Ann. Funct. Anal. 8 (2017), no. 2, 215–230
http://dx.doi.org/10.1215/20088752-0000007X
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

GEOMETRIC CONSTANTS OF $\pi/2$-ROTATION INVARIANT NORMS ON $\mathbb{R}^2$

YUKINO TOMIZAWA,[1] KEN-ICHI MITANI,[2]
KICHI-SUKE SAITO,[3*] and RYOTARO TANAKA[4]

Communicated by J. Chmieliński

Abstract. In this article, we study the (modified) von Neumann–Jordan constant and Zbăganu constant of $\pi/2$-rotation invariant norms on $\mathbb{R}^2$. Some estimations of these geometric constants are given. As an application, we construct various examples consisting of $\pi/2$-rotation invariant norms.

1. Introduction and preliminaries

This paper is concerned with geometric constants of Banach spaces, the (modified) von Neumann–Jordan constant, and the Zbăganu constant. For a Banach space X, let B_X and S_X be the unit ball and unit sphere, respectively. The von Neumann–Jordan constant $C_{NJ}(X)$ of X was defined in [8, Theorem II] by

$$C_{NJ}(X) = \sup\Big\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X, (x, y) \neq (0, 0)\Big\}.$$

The constant $C_{NJ}(X)$ can be viewed as a measure of the distortion of B_X from the viewpoint of the parallelogram law, and the estimation $1 \leq C_{NJ}(X) \leq 2$ holds for any Banach space X. Moreover, it is known that $C_{NJ}(X) = 1$ if and only if X is a Hilbert space (see [8]), and $C_{NJ}(X) < 2$ if and only if X is uniformly nonsquare (see [15]). To date, many works have been devoted to studying the von Neumann–Jordan constant of Banach spaces (see, e.g., [2], [4], [14], [16], [17]).

Received May 8, 2016; Accepted Aug. 25, 2016.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46B20.
Keywords. (modified) von Neumann–Jordan constant, Zbăganu constant, rotation invariant norm.

The *modified* von Neumann–Jordan constant $C'_{NJ}(X)$ of a Banach space X measures the distortion of S_X in the sense of the parallelogram law; that is, $C'_{NJ}(X)$ is given by

$$C'_{NJ}(X) = \sup\Big\{\frac{\|x+y\|^2 + \|x-y\|^2}{4} : x, y \in S_X\Big\}\big(\le C_{NJ}(X)\big).$$

This variation of $C_{NJ}(X)$ was introduced by Gao [5] (first in the form $\sup\{\|x+y\|^2 + \|x-y\|^2 : x, y \in S_X\}$), and its properties were studied, for example, in [5], [6], and [19].

As another variation of the von Neumann–Jordan constant, we have the Zbăganu constant $C_Z(X)$ of a Banach space X introduced in [18] as

$$C_Z(X) = \sup\Big\{\frac{\|x+y\|\|x-y\|}{\|x\|^2 + \|y\|^2} : x, y \in X, (x,y) \ne (0,0)\Big\}.$$

It follows from $2\|x+y\|\|x-y\| \le \|x+y\|^2 + \|x-y\|^2$ that $C_Z(X) \le C_{NJ}(X)$. However, in general, $C_Z(X)$ does not necessarily coincide with $C_{NJ}(X)$. There is a specific example of a (2-dimensional) normed space X with the property that $C_Z(X) < C_{NJ}(X)$ (see [1]).

In 2011, Mizuguchi and Saito [10] studied the relationship between the above-mentioned three constants in the case of absolute normalized norms on $\mathbb{R}^2$, and applied their results to the construction of some interesting new examples (see [7], [11], [12] for related results). A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be *absolute* if $\|(a,b)\| = \|(|a|,|b|)\|$ for each $(a,b) \in \mathbb{R}^2$, and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. Typical examples of such norms are the ℓ_p-norms $\|\cdot\|_p$ given by

$$\big\|(a,b)\big\|_p = \begin{cases} (|a|^p + |b|^p)^{1/p} & (1 \le p < \infty), \\ \max\{|a|,|b|\} & (p = \infty). \end{cases}$$

Let AN_2 be the collection of all absolute normalized norms on $\mathbb{R}^2$, and let Ψ_2 be the family of all convex functions ψ on $[0,1]$ satisfying $\max\{1-t, t\} \le \psi(t) \le 1$ for each $t \in [0,1]$. Then, as was shown in [3] (and in [14]), AN_2 is in a one-to-one correspondence with Ψ_2 under the equation $\psi(t) = \|(1-t,t)\|$ for each $t \in [0,1]$. The absolute normalized norm corresponding to $\psi \in \Psi_2$ is denoted by $\|\cdot\|_\psi$, and it satisfies the following equation:

$$\big\|(a,b)\big\|_\psi = \begin{cases} (|a|+|b|)\psi(\frac{|b|}{|a|+|b|}) & ((a,b) \ne (0,0)), \\ 0 & ((a,b) = (0,0)). \end{cases}$$

On the other hand, the convex function ψ_p corresponding to $\|\cdot\|_p$ is given by

$$\psi_p(t) = \begin{cases} ((1-t)^p + t^p)^{1/p} & (1 \le p < \infty), \\ \max\{1-t, t\} & (p = \infty). \end{cases}$$

It should be noted that $\psi_p(t) = \psi_p(1-t)$ for each $t \in [0,1]$ $(1 \le p \le \infty)$.

Recently, the notion of $\pi/2$-rotation invariant norms on $\mathbb{R}^2$ were investigated in [9], where a norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be $\pi/2$-*rotation invariant* if the $\pi/2$-rotation matrix

$$R(\pi/2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is an isometry on $(\mathbb{R}^2, \|\cdot\|)$ or, equivalently, $\|(a,b)\| = \|(-b,a)\|$ for each $(a,b) \in \mathbb{R}^2$. An example of a $\pi/2$-rotation invariant norm that is not isometrically isomorphic to any absolute normed space was given in [9, Theorem 5.13].

