COMPLEX HADAMARD MATRICES WITH NONCOMMUTATIVE ENTRIES

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Communicated by V. Runde

ABSTRACT. We axiomatize and study the matrices of type $H \in M_N(A)$ having unitary entries $H_{ij} \in U(A)$ and whose rows and columns are subject to orthogonality-type conditions. Here A can be any C^*-algebra, for instance $A = \mathbb{C}$, where we obtain the usual complex Hadamard matrices, or $A = C(X)$, where we obtain the continuous families of complex Hadamard matrices. Our formalism allows the construction of a quantum permutation group $G \subset S_N^+$, whose structure and computation are discussed here.

Introduction

A complex Hadamard matrix is a square matrix $H \in M_N(\mathbb{C})$ whose entries are on the unit circle $|H_{ij}| = 1$ and whose rows are pairwise orthogonal. The basic example of such a matrix is the Fourier one, $F_N = (w^{ij})$ with $w = e^{2\pi i/N}$:

$$F_N = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^{N-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & w^{N-1} & w^{2(N-1)} & \dots & w^{(N-1)^2} \end{pmatrix}.$$

In general, the theory of the complex Hadamard matrices can be regarded as a nonstandard branch of discrete Fourier analysis. (For a survey of the potential applications to quantum physics and quantum information theory questions, see [19].) The Hadamard matrices are also known to parameterize the pairs of

orthogonal maximal Abelian subalgebras in the simplest von Neumann algebra, $M_N(\mathbb{C})$. This discovery by Popa [18] has led to deep connections with subfactor theory and some related areas (see [13], [15]).

One way of understanding the operator algebra significance of the Hadamard matrices is via quantum permutation groups. The general theory here, which goes back to [2], was developed in a number of recent papers, including [3], [5]–[8], [10], and [22]. The idea is very simple. Given a Hadamard matrix $H \in M_N(\mathbb{C})$, with rows denoted $H_1, \ldots, H_N \in \mathbb{T}^N$, the rank 1 projections $P_{ij} = \text{Proj}(H_i/H_j)$ form a magic unitary in the sense of Wang [21], and so produce a representation $C(S_N^+) \to M_N(\mathbb{C})$. We can consider then the smallest quantum subgroup $G \subset S_N^+$ producing a factorization of type $C(S_N^+) \to C(G) \to M_N(\mathbb{C})$ and call it the *quantum group associated to H*.

As a basic example, for $H = F_N$ we obtain $G = \mathbb{Z}_N$. Several other things are known about the construction $H \to G$, including the important fact that the Jones planar algebra $P = (P_k)$ associated to H can be recovered from the knowledge of G (see [2]).

In this paper we develop an extension of this theory. We consider square matrices over a C^*-algebra $H \in M_N(A)$, whose entries are unitaries, which commute on all rows and columns, and which are such that the rows and columns are pairwise orthogonal. In the case $A = \mathbb{C}$, we obtain the usual Hadamard matrices. More generally, in the commutative case $A = C(X)$, we obtain the continuous families of Hadamard matrices.

In general now, the point with our axioms is that these allow the construction of a magic unitary matrix $P \in M_N(M_N(A))$, via the following formula:

$$(P_{ij})_{ab} = \frac{1}{N} H_{ia} H_{ja}^* H_{jb} H_{ib}^*.$$

Thus, associated to our Hadamard matrix $H \in M_N(A)$ is a certain quantum permutation group $G \subset S_N^+$. We will show here how certain basic Hadamard matrix results, mostly related to the correspondence $H \to G$, can be extended to the general case, and we will discuss as well some features brought by the noncommutative setting.

This paper is organized as follows. Section 1 is a preliminary section, while in Section 2 we present examples and classification results. In Section 3 we discuss the general properties of the associated quantum permutation groups, and in Section 4 we perform some explicit computations.

1. Definition and basic properties

Let A be a unital C^*-algebra. For most of the applications, A will be a commutative algebra, $A = C(X)$, with X being a compact space, or a matrix algebra, $A = M_K(\mathbb{C})$, with $K \in \mathbb{N}$. We will sometimes consider random matrix algebras, $A = M_K(C(X))$, with X being a compact space and with $K \in \mathbb{N}$. Our guiding example will be $A = \mathbb{C}$.

Two row or column vectors over A, say $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$, will be called *orthogonal* when $\sum_i a_i b_i^* = \sum_i a_i^* b_i = 0$. Observe that, by applying

the involution, we have as well $\sum_i b_i a_i^* = \sum_i b_i^* a_i = 0$. With this notion in hand, we can formulate the following.

Definition 1.1. A Hadamard matrix over A is a square matrix $H \in M_N(A)$ such that:

  (1) all the entries of H are unitaries, $H_{ij} \in U(A)$;
  (2) these entries commute on all rows and all columns of H;
  (3) the rows and columns of H are pairwise orthogonal.

As a first remark, in the simplest case $A = \mathbb{C}$, the unitary group is the unit circle in the complex plane, $U(\mathbb{C}) = \mathbb{T}$, and we obtain the usual complex Hadamard matrices. In the general commutative case $A = C(X)$ with X compact space, our Hadamard matrix must be formed of "fibers," one for each point $x \in X$. Therefore, we obtain the following.

