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HAUSDORFF OPERATORS ON MODULATION AND WIENER AMALGAM SPACES

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Abstract. We give sharp conditions for boundedness of Hausdorff operators on certain modulation and Wiener amalgam spaces.

1. Introduction and preliminaries

The study of Hausdorff operators, which originated from classical summation methods, has a long history in real and complex analysis. We refer the reader to [1] and [13] for a survey with some historical background and recent developments regarding Hausdorff operators.

For a suitable function Φ, one of the corresponding Hausdorff operators H_Φ can be defined by

$$H_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(y) f\Big(\frac{x}{|y|}\Big)\, dy. \tag{1.1}$$

Although there is a general definition, where $f(A(y)x)$, with matrix A, stays in place of $f(x/|y|)$ in (1.1), we only consider the special case in this article. However, we do not exclude the possibility that the general case will prove to be of interest as well.

There are many known results about the boundedness of Hausdorff operators on various function spaces (see [11], [12], [14], [15]). Unfortunately, sharp conditions

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on the boundedness of Hausdorff operators can be characterized in only a few cases. (We refer the reader to [20] for a sharp characterization of the boundedness of Hausdorff operators on L^p, and to [3] and [16] for a sharp characterization of the boundedness of Hausdorff operators on Hardy spaces H^1 and h^1.) We note that characterizations of the boundedness of Hausdorff operators have also been established in other function spaces (see [1], [6]). However, we find that these spaces have properties similar to those of L^p-spaces. Let us briefly describe this fact in the following.

In order to prove the necessity of boundedness of Hausdorff operators on L^p-spaces, we must choose a suitable function f and estimate $\|H_\Phi f\|_{L^p}$ from below by some integral involving Φ. The space L^p is suitable for this lower estimate, since for a function f, the norm $\|f\|_{L^p}$ depends only on the absolute value of f, and the L^p-norm has the scaling property $\|f(s\cdot)\|_{L^p} = s^{-n/p}\|f\|_{L^p}$. We note that the function spaces for which characterizations of the boundedness of Hausdorff operators have so far been established all have the above two properties as L^p-spaces, so that the proof of necessity follows the same line as that on L^p. However, in the case of frequency decomposition spaces, such as modulation spaces or Wiener amalgam spaces, the situation becomes quite different and complicated.

The modulation spaces $M^s_{p,q}$ were first introduced by Feichtinger [5] in 1983. As function spaces associated with uniform decomposition (see [18]), modulation spaces are closely linked to the topic of time-frequency analysis (see [7]) and have been regarded as appropriate function spaces for the study of partial differential equations (see [19]). We refer the reader to [4] for some motivations and historical remarks. Readers are also directed to our recent work [8], [9] for details on the properties of modulation spaces and Wiener amalgam spaces.

As a frequency decomposition space, the norm of f in a modulation space cannot be completely determined by the absolute value of the function. On the other hand, the scaling property of modulation spaces is not as simple as that of L^p-spaces (see [17]). Thus, we are interested in determining sharp conditions for boundedness of Hausdorff operators on modulation spaces, since in this case the method used in the L^p case is not adoptable.

We also consider the boundedness of Hausdorff operators on Wiener amalgam spaces $W^s_{p,q}$. In general, a Wiener amalgam space can be represented by $W(B,C)$, where B and C serve as the local and global component, respectively. In this article, we consider a special case $W(\mathscr{F}^{-1}L^s_q, L_p)$, which is closely related to modulation spaces. For notational simplicity, we also use $W^s_{p,q}$ to denote this function space. Before stating the main theorems, we establish the following by way of preparation.

We need to add some suitable assumptions on Φ. First, in order to establish sharp conditions for the boundedness of Hausdorff operators, we assume that $\Phi \geq 0$. In the proof of the necessity part, we must make some (pointwise) estimates from below. That is why the assumption $\Phi \geq 0$ is necessary in most of the known characterizations for the boundedness of Hausdorff operators on function spaces (see [3], [16], [20]).

Second, we make another assumption for Φ as follows:

$$\int_{B(0,1)} |y|^n \Phi(y)\,dy < \infty, \qquad \text{and} \qquad \int_{B(0,1)^c} \Phi(y)\,dy < \infty. \tag{1.2}$$

The following remarks are intended not only to explain the reasonableness of the assumption (1.2), but also to give some important properties of Hausdorff operators under the assumption (1.2).

Remark 1.1 (Assumption (1.2) is weakest). In fact, (1.2) is the weakest assumption ensuring that the Schwartz function can be mapped into a tempered distribution by the Hausdorff operator H_Φ.

On the one hand, if $H_\Phi f \in \mathscr{S}'$, it must be locally integrable, and since $\Phi \geq 0$, for a nonnegative function f we have

$$\begin{aligned}
\infty > \int_{B(0,1)} \big|H_\Phi f(x)\big|\,dx &= \int_{B(0,1)} \int_{\mathbb{R}^n} \Phi(y) f\big(x/|y|\big)\,dy\,dx \\
&= \int_{\mathbb{R}^n} \Phi(y) \int_{B(0,1)} f\big(x/|y|\big)\,dx\,dy = \int_{\mathbb{R}^n} \Phi(y)|y|^n \int_{B(0,\frac{1}{|y|})} f(x)\,dx\,dy \\
&= \int_{B(0,1)} \Phi(y)|y|^n \int_{B(0,\frac{1}{|y|})} f(x)\,dx\,dy + \int_{B^c(0,1)} \Phi(y)|y|^n \int_{B(0,\frac{1}{|y|})} f(x)\,dx\,dy.
\end{aligned}$$

On the other hand, for any nonnegative Schwartz function f satisfying $f = 1$ on $B(0,1)$, we have

$$\begin{aligned}
\int_{B(0,1)} \big|H_\Phi f(x)\big|\,dx &\geq \int_{B(0,1)} \Phi(y)|y|^n \int_{B(0,1)} f(x)\,dx\,dy \\
&\quad + \int_{B^c(0,1)} \Phi(y)|y|^n \int_{B(0,\frac{1}{|y|})} dx\,dy \\
&\sim \int_{B(0,1)} \Phi(y)|y|^n\,dy + \int_{B^c(0,1)} \Phi(y)\,dy.
\end{aligned}$$

