H)$. Using [3, Theorems 2.7 and 3.1] and the same arguments as above, we get that $\mathcal{P}(\mathcal{D}_{\mathrm{e}}(H)) \subseteq \mathcal{F}(H)$. $\square$

As a direct consequence of Theorem 2.6, we derive the following corollary.

Corollary 2.7. *Let H be an infinite-dimensional complex Hilbert space, and let $F \in \mathcal{B}(H)$. Then the following assertions are equivalent:*

(1) $F \in \mathcal{F}(H)$,
(2) $\sigma^{\mathrm{e}}_{\mathrm{asc}}(T + F) = \sigma^{\mathrm{e}}_{\mathrm{asc}}(T)$ *for every* $T \in \mathcal{B}(H)$,
(3) $\sigma^{\mathrm{e}}_{\mathrm{des}}(T + F) = \sigma^{\mathrm{e}}_{\mathrm{des}}(T)$ *for every* $T \in \mathcal{B}(H)$.

3. Commuting perturbation class of $\mathcal{A}(X)$

In this section, we focus on the commuting perturbations that leave invariant the set of algebraic operators. More precisely, the following theorem states the main result of this section.

Theorem 3.1. *We have* $\mathcal{P}_{\mathrm{c}}(\mathcal{A}(X)) = \mathcal{A}(X)$.

To prove this theorem, we need the following lemma.

Lemma 3.2. *Let $T \in \mathcal{B}(X)$ be an algebraic operator, and let $N \in \mathcal{B}(X)$ be a nilpotent operator such that $NT = TN$. Then $T + N$ is algebraic.*

Proof. Since T is algebraic, we can write $X = X_1 \oplus \cdots \oplus X_r$, where $X_k = \ker(T - \lambda_k)^{m_k}$, and the scalars λ_k are distinct. Clearly, the operators $T_k = (T - \lambda_k)_{|X_k}$, $1 \leq k \leq r$, are nilpotent and

$$T = (T_1 + \lambda_1) \oplus \cdots \oplus (T_n + \lambda_r).$$

Furthermore, the fact that $NT = TN$ implies that X_k, $1 \leq k \leq r$, are N-invariant subspaces. Hence, with respect to the decomposition of X, we can express N as $N = N_1 \oplus \cdots \oplus N_r$. Since N_k is nilpotent and $T_k N_k = N_k T_k$, it follows that $N_k + T_k$ is nilpotent for $1 \leq k \leq r$, and hence $N + T$ is algebraic. □

Proof of Theorem 3.1. Clearly, we have $\mathcal{P}_{\mathrm{c}}(\mathcal{A}(X)) \subseteq \mathcal{A}(X)$. Let $S, T \in \mathcal{A}(X)$ be such that $TS = ST$. Then, $X = X_1 \oplus \cdots \oplus X_n$, where $X_k = \ker(T - \lambda_k)^{m_k}$, and the scalars λ_k are distinct. Moreover, the operators $T_k = (T - \lambda_k)_{|X_k}$, $1 \leq k \leq n$, are nilpotent and

$$T = (T_1 + \lambda_1) \oplus \cdots \oplus (T_n + \lambda_n).$$

Since $TS = ST$, then X_k are S-invariant for $1 \leq k \leq r$. Hence, with respect to the decomposition of X, we can write $S = S_1 \oplus \cdots \oplus S_n$. One can easily see that S_k is algebraic and that $T_k S_k = S_k T_k$ for $1 \leq k \leq n$. Now, Lemma 3.2 infers that $T_k + S_k$ is algebraic for $1 \leq k \leq n$, so that $T + S$ is algebraic. This completes the proof. □

4. Topological interior of $\mathcal{A}(X)$

The main result of this section is the following theorem.

Theorem 4.1. *The topological interior of $\mathcal{A}(X)$ is empty.*

Before presenting the proof of this theorem, we first establish the following lemma.

Lemma 4.2. *Let $S \in \mathcal{B}(X)$ be a nilpotent operator. Then, for every $\varepsilon > 0$, there exists $T \in \mathcal{B}(X)$ such that $\|T\| < \varepsilon$ and $T + S \notin \mathcal{A}(X)$.*

Proof. Since S is nilpotent, then $\dim \ker(S) = \infty$. Let $x_0 \in \ker(S)$ be nonzero, and write $X = \mathrm{Span}\{x_0\} \oplus X_0$, where X_0 is a closed subspace. Choose a linear form $f_0 \in X^*$ such that

$$f_0(x_0) = 1 \quad \text{and} \quad f_0(x) = 0 \quad \text{for } x \in X_0.$$

Let $x_1 \in \ker(S) \cap X_0$ be nonzero, and write $X_0 = \mathrm{Span}\{x_1\} \oplus X_1$. In particular, the vectors x_0 and x_1 are linearly independent and $X = \mathrm{Span}\{x_0, x_1\} \oplus X_1$. Hence, there exists a linear form $f_1 \in X^*$ satisfying

$$f_1(x_i) = \delta_{i1} \quad \text{for } 0 \le i \le 1 \quad \text{and} \quad f_1(x) = 0 \quad \text{for } x \in X_1.$$

