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TRIPLE SOLUTIONS FOR QUASILINEAR ONE-DIMENSIONAL p-LAPLACIAN ELLIPTIC EQUATIONS IN THE WHOLE SPACE

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Abstract. In this paper, we establish the existence of three possibly nontrivial solutions for a Dirichlet problem on the real line without assuming on the nonlinearity asymptotic conditions at infinity. As a particular case, when the nonlinearity is superlinear at zero and sublinear at infinity, the existence of two nontrivial solutions is obtained. This approach is based on variational methods and, more precisely, a critical points theorem, which assumes a more general condition than the classical Palais–Smale condition, is exploited.

1. Introduction

Quasilinear elliptic equations in the whole space, as well as in the special case, on infinite intervals occur naturally in a variety of settings in physics and engineering, as in, for example, the study of non-Newtonian fluid flows, nonlinear mechanics, the theory of plasma, and electrical potential theory (see [2], [3], [8], and [12]). In particular, it highlights that they occur in dissipative quantum mechanics (see [7]), in condensed matter theory (see [11]), and in the theory of Heisenberg ferromagnets (see [13]). It is well known that the study of such equations presents great difficulties because of the lack of compactness of the involved operators. Indeed, in such cases the operators which solve the problem are not regular enough in comparison to operators which arise in problems on bounded domains. To be more precise, if we use operator theory and fixed point methods, then standard results

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such as the Ascoli–Arzelà theorem or the Rellich–Kondrachov theorem cannot be applied directly. An appropriate diagonalization process is usually needed, or, alternatively, one attempts to use fixed point results for monotone operators (see the monograph [1] for an overview on different approaches). On the other hand, if we use operator theory and variational methods, then the usual results such as the direct methods theorem or the classical mountain pass theorem cannot be directly applied due to lack of regularity of the operator. Other tools, such as the concentration compactness principle or the Nash–Moser-type implicit function theorem or the generalized linking theorem, are usually applied (see [3], [12], and [15, Chapter 6]). It is worth noticing that in this latest case, in particular, the Palais–Smale condition for the energy operator fails. Moreover, we also observe that in the study of such nonlinear equations in the whole space an appropriate behavior at infinity of the nonlinearity is usually assumed (see, for example, [6], [9], [10], [15], [16], and the references therein), and none of the previous cited results guarantees the multiplicity of solutions. Here, we obtain solutions for nonlinear elliptic equations in the whole one-dimensional space by showing that the corresponding operator satisfies a weaker Palais–Smale condition than the classical one. We apply a suitable version of the mountain pass theorem to establish the existence of multiple solutions possibly without asymptotic conditions at infinity. More precisely, we consider the following problem.

Find $u \in W^{1,p}(\mathbb{R})$ *with* $p \in]1,+\infty[$ *satisfying*

$$-\left(\left|u'(x)\right|^{p-2}u'(x)\right)' + B\left|u(x)\right|^{p-2}u(x) = \lambda\alpha(x)g\left(u(x)\right) \quad \text{a.e. in } \mathbb{R}, \qquad (P_\lambda)$$

where λ is a real positive parameter, B is a real positive number, $\alpha, g : \mathbb{R} \to \mathbb{R}$ are two functions such that $\alpha \in L^1(\mathbb{R})$, $\alpha(x) \geq 0$ a.e. $x \in \mathbb{R}$, $\alpha \not\equiv 0$, and g is continuous and nonnegative.

Therefore, the main aim of this article is to obtain the existence of three distinct solutions for problem (P_λ) without assuming an asymptotic condition, at infinity or at zero, of the nonlinearity g. Precisely, under suitable assumptions on the behavior of g, we prove the existence of three distinct possibly nontrivial solutions for problem (P_λ) in $W^{1,p}(\mathbb{R})$ (see Theorem 3.1 and Example 3.3). Indeed, in order to obtain three distinct solutions, we require, roughly speaking, a rapid growth of g in an interval $[c_1, d]$ (see (j) in Theorem 3.1) and a less rapid growth in a subsequent interval $[d, c_2]$ (see (jj) in Theorem 3.1) with g having arbitrary behavior after c_2 (see Example 3.3). Moreover, as a consequence, the existence of two nontrivial solutions is obtained if g is $(p-1)$-superlinear at zero and $(p-1)$-sublinear at infinity (see Corollary 3.4). Our main tool is a recent critical points result proved in [4] (see Theorem 2.6). We stress again that the application of such a theorem, which uses a nonstandard Palais–Smale condition (see Definition 2.2), allows us to study in an innovative way differential problems in unbounded domains, which, as it is well known, are characterized by the lack of compactness of the operator. Now we present as an example a special case of our main result.

