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L^p-INEQUALITIES AND PARSEVAL-TYPE RELATIONS FOR THE MEHLER–FOCK TRANSFORM OF GENERAL ORDER

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ABSTRACT. In this article we study new L^p-boundedness properties for the Mehler–Fock transform of general order on the spaces $L^p((0,\infty), e^{\alpha x}\,dx)$ and $L^p((0,\infty),(1+x)^\gamma\,dx)$, $1\le p\le\infty$, and $\alpha,\gamma\in\mathbb{R}$. We also obtain Parseval-type relations over these spaces.

1. INTRODUCTION AND PRELIMINARIES

In the following we consider the Mehler–Fock transform of general order of a suitable complex-valued function f defined on the interval $(0,\infty)$ given by

$$(\mathfrak{F}f)(y)=\int_0^\infty f(x)P^{-\mu}_{-\frac{1}{2}+iy}(\cosh x)\,dx,\quad y>0, \tag{1.1}$$

where $\Re\mu>-1/2$ and $P^{-\mu}_{-\frac{1}{2}+iy}$ is the associated Legendre function of the first kind and order $-\mu$ (for details, see [1, Chapter 3]) given in terms of the Gauss hypergeometric function ${}_2F_1$ by

$$P^{-\mu}_{\nu}(z)=\frac{1}{\Gamma(1+\mu)}\Big(\frac{z+1}{z-1}\Big)^{-\frac{\mu}{2}}{}_2F_1\Big(-\nu,\nu+1;1+\mu,\frac{1-z}{2}\Big). \tag{1.2}$$

This transform of general order (also called a *generalized Mehler–Fock transform*) is considered in [5], [6], and [10, Section 3.3].

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Our paper has the following organization. First, we study L^p-boundedness properties for the Mehler–Fock transform of general order (1.1) over the spaces $L^p((0,\infty),e^{\alpha x}\,dx)$ and $L^p((0,\infty),(1+x)^\gamma\,dx)$, $\alpha,\gamma\in\mathbb{R}$, and $1\le p\le\infty$. We also consider the integral operator

$$(\mathfrak{L}g)(x)=\int_0^\infty g(y)P^{-\mu}_{-\frac{1}{2}+iy}(\cosh x)\,dy,\quad x>0. \tag{1.3}$$

By using results of Section 2 of [2], we prove that the operator $\mathfrak{L}$ is bounded from the spaces $L^p((0,\infty),e^{\alpha x}\,dx)$ into $L^{p'}((0,\infty),e^{\alpha x}\,dx)$ if $1<p<\infty$, $p+p'=pp'$ whenever $0<\alpha<p'/2$ and $\Re\mu>-1/p'$. We also prove that the operator $\mathfrak{L}$ is bounded from the space $L^p((0,\infty),(1+x)^\gamma\,dx)$ into $L^{p'}((0,\infty),(1+x)^\gamma\,dx)$ if $1<p<\infty$, $p+p'=pp'$ whenever $\gamma>p-1$ and $\Re\mu>-1/p'$. This analysis is also extended for the case $p=1$. Using Section 3 of [2], we prove that the operator $\mathfrak{L}$ is bounded from $L^1((0,\infty),e^{\alpha x}\,dx)$ into $L^\infty((0,\infty),e^{\alpha x}\,dx)$ provided that $\alpha\ge0$ and $\Re\mu\ge0$. We also prove that the operator $\mathfrak{L}$ is bounded from $L^1((0,\infty),(1+x)^\gamma\,dx)$ into $L^\infty((0,\infty),(1+x)^\gamma\,dx)$ provided that $\gamma\ge0$ and $\Re\mu\ge0$.

Moreover, under these conditions, if $f,g\in L^p((0,\infty),e^{\alpha x}\,dx)$, $1\le p<\infty$, then we have the following Parseval-type relation:

$$\int_0^\infty(\mathfrak{F}f)(x)g(x)\,dx=\int_0^\infty f(x)(\mathfrak{L}g)(x)\,dx. \tag{1.4}$$

Also, under these conditions, if $f,g\in L^p((0,\infty),(1+x)^\gamma\,dx)$, $1\le p<\infty$, then we have the Parseval-type relation (1.4). Let $\mathfrak{L}'$ be the adjoint of the operator $\mathfrak{L}$; that is,

$$\langle\mathfrak{L}'f,g\rangle=\langle f,\mathfrak{L}g\rangle. \tag{1.5}$$

The aforementioned Parseval-type relation (1.4) allows us to obtain an interesting connection between the operator $\mathfrak{L}'$ and the operator $\mathfrak{F}$.