The purpose of the present article is to study the (modified) von Neumann–Jordan constant and the Zbăganu constant of $\pi/2$-rotation invariant norms on $\mathbb{R}^2$. In [9, Theorem 3.2], it was shown that any $\pi/2$-rotation invariant normed space is isometrically isomorphic to some Day–James space of the form $\ell^2_{\psi,\widetilde{\psi}}$, where $\widetilde{\psi}$ is the element of Ψ_2 defined by $\widetilde{\psi}(t) = \psi(1-t)$, and $\ell^2_{\psi,\widetilde{\psi}}$ is the space $\mathbb{R}^2$ endowed with the norm

$$\|(a,b)\|_{\psi,\widetilde{\psi}} = \begin{cases} (|a|+|b|)\psi(\frac{|a|}{|a|+|b|}) & (ab \geq 0), \\ (|a|+|b|)\widetilde{\psi}(\frac{|a|}{|a|+|b|}) & (ab \leq 0). \end{cases}$$

(See [13] for the general definition of Day–James spaces.) Of course, the norm $\|\cdot\|_{\psi,\widetilde{\psi}}$ is also $\pi/2$-rotation invariant for each $\psi \in \Psi_2$ (see [9, Proposition 3.4]). From this, since the (modified) von Neumann–Jordan constant and the Zbăganu constant are invariant under isometric isomorphisms, for our purpose, it is enough to consider Day–James spaces of the form $\ell^2_{\psi,\widetilde{\psi}}$; and hence, throughout this paper, $\pi/2$-rotation invariant normed spaces are assumed to be $\ell^2_{\psi,\widetilde{\psi}}$ for some $\psi \in \Psi_2$. Henceforth, fix an element ψ in Ψ_2 ($\psi \neq \psi_2$), and put $\|\cdot\| = \|\cdot\|_{\psi,\widetilde{\psi}}$ for short. The space $\ell^2_{\psi,\widetilde{\psi}}$ $(= (\mathbb{R}^2, \|\cdot\|_{\psi,\widetilde{\psi}}))$ will be simply denoted by Y_ψ. Under this hypothesis, we obtain some estimations of the abovementioned geometric constants that are similar to (but essentially different from) the results in [10]. As an application, we present various examples consisting of $\pi/2$-rotation invariant norms on $\mathbb{R}^2$.

2. Auxiliary results on Y_ψ

We start our argument with some auxiliary results.

Lemma 2.1. *Let $\varphi, \psi \in \Psi_2$. Then $\|\cdot\|_{\varphi,\widetilde{\varphi}} \leq M\|\cdot\|_{\psi,\widetilde{\psi}}$, where*

$$M = \max_{0\leq t\leq 1} \frac{\varphi(t)}{\psi(t)}.$$

Proof. We first note that $\varphi(t) \leq M\psi(t)$ for each $t \in [0,1]$, and hence $\widetilde{\varphi}(t) = \varphi(1-t) \leq M\psi(1-t) \leq M\widetilde{\psi}(t)$ for each $t \in [0,1]$. Now, for a nonzero element (a,b) of $\mathbb{R}^2$, we have

$$\|(a,b)\|_{\varphi,\widetilde{\varphi}} = \begin{cases} (|a|+|b|)\varphi(\frac{|a|}{|a|+|b|}) & (ab \geq 0), \\ (|a|+|b|)\widetilde{\varphi}(\frac{|a|}{|a|+|b|}) & (ab \leq 0), \end{cases}$$

and $\|(a,b)\|_{\psi,\widetilde{\psi}}$ has the same form (with ψ in place of φ). From this, it follows that $\|\cdot\|_{\varphi,\widetilde{\varphi}} \leq M\|\cdot\|_{\psi,\widetilde{\psi}}$. □

We note that $\|\cdot\|_2 = \|\cdot\|_{\psi_2} = \|\cdot\|_{\psi_2,\widetilde{\psi_2}}$. This, together with the preceding lemma, shows that $M_2^{-1}\|\cdot\|_2 \le \|\cdot\| \le M_1\|\cdot\|_2$, where

$$M_1 = \max_{0\le t\le 1} \frac{\psi(t)}{\psi_2(t)} \qquad \text{and} \qquad M_2 = \max_{0\le t\le 1} \frac{\psi_2(t)}{\psi(t)},$$

respectively.

As was mentioned in the last paragraph of Section 1, the norm $\|\cdot\|$ $(= \|\cdot\|_{\psi,\widetilde{\psi}})$ is $\pi/2$-rotation invariant. For such a norm, we obtain the following property.

Lemma 2.2. *If $x, y \in Y_\psi$ are such that $x \pm y \ne 0$ and $\|x\|_2 = \|y\|_2$, then*

$$\frac{\|x+y\|_2}{\|x+y\|} = \frac{\|x-y\|_2}{\|x-y\|}.$$

Proof. Since $\|x\|_2 = \|y\|_2$, we have that $\langle x+y, x-y\rangle = 0$, and so

$$x + y = \pm\frac{\|x+y\|}{\|x-y\|} R(\pi/2)(x-y).$$

By taking the Euclidean norms of both sides, one obtains

$$\|x+y\|_2 = \frac{\|x+y\|\|R(\pi/2)(x-y)\|_2}{\|x-y\|} = \frac{\|x+y\|\|x-y\|_2}{\|x-y\|}$$

since $R(\pi/2)$ is an isometry on the Euclidean space. Thus it follows that

$$\frac{\|x+y\|_2}{\|x+y\|} = \frac{\|x-y\|_2}{\|x-y\|}.$$

This proves the lemma. □

We now present a key to the proofs of our results in the sequel.

Theorem 2.3. *Let $c, d > 0$. Then the following two statements are equivalent:*

(i) *There exists a pair $x, y \in S_{Y_\psi}$ with $x \pm y \ne 0$ satisfying $\|x\|_2 = \|y\|_2 = 1/c$, $\|x+y\| = d\|x+y\|_2$ (and $\|x-y\| = d\|x-y\|_2$).*

(ii) *There exist $r, s, t \in [0,1]$ such that $\psi(s) = c\psi_2(s)$, $\psi(t) = c\psi_2(t)$, and $\psi(r) = d\psi_2(r)$, where r, s, t satisfy one of the following conditions:*

(a) *$s \ne t$ and*

$$r = \frac{\psi(t)s + \psi(s)t}{\psi(s) + \psi(t)}.$$

(b) *$(s,t) \notin \{(1,0),(0,1)\}$, $s + t \ge 1$, and*

$$r = \frac{\psi(t)s - \psi(s)(1-t)}{(2t-1)\psi(s) + \psi(t)}.$$

(c) *$(s,t) \notin \{(1,0),(0,1)\}$, $s + t < 1$, and*

$$r = \frac{\psi(t)(1-s) + \psi(s)t}{\psi(s) + (1-2s)\psi(t)}.$$

Proof. Suppose that (i) holds. Let x, y be the elements of S_{Y_ψ} having the properties set out in (i). Since $R(\pi/2)$ is an isometric isomorphism on Y_ψ, we may assume that x is in the first quadrant. Replacing y by $-y$ if necessary, we may also assume that y is in the first or fourth quadrant. Hence the argument separates into two parts.