This implies that

$$\int_{B(0,1)} \Phi(y)|y|^n\,dy + \int_{B^c(0,1)} \Phi(y)\,dy < \infty.$$

Remark 1.2 ($H_\Phi f$ is well defined as a tempered distribution). If Φ satisfies (1.2), then $H_\Phi f$ makes sense for all $f \in \mathscr{S}(\mathbb{R}^n)$ for the reason that for $x \neq 0$,

$$\begin{aligned}
\big|H_\Phi f(x)\big| &\leq \Big(\int_{B(0,1)} + \int_{B^c(0,1)}\Big) \Phi(y) f\big(x/|y|\big)\,dy \\
&\leq |x|^{-n} \int_{B(0,1)} |y|^n \Phi(y) \big(\big|x/|y|\big|^n f\big(x/|y|\big)\big) + \|f\|_{L^\infty} \int_{B^c(0,1)} \Phi(y)\,dy \\
&\lesssim C_f(1+|x|^{-n}) \Big(\int_{B(0,1)} |y|^n \Phi(y)\,dy + \int_{B^c(0,1)} \Phi(y)\,dy\Big) < \infty
\end{aligned}$$

and

$$
\begin{aligned}
\int_{B(0,1)} \big|H_\Phi f(x)\big|\,dx &\le \int_{B(0,1)} \int_{\mathbb{R}^n} \Phi(y)\big|f\big(x/|y|\big)\big|\,dy\,dx \\
&= \int_{B(0,1)} \int_{B(0,1)} \Phi(y)\big|f\big(x/|y|\big)\big|\,dy\,dx \\
&\quad + \int_{B(0,1)} \int_{B^c(0,1)} \Phi(y)\big|f\big(x/|y|\big)\big|\,dy\,dx \\
&\le \int_{B(0,1)} \Phi(y) \int_{\mathbb{R}^n} \big|f\big(x/|y|\big)\big|\,dx\,dy \\
&\quad + \big|B(0,1)\big| \cdot \|f\|_{L^\infty} \cdot \int_{B^c(0,1)} \Phi(y)\,dy \\
&\le \int_{B(0,1)} |y|^n \Phi(y)\,dy \|f\|_{L^1} \\
&\quad + \big|B(0,1)\big| \cdot \|f\|_{L^\infty} \cdot \int_{B^c(0,1)} \Phi(y)\,dy \\
&< \infty.
\end{aligned}
$$

Thus, for $f \in \mathscr{S}(\mathbb{R}^n)$, $H_\Phi f$ is a locally integrable function with polynomial growth at infinity. This implies that $H_\Phi f$ is a tempered distribution for $f \in \mathscr{S}(\mathbb{R}^n)$. Write

$$
\langle H_\Phi f, g\rangle = \int_{\mathbb{R}^n} H_\Phi f(x) g(x)\,dx,
$$

where $\langle u, f\rangle$ is the action of a tempered distribution u on a Schwartz function f.

Remark 1.3 ($H_\Phi : \mathscr{S} \to \mathscr{S}'$ is continuous). For $f, g \in \mathscr{S}(\mathbb{R}^n)$, we have that

$$
\int_{\mathbb{R}^n} \big|f\big(x/|y|\big) g(x)\big|\,dx \le \|f\|_{L^\infty} \|g\|_{L^1}
$$

and

$$
\int_{\mathbb{R}^n} \big|f\big(x/|y|\big) g(x)\big|\,dx \le \|g\|_{L^\infty} \big\|f\big(\cdot/|y|\big)\big\|_{L^1} \le |y|^n \|g\|_{L^\infty} \|f\|_{L^1}.
$$

It follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \big|\Phi(y)\big| \int_{\mathbb{R}^n} \big|f\big(x/|y|\big) g(x)\big|\,dx\,dy \\
&\quad \lesssim \big(\|f\|_{L^1} + \|f\|_{L^\infty}\big)\big(\|g\|_{L^1} + \|g\|_{L^\infty}\big) \int_{\mathbb{R}^n} \big|\Phi(y)\big| \min\big\{1, |y|^n\big\}\,dy.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\big|\langle H_\Phi f, g\rangle\big| &= \Big|\int_{\mathbb{R}^n} H_\Phi f(x) g(x)\,dx\Big| \\
&\le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(y)\big|f\big(x/|y|\big)\big|\,dy g(x)\,dx \\
&\le \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^n} \big|f\big(x/|y|\big)\big| \cdot \big|g(x)\big|\,dx\,dy
\end{aligned}
$$

$$\lesssim \big(\|f\|_{L^1} + \|f\|_{L^\infty}\big)\big(\|g\|_{L^1} + \|g\|_{L^\infty}\big) \\ \times \int_{\mathbb{R}^n} \big|\Phi(y)\big| \min\big\{1, |y|^n\big\}\, dy.$$

Using the definition of Schwartz function space, we have $|\langle H_\Phi f, g_l\rangle| \to 0$ for $f, g_l \in \mathscr{S}(\mathbb{R}^n)$ satisfying that $g_l \to 0$ as $l \to \infty$ in the topology of $\mathscr{S}$.