Theorem 1.1. *Let* $\alpha, g : \mathbb{R} \to \mathbb{R}$ *be two nonnegative continuous functions with* $\alpha \in L^1(\mathbb{R})$*,* $\alpha \not\equiv 0$*, and* $g(0) \neq 0$*. Assume that there exist two positive constants*

c and d with $c<d$ such that

$$\frac{\int_0^c g(\xi)\,d\xi}{c^2} < \Big(\frac{2}{7}\frac{\int_{-1}^{1}\alpha(x)\,dx}{\int_{-\infty}^{+\infty}\alpha(x)\,dx}\Big)\frac{\int_0^d g(\xi)\,d\xi}{d^2}$$

and

$$\lim_{\xi\to+\infty}\frac{g(\xi)}{\xi}=0.$$

Then, for each $\lambda\in\Big]\frac{1}{\int_{-1}^{1}\alpha(x)\,dx}\frac{7}{2}\frac{d^2}{\int_0^d g(\xi)\,d\xi},\frac{1}{\int_{-\infty}^{+\infty}\alpha(x)\,dx}\frac{c^2}{\int_0^c g(\xi)\,d\xi}\Big[$, *the problem*

$$\begin{cases}-u''+u=\lambda\alpha(x)g(u) & x\in\mathbb{R},\\ u(-\infty)=u(+\infty)=0\end{cases}$$

admits at least three distinct nontrivial classical solutions.

This article is organized as follows. In Section 2, we recall some preliminary results. In particular, we recall a critical points theorem proved in [4] that we will use to prove the main result of the paper. In Section 3, we present our main result (Theorems 3.1), and we point out a special case (Corollary 3.4). Furthermore, we give some examples to illustrate Theorem 3.1 and Corollary 3.4 (Examples 3.3 and 3.5).

2. Preliminaries

In this section, we introduce the notation and terminology, and we collect some well-known facts that we will use in the sequel.

Let $(X,\|\cdot\|)$ be a real Banach space, and let X^* be the dual space of X. A function $I:X\to\mathbb{R}$ is Gâteaux differentiable at $u\in X$ if there is a $I'(u)\in X^*$ such that

$$\lim_{t\to 0^+}\frac{I(u+tv)-I(u)}{t}=I'(u)(v)$$

for each $v\in X$. The function I is continuously Gâteaux differentiable if it is Gâteaux differentiable for any $u\in X$ and the function $u\to I'(u)$ is a continuous map from X to its dual X^*.

Let $\Phi,\Psi:X\to\mathbb{R}$ be two continuously Gâteaux differentiable functions, let λ be a real positive parameter, and let $I_\lambda=\Phi-\lambda\Psi$. Fix $r\in]-\infty,+\infty]$.

Definition 2.1. The function I_λ satisfies the Palais–Smale condition (in short, the (PS)-condition) if any sequence $\{u_n\}$ such that

(i) $I_\lambda(u_n)$ is bounded,
(ii) $\lim_{n\to+\infty}\|I'_\lambda(u_n)\|_{X^*}=0$

has a convergent subsequence.

Definition 2.2. The function I_λ satisfies the Palais–Smale condition cutoff upper at r (in short, the $(PS)^{[r]}$-condition) if any sequence $\{u_n\}$ that satisfies (i), (ii), and

(iii) $\Phi(u_n)<r$

has a convergent subsequence.

Definition 2.3. An element $u \in X$ is a critical point of I_λ if $I'_\lambda(u) = 0_{X^*}$; that is, $I'_\lambda(u)(v) = 0$ for each $v \in X$.