(A) If both x, y are in the first quadrant, then we have

$$x = \frac{1}{\psi(s)}(1-s, s) \qquad \text{and} \qquad y = \frac{1}{\psi(t)}(1-t, t)$$

for some $s, t \in [0,1]$. By (i), we obtain $1/c = \|x\|_2 = \psi_2(s)/\psi(s)$ and $1/c = \|y\|_2 = \psi_2(t)/\psi(t)$. Now, from the fact the function $t \mapsto t/\psi_2(t)$ is strictly increasing, and since

$$x + y = \Big(\frac{1-s}{\psi(s)} + \frac{1-t}{\psi(t)}, \frac{s}{\psi(s)} + \frac{t}{\psi(t)}\Big)$$

and

$$x - y = \Big(\frac{1-s}{\psi(s)} - \frac{1-t}{\psi(t)}, \frac{s}{\psi(s)} - \frac{t}{\psi(t)}\Big),$$

$x \pm y \neq 0$ is equivalent to $s \neq t$. Finally, one has that

$$\frac{\psi(s)+\psi(t)}{\psi(s)\psi(t)}\psi(r) = \|x+y\| = d\|x+y\|_2 = d\frac{\psi(s)+\psi(t)}{\psi(s)\psi(t)}\psi_2(r),$$

where r is given by the equation set out in (a), and so $\psi(r) = d\psi_2(r)$.

(B) Suppose that y is in the fourth quadrant. Then $y = \psi(t)^{-1}(t, -(1-t))$ for some $t \in [0,1]$ (and x has the same form as in (A)). As in the preceding paragraph, it follows that $\psi(t) = c\psi_2(t)$ for $1/c = \|y\|_2 = \psi_2(1-t)/\psi(t) = \psi_2(t)/\psi(t)$. We note that

$$x + y = \Big(\frac{1-s}{\psi(s)} + \frac{t}{\psi(t)}, \frac{s}{\psi(s)} - \frac{1-t}{\psi(t)}\Big)$$

and

$$x - y = \Big(\frac{1-s}{\psi(s)} - \frac{t}{\psi(t)}, \frac{s}{\psi(s)} + \frac{1-t}{\psi(t)}\Big).$$

In particular, $x + y = 0$ if and only if $(s,t) = (1,0)$, and $x - y = 0$ if and only if $(s,t) = (0,1)$. Moreover, from the fact that $\psi(s) = c\psi_2(s)$ and $\psi(t) = c\psi_2(t)$, one has that

$$\frac{s}{\psi(s)} - \frac{1-t}{\psi(t)} = \frac{1}{c}\Big(\frac{s}{\psi_2(s)} - \frac{1-t}{\psi_2(1-t)}\Big).$$

Since the function $t \mapsto t/\psi_2(t)$ is strictly increasing, it turns out that

$$\begin{aligned} r &= \Big|\frac{s}{\psi(s)} - \frac{1-t}{\psi(t)}\Big|\Big(\frac{1-s}{\psi(s)} + \frac{t}{\psi(t)} + \Big|\frac{s}{\psi(s)} - \frac{1-t}{\psi(t)}\Big|\Big)^{-1} \\ &= \begin{cases} \frac{\psi(t)s - \psi(s)(1-t)}{(2t-1)\psi(s)+\psi(t)} & (s+t \geq 1), \\ \frac{-\psi(t)s + \psi(s)(1-t)}{\psi(s)+(1-2s)\psi(t)} & (s+t < 1). \end{cases} \end{aligned}$$

Since $\|x+y\| = d\|x+y\|_2$, if $s+t \geq 1$, then an argument similar to that in (A) shows that $\psi(r) = d\psi_2(r)$. On the other hand, if $s+t<1$, then $x+y$ is in the forth quadrant, and hence

$$\frac{\psi(s)+(1-2s)\psi(t)}{\psi(s)\psi(t)}\widetilde{\psi}(r) = \|x+y\| = d\|x+y\|_2 = d\frac{\psi(s)+(1-2s)\psi(t)}{\psi(s)\psi(t)}\widetilde{\psi_2}(r),$$

which proves that $\psi(1-r) = d\psi_2(1-r)$. Noticing that

$$1-r = \frac{\psi(t)(1-s)+\psi(s)t}{\psi(s)+(1-2s)\psi(t)},$$

we have that (i) $\Rightarrow$ (ii).

For the converse, let r,s,t be elements of $[0,1]$ satisfying one of the three conditions set out in (ii). If r,s,t satisfy (a), then the vectors $x = \psi(s)^{-1}(1-s,s)$ and $y = \psi(t)^{-1}(1-t,t)$ have the desired properties. Similarly, in the cases of (b) and (c), it is enough to consider $x = \psi(s)^{-1}(1-s,s)$ and $y = \psi(t)^{-1}(t,-(1-t))$. The proof is complete. □

3. Geometric constants of Y_ψ

We first consider the (modified) von Neumann–Jordan constant $C_{NJ}(Y_\psi)$ (and $C'_{NJ}(Y_\psi)$) of Y_ψ when $\psi \leq \psi_2$. Then, as an application of Theorem 2.3, we have the following results.

Theorem 3.1. *Suppose that $\psi \neq \psi_2$ and that $\psi \leq \psi_2$. Then*

$$C'_{NJ}(Y_\psi) \leq C_{NJ}(Y_\psi) \leq \max_{0\leq t\leq 1}\frac{\psi_2(t)^2}{\psi(t)^2} (= M_2^2).$$

In particular, $C'_{NJ}(Y_\psi) = M_2^2$ if and only if there exist $r,s,t \in [0,1]$ such that $\psi_2(s)/\psi(s) = \psi_2(t)/\psi(t) = M_2$ and $\psi(r) = \psi_2(r)$, where r,s,t satisfy one of the following conditions:

(a) *$s \neq t$ and*

$$r = \frac{\psi(t)s+\psi(s)t}{\psi(s)+\psi(t)}.$$

(b) *$(s,t) \notin \{(1,0),(0,1)\}$, $s+t \geq 1$, and*

$$r = \frac{\psi(t)s-\psi(s)(1-t)}{(2t-1)\psi(s)+\psi(t)}.$$

(c) *$(s,t) \notin \{(1,0),(0,1)\}$, $s+t < 1$, and*

$$r = \frac{\psi(t)(1-s)+\psi(s)t}{\psi(s)+(1-2s)\psi(t)}.$$

Proof. Let $x,y \in Y_\psi$ with $(x,y) \neq (0,0)$. Then, by Lemma 2.1, it follows that

$$\begin{aligned}\|x+y\|^2+\|x-y\|^2 &\leq \|x+y\|_2^2+\|x-y\|_2^2\\ &= 2\big(\|x\|_2^2+\|y\|_2^2\big)\\ &\leq 2M_2^2\big(\|x\|^2+\|y\|^2\big),\end{aligned}$$

and hence $C'_{NJ}(Y_\psi) \leq C_{NJ}(Y_\psi) \leq M_2^2$.