We conclude that the operator $\mathfrak{L}'$ is the natural extension of the integral operator $\mathfrak{F}$; that is,

$$\mathfrak{L}'T_f=T_{\mathfrak{F}f},$$

where T_f is given by

$$\langle T_f,g\rangle=\int_0^\infty f(x)g(x)\,dx. \tag{1.6}$$

We also point out relevant connections of our work with various earlier related results (see also [4], [8], [9], [11], and [12]). From [1, p. 156, Entry 7], we have the integral representation

$$P^{-\mu}_{-\frac{1}{2}+iy}(\cosh x)=\frac{\pi^{-1/2}(\sinh x)^\mu}{2^\mu\Gamma(\frac{1}{2}+\mu)}\int_0^\pi(\cosh x+\sinh x\cos u)^{-\frac{1}{2}+iy-\mu}$$
$$\times(\sin u)^{2\mu}\,du,\quad x>0,y>0,\text{ and }\Re\mu>-\frac{1}{2}.$$

Now, observe that, for $x>0$ and $u\in[0,\pi]$, we have

$$\cosh x+\sinh x\cos u>0,$$

and so

$$\begin{aligned}
&\left|P^{-\mu}_{-\frac{1}{2}+iy}(\cosh x)\right| \\
&\quad\le \frac{\pi^{-1/2}(\sinh x)^{\Re\mu}}{2^{\Re\mu}|\Gamma(\frac{1}{2}+\mu)|}\int_0^\pi (\cosh x+\sinh x\cos u)^{-\frac{1}{2}-\Re\mu}(\sin u)^{2\Re\mu}\,du \\
&\quad= \frac{\Gamma(\frac{1}{2}+\Re\mu)}{|\Gamma(\frac{1}{2}+\mu)|}P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x).
\end{aligned}\tag{1.7}$$

From [7, p. 171, Entry 12.08], we have

$$P^{-\mu}_{-\frac{1}{2}}(\cosh x)\sim\frac{x^\mu}{2^\mu\Gamma(1+\mu)}\quad\text{for } x\to 0.\tag{1.8}$$

Also, from [7, p. 172, Entry 12.20], we have

$$P^{-\mu}_{-\frac{1}{2}}(\cosh x)\sim\frac{2}{\sqrt{\pi}\Gamma(\frac{1}{2}+\mu)}xe^{-x/2}\quad\text{for } x\to\infty.\tag{1.9}$$

Throughout this article, $\Re\mu>-1/2$.

2. The operator $\mathfrak{F}$ over the spaces $L^p((0,\infty),e^{\alpha x}\,dx)$ and $L^p((0,\infty),(1+x)^\gamma\,dx),\ 1<p<\infty$

In this section, we study the behavior of the operator $\mathfrak{F}$ on the spaces $L^p((0,\infty),e^{\alpha x}\,dx)$ and $L^p((0,\infty),(1+x)^\gamma\,dx), 1<p<\infty$, $\alpha,\gamma\in\mathbb{R}$.

Theorem 2.1. *Assume that $1<p<\infty$, $p+p'=pp'$. Then, for all $0<q<\infty$, we have the following.*

(i) *If $-p/2<\alpha<0$, $\Re\mu>-1/p'$, then the operator $\mathfrak{F}$ given by (1.1) is bounded from $L^p((0,\infty),e^{\alpha x}\,dx)$ into $L^q((0,\infty),e^{\alpha x}\,dx)$. Also, if $\alpha>-p/2$ and $\Re\mu>-1/p'$, then the operator $\mathfrak{F}$ is bounded from $L^p((0,\infty),e^{\alpha x}\,dx)$ into $L^\infty((0,\infty),e^{\alpha x}\,dx)$.*