Let $|\cdot|$ and $|\cdot|_t$ be the usual norms on $\mathbb{R}$ and $L^t(\mathbb{R})$ where $t \in [1, +\infty]$, respectively. Here, we denote by $W_0^{1,p}(\mathbb{R})$ the closure of $C_0^\infty(\mathbb{R})$ in $W^{1,p}(\mathbb{R})$ with respect to the norm

$$\|u\|_{1,p} := \big(|u'|_p^p + |u|_p^p\big)^{\frac{1}{p}},$$

and we consider $W^{1,p}(\mathbb{R})$ endowed with the norm

$$\|u\| = \Big(\int_{\mathbb{R}} \big(\big|u'(x)\big|^p + B\big|u(x)\big|^p\big)\,dx\Big)^{\frac{1}{p}}.$$

We note that $W^{1,p}(\mathbb{R}) \equiv W_0^{1,p}(\mathbb{R})$ and $W^{1,p}(\mathbb{R})$ is embedded in $L^t(\mathbb{R})$ for any $t \in [p, +\infty]$. Let $G : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$G(t) = \int_0^t g(\xi)\,d\xi \quad \text{for all } t \in \mathbb{R}.$$

We note that the hypotheses on g guarantee that $G \in C^1(\mathbb{R})$ and $G'(t) = g(t) \geq 0$ for all $t \in \mathbb{R}$, and so the function G is nondecreasing.

Now, let us denote by $\Phi, \Psi : W^{1,p}(\mathbb{R}) \to \mathbb{R}$ the functions defined by

$$\Phi(u) = \frac{1}{p}\|u\|^p \qquad \text{and} \qquad \Psi(u) = \int_{\mathbb{R}} \alpha(x) G\big(u(x)\big)\,dx \tag{2.1}$$

for all $u \in W^{1,p}(\mathbb{R})$. Moreover, by $I_\lambda : W^{1,p}(\mathbb{R}) \to \mathbb{R}$, we denote the energy function related to the problem (P_λ). Then, for each $u \in W^{1,p}(\mathbb{R})$, I_λ is defined by

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

with Φ and Ψ given by (2.1).

We recall that it is well known that Φ and Ψ are continuously Gâteaux differentiable. Furthermore, if u is a critical point of I_λ, then $I'_\lambda(u) \equiv 0$; that is,

$$\int_{\mathbb{R}} \big(\big|u'(x)\big|^{p-2} u'(x)v'(x) + B\big|u(x)\big|^{p-2} u(x)v(x) - \lambda\alpha(x)g\big(u(x)\big)v(x)\big)\,dx = 0$$

for all $v \in W^{1,p}(\mathbb{R})$, and hence u is a weak solution to (P_λ). Clearly, if in addition $\alpha \in C(\mathbb{R})$, then u is a classical solution to (P_λ).

We conclude this section with some results proved in [5] and [4] which we will need to prove the existence of three solutions to the problem (P_λ).

Proposition 2.4 ([5], Proposition 2.2)**.** *One has*

$$|u|_\infty \leq c_B\|u\|$$

for all $u \in W^{1,p}(\mathbb{R})$, where

$$c_B = 2^{\frac{p-2}{p}}\Big(\frac{p-1}{p}\Big)^{\frac{p-1}{p}}\Big(\frac{1}{B}\Big)^{\frac{p-1}{p^2}}.$$

Lemma 2.5 ([5], Lemma 2.8)**.** *Let Φ and Ψ be as defined in (2.1), and fix $\lambda > 0$. Then $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition for any $r > 0$.*

Finally, we state in a convenient form a three critical points theorem proved in [4].

Theorem 2.6 ([4], Theorem 7.3)**.** *Let X be a real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions with Φ bounded from below and convex such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist two positive constants r_1, r_2 and $\bar{u} \in X$ with $2r_1 < \Phi(\bar{u}) < \frac{r_2}{2}$ such that

(i) $\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} < \frac{2}{3}\frac{\Psi(\bar{u})}{\Phi(\bar{u})}$,
(ii) $\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(\bar{u})}{\Phi(\bar{u})}$.

Assume also that, for each

$$\lambda \in \Lambda = \left]\frac{3}{2}\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}, \frac{r_2}{2\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\right\}\right[,$$

the functional I_λ satisfies the $(PS)^{[r_2]}$-condition and

$$\inf_{t\in[0,1]} \Psi\big(tu_1 + (1-t)u_2\big) \geq 0$$

for each $u_1, u_2 \in X$, which are local minima for the functional I_λ with $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$.