Remark 1.4 (Fourier transform of $H_\Phi f$). Define

$$\widetilde{H_\Phi} f(x) = \int_{\mathbb{R}^n} \Phi(y)|y|^n f\big(|y|x\big)\, dy.$$

By a method similar to the one used before, we can verify that $\widetilde{H_\Phi} f$ is a tempered distribution and that the map $\widetilde{H_\Phi} : \mathscr{S} \to \mathscr{S}'$ is continuous. Moreover, we have

$$\widehat{H_\Phi f} = \widetilde{H_\Phi} \widehat{f}$$

in the distribution sense. Indeed, for $f, g \in \mathscr{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}
\langle \widehat{H_\Phi f}, g\rangle = \langle H_\Phi f, \widehat{g}\rangle &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(y) f\big(x/|y|\big)\, dy \widehat{g}(x)\, dx \\
&= \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^n} f\big(x/|y|\big)\widehat{g}(x)\, dx\, dy \\
&= \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^n} \widehat{f\big(\cdot/|y|\big)}(x) g(x)\, dx\, dy \\
&= \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^n} |y|^n \widehat{f}\big(|y|x\big) g(x)\, dx\, dy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y|^n \Phi(y) \widehat{f}\big(|y|x\big)\, dy\, g(x)\, dx \\
&= \langle \widetilde{H_\Phi} \widehat{f}, g\rangle.
\end{aligned}$$

Remark 1.5 (Adjoint operator of $H_\Phi f$). We define the complex inner product

$$\langle f | g\rangle = \int_{\mathbb{R}^n} f(x)\overline{g}(x)\, dx.$$

The adjoint operator of $H_\Phi f$ is defined by

$$\langle H_\Phi f | g\rangle = \langle f | H_\Phi^* g\rangle$$

for $f, g \in \mathscr{S}(\mathbb{R}^n)$. By a direct calculation, we have

$$\begin{aligned}
\langle H_\Phi f | g\rangle &= \int_{\mathbb{R}^n} H_\Phi f(x)\overline{g}(x)\, dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(y) f\big(x/|y|\big)\, dy \overline{g}(x)\, dx \\
&= \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^n} f\big(x/|y|\big)\overline{g}(x)\, dx\, dy \\
&= \int_{\mathbb{R}^n} |y|^n \Phi(y) \int_{\mathbb{R}^n} f(x)\overline{g}\big(|y|x\big)\, dx\, dy
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} \Phi(y)|y|^n \overline{g}\big(|y|x\big)\, dy\, dx \\
&= \langle f | \widetilde{H_\Phi g} \rangle.
\end{aligned}
$$

It follows that $H_\Phi^* g = \widetilde{H_\Phi g}$ in the distribution sense.

We turn now to give definitions of modulation and Wiener amalgam spaces. Let $\mathscr{S} := \mathscr{S}(\mathbb{R}^n)$ be the Schwartz space, and let $\mathscr{S}' := \mathscr{S}'(\mathbb{R}^n)$ be the space of tempered distributions. We define the Fourier transform $\mathscr{F} f$ and the inverse Fourier transform $\mathscr{F}^{-1} f$ of $f \in \mathscr{S}(\mathbb{R}^n)$ by

$$
\begin{aligned}
\mathscr{F} f(\xi) = \hat{f}(\xi) &= \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx, \\
\mathscr{F}^{-1} f(x) = f^{\vee}(x) &= \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x\cdot\xi}\, d\xi.
\end{aligned}
$$

The translation operator is defined as $T_{x_0} f(x) = f(x - x_0)$, and the modulation operator is defined as $M_\xi f(x) = e^{2\pi i \xi\cdot x} f(x)$, for $x, x_0, \xi \in \mathbb{R}^n$. Fix a nonzero function $\phi \in \mathscr{S}$. The short-time Fourier transform of $f \in \mathscr{S}'$ with respect to the window ϕ is given by

$$V_\phi f(x,\xi) = \langle f, M_\xi T_x \phi \rangle,$$

and that can be written as

$$V_\phi f(x,\xi) = \int_{\mathbb{R}^n} f(y) \overline{\phi(y-x)} e^{-2\pi i y\cdot\xi}\, dy$$

if $f \in \mathscr{S}$. We give the (continuous) definition of modulation space $\mathcal{M}_{p,q}^s$ as follows.

Definition 1.6. Let $s \in \mathbb{R}$, $0 < p, q \le \infty$. The (weighted) modulation space $\mathcal{M}_{p,q}^s$ consists of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that the (weighted) modulation space norm

$$
\begin{aligned}
\|f\|_{\mathcal{M}_{p,q}^s} &= \big\| \big\| V_\phi f(x,\xi) \big\|_{L_{x,p}} \big\|_{L_{\xi,q}^s} \\
&= \Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \big| V_\phi f(x,\xi) \big|^p\, dx \Big)^{q/p} \langle \xi \rangle^{sq}\, dx \Big)^{1/q}
\end{aligned}
$$

is finite, with the usual modifications when $p = \infty$ or $q = \infty$. This definition is independent of the choice of the window $\phi \in \mathscr{S}$.

Applying frequency-uniform localization techniques, one can give an alternative definition of modulation spaces (see [18] for details). We denote by Q_k the unit cube with center at k. Then the family $\{Q_k\}_{k\in\mathbb{Z}^n}$ constitutes a decomposition of $\mathbb{R}^n$. Let $\eta \in \mathscr{S}(\mathbb{R}^n)$, $\eta : \mathbb{R}^n \to [0,1]$ be a smooth function satisfying $\eta(\xi) = 1$ for $|\xi|_\infty \le 1/2$ and $\eta(\xi) = 0$ for $|\xi| \ge 3/4$. Let

$$\eta_k(\xi) = \eta(\xi - k), \quad k \in \mathbb{Z}^n,$$

be a translation of η. Since $\eta_k(\xi) = 1$ in Q_k, we have that $\sum_{k\in\mathbb{Z}^n} \eta_k(\xi) \ge 1$ for all $\xi \in \mathbb{R}^n$. Denote

$$\sigma_k(\xi) = \eta_k(\xi) \Big(\sum_{l\in\mathbb{Z}^n} \eta_l(\xi) \Big)^{-1}, \quad k \in \mathbb{Z}^n.$$

It is easy to see that $\{\sigma_k\}_{k\in\mathbb{Z}^n}$ constitutes a smooth partition of the unity, and $\sigma_k(\xi)=\sigma(\xi-k)$. The frequency-uniform decomposition operators can be defined by

$$\Box_k := \mathscr{F}^{-1}\sigma_k\mathscr{F}$$

for $k\in\mathbb{Z}^n$. Now, we give the (discrete) definition of modulation space $M^s_{p,q}$.