Next we consider restatements of $C'_{NJ}(Y_\psi) = M_2^2$. Since the set $S_{Y_\psi} \times S_{Y_\psi}$ with the product topology is compact and the function

$$S_{Y_\psi} \times S_{Y_\psi} \ni (x,y) \to \frac{\|x+y\|^2 + \|x-y\|^2}{4}$$

is continuous, $C'_{NJ}(Y_\psi) = M_2^2$ if and only if there exists a pair $(x,y) \in S_{Y_\psi} \times S_{Y_\psi}$ with $x \pm y \neq 0$ satisfying

$$\frac{\|x+y\|^2 + \|x-y\|^2}{4} = M_2^2; \tag{3.1}$$

for this, we note that if $x+y=0$ or $x-y=0$, then $M_2 = 1$, which contradicts $\psi \neq \psi_2$ since $\psi \leq \psi_2$. Moreover, since $\|x \pm y\| \leq \|x \pm y\|_2$, $\|x\|_2 \leq M_2\|x\| = M_2$, and $\|y\|_2 \leq M_2$, one has $C'_{NJ}(Y_\psi) = M_2^2$ if and only if there exists a pair $(x,y) \in S_{Y_\psi} \times S_{Y_\psi}$ with $x \pm y \neq 0$ satisfying $\|x \pm y\| = \|x \pm y\|_2$ and $\|x\|_2 = \|y\|_2 = M_2$. Hence Theorem 2.3 applies (for $c = M_2^{-1}$ and $d = 1$), and it turns out that $C'_{NJ}(Y_\psi) = M_2^2$ if and only if there exist r, s, t (satisfying one of the conditions (a), (b), (c) set out in Theorem 2.3) with $\psi(s) = M_2^{-1}\psi_2(s)$, $\psi(t) = M_2^{-1}\psi_2(t)$, and $\psi(r) = \psi_2(r)$. This completes the proof. □

Corollary 3.2. *Suppose that $\psi \neq \psi_2$ and that $\psi \leq \psi_2$. If there exists a $t_0 \in (0,1)$ satisfying $\psi(t_0) = \psi(1-t_0)$ and*

$$\frac{\psi_2(t_0)}{\psi(t_0)} = \max_{0 \leq t \leq 1} \frac{\psi_2(t)}{\psi(t)} (= M_2),$$

then $C'_{NJ}(Y_\psi) = C_{NJ}(Y_\psi) = M_2^2$.

Proof. Let $s = t_0$, and let $t = 1 - t_0$. Then $(s,t) \notin \{(1,0),(0,1)\}$, $s + t = 1$, and

$$r = \frac{\psi(1-t_0)t_0 - \psi(t_0)t_0}{(1-2t_0)\psi(t_0) + \psi(1-t_0)} = 0.$$

Thus we have $\psi_2(s)/\psi(s) = \psi_2(t)/\psi(t) = M_2$ and $\psi(r) = 1 = \psi_2(r)$; that is, r, s, t satisfy the condition (b) set out in Theorem 3.1. Hence one has that $C'_{NJ}(Y_\psi) = C_{NJ}(Y_\psi) = M_2^2$. □

Corollary 3.3. *Suppose that $\psi \neq \psi_2$ and that $\psi \leq \psi_2$. If*

$$\frac{\psi_2(1/2)}{\psi(1/2)} = \max_{0 \leq t \leq 1} \frac{\psi_2(t)}{\psi(t)} (= M_2),$$

then $C'_{NJ}(Y_\psi) = C_{NJ}(Y_\psi) = M_2^2$.

The case of $\psi \geq \psi_2$ is as follows.

Theorem 3.4. *Suppose that $\psi \neq \psi_2$ and that $\psi \geq \psi_2$. Then*

$$C'_{NJ}(Y_\psi) \leq C_{NJ}(Y_\psi) \leq \max_{0 \leq t \leq 1} \frac{\psi(t)^2}{\psi_2(t)^2} (= M_1^2).$$

In particular, $C'_{NJ}(Y_\psi) = M_1^2$ if and only if there exist $r, s, t \in [0,1]$ such that $\psi(s)/\psi_2(s) = \psi(t)/\psi_2(t) = 1$ and $\psi(r) = M_1\psi_2(r)$, where r, s, t satisfy one of the following conditions:

(a) $s \neq t$ *and*
$$r = \frac{\psi(t)s + \psi(s)t}{\psi(s) + \psi(t)}.$$
(b) $(s,t) \notin \{(1,0),(0,1)\}$, $s + t \geq 1$, *and*
$$r = \frac{\psi(t)s - \psi(s)(1-t)}{(2t-1)\psi(s) + \psi(t)}.$$
(c) $(s,t) \notin \{(1,0),(0,1)\}$, $s + t < 1$, *and*
$$r = \frac{\psi(t)(1-s) + \psi(s)t}{\psi(s) + (1-2s)\psi(t)}.$$

Proof. For each $x, y \in Y_\psi$ with $(x,y) \neq (0,0)$, we have
$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &\leq M_1^2\big(\|x+y\|_2^2 + \|x-y\|_2^2\big)\\ &= 2M_1^2\big(\|x\|_2^2 + \|y\|_2^2\big)\\ &\leq 2M_1^2\big(\|x\|^2 + \|y\|^2\big)\end{aligned}$$
by Lemma 2.1. This shows that $C'_{NJ}(Y_\psi) \leq C_{NJ}(Y_\psi) \leq M_1^2$.

Now, an argument similar to that in the proof of Theorem 3.1 shows that $C'_{NJ}(Y_\psi) = M_1^2$ if and only if there exists a pair $(x,y) \in S_{Y_\psi} \times S_{Y_\psi}$ with $x \pm y \neq 0$ satisfying $\|x \pm y\| = M_1\|x \pm y\|_2$ and $\|x\|_2 = \|y\|_2 = 1$. Thus Theorem 2.3 (applied for $c = 1$ and $d = M_1$) completes the proof. □

Corollary 3.5. *Suppose that* $\psi \neq \psi_2$ *and that* $\psi \geq \psi_2$. *If there exists a* $t_0 \in [0,1]$ *with* $t_0 \neq 1/2$ *satisfying* $\psi(t_0) = \psi(1-t_0) = \psi_2(t_0)$ *and*
$$\frac{\psi(1/2)}{\psi_2(1/2)} = \max_{0 \leq t \leq 1} \frac{\psi(t)}{\psi_2(t)} (= M_1),$$
then $C'_{NJ}(Y_\psi) = C_{NJ}(Y_\psi) = M_1^2$.