Then, for each $\lambda \in \Lambda$, the functional I_λ admits at least three critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.

We recall that a first version of a three critical points theorem for differentiable functionals depending on a real parameter was given by Ricceri in [14]. In that case, contrary to Theorem 2.6, regularity properties of the functional with respect to the weak topology are requested.

3. Main result

In this section we state and prove the main result of the paper. We also point out a special case of our main result, and we give some illustrative examples.

In the sequel, Φ and Ψ are the functions defined in (2.1), and we put

$$k = \frac{1}{3c_B^p(1 + B + \frac{B}{p+1})}\frac{\alpha_0}{|\alpha|_1},$$

where $\alpha_0 = \int_{-1}^1 \alpha(x)\,dx$ and c_B is given in Proposition 2.4.

Theorem 3.1. *Assume that there exist three positive constants c_1, c_2, d with $2c_1 < d < c_2$ such that*

(j) $\frac{G(c_1)}{c_1^p} < k\frac{G(d)}{d^p}$,
(jj) $\frac{G(c_2)}{c_2^p} < \frac{k}{2}\frac{G(d)}{d^p}$.

Then, for each

$$\lambda \in \bar{\Lambda} = \Big] \frac{1}{pc_B^p|\alpha|_1} \frac{1}{k} \frac{d^p}{G(d)}, \frac{1}{pc_B^p|\alpha|_1} \min\Big\{ \frac{c_1^p}{G(c_1)}, \frac{c_2^p}{2G(c_2)} \Big\} \Big[,$$

the problem (P_λ) *has at least three distinct weak solutions* $u_i \in W^{1,p}(\mathbb{R})$ *such that* $|u_i|_\infty < c_2$, $i = 1, 2, 3$.

Proof. We will apply Theorem 2.6. Fix three positive constants c_1, c_2, and d as in the statement. Then put $r_1 = \frac{1}{p}(\frac{c_1}{c_B})^p$, $r_2 = \frac{1}{p}(\frac{c_2}{c_B})^p$, and

$$\bar{u}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus]-2, 2[, \\ d & \text{if } x \in [-1, 1], \\ d(2 - |x|) & \text{if } x \in [-2, -1[\cup]1, 2]. \end{cases}$$

We notice that the regularity assumptions of Theorem 2.6 on Φ and Ψ are satisfied, and we have

$$\begin{aligned} \Phi(\bar{u}) = \frac{1}{p}\|\bar{u}\|^p &= \frac{1}{p}\int_{-2}^{2} \big|\bar{u}(x)'\big|^p\,dx + \frac{B}{p}\int_{-2}^{2}\big|\bar{u}(x)\big|^p\,dx \\ &= \frac{2}{p}\int_1^2 d^p\,dx + \frac{2B}{p}\Big[\int_0^1 d^p\,dx + \int_1^2 d^p(2-x)^p\,dx\Big] \\ &= \frac{2}{p}d^p\Big(1 + B + \frac{B}{p+1}\Big) \end{aligned}$$

and

$$\begin{aligned} \Psi(\bar{u}) &= \int_{\mathbb{R}} \alpha(x) G\big(\bar{u}(x)\big)\,dx \\ &\geq \int_{-1}^{1} \alpha(x) G(d)\,dx \\ &= \alpha_0 G(d). \end{aligned}$$

Now, taking into account that

$$\begin{aligned} \Big(\frac{1}{B}\Big)^{\frac{p-1}{p^2}}\Big(1 + B + \frac{B}{p+1}\Big)^{\frac{1}{p}} &= \Big(\Big(\frac{1}{B}\Big)^{\frac{p-1}{p}} + \frac{p+2}{p+1}B^{\frac{1}{p}}\Big)^{\frac{1}{p}} \\ &\geq \Big(\Big(\frac{p+2}{p^2-1}\Big)^{\frac{p-1}{p}} + \frac{p+2}{p+1}\Big(\frac{p^2-1}{p+2}\Big)^{\frac{1}{p}}\Big)^{\frac{1}{p}}, \end{aligned}$$

one has

$$c_B\Big(1 + B + \frac{B}{p+1}\Big)^{\frac{1}{p}} \geq L(p),$$

where

$$L(p) = 2^{\frac{p-2}{p}}\Big(\frac{p-1}{p}\Big)^{\frac{p-1}{p}}\Big(\Big(\frac{p+2}{p^2-1}\Big)^{\frac{p-1}{p}} + \frac{p+2}{p+1}\Big(\frac{p^2-1}{p+2}\Big)^{\frac{1}{p}}\Big)^{\frac{1}{p}}.$$