(ii) *If $\gamma<-1$, $\Re\mu>-1/p'$, then the operator $\mathfrak{F}$ given by (1.1) is bounded from $L^p((0,\infty),(1+x)^\gamma\,dx)$ into $L^q((0,\infty),(1+x)^\gamma\,dx)$. Also, if $\gamma\in\mathbb{R}$ and $\Re\mu>-1/p'$, then the operator $\mathfrak{F}$ is bounded from $L^p((0,\infty),(1+x)^\gamma\,dx)$ into $L^\infty((0,\infty),(1+x)^\gamma\,dx)$.*

Proof. (i) From (1.7), the condition (2.1) on Proposition 2.1 of [3] becomes

$$\begin{aligned}
&\int_0^\infty\Big(\int_0^\infty\left|P^{-\mu}_{-\frac{1}{2}+iy}(\cosh x)\right|^{p'}e^{-\alpha p'x/p}\,dx\Big)^{q/p'}e^{\alpha y}\,dy \\
&\quad\le\Big(\frac{\Gamma(\frac{1}{2}+\Re\mu)}{|\Gamma(\frac{1}{2}+\mu)|}\Big)^q \\
&\qquad\times\Big(\int_0^\infty\big(P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x)\big)^{p'}e^{-\alpha p'x/p}\,dx\Big)^{q/p'} \\
&\qquad\times\int_0^\infty e^{\alpha y}\,dy
\end{aligned}$$

$$= \Big(\frac{-1}{\alpha}\Big)\Big(\frac{\Gamma(\frac{1}{2}+\Re\mu)}{|\Gamma(\frac{1}{2}+\mu)|}\Big)^{q} \\ \times \Big(\int_0^\infty \big(P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x)\big)^{p'} e^{-\alpha p' x/p}\, dx\Big)^{q/p'}. \tag{2.1}$$

From (1.8) and (1.9), we obtain that, for $-p/2 < \alpha < 0$ and $\Re\mu > -1/p'$, the operator $\mathfrak{F}$ is bounded from $L^p((0,\infty), e^{\alpha x}\, dx)$ into $L^q((0,\infty), e^{\alpha x}\, dx)$. On the other hand, from (1.7) the condition (2.2) on Proposition 2.1 of [3] becomes

$$\operatorname*{ess\,sup}_{y\in(0,\infty)}\Big\{\int_0^\infty \big|P^{-\mu}_{-\frac{1}{2}+iy}(\cosh x)\big|^{p'} e^{-\alpha p' x/p}\, dx\Big\} \\ \le \Big(\frac{\Gamma(\frac{1}{2}+\Re\mu)}{|\Gamma(\frac{1}{2}+\mu)|}\Big)^{p'} \int_0^\infty \big(P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x)\big)^{p'} e^{-\alpha p' x/p}\, dx. \tag{2.2}$$

Now from (1.8) and (1.9), for $\alpha > -p/2$ and $\Re\mu > -1/p'$, the above integral converges, and therefore the operator $\mathfrak{F}$ is bounded from $L^p((0,\infty), e^{\alpha x}\, dx)$ into $L^\infty((0,\infty), e^{\alpha x}\, dx)$.

(ii) From (1.7), the condition (2.1) on Proposition 2.1 of [3] becomes

$$\int_0^\infty \Big(\int_0^\infty \big|P^{-\mu}_{-\frac{1}{2}+iy}(\cosh x)\big|^{p'} (1+x)^{-\gamma p'/p}\, dx\Big)^{q/p'} (1+y)^\gamma\, dy \\ \le \Big(\frac{\Gamma(\frac{1}{2}+\Re\mu)}{|\Gamma(\frac{1}{2}+\mu)|}\Big)^{q} \Big(\int_0^\infty \big(P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x)\big)^{p'} (1+x)^{-\gamma p'/p}\, dx\Big)^{q/p'} \times \int_0^\infty (1+y)^\gamma\, dy \\ = \Big(\frac{-1}{1+\gamma}\Big)\Big(\frac{\Gamma(\frac{1}{2}+\Re\mu)}{|\Gamma(\frac{1}{2}+\mu)|}\Big)^{q} \Big(\int_0^\infty \big(P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x)\big)^{p'} (1+x)^{-\gamma p'/p}\, dx\Big)^{q/p'}. \tag{2.3}$$