Proof. Let $s = t_0$, and let $t = 1 - t_0$. Then $s \neq t$ (since $t_0 \neq 1/2$) and
$$r = \frac{\psi(1-t_0)t_0 + \psi(t_0)(1-t_0)}{\psi(t_0) + \psi(1-t_0)} = \frac{1}{2}.$$
It follows from $\psi(s)/\psi_2(s) = \psi(t)/\psi_2(t) = 1$ and $\psi(r) = M_1\psi_2(r)$ that r, s, t satisfy the condition (a) set out in Theorem 3.4. Therefore, $C'_{NJ}(Y_\psi) = C_{NJ}(Y_\psi) = M_1^2$. □

We next provide similar results on the Zbăganu constant $C_Z(Y_\psi)$. For this purpose, it is convenient to transform the Zbăganu constant into the following form:
$$C_Z(X) = \sup\Big\{\frac{4\|x\|\|y\|}{\|x+y\|^2 + \|x-y\|^2} : x \in S_X, y \in B_X\Big\},$$
where X is a Banach space. To see this, it suffices to consider the transforms $x \to x + y$ and $y \to x - y$, and then divide the numerator and denominator by $\max\{\|x\|, \|y\|\}^2$.

As in the case of the modified von Neumann–Jordan constant, we consider the two cases of $\psi \leq \psi_2$ and $\psi \geq \psi_2$. Then Theorem 2.3 still works. We first consider the case $\psi \leq \psi_2$.

Theorem 3.6. *Suppose that $\psi \neq \psi_2$ and that $\psi \leq \psi_2$. Then*

$$C_Z(Y_\psi) \leq C_{NJ}(Y_\psi) \leq \max_{0\leq t\leq 1} \frac{\psi_2(t)^2}{\psi(t)^2} (= M_2^2).$$

In particular, $C_Z(Y_\psi) = M_2^2$ if and only if there exist $r, s, t \in [0,1]$ such that $\psi_2(s)/\psi(s) = \psi_2(t)/\psi(t) = 1$ and $\psi(r) = M_2^{-1}\psi_2(r)$, where r, s, t satisfy one of the following conditions:

(a) *$s \neq t$ and*

$$r = \frac{\psi(t)s + \psi(s)t}{\psi(s) + \psi(t)}.$$

(b) *$(s,t) \notin \{(1,0),(0,1)\}$, $s + t \geq 1$, and*

$$r = \frac{\psi(t)s - \psi(s)(1-t)}{(2t-1)\psi(s) + \psi(t)}.$$

(c) *$(s,t) \notin \{(1,0),(0,1)\}$, $s + t < 1$, and*

$$r = \frac{\psi(t)(1-s) + \psi(s)t}{\psi(s) + (1-2s)\psi(t)}.$$

Proof. The first statement is the consequence of $C_Z(Y_\psi) \leq C_{NJ}(Y_\psi)$ and Theorem 3.1. Now suppose that $C_Z(Y_\psi) = M_2^2$. Since Y_ψ has the finite dimension 2, there exists a pair $(x,y) \in S_{Y_\psi} \times B_{Y_\psi}$ satisfying

$$\frac{4\|x\|\|y\|}{\|x+y\|^2 + \|x-y\|^2} = M_2^2. \tag{3.2}$$

Then we note by Lemma 2.1 that

$$\begin{aligned} 4\|x\|\|y\| &\leq 2\big(\|x\|^2 + \|y\|^2\big) \\ &\leq 2\big(\|x\|_2^2 + \|y\|_2^2\big) \\ &= \|x+y\|_2^2 + \|x-y\|_2^2 \\ &\leq M_2^2\big(\|x+y\|^2 + \|x-y\|^2\big) \end{aligned}$$

which with (3.2) implies that $\|x\| = \|y\| = 1$ (that is, $x, y \in S_{Y_\psi}$), $\|x\|_2 = \|y\|_2 = 1$, and $\|x \pm y\| = M_2\|x \pm y\|_2$. In particular, $\psi \neq \psi_2$ ensures that $x \pm y \neq 0$. Conversely, if $x, y \in S_{Y_\psi}$ with $x \pm y \neq 0$ satisfy $\|x\|_2 = \|y\|_2 = 1$ and $\|x \pm y\| = M_2^{-1}\|x \pm y\|_2$, then we have (3.2). Hence $C_Z(Y_\psi) = M_2^2$ holds if and only if there exists a pair $(x,y) \in S_{Y_\psi} \times S_{Y_\psi}$ with $x \pm y \neq 0$ satisfying $\|x\|_2 = \|y\|_2 = 1$ and $\|x \pm y\| = M_2^{-1}\|x \pm y\|_2$. Now, applying Theorem 2.3 for $c = 1$ and $d = M_2^{-1}$ yields the theorem. □

Corollary 3.7. *Suppose that $\psi \neq \psi_2$ and that $\psi \leq \psi_2$. If there exists a $t_0 \in [0,1]$ with $t_0 \neq 1/2$ satisfying $\psi(t_0) = \psi(1-t_0) = \psi_2(t_0)$ and*

$$\frac{\psi_2(1/2)}{\psi(1/2)} = \max_{0\leq t\leq 1} \frac{\psi_2(t)}{\psi(t)} (= M_2),$$

then $C_Z(Y_\psi) = C_{NJ}(Y_\psi) = M_2^2$.

Proof. Putting $s = t_0$ and $t = 1 - t_0$ yields $s \neq t$ (since $t_0 \neq 1/2$) and

$$r = \frac{\psi(1-t_0)t_0 + \psi(t_0)(1-t_0)}{\psi(t_0) + \psi(1-t_0)} = \frac{1}{2}.$$

From these, one has that $\psi_2(s)/\psi(s) = \psi_2(t)/\psi(t) = 1$ and that $\psi(r) = M_2^{-1}\psi_2(r)$, which shows that r, s, t satisfy the condition (a) set out in Theorem 3.6. Hence it follows that $C_Z(Y_\psi) = C_{NJ}(Y_\psi) = M_2^2$. □

We next consider the case $\psi \geq \psi_2$.