Since $L(p) > \frac{1}{2}$ for all $p > 1$, one has

$$c_B\Big(1 + B + \frac{B}{p+1}\Big)^{\frac{1}{p}} > \frac{1}{2},$$

and then from $2c_1 < d$ we obtain

$$\frac{1}{c_B(1 + B + \frac{B}{p+1})^{\frac{1}{p}}} c_1 < d.$$

Thus it follows that

$$2\frac{1}{p}\frac{c_1^p}{c_B^p} < \frac{2}{p}\Big(1 + B + \frac{B}{p+1}\Big)d^p;$$

that is, $2r_1 < \Phi(\bar{u})$. Moreover, from (jj) taking into account that $G(d) \le G(c_2)$, one has

$$\frac{1}{c_2^p} < \frac{k}{2}\frac{G(d)}{G(c_2)}\frac{1}{d^p} \le \frac{1}{6c_B^p(1 + B + \frac{B}{p+1})}\frac{\alpha_0}{|\alpha|_1}\frac{1}{d^p} < \frac{1}{4c_B^p(1 + B + \frac{B}{p+1})}\frac{1}{d^p}.$$

Therefore, one has $\frac{2}{p}(1 + B + \frac{B}{p+1})d^p < \frac{1}{2}(\frac{1}{p}\frac{c_2^p}{c_B^p})$; that is, $\Phi(\bar{u}) < \frac{r_2}{2}$. Hence one has

$$2r_1 < \Phi(\bar{u}) < \frac{r_2}{2}.$$

Now, we notice that if $u \in W^{1,p}(\mathbb{R})$ is such that $\Phi(u) < r_i$, then from

$$\Phi(u) = \frac{1}{p}\|u\|^p < r_i = \frac{1}{p}\Big(\frac{c_i}{c_B}\Big)^p$$

we obtain that $c_B\|u\| < c_i$, and hence, by Proposition 2.4, we also obtain that $|u|_\infty < c_i$. Taking this into account, from (j) we have

$$\begin{aligned}
\frac{1}{r_1}\sup_{\Phi(u)<r_1}\Psi(u) &= \frac{1}{r_1}\sup_{\Phi(u)<r_1}\int_{\mathbb{R}}\alpha(x)G\big(u(x)\big)\,dx \\
&\le \frac{1}{r_1}\int_{\mathbb{R}}\alpha(x)\max_{|t|\le c_1}G(t)\,dx \\
&= pc_B^p|\alpha|_1\frac{G(c_1)}{c_1^p} \\
&< |\alpha|_1 pc_B^p\frac{\alpha_0}{3c_B^p|\alpha|_1(1 + B + \frac{B}{p+1})}\frac{G(d)}{d^p} \\
&\le \frac{2}{3}\frac{\Psi(\bar{u})}{\Phi(\bar{u})}.
\end{aligned}$$

Analogously, from (jj) we get

$$\begin{aligned}
\frac{1}{r_2}\sup_{\Phi(u)<r_2}\Psi(u) &\le \frac{1}{r_2}\int_{\mathbb{R}}\alpha(x)\max_{|t|\le c_2}G(t)\,dx \\
&= pc_B^p|\alpha|_1\frac{G(c_2)}{c_2^p}
\end{aligned}$$

$$< \frac{1}{2}|\alpha|_1 p c_B^p \frac{\alpha_0}{3c_B^p|\alpha|_1(1+B+\frac{B}{p+1})} \frac{G(d)}{d^p}$$
$$\leq \frac{1}{3}\frac{\Psi(\bar{u})}{\Phi(\bar{u})}.$$

Therefore, conditions (i) and (ii) of Theorem 2.6 are satisfied. In addition, we observe that

$$\bar{\Lambda} = \Big]\frac{1}{pc_B^p|\alpha|_1}\frac{1}{k}\frac{d^p}{G(d)}, \frac{1}{pc_B^p|\alpha|_1}\min\Big\{\frac{c_1^p}{G(c_1)}, \frac{c_2^p}{2G(c_2)}\Big\}\Big[\subset \Lambda,$$

and, furthermore, by Lemma 2.5, the functional I_λ satisfies the $(PS)^{[r_2]}$-condition for each $\lambda \in \bar{\Lambda}$.