Definition 1.7. Let $s\in\mathbb{R}$, $0<p,q\le\infty$. The modulation space $M^s_{p,q}$ consists of all $f\in\mathscr{S}'$ such that the (quasi)norm

$$\|f\|_{M^s_{p,q}} := \Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{sq}\|\Box_k f\|_p^q\Big)^{1/q}$$

is finite. We write $M_{p,q}:=M^0_{p,q}$ for short. We also recall that this definition is independent of the choice of $\{\sigma_k\}_{k\in\mathbb{Z}^n}$, and the definitions of $\mathcal{M}^s_{p,q}$ and $M^s_{p,q}$ are equivalent (see [19]).

Definition 1.8. Let $0<p,q\le\infty$, $s\in\mathbb{R}$. Given a window function $\phi\in\mathscr{S}\setminus\{0\}$, the Wiener amalgam space $W^s_{p,q}$ consists of all $f\in\mathscr{S}'(\mathbb{R}^n)$ such that the norm

$$\begin{aligned}\|f\|_{W^s_{p,q}} &= \big\|\big\|V_\phi f(x,\xi)\big\|_{L^s_{\xi,q}}\big\|_{L_{x,p}}\\ &= \Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}|V_\phi f(x,\xi)|^q\langle\xi\rangle^{sq}\,d\xi\Big)^{p/q}dx\Big)^{1/p}\end{aligned}$$

is finite, with the usual modifications when $p=\infty$ or $q=\infty$. We write $W_{p,q}:=W^0_{p,q}$ for short.

Now, we state our main results as follows.

Theorem 1.9. *Let* $1\le p,q\le\infty$, $(1/p-1/2)(1/q-1/p)\ge 0$, Φ *be a nonnegative function satisfying the basic assumption (1.2). Then* H_Φ *is bounded on* $M_{p,q}$ *if and only if*

$$\int_{\mathbb{R}^n}\big(|y|^{n/p}+|y|^{n/q'}\big)\Phi(y)\,dy<\infty.$$

Theorem 1.10. *Let* $1\le p,q\le\infty$, $(1/q-1/2)(1/q-1/p)\le 0$, Φ *be a nonnegative function satisfying the basic assumption (1.2). Then* H_Φ *is bounded on* $W_{p,q}$ *if and only if*

$$\int_{\mathbb{R}^n}\big(|y|^{n/p}+|y|^{n/q'}\big)\Phi(y)\,dy<\infty.$$

Our article is organized as follows. In Section 2, we collect some basic properties of modulation and Wiener amalgam spaces, and we give proofs of Theorems 1.9 and 1.10. We also adopt the following notation throughout this article. We use $X\lesssim Y$ to denote the statement that $X\le CY$, with a positive constant C that may depend on n, p, but that might be different from line to line. The notation $X\sim Y$ means the statement $X\lesssim Y\lesssim X$. We use $X\lesssim_\lambda Y$ to denote $X\le C_\lambda Y$, meaning that the implied constant C_λ depends on the parameter λ. For a multi-index $k=(k_1,k_2,\dots,k_n)\in\mathbb{Z}^n$, we denote $|k|_\infty:=\max_{i=1,2,\dots,n}|k_i|$ and $\langle k\rangle:=(1+|k|^2)^{1/2}$.

2. Proofs of main theorems

First, we list some basic properties about modulation spaces as follows.

Lemma 2.1 (Time-frequency symmetry). *We have* $\|\mathscr{F}^{-1}f\|_{M_{p,q}} \sim \|f\|_{W_{q,p}} \sim \|\mathscr{F}f\|_{M_{p,q}}$.

Proof. In view of the fact that

$$\left|V_\phi f(x,\xi)\right| = \left|V_{\hat{\phi}}\hat{f}(\xi,-x)\right|,$$

the conclusion follows by the definition of modulation and Wiener amalgam spaces. □

Lemma 2.2 (Dilation property of modulation space [17, Theorem 1.1]). *Let* $1 \le p,q \le \infty$, $(1/p-1/2)(1/q-1/p) \ge 0$. *Set* $f_\lambda(x) = f(\lambda x)$. *Then*

$$\|f_\lambda\|_{M_{p,q}} \lesssim \max\{\lambda^{-n/p}, \lambda^{-n/q'}\}\|f\|_{M_{p,q}}.$$

Lemma 2.3 (Embedding relations between modulation and Lebesgue spaces [10, Theorems 1.3–1.4]). *The following embedding relations are right:*

(1) $M_{p,q} \hookrightarrow L^p$ *for* $1/q \ge 1/p \ge 1/2$;
(2) $L^p \hookrightarrow M_{p,q}$ *for* $1/q \le 1/p \le 1/2$.

Lemma 2.4 (Embedding relations between Wiener amalgam and Lebesgue spaces [2, Theorems 1.1–1.2]). *The following embedding relations are right:*

(1) $W_{p,q} \hookrightarrow L^p$ *for* $1/p \ge 1/q \ge 1/2$;
(2) $L^p \hookrightarrow W_{p,q}$ *for* $1/p \le 1/q \le 1/2$.