Proposition 1.2. *The Hadamard matrices $H \in M_N(A)$ over a commutative algebra $A = C(X)$ are exactly the families of complex Hadamard matrices of type*

$$H = \{H^x \mid x \in X\}$$

with H^x depending continuously on the parameter $x \in X$.

Proof. This follows indeed by combining the above two observations. Observe that, when we wrote $A = C(X)$ in the above statement, we used the Gelfand theorem. $\square$

Let us comment now on the above axioms. Since for $U, V \in U(A)$ the commutation relation $UV = VU$ implies as well the commutation relations $UV^* = V^*U$, $U^*V = VU^*$, $U^*V^* = U^*V^*$, the axiom (2) tells us that the C^*-algebras $R_1, \ldots, R_N$ and $C_1, \ldots, C_N$ generated by the rows and the columns of A must be all commutative. We will be particulary interested in the following type of matrices.

Definition 1.3. A Hadamard matrix $H \in M_N(A)$ is called *nonclassical* if the C^*-algebra generated by its coefficients is not commutative.

Let us comment now on the axiom (3). According to our definition of orthogonality there are four sets of relations to be satisfied, namely, for any $i \neq k$ we must have

$$\sum_j H_{ij} H_{kj}^* = \sum_j H_{ij}^* H_{kj} = \sum_j H_{ji} H_{jk}^* = \sum_j H_{ji}^* H_{jk} = 0.$$

Now since by axiom (1) all the entries H_{ij} are known to be unitaries, we can replace this formula by the following more general equation, valid for any i, k:

$$\sum_j H_{ij} H_{kj}^* = \sum_j H_{ij}^* H_{kj} = \sum_j H_{ji} H_{jk}^* = \sum_j H_{ji}^* H_{jk} = N\delta_{ik}.$$

The point now is that, in terms of the matrices $H = (H_{ij})$, $H^* = (H_{ji}^*)$, $H^t = (H_{ji})$, and $\bar{H} = (H_{ij}^*)$, these equations simply read

$$HH^* = H^*H = H^t\bar{H} = \bar{H}H^t = N1_N.$$

So, let us recall now that a square matrix $H \in M_N(A)$ is called *biunitary* if both H and H^t are unitaries. In the particular case where A is commutative, $A = C(X)$, we have H unitary $\implies$ H^t unitary, so in this case "biunitary" means of course "unitary."

In terms of this notion, we have the following reformulation of Definition 1.1.

Proposition 1.4. *Assume that $H \in M_N(A)$ has unitary entries, which commute on all rows and all columns of H. Then the following are equivalent:*

(1) *H is Hadamard,*
(2) *$H/\sqrt{N}$ is biunitary,*
(3) *$HH^* = H^t\bar{H} = N1_N$.*

Proof. We know that (1) happens if and only if the axiom (3) in Definition 1.1 is satisfied, and by the preceding discussion, this axiom (3) is equivalent to (2). Regarding now the equivalence with (3), this follows from the commutation axiom (2) in Definition 1.1. $\square$

Observe that if $H = (H_{ij})$ is Hadamard, so too are the matrices $\bar{H} = (H_{ij}^*)$, $H^t = (H_{ji})$, and $H^* = (H_{ji}^*)$. In addition, we have the following result.

Proposition 1.5. *The class of Hadamard matrices $H \in M_N(A)$ is stable under the following two operations:*

(1) *permuting the rows or columns,*
(2) *multiplying the rows or columns by central unitaries.*

When successively combining these two operations, we obtain an equivalence relation on the class of Hadamard matrices $H \in M_N(A)$.

Proof. This is clear from definitions. Note that in the commutative case $A = C(X)$ any unitary is central, so we can multiply the rows or columns by any unitary. In particular in this case we can always "dephase" the matrix (i.e., assume that its first row and column consist of one entry; note that this operation is not allowed in the general case). $\square$

Let us discuss now the tensor product operation.

Proposition 1.6. *Let $H \in M_N(A)$ and $K \in M_M(A)$ be Hadamard matrices, and assume that $\langle H_{ij} \rangle$ commutes with $\langle K_{ab} \rangle$. Then the "tensor product" $H \otimes K \in M_{NM}(A)$, given by $(H \otimes K)_{ia,jb} = H_{ij}K_{ab}$, is a Hadamard matrix.*

Proof. This follows from definitions, and is as well a consequence of the more general Theorem 1.7 below, which will be proved with full details. $\square$

Following [12], the deformed tensor products are constructed as follows.

Theorem 1.7. *Let $H \in M_N(A)$ and $K \in M_M(A)$ be Hadamard matrices, and let $Q \in M_{N \times M}(A)$ be a rectangular matrix of unitaries whose entries commute on rows and columns. Assume in addition that the algebras $\langle H_{ij} \rangle$, $\langle K_{ab} \rangle$, and $\langle Q_{ib} \rangle$ pairwise commute. Then the "deformed tensor product" $H \otimes_Q K \in M_{NM}(A)$, given by*

$$(H \otimes_Q K)_{ia,jb} = Q_{ib}H_{ij}K_{ab},$$

is a Hadamard matrix as well.