Note that $f_0(x_1) = 0$. Repeating the same argument, we get two linearly independent sets $\{x_n\}_{n\ge 0} \subset \ker(S)$ and $\{f_n\}_{n\ge 0} \subset X^*$ such that $f_i(x_j) = \delta_{ij}$ for all $i, j \ge 0$. Now, let $\varepsilon > 0$, and consider the operator $T \in \mathcal{B}(X)$ given by

$$T = \sum_{n=0}^{\infty} \alpha_n x_{n+1} \otimes f_n,$$

where α_n are nonzero complex numbers for which $\sum_{n\ge 0} |\alpha_n| \|x_{n+1} \otimes f_n\| < \varepsilon$. For every integer $n \ge 1$, we have

$$\begin{aligned}
(T+S)x_0 &= \alpha_0 x_1,\\
(T+S)^2 x_0 &= \alpha_0\alpha_1 x_2,\\
(T+S)^3 x_0 &= \alpha_0\alpha_1\alpha_2 x_3,\\
&\vdots\\
(T+S)^n x_0 &= \alpha_0 \cdots \alpha_{n-1} x_n,
\end{aligned}$$

so that $\{(T+S)^k x_0 : 0 \le k \le n\}$ is a linearly independent set. Hence, $T+S$ is nonalgebraic. This completes the proof. $\square$

It is well known that if Y is an infinite-dimensional Banach space, then the algebra $\mathcal{B}(Y)$ contains a nonalgebraic operator (see [5, Corollary 1.10]). We recapture this result as an immediate consequence of Lemma 4.2.

Corollary 4.3. *Let Y be a Banach space. Then the following assertions are equivalent:*

(1) *Y is infinite-dimensional,*
(2) *$\mathcal{B}(Y)$ contains a nonalgebraic operator.*

Proof of Theorem 4.1. Let $S \in \mathcal{A}(X)$ and $\varepsilon > 0$. We claim that there exists $T \in \mathcal{B}(X)$ such that $\|T\| < \varepsilon$ and $T + S \notin \mathcal{A}(X)$. Write $X = X_1 \oplus \cdots \oplus X_n$, where $X_k = \ker(S - \lambda_k)^{m_k}$ for $1 \le k \le n$, and the scalars λ_k are distinct. Clearly, the operators $N_k = (S - \lambda_k)_{|X_k}$, $1 \le k \le n$, are nilpotent and

$$S = (N_1 + \lambda_1) \oplus \cdots \oplus (N_n + \lambda_n).$$

Without loss of generality, we may assume that $\dim X_1 = \infty$. Let $P \in \mathcal{B}(X)$ be the idempotent operator given by $P = I \oplus 0 \oplus \cdots \oplus 0$ with respect to the decomposition of X. From Lemma 4.2, there exists a bounded operator T_1 on X_1 such that $\|T_1\| < \varepsilon \|P\|^{-1}$ and $T_1 + N_1$ is nonalgebraic. If we set $T = T_1 \oplus 0 \oplus \cdots \oplus 0$, then we get $\|T\| < \varepsilon$ and $T + S \notin \mathcal{A}(X)$. This finishes the proof. $\square$

We end this article with the following question.

Question 4.4. What can we say about the topological closure of $\mathcal{A}(X)$?

References

1. P. Aiena and M. González, *Intrinsic characterizations of perturbation classes on some Banach spaces*, Arch. Math. (Basel) **94** (2010), no. 4, 373–381. Zbl 1187.47014. MR2643971. DOI 10.1007/s00013-010-0103-7. 427
2. B. Aupetit, *A Primer on Spectral Theory*, Universitext, Springer, New York, 1991. Zbl 0715.46023. MR1083349. DOI 10.1007/978-1-4612-3048-9. 427, 429
3. O. Bel Hadj Fredj, *Essential descent spectrum and commuting compact perturbations*, Extracta Math. **21** (2006), no. 3, 261–271. Zbl 1131.47002. MR2332074. 427, 431
4. O. Bel Hadj Fredj, M. Burgos, and M. Oudghiri, *Ascent spectrum and essential ascent spectrum*, Studia Math. **187** (2008), no. 1, 59–73. Zbl 1160.47007. MR2410883. DOI 10.4064/sm187-1-3. 427, 431
5. M. Burgos, A. Kaidi, M. Mbekhta, and M. Oudghiri, *The descent spectrum and perturbations*, J. Operator Theory **56** (2006), no. 2, 259–271. Zbl 1117.47008. MR2282682. 427, 433
6. M. González, *The perturbation classes problem in Fredholm theory*, J. Funct. Anal. **200** (2003), no. 1, 65–70. Zbl 1036.47007. MR1974088. DOI 10.1016/S0022-1236(02)00071-X. 427
7. M. González, A. Martínez-Abejón, and J. Pello, *A survey on the perturbation classes problem for semi-Fredholm and Fredholm operators*, Funct. Anal. Approx. Comput. **7** (2015), no. 2, 75–87. Zbl 1353.47019. MR3313269. 427
8. S. Grabiner, *Ranges of products of operators*, Canad. J. Math. **26** (1974), no. 6, 1430–1441. Zbl 0259.47003. MR0380449. DOI 10.4153/CJM-1974-137-7. 427
9. A. Lebow and M. Schechter, *Semigroups of operators and measures of noncompactness*, J. Funct. Anal. **7** (1971), no. 1, 1–26. Zbl 0209.45002. MR0273422. DOI 10.1016/0022-1236(71)90041-3. 427
10. M. Mbekhta and V. Müller, *On the axiomatic theory of spectrum, II*, Studia Math. **119** (1996), no. 2, 129–147. Zbl 0857.47002. MR1391472. DOI 10.4064/sm-119-2-129-147. 427, 431
11. M. Oudghiri and K. Souilah, *Linear preservers of quadratic operators*, Mediterr. J. Math. **13** (2016), no. 6, 4929–4938. Zbl 1362.47027. MR3564542. DOI 10.1007/s00009-016-0783-8. 427
12. S. Č. Živković-Zlatanović, D. S. Djordjević, and R. Harte, *Ruston, Riesz and perturbation classes*, J. Math. Anal. Appl. **389** (2012), no. 2, 871–886. Zbl 1253.47029. MR2879265. DOI 10.1016/j.jmaa.2011.12.030. 427

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