Lemma 2.5. *Let* $1 \le p,q \le \infty$. *We have*

(1)
$$\left|\int_{\mathbb{R}^n} f(x)\bar{g}(x)\,dx\right| \le \|f\|_{M_{p',q'}}\|g\|_{M_{p,q}},$$

(2)
$$\left|\int_{\mathbb{R}^n} f(x)\bar{g}(x)\,dx\right| \le \|f\|_{W_{q',p'}}\|g\|_{W_{q,p}}.$$

Proof. By Lemma 2.1, we only give the proof of the first inequality. Denote $\eta_k^* = \sum_{l\in\mathbb{Z}^n:\eta_k\eta_l\neq 0}\eta_k$ and $\Box_k^* = \mathscr{F}^{-1}\eta_k^*\mathscr{F}$. By the definition of modulation spaces and Plancherel's equality, we get

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} f(x)\bar{g}(x)\,dx\right| &= \left|\int_{\mathbb{R}^n}\hat{f}\hat{g}\,d\xi\right| = \left|\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}^n}\sigma_k\hat{f}\cdot\sum_{l\in\mathbb{Z}^n}\sigma_l\hat{g}\,dx\right| \\
&= \left|\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}^n}\sigma_k\hat{f}\cdot\sigma_k^*\hat{g}\,dx\right| = \left|\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}^n}\Box_k f\cdot\Box_k^* g\,dx\right| \\
&\le \sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|\Box_k f\cdot\Box_k^* g|\,dx \le \sum_{k\in\mathbb{Z}^n}\|\Box_k f\|_{L^{p'}}\|\Box_k^* g\|_{L^p} \\
&\le \Big(\sum_{k\in\mathbb{Z}^n}\|\Box_k f\|_{L^{p'}}^{q'}\Big)^{1/q'}\Big(\sum_{k\in\mathbb{Z}^n}\|\Box_k^* g\|_{L^p}^{q}\Big)^{1/q} \le \|f\|_{M_{p',q'}}\|g\|_{M_{p,q}},
\end{aligned}$$

where we use Hölder's inequality in the last two lines and the fact that the definition of modulation space is independent of the decomposition function. □

In order to make the proof clearer, we give the following technical proposition.

Proposition 2.6 (For technique). *Let $1/2 \le 1/p \le 1/q \le 1$, and let Φ be a nonnegative function satisfying the basic assumption (1.2). Then*

(1) *if $H_\Phi : M_{p,q} \to L^p$ is bounded, then we have*

$$\int_{\mathbb{R}^n} |y|^{n/p}\Phi(y)\,dy < \infty;$$

(2) *if $H^*_\Phi : W_{q,p} \to L^q$ is bounded, then we have*

$$\int_{\mathbb{R}^n} |y|^{n/q'}\Phi(y)\,dy < \infty;$$

(3) *if $H^*_\Phi : M_{p,q} \to L^p$ is bounded, then we have*

$$\int_{\mathbb{R}^n} |y|^{n/p'}\Phi(y)\,dy < \infty;$$

(4) *if $H_\Phi : W_{q,p} \to L^q$ is bounded, then we have*

$$\int_{\mathbb{R}^n} |y|^{n/q}\Phi(y)\,dy < \infty.$$

Proof. We only prove statements (1) and (2) since the other cases can be handled similarly. Suppose that $H_\Phi : M_{p,q} \to L^p$ is bounded. Let $\psi : \mathbb{R}^n \to [0,1]$ be a smooth bump function supported in the ball $\{\xi : |\xi| < \frac{3}{2}\}$, and let it be equal to 1 on the ball $\{\xi : |\xi| \le \frac{4}{3}\}$. Let $\rho(\xi) = \psi(\xi) - \psi(2\xi)$. Then ρ is a positive smooth function supported in the annulus $\{\xi : \frac{2}{3} < |\xi| < \frac{3}{2}\}$, satisfying $\rho(\xi) = 1$ on a smaller annulus $\{\xi : \frac{3}{4} \le |\xi| \le \frac{4}{3}\}$. Denote $\rho_j(\xi) := \rho(\xi/2^j)$. We have $\operatorname{supp}\rho_j \subset \{\xi : \frac{2}{3}\cdot 2^j \le |\xi| \le \frac{3}{2}\cdot 2^j\}$ and $\rho_j(\xi) = 1$ on $\{\xi : \frac{3}{4}\cdot 2^j \le |\xi| \le \frac{4}{3}\cdot 2^j\}$. Thus, we have $\operatorname{supp}\sum_{j=1}^N \rho_j(\xi) \subset \{\xi : \frac{4}{3} \le |\xi| \le \frac{3}{2}\cdot 2^N\}$ and $\sum_{j=1}^N \rho_j(\xi) = 1$ on $\{\xi : \frac{3}{2} \le |\xi| \le \frac{4}{3}\cdot 2^N\}$.

Take φ to be a nonnegative smooth function satisfying that $\operatorname{supp}\widehat{\varphi} \subset B(0,1/2)$, $\varphi(0) = 1$. Choose $f_N(x) = (\sum_{j=0}^{N+1} \rho_j(x)\cdot |x|^{-n/p}) * \varphi$. So we have

$$\operatorname{supp}\widehat{f_N} \subset B(0,1/2) \qquad \text{and} \qquad f_N(x) \gtrsim \sum_{j=1}^{N} \rho_j(x)\cdot |x|^{-n/p}. \tag{2.1}$$

In the above, the previous inclusion relation follows from the support condition of $\widehat{\varphi}$. We interpret the latter inequality. We only need to prove it when the right-hand side is nonzero, that is, $x \in \{\frac{4}{3} \le |x| \le \frac{3}{2}\cdot 2^N\}$. For the nonnegative function φ satisfying $\varphi(0) = 1$, there exists a positive constant $\delta < \min\{4/3, 1/12\}$ such that $\varphi(x) > 1/2$ when $|x| < \delta$. By the triangle inequality and the properties of φ, we have that

$$
\begin{aligned}
f_N(x) &= \Big(\sum_{j=0}^{N+1} \rho_j(x)\cdot |x|^{-n/p}\Big) * \varphi \\
&= \int_{\mathbb{R}^n} \Big(\sum_{j=0}^{N+1} \rho_j(x-y)\cdot |x-y|^{-n/p}\Big)\varphi(y)\,dy \\
&\gtrsim \int_{\substack{\frac{4}{3}\le |x-y|\le \frac{3}{2}\cdot 2^N \\ |y|<\delta}} \sum_{j=0}^{N+1} \rho_j(x-y)\cdot |x|^{-n/p}\,dy \gtrsim \sum_{j=1}^{N} \rho_j(x)\cdot |x|^{-n/p},
\end{aligned}
$$