Proof. First, the entries of $L = H \otimes_Q K$ are unitaries, and its rows are orthogonal:

$$\sum_{jb} L_{ia,jb} L^*_{kc,jb} = \sum_{jb} Q_{ib} H_{ij} K_{ab} \cdot Q^*_{kb} K^*_{cb} H^*_{kj} = N\delta_{ik} \sum_b Q_{ib} K_{ab} \cdot Q^*_{kb} K^*_{cb}$$

$$= N\delta_{ik} \sum_j K_{ab} K^*_{cb} = NM \cdot \delta_{ik}\delta_{ac}.$$

The orthogonality of columns can be checked as follows:

$$\sum_{ia} L_{ia,jb} L^*_{ia,kc} = \sum_{ia} Q_{ib} H_{ij} K_{ab} \cdot Q^*_{ic} K^*_{ac} H^*_{ik} = M\delta_{bc} \sum_i Q_{ib} H_{ij} \cdot Q^*_{ic} H^*_{ik}$$

$$= M\delta_{bc} \sum_i H_{ij} H^*_{ik} = NM \cdot \delta_{jk}\delta_{bc}.$$

For the commutation on rows, we use in addition the commutation on rows for Q:

$$L_{ia,jb} L_{kc,jb} = Q_{ib} H_{ij} K_{ab} \cdot Q_{kb} H_{kj} K_{cb} = Q_{ib} Q_{kb} \cdot H_{ij} H_{kj} \cdot K_{ab} K_{cb}$$

$$= Q_{kb} Q_{ib} \cdot H_{kj} H_{ij} \cdot K_{cb} K_{ab} = Q_{kb} H_{kj} K_{cb} \cdot Q_{ib} H_{ij} K_{ab} = L_{kc,jb} L_{ia,jb}.$$

The commutation on columns is similar, using the commutation on columns for Q:

$$L_{ia,jb} L_{ia,kc} = Q_{ib} H_{ij} K_{ab} \cdot q_{ic} H_{ik} K_{ac} = Q_{ib} Q_{ic} \cdot H_{ij} H_{ik} \cdot K_{ab} K_{ac}$$

$$= Q_{ic} Q_{ib} \cdot H_{ik} H_{ij} \cdot K_{ac} K_{ab} = Q_{ic} H_{ik} K_{ac} \cdot Q_{ib} H_{ij} K_{ab} = L_{ia,kc} L_{ia,jb}.$$

Thus all the axioms are satisfied, and L is indeed Hadamard. $\qquad\square$

As a basic example, we have the following construction.

Proposition 1.8. *The matrix*

$$M = \begin{pmatrix} x & y & x & y \\ x & -y & x & -y \\ z & t & -z & -t \\ z & -t & -z & t \end{pmatrix}$$

is Hadamard for any unitaries x, y, z, t satisfying $[x, y] = [x, z] = [y, t] = [z, t] = 0$.

Proof. This follows indeed from Theorem 1.7, because we have

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes_{\begin{pmatrix} x & y \\ z & t \end{pmatrix}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} x & y & x & y \\ x & -y & x & -y \\ z & t & -z & -t \\ z & -t & -z & t \end{pmatrix}.$$

Observe that M is generically nonclassical, in the sense of Definition 1.3. $\qquad\square$

2. Classification results and examples

The usual complex Hadamard matrices were classified in [13] at $N = 2, 3, 4, 5$. In this section we investigate the case of the general Hadamard matrices. We use the equivalence relation constructed in Proposition 1.5 above. We first have the following.

Proposition 2.1. *The 2×2 Hadamard matrices are all classical, and are all equivalent to the Fourier matrix F_2.*

Proof. Consider indeed an arbitrary 2×2 Hadamard matrix:

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We already know that A, D each commute with B, C. Now since we have $AB^* + CD^* = 0$, we deduce that $A = -CD^*B$ commutes with D, and that $C = -AB^*D$ commutes with B. Thus our matrix is classical, and since all unitaries are now central, we can dephase our matrix, which therefore holds to be the Fourier matrix F_2. $\square$

Let us discuss now the case $N = 3$. Here the classification in the classical case uses the key fact that any formula of type $a + b + c = 0$, with $|a| = |b| = |c| = 1$, must be, up to a permutation of terms, a trivial formula of type $a + ja + j^2a = 0$, with $j = e^{2\pi i/3}$.

Here is the noncommutative analogue of this simple fact.

Proposition 2.2. *Assume that $a + b + c = 0$ is a vanishing sum of unitaries. Then this sum must be of type $a + wa + w^2a = 0$, with w unitary satisfying $1 + w + w^2 = 0$.*

Proof. Since $-c = a + b$ is unitary, we have $(a+b)(a+b)^* = 1$, so $ab^* + ba^* = -1$. Thus $ab^*ba^* + (ba^*)^2 = -ba^*$, and with $w = ba^*$ we obtain $1 + w^2 = -w$, and we are done. $\square$

With this result in hand, we can start the $N = 3$ classification. We first have the following technical result, which we will improve later.