From (1.8) and (1.9) we get that, for $\gamma < -1$ and $\Re\mu > -1/p'$, the operator $\mathfrak{F}$ is bounded from $L^p((0,\infty), (1+x)^\gamma\, dx)$ into $L^q((0,\infty), (1+x)^\gamma\, dx)$. On the other hand, from (1.7) the condition (2.2) on Proposition 2.1 of [3] becomes

$$\operatorname*{ess\,sup}_{y\in(0,\infty)}\Big\{\int_0^\infty \big|P^{-\mu}_{-\frac{1}{2}+iy}(\cosh x)\big|^{p'} (1+x)^{-\gamma p'/p}\, dx\Big\} \\ \le \Big(\frac{\Gamma(\frac{1}{2}+\Re\mu)}{|\Gamma(\frac{1}{2}+\mu)|}\Big)^{p'} \int_0^\infty \big(P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x)\big)^{p'} (1+x)^{-\gamma p'/p}\, dx. \tag{2.4}$$

Now from (1.8) and (1.9), for $\gamma \in \mathbb{R}$ and $\Re\mu > -1/p'$, the above integral converges, and therefore the operator $\mathfrak{F}$ is bounded from $L^p((0,\infty), (1+x)^\gamma\, dx)$ into $L^\infty((0,\infty), (1+x)^\gamma\, dx)$. □

3. The operator $\mathfrak{F}$ over the spaces $L^1((0,\infty), e^{\alpha x}\, dx)$ and $L^1((0,\infty), (1+x)^\gamma\, dx)$

We now prove corresponding results for the case when $p = 1$.

Theorem 3.1. *For all $0 < q < \infty$, we get the following.*

(i) *For* $-1/2 < \alpha < 0$, $\Re\mu \geq 0$, *the operator* $\mathfrak{F}$ *given by (1.1) is bounded from* $L^1((0,\infty), e^{\alpha x}\,dx)$ *into* $L^q((0,\infty), e^{\alpha x}\,dx)$. *Also, for* $\alpha > -1/2$ *and* $\Re\mu \geq 0$, *then the operator* $\mathfrak{F}$ *is bounded from* $L^1((0,\infty), e^{\alpha x}\,dx)$ *into* $L^\infty((0,\infty), e^{\alpha x}\,dx)$.

(ii) *For* $\gamma < -1$, $\Re\mu \geq 0$, *the operator* $\mathfrak{F}$ *given by (1.1) is bounded from* $L^1((0,\infty), (1+x)^\gamma\,dx)$ *into* $L^q((0,\infty), (1+x)^\gamma\,dx)$. *Also, for* $\gamma \in \mathbb{R}$ *and* $\Re\mu \geq 0$, *then the operator* $\mathfrak{F}$ *is bounded from* $L^1((0,\infty), (1+x)^\gamma\,dx)$ *into* $L^\infty((0,\infty), (1+x)^\gamma\,dx)$.

Proof. (i) From (1.7) the condition (3.1) on Proposition 3.1 of [3] becomes

$$\begin{aligned}\int_0^\infty \Big(\operatorname*{ess\,sup}_{x\in(0,\infty)}\Big\{\frac{|P^{-\mu}_{-\frac12+iy}(\cosh x)|}{e^{\alpha x}}\Big\}\Big)^q e^{\alpha y}\,dy &\leq \Big(\frac{\Gamma(\frac12+\Re\mu)}{|\Gamma(\frac12+\mu)|}\Big)^q\Big(\operatorname*{ess\,sup}_{x\in(0,\infty)}\Big\{\frac{P^{-\Re\mu}_{-\frac12}(\cosh x)}{e^{\alpha x}}\Big\}\Big)^q \times \int_0^\infty e^{\alpha y}\,dy\\ &= \Big(\frac{-1}{\alpha}\Big)\Big(\frac{\Gamma(\frac12+\Re\mu)}{|\Gamma(\frac12+\mu)|}\Big)^q\Big(\operatorname*{ess\,sup}_{x\in(0,\infty)}\Big\{\frac{P^{-\Re\mu}_{-\frac12}(\cosh x)}{e^{\alpha x}}\Big\}\Big)^q. \qquad (3.1)\end{aligned}$$

Now from (1.8) and (1.9), for $-1/2 < \alpha < 0$ and $\Re\mu \geq 0$, the operator $\mathfrak{F}$ is bounded from $L^1((0,\infty), e^{\alpha x}\,dx)$ into $L^q((0,\infty), e^{\alpha x}\,dx)$.

