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HLAWKA'S FUNCTIONAL INEQUALITY ON TOPOLOGICAL GROUPS

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Abstract. Let $(X,+)$ be a topological abelian group. We discuss regularity of solutions $f\colon X \to \mathbb{R}$ of Hlawka's functional inequality

$$f(x+y)+f(y+z)+f(x+z) \leq f(x+y+z)+f(x)+f(y)+f(z),$$

postulated for all $x,y,z \in X$. We study the lower and upper hull of f. Moreover, we provide conditions which imply continuity of f. We prove, in particular, that if X is generated by any neighborhood of zero, f is continuous at zero, and $f(0)=0$, then f is continuous on X.

1. Introduction

Assume that $(X,+)$ is an abelian group and that $f\colon X \to \mathbb{R}$ is an arbitrary mapping. We say that f is *additive* if

$$f(x+y)=f(x)+f(y)$$

for all $x,y \in X$, and we say that f is *quadratic* if

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x,y \in X$. Further, f is *subadditive* if

$$f(x+y) \leq f(x)+f(y)$$

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for all $x, y \in X$, and f is *subquadratic* if

$$f(x+y) + f(x-y) \leq 2f(x) + 2f(y)$$

for all $x, y \in X$. If one of the above inequalities holds in reverse, then we say that f is *superadditive* or *superquadratic*, respectively. Finally, two other inequalities which are crucial for our studies are the *Drygas functional inequality*

$$f(x+y) + f(x-y) \leq 2f(x) + f(y) + f(-y),$$

postulated for all $x, y \in X$, and *Hlawka's functional inequality*

$$f(x+y) + f(y+z) + f(x+z) \leq f(x+y+z) + f(x) + f(y) + f(z),$$

postulated for all $x, y, z \in X$.

Subquadratic functions (sometimes also called *weakly* subquadratic functions) were studied by Gilányi, Kézi, and Troczka-Pawelec [5], Gilányi and Troczka-Pawelec [6], Kominek and Troczka-Pawelec [10], [11], and Troczka-Pawelec [14]. The Drygas inequality was discussed by Kominek in [9] and Hlawka's inequality by the author of the present article in [3]. For other basic notions and results of the theory of functional equations and inequalities, the reader is referred to the monograph by Kuczma [12].

In the theory of functional inequalities, the usual situation is that a satisfactory description of solutions exists under an additional regularity assumption. This is also the case of Hlawka's inequality (see [3, Theorem 8]). Therefore, it is of interest to provide possibly weak conditions upon solutions of a functional inequality which imply stronger regularity. A striking example of such situation is the celebrated Bernstein–Doetsch theorem (see [12, Chapter 6.4]). It states that if a Jensen (midpoint) convex function defined on an open and convex set is bounded above at a neighborhood of a point, then it is necessarily continuous (on the whole domain). However, for other functional inequalities, in particular for subadditive or subquadratic functions, the situation is much less comfortable. Namely, one needs to impose considerably stronger assumptions to obtain the continuity. Usually, it is enough to assume continuity at zero, together with $f(0) = 0$ (for subadditive functions, see Hille and Phillips [7, Theorem 7.8.2] and [12, Theorem 16.2.1]; for subquadratic functions, see [6, Theorem 5.3], [11, Theorem 3.1]; and for the Drygas inequality, see [9, Theorem 3]). Our purpose here is to provide analogous conditions which guarantee the continuity of solutions of Hlawka's inequality.

The main motivation for studying Hlawka's functional inequality comes from functional analysis (see [3, Introduction]) and, as a consequence, this inequality is well motivated for functions defined on normed spaces, or more generally, on linear topological spaces. However, for technical reasons, a natural framework for this study can (and should) be even more general. Solutions of the Drygas inequality and subquadratic functions have been discussed in cases where the domain is a topological group, or sometimes one uniquely divisible by 2.

We will exhibit two assumptions upon topological groups which were introduced in [11]. We say that a topological group uniquely divisible by 2 has the property $(\frac{1}{2})$ if, for every neighborhood of zero V in X, there exists a neighborhood of zero W such that $\frac{1}{2}W \subset W \subset V$. Note that $(\frac{1}{2})$ is trivially satisfied in arbitrary linear topological spaces. Next, we say that a topological group X has the property (2^n) if $X = \bigcup\{2^n U : n \in \mathbb{N}\}$ for every neighborhood of zero U.