so we prove (2.1). We have that

$$
\begin{aligned}
\|H_\Phi f_N\|_{L^p} &= \Big\|\int_{\mathbb{R}^n} \Phi(y) f_N\big(x/|y|\big)\,dy\Big\|_{L^p} \\
&\gtrsim \Big\|\int_{\mathbb{R}^n} \Phi(y)|y|^{n/p}\cdot \sum_{j=1}^{N} \rho_j\big(x/|y|\big)\cdot |x|^{-n/p}\,dy\Big\|_{L^p} \\
&\ge \Big\|\int_{B(0,\frac{2}{3}\cdot 2^M)\setminus B(0,\frac{3}{4}\cdot 2^{-M})} \Phi(y)|y|^{n/p}\cdot \sum_{j=1}^{N} \rho_j\big(x/|y|\big)\cdot |x|^{-n/p}\,dy\Big\|_{L^p} \\
&\ge \Big\|\int_{B(0,\frac{2}{3}\cdot 2^M)\setminus B(0,\frac{3}{4}\cdot 2^{-M})} \Phi(y)|y|^{n/p}\cdot \chi_{\{2^M<|x|<2^{N-M}\}}(x)\cdot |x|^{-n/p}\,dy\Big\|_{L^p} \\
&= \int_{B(0,\frac{2}{3}\cdot 2^M)\setminus B(0,\frac{3}{4}\cdot 2^{-M})} \Phi(y)|y|^{n/p}\,dy\cdot \Big\||x|^{-n/p}\chi_{\{2^M<|x|<2^{N-M}\}}(x)\Big\|_{L^p} \\
&\gtrsim \int_{B(0,\frac{2}{3}\cdot 2^M)\setminus B(0,\frac{3}{4}\cdot 2^{-M})} \Phi(y)|y|^{n/p}\,dy\cdot (\lg 2^{N-2M})^{1/p},
\end{aligned}
$$

where we use the fact that $\sum_{j=1}^{N} \rho_j(x/|y|) = 1$ for $y \in B(0,\frac{2}{3}\cdot 2^M)\setminus B(0,\frac{3}{4}\cdot 2^{-M})$ and $x \in B(0,2^M)\setminus B(0,2^{N-M})$. On the other hand, observing that $\operatorname{supp}\widehat{f_N} \subset B(0,1/2)$, we have

$$
\begin{aligned}
\|f_N\|_{M_{p,q}} &= \Big(\sum_{\substack{\sigma_k\widehat{f_N}\neq 0\\ k\in\mathbb{Z}^n}} \|\mathscr{F}^{-1}(\sigma_k\widehat{f_N})\|_{L^p}^q\Big)^{1/q} \lesssim \Big(\sum_{\substack{\sigma_k\widehat{f_N}\neq 0\\ k\in\mathbb{Z}^n}} \|f_N\|_{L^p}^q\Big)^{1/q} \\
&\lesssim \|f_N\|_{L^p} \lesssim \Big\|\sum_{j=0}^{N+1} \rho_j(x)\cdot |x|^{-n/p}\Big\|_{L^p} \sim (\ln 2^N)^{1/p}.
\end{aligned}
$$

Using the boundedness of H_Φ and the above estimates for $H_\Phi f_N$ and f_N, we have that

$$
\|H_\Phi\|_{M_{p,q}\to L^p} \ge \frac{\|H_\Phi f_N\|_{L^p}}{\|f_N\|_{M_{p,q}}} \gtrsim \int_{B(0,\frac{2}{3}\cdot 2^M)\setminus B(0,\frac{3}{4}\cdot 2^{-M})} \Phi(y)|y|^{n/p}\,dy\Big(\frac{\lg 2^{N-2M}}{\lg 2^N}\Big)^{1/p}.
$$

Letting $N \to \infty$, we have

$$\int_{B(0,\frac{2}{3}\cdot 2^M)\setminus B(0,\frac{3}{4}\cdot 2^{-M})} \Phi(y)|y|^{n/p}\,dy \lesssim \|H_\Phi\|_{M_{p,q}\to L^p}.$$

By the arbitrariness of M, we let $M \to \infty$ and obtain that $\int_{\mathbb{R}^n} \Phi(y)|y|^{n/p}\,dy \lesssim \|H_\Phi\|_{M_{p,q}\to L^p}$.

Now we turn to give the proof for the second conclusion. Suppose that $H_\Phi^* : W^{q,p} \to L^q$ is bounded. As in the proof of conclusion (1), we take $g_N(x) = \sum_{j=1}^N \rho_j(x)\cdot|x|^{-n/q}$. A direction calculation yields that