Theorem 3.8. *Suppose that $\psi \neq \psi_2$ and that $\psi \geq \psi_2$. Then*

$$C_Z(Y_\psi) \leq C_{NJ}(Y_\psi) \leq \max_{0\leq t\leq 1} \frac{\psi(t)^2}{\psi_2(t)^2} (= M_1^2).$$

In particular, $C_Z(Y_\psi) = M_1^2$ if and only if there exist $r, s, t \in [0,1]$ such that $\psi(s)/\psi_2(s) = \psi(t)/\psi_2(t) = M_1$ and $\psi(r) = \psi_2(r)$, where r, s, t satisfy one of the following conditions:

(a) *$s \neq t$ and*

$$r = \frac{\psi(t)s + \psi(s)t}{\psi(s) + \psi(t)}.$$

(b) *$(s,t) \notin \{(1,0),(0,1)\}$, $s + t \geq 1$, and*

$$r = \frac{\psi(t)s - \psi(s)(1-t)}{(2t-1)\psi(s) + \psi(t)}.$$

(c) *$(s,t) \notin \{(1,0),(0,1)\}$, $s + t < 1$, and*

$$r = \frac{\psi(t)(1-s) + \psi(s)t}{\psi(s) + (1-2s)\psi(t)}.$$

Proof. The proof is almost the same as that of Theorem 3.6, but, in this case, the key is the following inequalities:

$$\begin{aligned} 4\|x\|\|y\| &\leq 2\big(\|x\|^2 + \|y\|^2\big) \\ &\leq 2M_1^2\big(\|x\|_2^2 + \|y\|_2^2\big) \\ &= M_1^2\big(\|x+y\|_2^2 + \|x-y\|_2^2\big) \\ &\leq M_1^2\big(\|x+y\|^2 + \|x-y\|^2\big). \end{aligned}$$

As in the proof of Theorem 3.6, we have that $C_Z(Y_\psi) = M_1^2$ if and only if there exists a pair $(x,y) \in S_{Y_\psi} \times S_{Y_\psi}$ with $x \pm y \neq 0$ satisfying $\|x\|_2 = \|y\|_2 = M_1^{-1}$ and $\|x \pm y\| = \|x \pm y\|_2$. Thus, for $c = M_1$ and $d = 1$, Theorem 2.3 applies, and we have the theorem. □

Corollary 3.9. *Suppose that $\psi \neq \psi_2$ and that $\psi \geq \psi_2$. If there exists a $t_0 \in (0,1)$ satisfying $\psi(t_0) = \psi(1-t_0)$ and*

$$\frac{\psi(t_0)}{\psi_2(t_0)} = \max_{0\leq t\leq 1} \frac{\psi(t)}{\psi_2(t)} (= M_1),$$

then $C_Z(Y_\psi) = C_{NJ}(Y_\psi) = M_1^2$.

Proof. Suppose that $s = t_0$ and $t = 1-t_0$. Since $(s,t) \notin \{(1,0),(0,1)\}$, $s+t=1$, and

$$r = \frac{\psi(1-t_0)t_0 - \psi(t_0)t_0}{(1-2t_0)\psi(t_0) + \psi(1-t_0)} = 0,$$

we obtain $\psi(s)/\psi_2(s) = \psi(t)/\psi_2(t) = M_1$ and $\psi(r) = 1 = \psi_2(r)$. This proves that r, s, t satisfy the condition (b) set out in Theorem 3.8, and $C_Z(Y_\psi) = C_{NJ}(Y_\psi) = M_1^2$. □

Corollary 3.10. *Suppose that $\psi \neq \psi_2$ and that $\psi \geq \psi_2$. If*

$$\frac{\psi(1/2)}{\psi_2(1/2)} = \max_{0\leq t\leq 1} \frac{\psi(t)}{\psi_2(t)} (= M_1),$$

then $C_Z(Y_\psi) = C_{NJ}(Y_\psi) = M_1^2$.

4. Examples

As an application of the results in the preceding section, we will construct examples of $\pi/2$-rotation invariant norms on $\mathbb{R}^2$ with the following properties:

(i) $C'_{NJ}(Y_\psi) = M_2^2$ can be deduced by Corollary 3.3 (a part of Corollary 3.2),
(ii) $C'_{NJ}(Y_\psi) = M_2^2$ can be deduced by Theorem 3.1, but Corollary 3.2 does not apply,
(iii) $C'_{NJ}(Y_\psi) < M_2^2$.

While all of these examples are concerned with the case $C'_{NJ}(Y_\psi) = M_2^2$ under the hypothesis $\psi \leq \psi_2$, appropriate examples for other cases can be constructed in the same spirit.

Example 4.1. To construct an example satisfying (i), we put

$$\varphi_1(t) = \begin{cases} 1-t & (t \in [0,1/3]), \\ 2\sqrt{5}\psi_2(t)/5 & (t \in [1/3,1/2]), \\ \frac{(20-6\sqrt{10})t+4\sqrt{10}-10}{5} & (t \in [1/2,2/3]), \\ t & (t \in [2/3,1]). \end{cases}$$

See Figures 1 and 2 for its graph and corresponding unit sphere.

Then it is easy to check that $\varphi_1 \in \Psi_2$ and that

$$\varphi_1 \leq \max\{1-t, t, 2/3\} \leq \max\{1-t, t, 1/\sqrt{2}\} \leq \psi_2(t)$$

for each $t \in [0,1]$. From this one has that $\varphi_1 \neq \psi_2$. Since the functions $t \mapsto \psi_2(t)/(1-t)$ and $t \mapsto \psi_2(t)/t$ are strictly increasing and strictly decreasing,

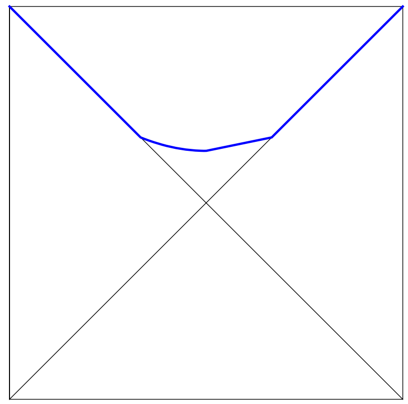

FIGURE 1. The graph of φ_1.

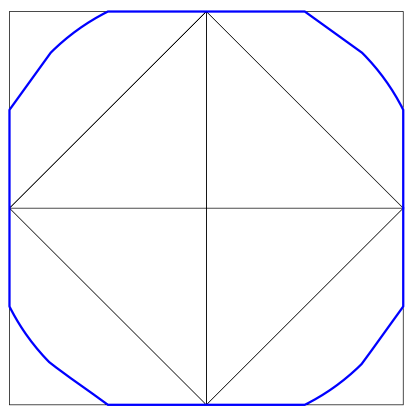

FIGURE 2. The unit sphere of Y_{φ_1}.

respectively, it follows that

$$\frac{\psi_2(1/2)}{\varphi_1(1/2)} = \max_{0\le t\le 1} \frac{\psi_2(t)}{\varphi_1(t)} (= M_2).$$

Hence, by Corollary 3.3, we have $C'_{NJ}(Y_{\varphi_1}) = M_2^2$.