One can note that $(\frac{1}{2})$ and (2^n) together imply that for every $x \in X$ the sequence $(\frac{1}{2^n}x)_{n\in\mathbb{N}}$ is convergent to zero. Moreover, $(\frac{1}{2})$ implies that division by 2 is a continuous mapping on X. An example of a topological abelian group uniquely divisible by 2 on which division by 2 is discontinuous was provided by Chmieliński in [2]. A subgroup of the complex unit sphere with the multiplication of complex numbers as the group operation, which is generated by a single element that is not commensurate with π, is an example of a topological group uniquely divisible by 2 with continuous division by 2 but without property $(\frac{1}{2})$.

2. Preliminary observations

We begin with an easy observation, which has immediate consequences for the regularity behavior of solutions of Hlawka's functional inequality.

Proposition 1. *Assume that $(X, +)$ is an abelian group. If $f\colon X \to \mathbb{R}$ satisfies Hlawka's inequality, then $f - \frac{1}{2}f(0)$ satisfies the Drygas inequality.*

Proof. It is enough to substitute $z = -y$ in Hlawka's inequality. □

On joining this proposition with results of Kominek [9, Theorems 3, 4], we obtain the following corollary.

Corollary 1. *Let X be a topological abelian group uniquely divisible by 2 with the properties $(\frac{1}{2})$ and (2^n). Assume that $f\colon X \to \mathbb{R}$ satisfies Hlawka's inequality and that $f(0) = 0$. Then, any two of the following conditions are equivalent:*

(i) *f is upper-semicontinuous at zero and locally bounded below at some point,*
(ii) *f is continuous at zero,*
(iii) *f is continuous on X.*

The next example shows that the converse of Proposition 1 does not hold in general. Even more, there exist solutions f of the Drygas inequality for which every shift $f + c$, for any $c \in \mathbb{R}$, does not satisfy Hlawka's inequality. Moreover, the same is true in the case of even solutions of Hlawka's inequality and subquadratic functions. Therefore, one can expect that it is possible to establish better regularity properties for Hlawka's inequality than those which are known for the Drygas inequality or for subquadratic functions.

Example 1. Consider the function $f_0\colon \mathbb{R} \to \mathbb{R}$ given by $f_0(x) = -x^4$ for all $x \in \mathbb{R}$. Clearly, f_0 is even and $f_0(0) = 0$. Moreover,

$$2f_0(x) + 2f_0(y) - f_0(x+y) - f_0(x-y) = 12x^2y^2.$$

Thus, f_0 is subquadratic and satisfies the Drygas inequality. On the other hand, the difference

$$\begin{aligned} &f_0(x+y+z) + f_0(x) + f_0(y) + f_0(z) - f_0(x+y) - f_0(y+z) - f_0(x+z) \\ &\quad = -12xyz(x+y+z) \end{aligned}$$

is unbounded (from both sides). Therefore, for any $c \in \mathbb{R}$, the function $f_0 + c$ does not solve Hlawka's inequality.

Moreover, it is evident that every function which is subadditive and even is also subquadratic (and consequently satisfies the Drygas inequality). Whereas, it is easy to find a subadditive even function which does not satisfy Hlawka's inequality (one can consider the norm in a normed space which is not a Hlawka space).

In the next example, we construct solutions of Hlawka's inequality on the real line which vanish at zero, can be discontinuous for all $x \in \mathbb{R}$, and can be globally bounded above or below.

Example 2. Assume that $a\colon \mathbb{R} \to \mathbb{R}$ is an additive mapping. Then its absolute value $|a|$, as well as a and a^2, satisfies Hlawka's inequality, and the last two satisfy it with an equality. Consequently, $f_1 = |a| - a$, $f_2 = |a| - a^2$, and $f_3 = -2a + a^2$ solve this inequality as well. Further, $f_1(0) = f_2(0) = f_3(0) = 0$, $f_1 \geq 0$, $f_2 \leq \frac{1}{4}$, and $f_3 \geq -1$. Also note that

$$\big\{x \in \mathbb{R} : f_1(x) = 0\big\} = \big\{x \in \mathbb{R} : a(x) \leq 0\big\},$$
$$\big\{x \in \mathbb{R} : f_2(x) = 0\big\} = \big\{x \in \mathbb{R} : a(x) \in \{-1, 0, 1\}\big\},$$

and

$$\big\{x \in \mathbb{R} : f_3(x) = 0\big\} = \big\{x \in \mathbb{R} : a(x) \in \{0, 2\}\big\}.$$

Therefore, if a is chosen as a discontinuous additive function, then $f_1 = 0$ on a relatively large set (although nonmeasurable and without the Baire property). Moreover, a can be chosen in such a way that $f_2 = 0$ or $f_3 = 0$ on a set consisting of exactly three (resp., two) points or of zero only. Function f_3 will be also utilized later (see Remark 2 below). For a detailed discussion of Hamel bases and discontinuous additive functions, we refer the reader to [12, Chapter 11].