Now, let u be a local minima of I_λ. We notice that $u(x) \geq 0$ for each x. In fact, since u is a critical point of I_λ, we have that

$$\int_{\mathbb{R}} \big(\big|u'(x)\big|^{p-2}u'(x)v'(x) + B\big|u(x)\big|^{p-2}u(x)v(x)\big)\,dx - \int_{\mathbb{R}} \lambda\alpha(x)g\big(u(x)\big)v(x)\,dx = 0$$

for all $v \in W^{1,p}(\mathbb{R})$. Then, if we choose $v(x) = \max\{-u(x), 0\}$, we have

$$\begin{aligned}
0 &= \int_{\mathbb{R}} \lambda\alpha(x)g\big(u(x)\big)v(x)\,dx - \int_{\mathbb{R}} \big(\big|u'(x)\big|^{p-2}u'(x)v'(x) + B\big|u(x)\big|^{p-2}u(x)v(x)\big)\,dx \\
&= \int_{\mathbb{R}} \lambda\alpha(x)g\big(u(x)\big)v(x)\,dx - \int_{\mathbb{R}} \big(-\big|v'(x)\big|^{p-2}\big(v'(x)\big)^2 - B\big|v(x)\big|^{p-2}\big(v(x)\big)^2\big)\,dx \\
&= \int_{\mathbb{R}} \lambda\alpha(x)g\big(u(x)\big)v(x)\,dx + \int_{\mathbb{R}} \big(\big|v'(x)\big|^{p} + B\big|v(x)\big|^{p}\big)\,dx \\
&= \int_{\mathbb{R}} \lambda\alpha(x)g\big(u(x)\big)v(x)\,dx + \|v\|^p \\
&\geq \|v\|^p.
\end{aligned}$$

From the previous inequality it follows that $v(x) = 0$ for each x and, consequently, $u(x) \geq 0$ for each x.

Finally, let $u_1, u_2 \in W^{1,p}(\mathbb{R})$ be two local minima of the functional I_λ with $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$. Since u_1, u_2, as seen before, are nonnegative, we get

$$\min_{t\in[0,1]} \Psi\big(tu_1 + (1-t)u_2\big) \geq 0.$$

Then we can apply Theorem 2.6. Thus, for each $\lambda \in \bar{\Lambda}$, the functional I_λ admits at least three critical points u_i which lie in $\Phi^{-1}(]-\infty, r_2[)$, and hence problem (P_λ) has three weak solutions u_i such that $|u_i|_\infty < c_2$, $i = 1, 2, 3$. □

Remark 3.2. If $p \geq 2$, then in Theorem 3.1 it is enough to assume that there exist three positive constants c_1, d, c_2, with $c_1 < d < c_2$ such that (j) and (jj) hold true. Indeed, a simple computation shows that $L(p) > 1$ for all $p \geq 2$, and from the same proof in Theorem 3.1 our assertion is proved.

We now give an example to illustrate Theorem 3.1.

Example 3.3. Let $\alpha, g : \mathbb{R} \to \mathbb{R}$ be the functions defined as follows:

$$\alpha(x) = \begin{cases} 1 & \text{if } x \in [-1,1], \\ \frac{1}{x^2} & \text{otherwise} \end{cases}$$

and

$$g(t) = \begin{cases} 1 & \text{if } t \in]-\infty, 1], \\ t^7 & \text{if } t \in]1,2], \\ 2^7 & \text{if } t \in]2,300], \\ \bar{g}(t) & \text{if } t \in]300, +\infty[, \end{cases}$$

where $\bar{g}(t) :]300, +\infty[\to \mathbb{R}$ is an arbitrary function. Therefore, Theorem 3.1 ensures that the problem

$$\begin{cases} -u'' + u = \lambda \alpha(x) g(u) & x \in \mathbb{R}, \\ u(-\infty) = u(+\infty) = 0 \end{cases}$$

admits at least three distinct nontrivial classical solutions whose norms in $C^0(\mathbb{R})$ are less than 300 for each $\lambda \in]\frac{56}{263}, \frac{1}{4}[$. In fact, it is sufficient to choose $p = 2, c_1 = 1, d = 2$, and $c_2 = 300$ in order to apply Theorem 3.1 (see also Remark 3.2), and so we obtain the conclusion.