Proposition 2.3. *Any 3×3 Hadamard matrix must be of the form*

$$H = \begin{pmatrix} a & b & c \\ ua & uv^*w^2vb & uv^*wvc \\ va & wvb & w^2vc \end{pmatrix}$$

with w being subject to the equation $1 + w + w^2 = 0$.

Proof. Consider a Hadamard matrix $H \in M_3(A)$. We define a, b, c, u, v, w as for that part of the matrix to be exactly as in the statement, as follows:

$$H = \begin{pmatrix} a & b & c \\ ua & x & y \\ va & wvb & z \end{pmatrix}.$$

Let us look first at the scalar product between the first and third row:

$$vaa^* + wvbb^* + zc^* = 0.$$

By simplifying we obtain $v + wv + zc^* = 0$, and now by using Proposition 2.2 we conclude that we have $1 + w + w^2 = 0$, and that $zc^* = w^2v$, and so $z = w^2vc$, as claimed.

The scalar products of the first column with the second and third ones are

$$a^*b + a^*u^*x + a^*v^*wvb = 0,$$
$$a^*c + a^*u^*y + a^*v^*w^2vc = 0.$$

By multiplying to the left by va, and to the right by b^*v^* and c^*v^*, we obtain

$$1 + vu^*xb^*v^* + w = 0,$$
$$1 + vu^*yc^*v^* + w^2 = 0.$$

Now by using Proposition 2.2 again, we obtain $vu^*xb^*v^* = w^2$ and $vu^*yc^*v^* = w$, and so $x = uv^*w^2vb$ and $y = uv^*wvc$, and we are done. $\square$

We can already deduce now a first classification result, as follows.

Proposition 2.4. *There is no Hadamard matrix $H \in M_3(A)$ with self-adjoint entries.*

Proof. We use Proposition 2.3. Since the entries are idempotents, we have

$$a^2 = b^2 = c^2 = u^2 = v^2 = (uw)^2 = (vw)^2 = 1.$$

It follows that our matrix is in fact of the following form:

$$H = \begin{pmatrix} a & b & c \\ ua & uwb & uw^2c \\ va & wvb & w^2vc \end{pmatrix}.$$

The commutation between H_{22}, H_{23} reads

$$[uwb, wvb] = 0 \implies [uw, wv] = 0 \implies uwwv = wvuw$$
$$\implies uvw = vuw^2 \implies w = 1.$$

Thus we have reached a contradiction, and we are done. $\square$

Let us go back now to the general case. We have the following technical result, which refines Proposition 2.3 above, and which will be in turn further refined later.

Proposition 2.5. *Any 3×3 Hadamard matrix must be of the form*

$$H = \begin{pmatrix} a & b & c \\ ua & w^2ub & wuc \\ va & wvb & w^2vc \end{pmatrix},$$

where (a, b, c) and (u, v, w) are triples of commuting unitaries, and $1 + w + w^2 = 0$.

Proof. We use Proposition 2.3. With $e = uv^*$, the matrix there becomes

$$H = \begin{pmatrix} a & b & c \\ eva & ew^2vb & ewvc \\ va & wvb & w^2vc \end{pmatrix}.$$

The commutation relation between H_{22}, H_{32} reads

$$[ew^2vb, wvb] = 0 \implies [ew^2v, wv] = 0 \implies ew^2vwv = wvew^2v$$
$$\implies ew^2v = wvew \implies [ew, wv] = 0.$$

Similarly, the commutation between H_{23}, H_{33} reads

$$[ewvc, w^2vc] = 0 \implies [ewv, w^2v] = 0 \implies ewvw^2v = w^2vewv$$
$$\implies ewv = w^2vew^2 \implies [ew^2, w^2v] = 0.$$

We can rewrite this latter relation by using the formula $w^2 = -1 - w$, and then, by further processing it by using the first relation, we obtain

$$\big[e(1+w), (1+w)v\big] = 0 \implies [e, wv] + [ew, v] = 0$$
$$\implies 2ewv - wve - vew = 0$$
$$\implies ewv = \frac{1}{2}(wve + vew).$$

We use now the key fact that when an average of two unitaries is unitary, then the three unitaries involved are in fact all equal. This gives

$$ewv = wve = vew.$$

Thus we obtain $[w, e] = [w, v] = 0$, so w, e, v commute. Our matrix becomes

$$H = \begin{pmatrix} a & b & c \\ eva & w^2evb & wevc \\ va & wvb & w^2vc \end{pmatrix}.$$

Now by remembering that $u = ev$, this gives the formula in the statement. $\qquad \square$

We can now formulate our main classification result, as follows.

Theorem 2.6. *The 3×3 Hadamard matrices are all classical, and are all equivalent to the Fourier matrix F_3.*

Proof. We know from Proposition 2.5 that we can write our matrix in the following way, where (a, b, c) and (u, v, w) pairwise commute, and where $1 + w + w^2 = 0$:

$$H = \begin{pmatrix} a & b & c \\ au & buw & cuw^* \\ av & bvw^* & cvw \end{pmatrix}.$$

We also know that (a, u, v), (b, uw, vw^*), (c, uw^*, vw), and (ab, ac, bc, w) have entries which pairwise commute. We first show that uv is central. Indeed, we have

$$buv = buvww^* = b(uw)(vw^*) = (uw)(vw^*)b = uvb.$$

Similarly, $cuv = uvc$. It follows that we may in fact suppose that uv is a scalar. But since our relations are homogeneous, we may assume in fact that $u = v^*$.