Likewise, from (1.7) the condition (3.2) on Proposition 3.1 of [3] becomes

$$\begin{aligned}\operatorname*{ess\,sup}_{y\in(0,\infty)}\operatorname*{ess\,sup}_{x\in(0,\infty)}\Big\{\frac{|P^{-\mu}_{-\frac12+iy}(\cosh x)|}{e^{\alpha x}}\Big\} &\leq \frac{\Gamma(\frac12+\Re\mu)}{|\Gamma(\frac12+\mu)|}\operatorname*{ess\,sup}_{x\in(0,\infty)}\Big\{\frac{P^{-\Re\mu}_{-\frac12}(\cosh x)}{e^{\alpha x}}\Big\}. \qquad (3.2)\end{aligned}$$

From (1.8) and (1.9) one obtains that, for $\alpha > -1/2$ and $\Re\mu \geq 0$, the operator $\mathfrak{F}$ is bounded from $L^1((0,\infty), e^{\alpha x}\,dx)$ into $L^\infty((0,\infty), e^{\alpha x}\,dx)$.

(ii) From (1.7) the condition (3.1) on Proposition 3.1 of [3] becomes

$$\begin{aligned}\int_0^\infty \Big(\operatorname*{ess\,sup}_{x\in(0,\infty)}\Big\{\frac{|P^{-\mu}_{-\frac12+iy}(\cosh x)|}{(1+x)^\gamma}\Big\}\Big)^q (1+y)^\gamma\,dy &\leq \Big(\frac{\Gamma(\frac12+\Re\mu)}{|\Gamma(\frac12+\mu)|}\Big)^q\Big(\operatorname*{ess\,sup}_{x\in(0,\infty)}\Big\{\frac{P^{-\Re\mu}_{-\frac12}(\cosh x)}{(1+x)^\gamma}\Big\}\Big)^q \times \int_0^\infty (1+y)^\gamma\,dy\\ &= \Big(\frac{-1}{1+\gamma}\Big)\Big(\frac{\Gamma(\frac12+\Re\mu)}{|\Gamma(\frac12+\mu)|}\Big)^q\Big(\operatorname*{ess\,sup}_{x\in(0,\infty)}\Big\{\frac{P^{-\Re\mu}_{-\frac12}(\cosh x)}{(1+x)^\gamma}\Big\}\Big)^q. \qquad (3.3)\end{aligned}$$

Now from (1.8) and (1.9), for $\gamma < -1$ and $\Re\mu \geq 0$, the operator $\mathfrak{F}$ is bounded from $L^1((0,\infty), (1+x)^\gamma\,dx)$ into $L^q((0,\infty), (1+x)^\gamma\,dx)$. Likewise, from (1.7) the condition (3.2) on Proposition 3.1 of [3] becomes

$$\begin{aligned} &\operatorname*{ess\,sup}_{y\in(0,\infty)} \operatorname*{ess\,sup}_{x\in(0,\infty)} \Big\{ \frac{|P^{-\mu}_{-\frac{1}{2}+iy}(\cosh x)|}{(1+x)^\gamma} \Big\} \\ &\quad \leq \frac{\Gamma(\frac{1}{2}+\Re\mu)}{|\Gamma(\frac{1}{2}+\mu)|} \operatorname*{ess\,sup}_{x\in(0,\infty)} \Big\{ \frac{P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x)}{(1+x)^\gamma} \Big\}. \end{aligned} \tag{3.4}$$

From (1.8) and (1.9), we get that, for all $\gamma \in \mathbb{R}$ and $\Re\mu \geq 0$, the operator $\mathfrak{F}$ is bounded from $L^1((0,\infty),(1+x)^\gamma\,dx)$ into $L^\infty((0,\infty),(1+x)^\gamma\,dx)$. □

4. The operator $\mathfrak{F}$ over the spaces $L^\infty((0,\infty),e^{\alpha x}\,dx)$ and $L^\infty((0,\infty),(1+x)^\gamma\,dx)$

We now prove corresponding results for the case when $p=\infty$.