Next, we present a consequence of Theorem 3.1.

Corollary 3.4. *Assume that $\alpha_0 \neq 0$ and $g \not\equiv 0$. Moreover, assume that*

$$\lim_{\xi \to 0^+} \frac{g(\xi)}{\xi^{p-1}} = 0 \qquad \text{and} \qquad \lim_{\xi \to +\infty} \frac{g(\xi)}{\xi^{p-1}} = 0.$$

Then, for each $\lambda \in]\frac{1}{pc_B^p |\alpha|_1} \frac{1}{k} \inf_{d>0} \frac{d^p}{\int_0^d g(\xi)\, d\xi}, +\infty[$, problem (P_λ) admits at least two distinct nontrivial weak solutions.

Proof. First observe that $\frac{1}{pc_B^p |\alpha|_1} \frac{1}{k} \inf_{d>0} \frac{d^p}{G(d)} < +\infty$ since $\alpha_0 \neq 0$ and $g \not\equiv 0$. Then fix $\lambda > \frac{1}{pc_B^p |\alpha|_1} \frac{1}{k} \inf_{d>0} \frac{d^p}{G(d)}$. Now, let d be a positive constant such that

$$\lambda > \frac{1}{pc_B^p |\alpha|_1} \frac{1}{k} \frac{d^p}{G(d)}. \tag{3.1}$$

Therefore, from $\lim_{\xi \to 0^+} pc_B^p |\alpha|_1 \frac{G(\xi)}{\xi^p} = 0 < \frac{1}{\lambda}$, there exists $c_1 > 0$ with $c_1 < \frac{1}{2} d$ such that

$$pc_B^p |\alpha|_1 \frac{G(c_1)}{c_1^p} < \frac{1}{\lambda}. \tag{3.2}$$

Moreover, from $\lim_{\xi \to +\infty} pc_B^p |\alpha|_1 \frac{2G(\xi)}{\xi^p} = 0 < \frac{1}{\lambda}$, there exists $c_2 > d$ such that

$$pc_B^p |\alpha|_1 \frac{2G(c_2)}{c_2^p} < \frac{1}{\lambda}. \tag{3.3}$$

Hence, taking (3.1), (3.2), and (3.3) into account, from Theorem 3.1 we have the result. □

Example 3.5. The problem

$$\begin{cases} -(|u'|u')' + |u|u = 15\frac{e^{-u}u^3}{1+x^2} & x \in \mathbb{R}, \\ u(-\infty) = u(+\infty) = 0 \end{cases}$$

admits at least two nontrivial classical solutions. It is enough to apply Corollary 3.4 by choosing $p = 3$, $g(u) = e^{-u}u^3$, $\alpha(x) = \frac{1}{1+x^2}$ and by observing that $G(u) = 6 - e^{-u}(u^3 + 3u^2 + 6u + 6)$ and $15 > \frac{1}{pc_B^p|\alpha|_1}\frac{1}{k}\frac{1^p}{G(1)} = \frac{9}{2\pi}\frac{e}{6e-16}$.

Remark 3.6. Theorem 1.1 in the Introduction is a consequence of Theorem 3.1. Indeed, it is enough to choose $p = 2$, taking Remark 3.2 into account, and to argue as in the proof of Corollary 3.4 to prove (jj). Therefore, Theorem 3.1 ensures the existence of three weak solutions which, since $g(0) \neq 0$ and $\alpha \in C(\mathbb{R})$, are nontrivial classical solutions.

Remark 3.7. We recall that existence of at least one nontrivial nonnegative solution for problem (P_λ) without requiring any asymptotic condition on g either at zero or at infinity was established in [5]. In that case the requested condition is of an opposite type with respect to (j) of Theorem 3.1. Also, in [5], under a suitable condition at infinity on g, the authors guaranteed the existence of two nontrivial solutions for problem (P_λ).

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