Let us now show that $[abc, vw^*] = 0$. Indeed, we have

$$abc = a(bc)ww^* = aw(bc)w^* = av(wv^*)bcw^* = avb(wv^*)cw^* = v(ab)wv^*cw^*$$
$$= vw(ab)v^*cw^* = vw(ab)w(w^*v^*)cw^* = vw^2(ab)c(w^*v^*)w^* = vw^*abcv^*w.$$

We know also that $[b, vw^*] = 0$. Hence $[ac, vw^*] = 0$. But $[ac, w^*] = 0$. Hence $[ac, v] = 0$. But $[a, v] = 0$. Hence $[c, v] = 0$. But $[c, vw] = 0$. So $[c, w] = 0$. But $[bc, w] = 0$. So $[b, w] = 0$. But $[b, v^*w] = 0$ and $[ab, w] = 0$, so respectively $[b, v] = 0$ and $[a, w] = 0$. Thus all operators a, b, c, v, w pairwise commute, and we are done.   □

At $N = 4$, the classical theory uses the fact that an equation of type $a + b + c + d = 0$ with $|a| = |b| = |c| = |d| = 1$ must be, up to a permutation of the terms, a trivial equation of the form $a - a + b - b = 0$. In our setting, however, we have, for instance,

$$\begin{pmatrix} a & 0 \\ 0 & x \end{pmatrix} + \begin{pmatrix} -a & 0 \\ 0 & y \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & -x \end{pmatrix} + \begin{pmatrix} -b & 0 \\ 0 & -y \end{pmatrix} = 0.$$

It is probably possible to further complicate this kind of identity, and this makes the $N = 4$ classification a quite difficult task. However, we have the following statement.

Conjecture 2.7. *The Hadamard matrices of small size are as follows:*

(1) *the matrices in Proposition 1.8 are the only ones at $N = 4$,*
(2) *there is no 5×5 Hadamard matrix with self-adjoint entries.*

We have no computer-assisted evidence for this statement because all the problematics here are quite hard to implement on a computer; but we have instead some encouraging computations at $A = M_2(\mathbb{C})$ that we will not detail here, along with some supporting theoretical evidence, coming from [4] and [16]. Indeed, as explained in Section 3 below, any Hadamard matrix $H \in M_N(A)$ produces a quantum permutation group $G \subset S_N^+$. The point now is that the quantum subgroups $G \subset S_4^+$ were classified in [4], and the classification of the quantum subgroups $G \subset S_5^+$ is something that in principle can be deduced from [16]; looking at the list of the possible candidates gives some evidence for Conjecture 2.7 above. Of course, some serious work remains to be done here.

Summarizing, the situation in our generalized Hadamard setting is more complicated than the one in the classical case, as fully discussed by Haagerup in [13], up to $N = 5$. There are many interesting classes of special Hadamard matrices, such as the circulant ones (see [11], [14]), the master ones (see [1]), and the McNulty–Weigert ones (see [17]), whose theory waits to be extended to the noncommutative setting. Equally interesting are the defect problematics in [20], but an extension here would probably require some geometric conditions, such as assuming that we have a random matrix algebra $A = M_K(C(X))$.

3. Quantum permutation groups

We recall that a magic unitary matrix is a square matrix over a C^*-algebra, $u \in M_N(A)$, whose entries are projections ($p^2 = p^* = p$), summing up to 1 on each row and each column. The following key definition is due to Wang [21].

Definition 3.1. Let $C(S_N^+)$ be the universal C^*-algebra generated by the entries of a $N \times N$ magic unitary matrix $u = (u_{ij})$, with the morphisms given by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \qquad \varepsilon(u_{ij}) = \delta_{ij}, \qquad S(u_{ij}) = u_{ji}$$

as comultiplication, counit, and antipode.

This algebra satisfies Woronowicz's axioms in [23] and [24], and the underlying noncommutative space S_N^+ is therefore a compact quantum group designated as the quantum permutation group. As a first observation, we have a compact quantum group inclusion $S_N \subset S_N^+$ coming from the representation $\pi : C(S_N^+) \to C(S_N)$ constructed as follows:

$$u_{ij} \to \chi\big(\sigma \in S_N \mid \sigma(j) = i\big).$$

It is known that this inclusion is an isomorphism at $N = 2, 3$, but not at $N \geq 4$, where S_N^+ is nonclassical and infinite. Moreover, it is known that we have $S_4^+ \simeq SO_3^{-1}$, and that any S_N^+ with $N \geq 4$ has the same fusion semiring as SO_3 (see [6], [21]).

The complex Hadamard matrices are known to produce magic matrices, which in turn produce quantum permutation groups. In our generalized setting, we have the following.