Our last example shows in particular that in Corollary 1, one cannot replace zero by another point. That is, there exists a solution of Hlawka's inequality which vanishes at zero, is continuous everywhere except at zero, and is not upper-semicontinuous.

Example 3. By χ_B we denote the indicator function of a set B; that is, $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ if $x \notin B$. Note that the indicator function of an open set is lower-semicontinuous, whereas the indicator function of a closed set is upper-semicontinuous.

Assume that $D \subset \mathbb{R}$ is an additive subgroup of the real line. Then, it is straightforward to check that the function $f_4 = \chi_{\mathbb{R}\setminus D}$ solves Hlawka's inequality. Clearly, f_4 is bilaterally bounded and $f_4(0) = 0$. If one takes, for example, $D = \{0\}$, then f_4 is lower-continuous, but it is not upper-semicontinuous (at zero). And for $D = \mathbb{Q}$, the function f_4 is neither lower-continuous nor upper-semicontinuous.

Next, we will state and prove a decomposition lemma, which is an extension of our earlier result in [3, Lemma 1].

Lemma 1. *Assume that $(X, +)$ is an abelian group and that $f\colon X \to \mathbb{R}$ satisfies Hlawka's inequality for all $x, y, z \in X$. Then there exist mappings $\varphi, g\colon X \to \mathbb{R}$ such that*

(a) *φ is odd,*
(b) *g is even,*
(c) *$f = \varphi + g + \frac{1}{2}f(0)$,*
(d) *$|\varphi(x+y) - \varphi(x) - \varphi(y)| \le \frac{1}{2}f(0)$ for all $x, y \in X$,*
(e) *g is subquadratic.*

Proof. Let us define mappings $\varphi, g\colon X \to \mathbb{R}$ by the formulas

$$\varphi(x) = \frac{f(x) - f(-x)}{2}, \qquad g(x) = f(x) - \varphi(x) - \frac{1}{2}f(0)$$

for all $x \in X$. It is straightforward to check that conditions (a), (b), and (c) are satisfied.

Substitute in Hlawka's inequality $z = -x - y$. We get

$$f(x+y) + f(-y) + f(-x) \le f(0) + f(x) + f(y) + f(-x-y)$$

for all $x, y \in X$. This means that

$$2\varphi(x+y) \le 2\varphi(x) + 2\varphi(y) + f(0), \quad x, y \in X.$$

Replace x by $-x$ and y by $-y$ to obtain

$$-2\varphi(x+y) = 2\varphi(-x-y) \le 2\varphi(-x) + 2\varphi(-y) + f(0) = -2\varphi(x) - 2\varphi(y) + f(0)$$

for all $x, y \in X$. The last two inequalities added side by side imply (d).

Finally, put in Hlawka's inequality $z = -x$. We see that

$$f(x+y) + f(0) + f(y-x) \le 2f(y) + f(x) + f(-x), \quad x, y \in X.$$

Replace x by $-x$ and y by $-y$ to get

$$f(-x-y) + f(0) + f(x-y) \le 2f(-y) + f(x) + f(-x), \quad x, y \in X.$$

Note that

$$g(x) = \frac{f(x) + f(-x) - f(0)}{2}, \quad x \in X.$$

It is enough to add the last two inequalities side by side to derive that g is subquadratic. □

In what follows, we will show that mappings φ and g have a stronger regularity behavior than can be deduced from Corollary 1. In particular, we will avoid the assumption that $f(0) = 0$ and, what is more, we will not need assumptions $(\frac{1}{2})$ and (2^n), at least in their full strengths.