Example 4.2. Let φ_2 be the element of Ψ_2 given by

$$\varphi_2(t) = \begin{cases} \psi_2(t) & (t \in [0, 1/2]), \\ 1/\sqrt{2} & (t \in [1/2, 1/\sqrt{2}]), \\ t & (t \in [1/\sqrt{2}, 1]). \end{cases}$$

Figures 3 and 4 provide images of the function φ_2 and unit sphere of Y_{φ_2}.

Then $\varphi_2 \neq \psi_2$ and $\varphi_2 \le \psi_2$. Since the function $\psi_2(t)/t$ is strictly decreasing, it follows that

$$\frac{\psi_2(t_0)}{\varphi_2(t_0)} = \max_{0\le t\le 1} \frac{\psi_2(t)}{\varphi_2(t)} (= M_2 > 1)$$

if and only if $t_0 = 1/\sqrt{2}$. This, together with the fact that $\varphi_2(t) = \varphi_2(1-t)$ if and only if $t = 0, 1, 1/2$, shows that we can not take a $t_0 \in (0, 1)$ satisfying

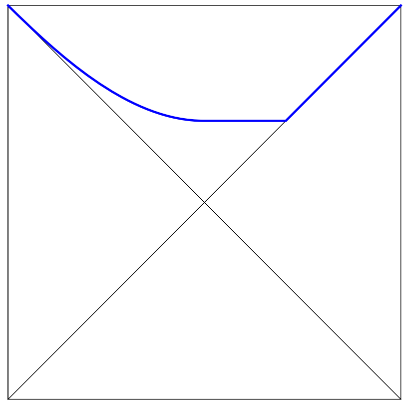

FIGURE 3. The graph of φ_2.

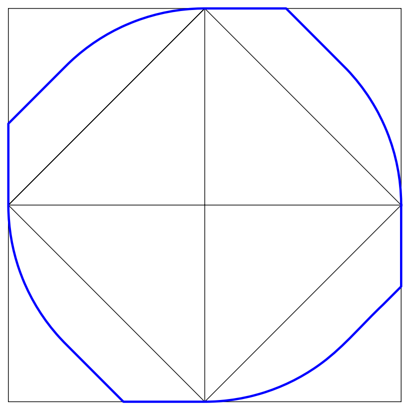

FIGURE 4. The unit sphere of Y_{φ_2}.

$\varphi_2(t_0) = \varphi_2(1-t_0)$ and

$$\frac{\psi_2(t_0)}{\varphi_2(t_0)} = M_2,$$

and so Corollary 3.2 does not apply. On the other hand, putting $s = t = 1/\sqrt{2}$ yields that $s + t = \sqrt{2} > 1$, $\psi_2(s)/\varphi_2(s) = \psi_2(t)/\varphi_2(t) = M_2$ and

$$r = \frac{\varphi_2(t)s - \varphi_2(s)(1-t)}{(2t-1)\varphi_2(s) + \varphi_2(t)} = \frac{2s-1}{2s} = 1 - \frac{1}{\sqrt{2}} \in [0, 1/2],$$

which implies that $\varphi_2(r) = \psi_2(r)$. Thus Theorem 3.1(b) applies, and we have $C'_{NJ}(Y_{\varphi_2}) = M_2^2$.

Example 4.3. Define an element φ_3 of Ψ_2 by

$$\varphi_3(t) = \begin{cases} 1-t & (t \in [0, 1/3]), \\ -t/2 + 5/6 & (t \in [1/3, 5/9]), \\ t & (t \in [5/9, 1]). \end{cases}$$

For the graph of φ_3 and image of $S_{Y_{\varphi_3}}$, see Figures 5 and 6.

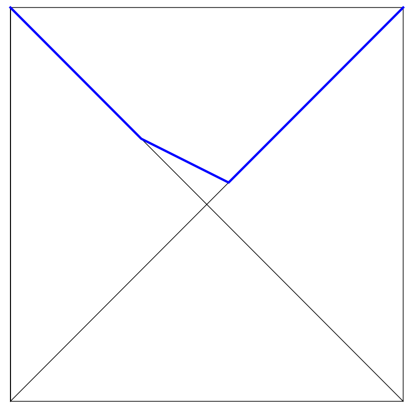

FIGURE 5. The graph of φ_3.

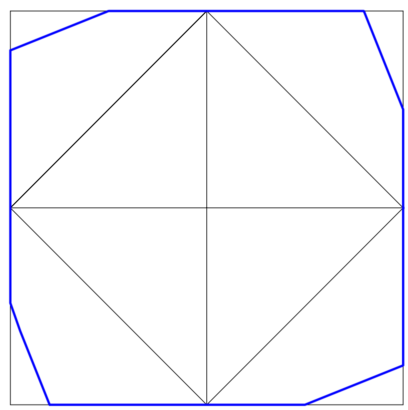

FIGURE 6. The unit sphere of Y_{φ_3}.

Then $\varphi_3(t) \leq \max\{1-t, t, 1/\sqrt{2}\} \leq \psi_2(t)$ for each $t \in [0,1]$, and so $\varphi_3 \neq \psi_2$. Moreover, the functions $t \mapsto \psi_2(t)/(1-t)$ and $t \mapsto \psi_2(t)/t$ are strictly increasing and strictly decreasing, respectively, while the function

$$t \mapsto \frac{\psi_2(t)}{-t/2+5/6}$$

is strictly increasing on [1/3,5/9]. Hence one has that

$$\frac{\psi_2(t_0)}{\varphi_3(t_0)} = \max_{0\leq t\leq 1} \frac{\psi_2(t)}{\varphi_3(t)} (= M_2)$$

if and only if $t_0 = 5/9$. To see that $C'_{NJ}(Y_{\varphi_3}) < M_2^2$, by Theorem 3.1, it is enough to show that there is no r, s, t satisfying one of the conditions (a), (b), (c) set out in that theorem. However, now, the only candidate for s, t is $5/9$ since s, t must satisfy $\psi_2(s)/\varphi_3(s) = \psi_2(t)/\varphi_3(t) = M_2$. This eliminates the possibility of (a). Since $s+t = 10/9 > 1$, it suffices to consider (b). Then it follows from $s = t = 5/9$ that

$$r = \frac{\varphi_3(t)s - \varphi_3(s)(1-t)}{(2t-1)\varphi_3(s) + \varphi_3(t)} = \frac{2s-1}{2s} = \frac{1}{10} \in (0,1),$$

which, together with the fact that $\varphi_3(u) < \psi_2(u)$ for each $u \in (0,1)$, proves that $\varphi_3(r) \neq \psi_2(r)$. Thus r, s, t can not satisfy any of the conditions set out in Theorem 3.1, from which we obtain $C'_{NJ}(Y_{\varphi_3}) < M_2^2$.