Theorem 3.2. *If $H \in M_N(A)$ is Hadamard, then the matrices $P_{ij} \in M_N(A)$ defined by*

$$(P_{ij})_{ab} = \frac{1}{N} H_{ia} H_{ja}^* H_{jb} H_{ib}^*$$

form altogether a magic matrix $P = (P_{ij})$ over the algebra $M_N(A)$.

Proof. The magic condition can be checked in the following three steps.

(1) Let us first check that each P_{ij} is a projection (i.e., that we have $P_{ij} = P_{ij}^* = P_{ij}^2$). Regarding the first condition, namely $P_{ij} = P_{ij}^*$, this simply follows from

$$(P_{ij})_{ba}^* = \frac{1}{N}(H_{ib} H_{jb}^* H_{ja} H_{ia}^*)^* = \frac{1}{N} H_{ia} H_{ja}^* H_{jb} H_{ib}^* = (P_{ij})_{ab}.$$

As for the second condition, $P_{ij} = P_{ij}^2$, this follows from the fact that all the entries H_{ij} are assumed to be unitaries; that is, this follows from axiom (1) in Definition 1.1:

$$(P_{ij}^2)_{ab} = \sum_c (P_{ij})_{ac}(P_{ij})_{cb} = \frac{1}{N^2} \sum_c H_{ia} H_{ja}^* H_{jc} H_{ic}^* H_{ic} H_{jc}^* H_{jb} H_{ib}^*$$

$$= \frac{1}{N} H_{ia} H_{ja}^* H_{jb} H_{ib}^* = (P_{ij})_{ab}.$$

(2) Now let us check that fact that the entries of P sum up to 1 on each row. For this purpose we use the equality $H^*H = N1_N$, coming from the axiom (3), which gives

$$\left(\sum_j P_{ij}\right)_{ab} = \frac{1}{N}\sum_j H_{ia}H^*_{ja}H_{jb}H^*_{ib} = \frac{1}{N}H_{ia}(H^*H)_{ab}H^*_{ib} = \delta_{ab}H_{ia}H^*_{ib} = \delta_{ab}.$$

(3) Finally, let us check that the entries of P sum up to 1 on each column. This is the trickiest check, because it involves, besides axiom (1) and the formula $H^t\bar{H} = N1_N$ coming from axiom (3), the commutation on the columns of H, coming from axiom (2):

$$\left(\sum_i P_{ij}\right)_{ab} = \frac{1}{N}\sum_i H_{ia}H^*_{ja}H_{jb}H^*_{ib} = \frac{1}{N}\sum_i H^*_{ja}H_{ia}H^*_{ib}H_{jb}$$

$$= \frac{1}{N}H^*_{ja}(H^t\bar{H})_{ab}H_{jb} = \delta_{ab}H^*_{ja}H_{jb} = \delta_{ab}.$$

Thus P is indeed a magic matrix in the above sense, and we are done. $\qquad\square$

Now, by using the general theory of Hopf images from [5], we can formulate the following.

Definition 3.3. The quantum permutation group $G \subset S_N^+$ associated to a Hadamard matrix $H \in M_N(A)$ is constructed as follows:

(1) we construct a magic matrix $P \in M_N(M_N(A))$ as above;
(2) we let $\pi : C(S_N^+) \to M_N(A)$ be the representation associated to P;
(3) we factorize $\pi : C(S_N^+) \to C(G) \to M_N(A)$, with $G \subset S_N^+$ chosen minimal.

As an illustration, consider a usual Hadamard matrix $H \in M_N(\mathbb{C})$. If we denote its rows by $H_1, \ldots, H_N$ and we consider the vectors $\xi_{ij} = H_i/H_j$, then we have

$$\xi_{ij} = \left(\frac{H_{i1}}{H_{j1}}, \ldots, \frac{H_{iN}}{H_{jN}}\right).$$

Thus the orthogonal projection on this vector ξ_{ij} is given by

$$(P_{\xi_{ij}})_{ab} = \frac{1}{\|\xi_{ij}\|^2}(\xi_{ij})_a\overline{(\xi_{ij})_b} = \frac{1}{N}H_{ia}H^*_{ja}H_{jb}H^*_{ib} = (P_{ij})_{ab}.$$

We conclude that we have $P_{ij} = P_{\xi_{ij}}$ for any i, j, so our construction from Definition 3.3 is compatible with the construction for the usual complex Hadamard matrices.

At the general level, there are many interesting questions, and notably that of understanding how the idempotent state theory from [7] and [22] can be applied in this setting, in order to obtain an explicit formula for the moments of the main character of G.

4. Free wreath products

We discuss now the computation of the quantum permutation groups associated to the deformed tensor products of Hadamard matrices. In the classical case this is a key problem, and several results on the subject were obtained in [3], [6], [10]. We begin with a study of the associated magic unitary. Regarding the deformed tensor products of Hadamard matrices, we use here the general notation from Theorem 1.7 above. Regarding the quantum algebra part, we use here the various notions introduced in Section 3 above, and notably the magic unitary matrix $P = (P_{ij})$ introduced in Theorem 3.2, which we will write now with double indices, $P = (P_{ia,jb})$. We have the following.