Lemma 2. *Assume that X is a topological abelian group, $f\colon X \to \mathbb{R}$ satisfies Hlawka's inequality for all $x, y, z \in X$, and mapping $g\colon X \to \mathbb{R}$ is as in Lemma 1. If f is locally bounded below at a point, then g is locally bounded below at zero.*

Proof. Assume that f is locally bounded below at $x_0 \in X$. That is, there exists a neighborhood of this point, say, U_{x_0} and a constant $M \in \mathbb{R}$ such that $f(x) \ge M$

for every $x \in U_{x_0}$. Let $V = (U_{x_0} - x_0) \cap (-U_{x_0} + x_0)$. Then V is a symmetric neighborhood of zero. Take arbitrary $y \in V$. By Proposition 1 we have

$$\begin{aligned} 2g(y) &= f(y) + f(-y) - f(0) \geq f(x_0 + y) + f(x_0 - y) - 2f(x_0) - 2f(0) \\ &\geq 2M - 2f(x_0) - 2f(0). \end{aligned}$$

Note that the right-hand side of the above inequality does not depend upon y. Therefore, g is bounded below at V. □

Lemma 3. *Assume that X is a topological abelian group which is generated by any neighborhood of zero. Next, assume that $f\colon X \to \mathbb{R}$ satisfies Hlawka's inequality for all $x, y, z \in X$ and that mapping $\varphi\colon X \to \mathbb{R}$ is as in Lemma 1. If f is locally bounded (bilaterally) at zero, then there exists a unique continuous group homomorphism $a\colon X \to \mathbb{R}$ such that $|\varphi(x) - a(x)| \leq \frac{1}{2}f(0)$ for all $x \in X$.*

Proof. We will apply Hyers's theorem in [8, Theorem 1] (see also Forti's survey [4], where generalizations of Hyers's theorem for mappings defined on semigroups are provided; cf. the recent survey by Brzdęk et al. [1]), which says that inequality (d) of Lemma 1 implies the existence of an additive function $a\colon X \to \mathbb{R}$ such that

$$\big|\varphi(x) - a(x)\big| \leq \frac{1}{2}f(0), \quad x \in X.$$

By Lemma 2, function g is locally bounded below at zero. Thus, φ is locally bounded at zero. Therefore, a is locally bounded at zero and, consequently, a is a continuous mapping (see, e.g., Székelyhidi [13, Theorem 3.7]). □

A result of Gilányi and Troczka-Pawelec [6, Theorem 5.2] says that every subquadratic function defined on a topological group uniquely divisible by 2 which is generated by any neighborhood of zero and which is locally bounded from above at a point and locally bounded below at a point is locally bounded at every point. This theorem, together with our previous two lemmas, implies the next corollary.

Corollary 2. *Assume that X is a topological abelian group uniquely divisible by 2 which is generated by any neighborhood of zero. Next, assume that $f\colon X \to \mathbb{R}$ satisfies Hlawka's inequality for all $x, y, z \in X$. If f is locally bounded (bilaterally) at zero, then it is locally bounded (bilaterally) at every point.*

Remark 1. Functions f_1, f_2, f_3 constructed in Example 2 show that in Corollary 2, one cannot replace the assumption of bilateral boundedness of f at zero by one-sided boundedness. Moreover, in view of Example 3, neither Lemma 2 nor Corollary 2 can be improved in a way that f or φ of Lemma 1 are continuous. In fact, both functions can be discontinuous everywhere.

Another result of Gilányi and Troczka-Pawelec [6, Theorem 5.3] states that every subquadratic function defined on a topological group uniquely divisible by 2 which is generated by any neighborhood of zero and which is continuous at zero and which vanishes at zero is continuous. This, together with Lemmas 2 and 3, leads to the next corollary.

Corollary 3. *Assume that X is a topological abelian group uniquely divisible by 2 which is generated by any neighborhood of zero. Next, assume that $f\colon X \to \mathbb{R}$ satisfies Hlawka's inequality for all $x, y, z \in X$. If f is continuous at zero and $f(0) = 0$, then it is continuous on X.*

3. Main results

In this section, we will state and prove more general results than Corollaries 1, 2, and 3. In particular, we will impose considerably weaker assumptions upon the domain.

We will employ the limit functions, which will be studied for solutions of Hlawka's inequality. If X is a topological space and $f\colon X \to \mathbb{R}$ is a function, then we define $\underline{f}\colon X \to \mathbb{R} \cup \{-\infty\}$ and $\overline{f}\colon X \to \mathbb{R} \cup \{+\infty\}$ by

$$\underline{f}(x) = \sup\bigl\{\inf_{t\in U_x} f(t) : U_x\text{-neighborhood of } x\bigr\},$$
$$\overline{f}(x) = \inf\bigl\{\sup_{t\in U_x} f(t) : U_x\text{-neighborhood of } x\bigr\},$$

for all $x \in X$. It is well known that $\underline{f}$ is lower-semicontinuous, $\overline{f}$ is upper-semicontinuous, and $\underline{f}(x) \le f(x) \le \overline{f}(x)$ for every $x \in X$ (see [7, Section 7.8]). Therefore, f is continuous at some $x \in X$ if and only if $\underline{f}(x) = \overline{f}(x)$.