$$\begin{aligned}
\|H_\Phi^* g_N\|_{L^q} &= \Big\|\int_{\mathbb{R}^n} \Phi(y)|y|^n g_N\big(|y|x\big)\,dy\Big\|_{L^q} \\
&= \Big\|\int_{\mathbb{R}^n} \Phi(y)|y|^{n/q'}\cdot\sum_{j=1}^N \rho_j\big(|y|x\big)\cdot|x|^{-n/q}\,dy\Big\|_{L^p} \\
&\geq \Big\|\int_{B(0,4/3\cdot 2^M)\setminus B(0,3/2\cdot 2^{-M})} \Phi(y)|y|^{n/q'}\cdot\sum_{j=1}^N \rho_j\big(|y|x\big)\cdot|x|^{-n/q}\,dy\Big\|_{L^q} \\
&\geq \int_{B(0,4/3\cdot 2^M)\setminus B(0,3/2\cdot 2^{-M})} \Phi(y)|y|^{n/q'}\,dy\cdot\big\||x|^{-n/q}\chi_{\{2^M<|x|<\cdot 2^{N-M}\}}(x)\big\|_{L^q} \\
&\gtrsim \int_{B(0,4/3\cdot 2^M)\setminus B(0,3/2\cdot 2^{-M})} \Phi(y)|y|^{n/q'}\,dy\cdot(\lg 2^{N-2M})^{1/q}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\big\|\mathscr{F}^{-1}(\sigma_k g_N)\big\|_{L^p} &= \Big\|\mathscr{F}^{-1}\Big(\sigma_k\sum_{j=1}^N \rho_j(x)\cdot|x|^{-n/q}\Big)\Big\|_{L^p} \\
&\lesssim \langle k\rangle^{-n/q}\Big\|\mathscr{F}^{-1}\Big(\sigma_k\sum_{j=1}^N \rho_j(x)\Big)\Big\|_{L^p} \\
&\leq \langle k\rangle^{-n/q}\|\mathscr{F}^{-1}\sigma_k\|_{L^p}\cdot\sum_{1\leq j\leq N:\sigma_k\rho_j\neq 0}\big\|\mathscr{F}^{-1}\big(\rho_j(x)\big)\big\|_{L^1} \\
&\lesssim \langle k\rangle^{-n/q}.
\end{aligned}$$

Using Lemma 2.1, we obtain that

$$\begin{aligned}
\|g_N\|_{W_{q,p}} = \|\mathscr{F}^{-1}g_N\|_{M_{p,q}} &= \Big(\sum_{k\in\mathbb{Z}^n}\big\|\mathscr{F}^{-1}(\sigma_k g_N)\big\|_{L^p}^q\Big)^{1/q} \\
&\lesssim \Big(\sum_{\substack{|k|<2^{N+1}\\ k\in\mathbb{Z}^n}}\langle k\rangle^{-n}\Big)^{1/q} \sim (\lg 2^N)^{1/q}.
\end{aligned}$$

We deduce that

$$\|H_\Phi^*\|_{W_{q,p}\to L^q} \geq \frac{\|H_\Phi^* f_N\|_{L^q}}{\|f_N\|_{W_{q,p}}} \gtrsim \int_{B(0,\frac{4}{3}\cdot 2^M)\setminus B(0,\frac{3}{2}\cdot 2^{-M})} \Phi(y)|y|^{n/q'}\,dy\Big(\frac{\lg 2^{N-2M}}{\lg 2^N}\Big)^{1/q}.$$

Letting $N \to \infty$, we have

$$\int_{B(0,\frac{4}{3}\cdot 2^M)\setminus B(0,\frac{3}{2}\cdot 2^{-M})} \Phi(y)|y|^{n/q'}\, dy \lesssim \|H_\Phi^*\|_{W_{q,p}\to L^q}.$$

By the arbitrariness of M, we let $M \to \infty$ and obtain that $\int_{\mathbb{R}^n} \Phi(y)|y|^{n/q'}\, dy \lesssim \|H_\Phi^*\|_{W_{q,p}\to L^q}$. □

Next, we establish the following two propositions for reduction.

Proposition 2.7 (For reduction of modulation space)**.** *Let $1/2 \le 1/p \le 1/q \le 1$, and let Φ be a nonnegative function satisfying (1.2). If the Hausdorff operator H_Φ is bounded on $M_{p,q}$, we have that*

(1) *$H_\Phi : M_{p,q} \to L^p$ is bounded,*
(2) *$H_\Phi^* : W_{q,p} \to L^q$ is bounded.*

Proof. The first conclusion can be deduced by the embedding relation $M_{p,q} \hookrightarrow L^p$ (see Lemma 2.3) directly. We turn to prove the second conclusion. For any Schwartz function f, by the property of H_Φ and Lemma 2.1, we have $\|f\|_{M_{p,q}} = \|\widehat{f}\|_{W_{q,p}}$ and

$$\|H_\Phi f\|_{M_{p,q}} = \|\widehat{H_\Phi f}\|_{W_{q,p}} = \|\widetilde{H_\Phi}\widehat{f}\|_{W_{q,p}} = \|H_\Phi^* \widehat{f}\|_{W_{q,p}}.$$

Thus, if H_Φ is bounded on $M_{p,q}$, we have

$$\|H_\Phi^* \widehat{f}\|_{W_{q,p}} \lesssim \|\widehat{f}\|_{W_{q,p}}.$$

The embedding relation $W_{q,p} \hookrightarrow L^q$ then yields that

$$\|H_\Phi^* f\|_{L^q} \lesssim \|f\|_{W_{q,p}}$$

for all $f \in \mathscr{S}(\mathbb{R}^n)$. □

Proposition 2.8 (For reduction of Wiener amalgam space)**.** *Let $1/2 \le 1/q \le 1/p \le 1$, and let Φ be a nonnegative function satisfying (1.2). If the Hausdorff operator H_Φ is bounded on $W_{p,q}$, we have that*

(1) *$H_\Phi : W_{p,q} \to L^p$ is bounded,*
(2) *$H_\Phi^* : M_{q,p} \to L^q$ is bounded.*

Proof. The first conclusion can be deduced by the embedding relation $W_{p,q} \hookrightarrow L^p$ (see Lemma 2.4) directly. We turn to prove the second conclusion. For any Schwartz function f, by the property of H_Φ and Lemma 2.1, we have $\|f\|_{W_{p,q}} = \|\widehat{f}\|_{M_{q,p}}$ and

$$\|H_\Phi f\|_{W_{p,q}} = \|\widehat{H_\Phi f}\|_{M_{q,p}} = \|\widetilde{H_\Phi}\widehat{f}\|_{M_{q,p}} = \|H_\Phi^* \widehat{f}\|_{M_{q,p}}.$$

Thus, if H_Φ is bounded on $W_{p,q}$, we have

$$\|H_\Phi^* \widehat{f}\|_{M_{q,p}} \lesssim \|\widehat{f}\|_{M_{q,p}}.$$

The embedding relation $M_{q,p} \hookrightarrow L^q$ then yields that

$$\|H_\Phi^* f\|_{L^q} \lesssim \|f\|_{M_{q,p}}$$

for all $f \in \mathscr{S}(\mathbb{R}^n)$. □

We are now ready to prove Theorem 1.9.