Theorem 4.1. *For all $0<q<\infty$, we get the following.*

(i) *If $\alpha<0$, then the operator $\mathfrak{F}$ given by (1.1) is bounded from $L^\infty((0,\infty),e^{\alpha x}\,dx)$ into $L^q((0,\infty),e^{\alpha x}\,dx)$. Also, for all $\alpha\in\mathbb{R}$, then the operator $\mathfrak{F}$ is bounded from $L^\infty((0,\infty),e^{\alpha x}\,dx)$ into $L^\infty((0,\infty),e^{\alpha x}\,dx)$.*

(ii) *If $\gamma<-1$, then the operator $\mathfrak{F}$ given by (1.1) is bounded from $L^\infty((0,\infty),(1+x)^\gamma\,dx)$ into $L^q((0,\infty),(1+x)^\gamma\,dx)$. Also, for all $\gamma\in\mathbb{R}$, the operator $\mathfrak{F}$ is bounded from $L^\infty((0,\infty),(1+x)^\gamma\,dx)$ into $L^\infty((0,\infty),(1+x)^\gamma\,dx)$.*

Proof. (i) From (1.7) the condition (4.1) on Proposition 4.1 of [3] becomes

$$\begin{aligned} &\int_0^\infty \Big(\int_0^\infty \big|P^{-\mu}_{-\frac{1}{2}+iy}(\cosh x)\big|\,dx \Big)^q e^{\alpha y}\,dy \\ &\quad \leq \Big(\frac{\Gamma(\frac{1}{2}+\Re\mu)}{|\Gamma(\frac{1}{2}+\mu)|} \Big)^q \Big(\int_0^\infty P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x)\,dx \Big)^q \times \int_0^\infty e^{\alpha y}\,dy \\ &\quad = \Big(\frac{-1}{\alpha} \Big) \Big(\frac{\Gamma(\frac{1}{2}+\Re\mu)}{|\Gamma(\frac{1}{2}+\mu)|} \Big)^q \Big(\int_0^\infty P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x)\,dx \Big)^q. \end{aligned} \tag{4.1}$$

From (1.8) and (1.9) one obtains that, for $\alpha<0$, the above integral converges. Therefore, the operator $\mathfrak{F}$ is bounded from $L^\infty((0,\infty),e^{\alpha x}\,dx)$ into $L^q((0,\infty),e^{\alpha x}\,dx)$. Also, from (1.7) the condition (4.2) on Proposition 4.1 of [3] becomes

$$\begin{aligned} &\operatorname*{ess\,sup}_{y\in(0,\infty)} \Big\{ \int_0^\infty \big|P^{-\mu}_{-\frac{1}{2}+iy}(\cosh x)\big|\,dx \Big\} \\ &\quad \leq \frac{\Gamma(\frac{1}{2}+\Re\mu)}{|\Gamma(\frac{1}{2}+\mu)|} \int_0^\infty P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x)\,dx. \end{aligned} \tag{4.2}$$

From (1.8) and (1.9) one obtains that, for all $\alpha\in\mathbb{R}$, the above integral converges. Therefore, the operator $\mathfrak{F}$ is bounded from $L^\infty((0,\infty),e^{\alpha x}\,dx)$ into $L^\infty((0,\infty),e^{\alpha x}\,dx)$.

(ii) From (1.7) the condition (4.1) on Proposition 4.1 of [3] becomes

$$\begin{aligned}\int_0^\infty \Big(\int_0^\infty \big|P^{-\mu}_{-\frac{1}{2}+iy}(\cosh x)\big|\,dx\Big)^q (1+y)^\gamma\,dy \\ \le \Big(\frac{\Gamma(\frac{1}{2}+\Re\mu)}{|\Gamma(\frac{1}{2}+\mu)|}\Big)^q \Big(\int_0^\infty P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x)\,dx\Big)^q \times \int_0^\infty (1+y)^\gamma\,dy \\ = \Big(\frac{-1}{1+\gamma}\Big)\Big(\frac{\Gamma(\frac{1}{2}+\Re\mu)}{|\Gamma(\frac{1}{2}+\mu)|}\Big)^q \Big(\int_0^\infty P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x)\,dx\Big)^q. \end{aligned} \tag{4.3}$$