Proposition 4.1. *The magic unitary associated to $H \otimes_Q K$ is given by*

$$P_{ia,jb} = R_{ij} \otimes \frac{1}{N}(Q_{ic}Q_{jc}^*Q_{jd}Q_{id}^* \cdot K_{ac}K_{bc}^*K_{bd}K_{ad}^*)_{cd},$$

where R_{ij} is the magic unitary matrix associated to H.

Proof. With the above-mentioned conventions for deformed tensor products and for double indices, the entries of $L = H \otimes_Q K$ are by definition the following elements:

$$L_{ia,jb} = Q_{ib}H_{ij}K_{ab}.$$

Thus the projections $P_{ia,jb}$ constructed in Theorem 3.2 are given by

$$
\begin{aligned}
(P_{ia,jb})_{kc,ld} &= \frac{1}{MN}L_{ia,kc}L_{jb,kc}^*L_{jb,ld}L_{ia,ld}^* \\
&= \frac{1}{MN}(Q_{ic}H_{ik}K_{ac})(Q_{jc}H_{jk}K_{bc})^*(Q_{jd}H_{jl}K_{bd})(Q_{id}H_{il}K_{ad})^* \\
&= \frac{1}{MN}(Q_{ic}Q_{jc}^*Q_{jd}Q_{id}^*)(H_{ik}H_{jk}^*H_{jl}H_{il}^*)(K_{ac}K_{bc}^*K_{bd}K_{ad}^*).
\end{aligned}
$$

In terms now of the standard matrix units e_{kl}, e_{cd}, we have

$$
\begin{aligned}
P_{ia,jb} &= \frac{1}{MN}\sum_{kcld} e_{kl} \otimes e_{cd} \otimes (Q_{ic}Q_{jc}^*Q_{jd}Q_{id}^*)(H_{ik}H_{jk}^*H_{jl}H_{il}^*)(K_{ac}K_{bc}^*K_{bd}K_{ad}^*) \\
&= \frac{1}{MN}\sum_{kcld}(e_{kl} \otimes 1 \otimes H_{ik}H_{jk}^*H_{jl}H_{il}^*) \\
&\quad \times (1 \otimes e_{cd} \otimes Q_{ic}Q_{jc}^*Q_{jd}Q_{id}^* \cdot K_{ac}K_{bc}^*K_{bd}K_{ad}^*).
\end{aligned}
$$

Since the quantities on the right commute, this gives the formula in the statement. $\square$

In order to investigate the Diţă deformations, we use the following.

Definition 4.2. Let $C(S_M^+) \to A$ and $C(S_N^+) \to B$ be Hopf algebra quotients, with fundamental corepresentations denoted u, v. We let

$$A *_w B = A^{*N} * B/\big\langle [u_{ab}^{(i)}, v_{ij}] = 0 \big\rangle$$

with the Hopf algebra structure making $w_{ia,jb} = u_{ab}^{(i)}v_{ij}$ a corepresentation.

The fact that we have indeed a Hopf algebra follows from the fact that w is magic. In terms of quantum groups, if $A = C(G)$, $B = C(H)$, we write $A *_w B = C(G \wr_* H)$:

$$C(G) *_w C(H) = C(G \wr_* H).$$

The $\wr_*$ operation is the free analogue of $\wr$, the usual wreath product (see [9]). With this convention, we have the following result.

Theorem 4.3. *The representation associated to $L = H \otimes_Q K$ factorizes as*

$$C(S_{NM}^+) \qquad\qquad \overset{\pi_L}{} \qquad\qquad M_{NM}(\mathbb{C})$$

$$C(S_M^+ \wr_* G_H)$$

and so the quantum group associated to L appears as a subgroup $G_L \subset S_M^+ \wr_* G_H$.

Proof. We use the formula in Proposition 4.1. For simplifying the writing, we agree to use fractions of type $\frac{H_{ia}H_{jb}}{H_{ja}H_{ib}}$ instead of expressions of type $H_{ia}H_{ja}^* H_{jb}H_{ib}^*$, by keeping in mind that the variables are only subject to the commutation relations in Definition 1.1. Our claim is that the factorization can be indeed constructed, as follows:

$$U_{ab}^{(i)} = \sum_j P_{ia,jb}, \qquad V_{ij} = \sum_a P_{ia,jb}.$$

Indeed, we have three verifications to be made:

(1) We must prove that the elements $V_{ij} = \sum_a P_{ia,jb}$ do not depend on b, and generate a copy of $C(G_H)$. But if we denote by (R_{ij}) the magic for H, then we have indeed

$$V_{ij} = \frac{1}{N}\left(\frac{Q_{ic}Q_{jd}}{Q_{id}Q_{jc}} \cdot \frac{H_{ik}H_{jl}}{H_{il}H_{jk}} \cdot \delta_{cd}\right)_{kc,ld} = \left((R_{ij})_{kl}\delta_{cd}\right)_{kc,ld} = R_{ij} \otimes 1.$$