Proof of Theorem 1.9. We divide this proof into two parts.

"IF" PART: Using the Minkowski inequality, we deduce that

$$\begin{aligned}\|H_\Phi f\|_{M_{p,q}} &\lesssim \Big\| \int_{\mathbb{R}^n} \Phi(y) f\big(x/|y|\big)\,dy \Big\|_{M_{p,q}} \\ &\lesssim \int_{\mathbb{R}^n} \Phi(y) \big\| f\big(x/|y|\big)\big\|_{M_{p,q}}\,dy.\end{aligned}$$

Recalling the dilation properties of modulation space (see Lemma 2.2), we obtain that

$$\begin{aligned}\|H_\Phi f\|_{M_{p,q}} &\lesssim \int_{\mathbb{R}^n} \Phi(y) \max\big\{|y|^{n/p}, |y|^{n/q'}\big\}\,dy \|f\|_{M_{p,q}} \\ &\lesssim \int_{\mathbb{R}^n} \big(|y|^{n/p} + |y|^{n/q'}\big)\Phi(y)\,dy\|f\|_{M_{p,q}}.\end{aligned}$$

This implies the boundedness of H_Φ on $M_{p,q}$.

"ONLY IF" PART: Suppose that H_Φ is bounded on $M_{p,q}$. If $1/2 \le 1/p \le 1/q \le 1$, then the conclusion can be verified directly by Propositions 2.6 and 2.7.

We only need to deal with the case in which $1/q \le 1/p \le 1/2$. We use a dual argument to deal with this case. Recalling that

$$\langle H_\Phi^* f | g\rangle = \langle f | H_\Phi g\rangle$$

for all $f, g \in \mathscr{S}(\mathbb{R}^n)$, by Lemmas 2.5 and 2.3 we deduce that

$$\begin{aligned}\big|\langle H_\Phi^* f | g\rangle\big| &= \big|\langle f | H_\Phi g\rangle\big| \\ &\le \|f\|_{M_{p',q'}} \|H_\Phi g\|_{M_{p,q}} \\ &\lesssim \|f\|_{M_{p',q'}} \|g\|_{M_{p,q}} \\ &\lesssim \|f\|_{M_{p',q'}} \|g\|_{L^p},\end{aligned}$$

which implies that

$$\|H_\Phi^* f\|_{L^{p'}} \lesssim \|f\|_{M_{p',q'}} \tag{2.2}$$

for all $f \in \mathscr{S}(\mathbb{R}^n)$. In addition, by the boundedness of H_Φ on $M_{p,q}$, we use Lemma 2.1 to deduce that H_Φ^* is also bounded on $W_{q,p}$. Thus, by Lemmas 2.5 and 2.4 we have

$$\begin{aligned}\big|\langle H_\Phi f | g\rangle\big| &= \big|\langle f | H_\Phi^* g\rangle\big| \\ &\le \|f\|_{W_{q',p'}} \|H_\Phi^* g\|_{W_{q,p}} \\ &\lesssim \|f\|_{W_{q',p'}} \|g\|_{W_{q,p}} \\ &\lesssim \|f\|_{W_{q',p'}} \|g\|_{L^q},\end{aligned}$$

which implies that

$$\|H_\Phi f\|_{L^{q'}} \lesssim \|f\|_{W_{q',p'}} \tag{2.3}$$

for all $f \in \mathscr{S}(\mathbb{R}^n)$.

Combining (2.2) and (2.3), and observing that $1/2 \le 1/p' \le 1/q'$, we use Proposition 2.6 to get the conclusion. □

Proof of Theorem 1.10. We divide this proof into two parts.

"IF" PART: Using Lemma 2.1 and the Minkowski inequality, we deduce that

$$\begin{aligned}\|H_\Phi f\|_{W_{p,q}} &\sim \|\widehat{H_\Phi f}\|_{M_{q,p}} \sim \|H_\Phi^* \widehat{f}\|_{M_{q,p}} \sim \Big\| \int_{\mathbb{R}^n} \Phi(y)|y|^n \widehat{f}\big(|y|x\big)\, dy \Big\|_{M_{q,p}} \\ &\lesssim \int_{\mathbb{R}^n} \Phi(y)|y|^n \big\|\widehat{f}\big(|y|x\big)\big\|_{M_{q,p}}\, dy.\end{aligned}$$

Recalling the dilation properties of modulation space (see Lemma 2.2), we obtain that

$$\begin{aligned}\|H_\Phi f\|_{W_{p,q}} &\lesssim \int_{\mathbb{R}^n} \Phi(y)|y|^n \max\big\{|y|^{-n/q}, |y|^{-n/p'}\big\}\, dy \|\widehat{f}\|_{M_{q,p}} \\ &\lesssim \int_{\mathbb{R}^n} \big(|y|^{n/p} + |y|^{n/q'}\big)\Phi(y)\, dy \|f\|_{W_{p,q}}.\end{aligned}$$

This implies the boundedness of H_Φ on $W_{p,q}$.

"ONLY IF" PART: Suppose that H_Φ is bounded on $W_{p,q}$. If $1/2 \le 1/q \le 1/p \le 1$, then the conclusion can be verified directly by Proposition 2.6 and 2.8.

For the case $1/p \le 1/q \le 1/2$, the desired conclusion follows by a dual argument as in the proof of Theorem 1.9. □

Remark 2.9. For various technical reasons, our main theorems only characterize the boundedness of Hausdorff operators on $M_{p,q}$ and $W_{p,q}$ in some special cases. Our theorems remain an open problem for the characterization of Hausdorff operators on the full range $1 \le p, q \le \infty$.

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