From (1.8) and (1.9) one obtains that, for $\gamma < -1$, the above integral converges. Therefore, the operator $\mathfrak{F}$ is bounded from $L^\infty((0,\infty),(1+x)^\gamma\,dx)$ into $L^q((0,\infty),(1+x)^\gamma\,dx)$. Also, from (1.7) the condition (4.2) on Proposition 4.1 of [3] becomes

$$\begin{aligned}\operatorname*{ess\,sup}_{y\in(0,\infty)}\Big\{\int_0^\infty \big|P^{-\mu}_{-\frac{1}{2}+iy}(\cosh x)\big|\,dx\Big\} \\ \le \frac{\Gamma(\frac{1}{2}+\Re\mu)}{|\Gamma(\frac{1}{2}+\mu)|}\int_0^\infty P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x)\,dx. \end{aligned} \tag{4.4}$$

From (1.8) and (1.9) one obtains that, for all $\gamma \in \mathbb{R}$, the above integral converges. Therefore, the operator $\mathfrak{F}$ is bounded from $L^\infty((0,\infty),(1+x)^\gamma\,dx)$ into $L^\infty((0,\infty),(1+x)^\gamma\,dx)$. □

5. The operator $\mathfrak{L}$ over the spaces $L^p((0,\infty),e^{\alpha x}\,dx)$ and $L^p((0,\infty),(1+x)^\gamma\,dx)$, $1<p<\infty$

In this section, we study the behavior of the operator $\mathfrak{L}$ on the spaces $L^p((0,\infty), e^{\alpha x}\,dx)$ and $L^p((0,\infty),(1+x)^\gamma\,dx)$, $\alpha,\gamma\in\mathbb{R}$, and $1<p<\infty$.

Theorem 5.1. *Assume that $1<p<\infty$, $p+p'=pp'$. Then we have the following.*

(i) *For all $0<\alpha<p'/2$ and $\Re\mu > -1/p'$, the operator $\mathfrak{L}$ given by (1.3) is bounded from $L^p((0,\infty),e^{\alpha x}\,dx)$ into $L^{p'}((0,\infty),e^{\alpha x}\,dx)$.*

(ii) *For all $\gamma > p-1$ and $\Re\mu > -1/p'$, the operator $\mathfrak{L}$ given by (1.3) is bounded from $L^p((0,\infty),(1+x)^\gamma\,dx)$ into $L^{p'}((0,\infty),(1+x)^\gamma\,dx)$.*

Proof. (i) Note that, for $0<\alpha<p'/2$ and $\Re\mu > -1/p'$, and using (1.8) and (1.9), we have

$$\int_0^\infty e^{-\alpha p' y/p}\,dy = \frac{p}{\alpha p'}$$

and $P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x) \in L^{p'}((0,\infty),e^{\alpha x}\,dx)$. Then from Proposition 2.1 in [2] the result holds.

(ii) Note that, for $\gamma > p-1$ and $\Re\mu > -1/p'$, and using (1.8) and (1.9), we have

$$\int_0^\infty (1+y)^{-\gamma p'/p}\,dy = \frac{p}{\gamma p'}$$

and $P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x) \in L^{p'}((0,\infty),(1+y)^\gamma\,dy)$. Then from Proposition 2.1 in [2] the result holds. □

As a consequence of Proposition 2.2 in [2], we have the following.

Theorem 5.2. *Assume that* $1 < p < \infty$, $p + p' = pp'$. *Then the following Parseval-type relation holds:*

$$\int_0^\infty (\mathfrak{F}f)(x)g(x)\,dx = \int_0^\infty f(x)(\mathfrak{L}g)(x)\,dx \tag{5.1}$$

(i) *for* $f, g \in L^p((0,\infty), e^{\alpha x}\,dx)$ *with* $0 < \alpha < p'/2$ *and* $\Re\mu > -1/p'$ *or, alternatively,*
(ii) *for* $f, g \in L^p((0,\infty), (1+x)^\gamma\,dx)$ *with* $\alpha > p - 1$ *and* $\Re\mu > -1/p'$.