(2) We prove now that for any i, the elements $U_{ab}^{(i)} = \sum_j P_{ia,jb}$ form a magic matrix. Since $P = (P_{ia,jb})$ is magic, the elements $U_{ab}^{(i)} = \sum_j P_{ia,jb}$ are self-adjoint, and we have $\sum_b U_{ab}^{(i)} = \sum_{bj} P_{ia,jb} = 1$. The fact that each $U_{ab}^{(i)}$ is an idempotent follows from

$$\left((U_{ab}^{(i)})^2\right)_{kc,ld}$$

$$= \frac{1}{N^2M^2}\sum_{mejn} \frac{Q_{ic}Q_{je}}{Q_{ie}Q_{jc}} \cdot \frac{H_{ik}H_{jm}}{H_{im}H_{jk}} \cdot \frac{K_{ac}K_{be}}{K_{ae}K_{bc}} \cdot \frac{Q_{ie}Q_{nd}}{Q_{id}Q_{ne}} \cdot \frac{H_{im}H_{nl}}{H_{il}H_{nm}} \cdot \frac{K_{ae}K_{bd}}{K_{ad}K_{be}}$$

$$= \frac{1}{NM^2}\sum_{ejn} \frac{Q_{ic}Q_{je}Q_{nd}}{Q_{jc}Q_{id}Q_{ne}} \cdot \frac{H_{ik}H_{nl}}{H_{jk}H_{il}} \delta_{jn} \cdot \frac{K_{ac}K_{bd}}{K_{bc}K_{ad}}$$

$$= \frac{1}{NM^2}\sum_{ej} \frac{Q_{ic}Q_{je}Q_{jd}}{Q_{jc}Q_{id}Q_{je}} \cdot \frac{H_{ik}H_{jl}}{H_{jk}H_{il}} \cdot \frac{K_{ac}K_{bd}}{K_{bc}K_{ad}}$$

$$= \frac{1}{NM}\sum_j \frac{Q_{ic}Q_{jd}}{Q_{jc}Q_{id}} \cdot \frac{H_{ik}H_{.jl}}{H_{jk}H_{il}} \cdot \frac{K_{ac}K_{bd}}{K_{bc}K_{ad}} = (U_{ab}^{(i)})_{kc,ld}.$$

Finally, the condition $\sum_a U_{ab}^{(i)} = 1$ can be checked, as follows:

$$\sum_a U_{ab}^{(i)} = \frac{1}{N}\left(\sum_j \frac{Q_{ic}Q_{jd}}{Q_{id}Q_{jc}} \cdot \frac{H_{ik}H_{jl}}{H_{il}H_{jk}} \cdot \delta_{cd}\right)_{kc,ld} = \frac{1}{N}\left(\sum_j \frac{H_{ik}H_{jl}}{H_{il}H_{jk}} \cdot \delta_{cd}\right)_{kc,ld} = 1.$$

(3) It remains to prove that we have $U_{ab}^{(i)}V_{ij} = V_{ij}U_{ab}^{(i)} = P_{ia,jb}$. First, we have

$$(U_{ab}^{(i)}V_{ij})_{kc,ld} = \frac{1}{N^2M}\sum_{mn} \frac{Q_{ic}Q_{nd}}{Q_{id}Q_{nc}} \cdot \frac{H_{ik}H_{nm}}{H_{im}H_{nk}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} \cdot \frac{H_{im}H_{jl}}{H_{il}H_{jm}}$$

$$= \frac{1}{NM}\sum_{n} \frac{Q_{ic}Q_{nd}}{Q_{id}Q_{nc}} \cdot \frac{H_{ik}H_{jl}}{H_{nk}H_{il}}\delta_{nj} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}}$$

$$= \frac{1}{NM} \cdot \frac{Q_{ic}Q_{jd}}{Q_{id}Q_{jc}} \cdot \frac{H_{ik}H_{jl}}{H_{jk}H_{il}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} = (P_{ia,jb})_{kc,ld}.$$

The remaining computation is similar:

$$(V_{ij}U_{ab}^{(i)})_{kc,ld} = \frac{1}{N^2M}\sum_{mn} \frac{H_{ik}H_{jm}}{H_{im}H_{jk}} \cdot \frac{Q_{ic}Q_{nd}}{Q_{id}Q_{nc}} \cdot \frac{H_{im}H_{nl}}{H_{il}H_{nm}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}}$$

$$= \frac{1}{NM}\sum_{n} \frac{Q_{ic}Q_{nd}}{Q_{id}Q_{nc}} \cdot \frac{H_{ik}H_{nl}}{H_{jk}H_{il}}\delta_{jn} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}}$$

$$= \frac{1}{NM} \cdot \frac{Q_{ic}Q_{jd}}{Q_{id}Q_{jc}} \cdot \frac{H_{ik}H_{jl}}{H_{jk}H_{il}} \cdot \frac{K_{ac}K_{bd}}{K_{ad}K_{bc}} = (P_{ia,jb})_{kc,ld}.$$

Thus we have checked all the relations, and we are done. □

In general, the problem of further factorizing the above representation is a quite difficult one, even in the classical case. For a number of results here, we refer to [3], [6], and [10].

Acknowledgment. I would like to thank K. De Commer for useful discussions.

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