Acknowledgments. The work of Saito and Tanaka was supported in part by Grants-in-Aid for Scientific Research 15K04920 and 16J01162 from the Japan Society for the Promotion of Science (JSPS).

References

1. J. Alonso and P. Martín, *A counterexample to a conjecture of G. Zbăganu about the Neumann–Jordan constant*, Rev. Roumaine Math. Pures Appl. **51** (2006), no. 2, 135–141. Zbl 1113.46007. MR2275298. 216
2. J. Alonso, P. Martín, and P. L. Papini, *Wheeling around von Neumann–Jordan constant in Banach spaces*, Studia Math. **188** (2008), no. 2, 135–150. Zbl 1170.46016. MR2430999. DOI 10.4064/sm188-2-3. 215
3. F. F. Bonsall and J. Duncan, *Numerical Ranges, II*, London Math. Soc. Lecture Note Ser. **10**, Cambridge Univ. Press, Cambridge, 1973. Zbl 0262.47001. MR0442682. 216
4. J. A. Clarkson, *The von Neumann–Jordan constant for the Lebesgue spaces*, Ann. of Math. (2) **38** (1937), no. 1, 114–115. Zbl 0016.03002. MR1503327. DOI 10.2307/1968512. 215
5. J. Gao, *A Pythagorean approach in Banach spaces*, J. Inequal. Appl. **2006**, art. ID 94982. Zbl 1171.46304. MR2215461. DOI 10.1155/JIA/2006/94982. 216
6. J. Gao and S. Saejung, *Remarks on a Pythagorean approach in Banach spaces*, Math. Inequal. Appl. **11** (2008), no. 2, 213–220. Zbl 1156.46019. MR2410273. DOI 10.7153/mia-11-15. 216
7. T. Ikeda and M. Kato, *Notes on von Neumann–Joredan and James constants for absolute norms on* $\mathbb{R}^2$, Mediterr. J. Math. **11** (2014), no. 2, 633–642. Zbl 1308.46023. MR3198629. DOI 10.1007/s00009-013-0370-1. 216
8. P. Jordan and J. von Neumann, *On inner products in linear, metric spaces*, Ann. of Math. (2) **36** (1935), no. 3, 719–723. JFM61.0435.05 Zbl JFM61.0435.05. MR1503247. DOI 10.2307/1968653. 215
9. N. Komuro, K. -S. Saito, and R. Tanaka, *On the class of Banach spaces with James constant* $\sqrt{2}$*, II*, Mediterr. J. Math. **13** (2016), no. 6, 4039–4061. Zbl 06657677. MR3564490. DOI 10.1007/s00009-016-0731-7. 216, 217
10. H. Mizuguchi and K. -S. Saito, *Some geometric constant of absolute normalized norms on* $\mathbb{R}^2$, Ann. Funct. Anal. **2** (2011), no. 2, 22–33. Zbl 1256.46006. MR2855283. DOI 10.15352/afa/1399900191. 216, 217
11. H. Mizuguchi and K. -S. Saito, *On the upper bound of some geometric constants of absolute normalized norms on* $\mathbb{R}^2$, Mediterr. J. Math. **13** (2016), no. 1, 309–322. Zbl 1347.46013. MR3456924. DOI 10.1007/s00009-014-0485-z. 216
12. L. Nikolova and S. Varošanec, *Geometric constants for Banach spaces with absolute normalized norms which corresponding functions are comparable*, J. Nonlinear Convex Anal. **14** (2013), no. 3, 529–549. Zbl 1276.46007. MR3098568. 216
13. W. Nilsrakoo and S. Saejung, *The James constant of normalized norms on* $\mathbb{R}^2$, J. Inequal. Appl. **2006**, art. ID 26265. Zbl 1104.46007. MR2221225. 217
14. K. -S. Saito, M, Kato, and Y. Takahashi, *Von Neumann–Jordan constant of absolute normalized norms on* $\mathbb{C}^2$, J. Math. Anal. Appl. **244** (2000), no. 2, 515–532. Zbl 0961.46008. MR1753376. DOI 10.1006/jmaa.2000.6727. 215, 216
15. Y. Takahashi and M. Kato, *von Neumann–Jordan constant and uniformly non-square Banach spaces*, Nihonkai Math. J. **9** (1988), no. 2, 155–169. Zbl 1015.46010. MR1661080. 215

16. Y. Takahashi and M. Kato, *A simple inequality for the von Neumann–Jordan and James constants of a Banach space*, J. Math. Anal. Appl. **359** (2009), no. 2, 602–609. Zbl 1176.46022. MR2546776. DOI 10.1016/j.jmaa.2009.05.051. 215
17. C. Yang and H. Li, *An inequality between Jordan–von Neumann constant and James constant*, Appl. Math. Lett. **23** (2010), no. 3, 277–281. Zbl 1194.46029. MR2565190. DOI 10.1016/j.aml.2009.09.023. 215
18. G. Zbăganu, *An inequality of M. Rădulescu and S. Rădulescu which characterizes inner product spaces*, Rev. Roumaine Math. Pures Appl. **47** (2002), no. 2, 253–257. Zbl 1052.46009. MR1979046. 216
19. Z. Zuo and Y. Cui, *Some modulus and normal structure in Banach space*, J. Inequal. Appl. **2009**, art. ID 676373. Zbl 1178.46013. MR2500739. DOI 10.1155/2009/676373. 216

[1]Department of Mathematics, Faculty of Science and Engineering, Chuo University, Tokyo 112-8551, Japan.
E-mail address: tomizawa@gug.math.chuo-u.ac.jp

[2]Department of Information and Communication Engineering, Faculty of Computer Science and System Engineering, Okayama Prefectural University, Okayama, 719-1197, Japan.
E-mail address: mitani@cse.oka-pu.ac.jp

[3]Department of Mathematical Sciences, Institute of Science and Technology, Niigata University, Niigata 950-2181, Japan.
E-mail address: saito@math.sc.niigata-u.ac.jp

[4]Faculty of Mathematics, Kyushu University, Fukuoka, 819-0395, Japan.
E-mail address: r-tanaka@math.kyushu-u.ac.jp