Also, as a consequence of Corollary 2.1 in [2], we have the following.

Corollary 5.3. *Assume that* $1 < p < \infty$, $p+p' = pp'$. *Then we have the following.*

(i) *For* $f \in L^p((0,\infty), e^{\alpha x}\,dx)$, $0 < \alpha < p'/2$, *and* $\Re\mu > -1/p'$, *we have*

$$\mathfrak{L}' T_f = T_{\mathfrak{F}f} \tag{5.2}$$

on $(L^p((0,\infty), e^{\alpha x}\,dx))'$.

(ii) *For* $f \in L^p((0,\infty), (1+x)^\gamma\,dx)$, $\gamma > p - 1$, *and* $\Re\mu > -1/p'$, *we have*

$$\mathfrak{L}' T_f = T_{\mathfrak{F}f} \tag{5.3}$$

on $(L^p((0,\infty), (1+x)^\gamma\,dx))'$.

6. The operator $\mathfrak{L}$ over the spaces $L^1((0,\infty), e^{\alpha x}\,dx)$ and $L^1((0,\infty), (1+x)^\gamma\,dx)$

In this section, we study the behavior of the operator $\mathfrak{L}$ on the spaces $L^1((0,\infty), e^{\alpha x}\,dx)$ and $L^1((0,\infty), (1+x)^\gamma\,dx)$, $\alpha, \gamma \in \mathbb{R}$.

Theorem 6.1. *We have the following.*

(i) *For all* $\alpha \geq 0$ *and* $\Re\mu \geq 0$, *the operator* $\mathfrak{L}$ *given by (1.3) is bounded from* $L^1((0,\infty), e^{\alpha x}\,dx)$ *into* $L^\infty((0,\infty), e^{\alpha x}\,dx)$.
(ii) *For all* $\gamma \geq 0$ *and* $\Re\mu \geq 0$, *the operator* $\mathfrak{L}$ *given by (1.3) is bounded from* $L^1((0,\infty), (1+x)^\gamma\,dx)$ *into* $L^\infty((0,\infty), (1+x)^\gamma\,dx)$.

Proof. (i) Note that, for $\alpha \geq 0$ and $\Re\mu \geq 0$, and using (1.8) and (1.9), we get that $P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x)$ is essentially bounded on $(0,\infty)$. Then from Proposition 3.1 in [2] the result holds.

(ii) Note that, for $\gamma \geq 0$ and $\Re\mu \geq 0$, and using (1.8) and (1.9), we get that $P^{-\Re\mu}_{-\frac{1}{2}}(\cosh x)$ is essentially bounded on $(0,\infty)$. Then from Proposition 3.1 in [2] the result holds. □

As a consequence of Proposition 3.1 in [2], we get the following.

Theorem 6.2. *The following Parseval-type relation holds:*

$$\int_0^\infty (\mathfrak{F}f)(x)g(x)\,dx = \int_0^\infty f(x)(\mathfrak{L}g)(x)\,dx \tag{6.1}$$

(i) *for* $f, g \in L^1((0,\infty), e^{\alpha x}\,dx)$ *with* $\alpha \geq 0$ *and* $\Re\mu \geq 0$ *or, alternatively,*

(ii) *for* $f, g \in L^1((0,\infty),(1+x)^\gamma\,dx)$ *with* $\gamma \geq 0$ *and* $\Re\mu \geq 0$.

Also, as a consequence of Corollary 3.2 in [2], we have the following.

Corollary 6.3. (i) *For* $f \in L^1((0,\infty), e^{\alpha x}\,dx)$, $\alpha \geq 0$, *and* $\Re\mu \geq 0$, *it holds that*

$$\mathfrak{L}' T_f = T_{\mathfrak{F} f} \tag{6.2}$$

on $(L^1((0,\infty), e^{\alpha x}\,dx))'$.

(ii) *For* $f \in L^1((0,\infty),(1+x)^\gamma\,dx)$, $\gamma \geq 0$, *and* $\Re\mu \geq 0$, *it holds that*

$$\mathfrak{L}' T_f = T_{\mathfrak{F} f} \tag{6.3}$$

on $(L^1((0,\infty),(1+x)^\gamma\,dx))'$.

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