If a real mapping f defined on a topological group satisfies some functional inequality, then it is a frequent situation that both limit functions satisfy this inequality, as well. This is the case of subadditive functions (see [7, Theorem 7.8.1]) and subquadratic functions (see [6, Theorems 4.1 and 4.2]). For solutions to Hlawka's inequality, we will prove a weaker inequality for the lower-limit function; moreover, under the assumption that the domain is uniquely divisible by 2 with continuous division by 2, we show that the upper-limit function satisfies Hlawka's inequality.

Theorem 1. *Let X be a topological abelian group, and let $f\colon X \to \mathbb{R}$ satisfy Hlawka's inequality for all $x, y, z \in X$. Then*

$$\underline{f}(x+y) + \underline{f}(y+z) + \underline{f}(x+z) - \overline{f}(x+y+z) \le \underline{f}(x) + \underline{f}(y) + \underline{f}(z),$$

for all $x, y, z \in X$.

Proof. Fix $x, y, z \in X$. If any of the values $\underline{f}(x+y)$, $\underline{f}(y+z)$, $\underline{f}(x+z)$ is equal to $-\infty$ or $\overline{f}(x+y+z) = +\infty$, then the desired inequality holds true. Therefore, we will assume that they are all finite. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ be such that

$$\alpha < \underline{f}(x+y), \qquad \beta < \underline{f}(y+z),$$
$$\gamma < \underline{f}(x+z), \qquad \delta > \overline{f}(x+y+z).$$

Then, there exist open sets U_{x+y}, U_{y+z}, U_{x+z}, U_{x+y+z} which are neighborhoods of $x+y$, $y+z$, $x+z$, and $x+y+z$, respectively, such that

$$\alpha < \inf_{t\in U_{x+y}} f(t), \qquad \beta < \inf_{t\in U_{y+z}} f(t),$$
$$\gamma < \inf_{t\in U_{x+z}} f(t), \qquad \delta > \sup_{t\in U_{x+y+z}} f(t).$$

Using the continuity of group operation, we derive that there exist open sets U_x, U_y, U_z which are neighborhoods of x, y, and z, respectively, such that

$$\begin{aligned} U_x + U_y &\subseteq U_{x+y}, & U_y + U_z &\subseteq U_{y+z}, \\ U_x + U_z &\subseteq U_{x+z}, & U_x + U_y + U_z &\subseteq U_{x+y+z}. \end{aligned}$$

Fix arbitrary $u \in U_x$, $v \in U_y$, $w \in U_z$. Note that $u + v \in U_{x+y}$, $v + w \in U_{y+z}$, $u + w \in U_{x+z}$, and $u + v + w \in U_{x+y+z}$. Therefore, we have

$$\begin{aligned} \alpha + \beta + \gamma - \delta &< f(u + v) + f(v + w) + f(u + w) - f(u + v + w) \\ &\leq f(u) + f(v) + f(w). \end{aligned}$$

Observe that the arguments of the last three elements on the right-hand side are independent and are chosen arbitrarily from the respective open sets. Thus, we infer that

$$\begin{aligned} \alpha + \beta + \gamma - \delta &\leq \inf_{u \in U_x} f(u) + \inf_{v \in U_y} f(v) + \inf_{w \in U_z} f(w) \\ &\leq \underline{f}(x) + \underline{f}(y) + \underline{f}(z). \end{aligned}$$

From the choice of α, β, γ, δ we finally obtain the desired inequality. □

Remark 2. Function f_3 constructed in the previous section satisfies assumptions of Theorem 1, whereas $\underline{f_3} = -1$ on $\mathbb{R}$, so $\underline{f_3}$ does not satisfy Hlawka's inequality (the equality $\underline{f_3} = -1$ can be derived from the fact that the graph of a discontinuous additive function is dense; see [12, Theorem 12.1.2]). Therefore, Theorem 1 cannot be strengthened in that direction ($\underline{f}$ need not satisfy Hlawka's inequality).

The next theorem states that the roles of $\underline{f}$ and $\overline{f}$ are not fully symmetric.

Theorem 2. *Let X be a topological abelian group uniquely divisible by 2 on which division by 2 is continuous, and let $f\colon X \to \mathbb{R}$ satisfy Hlawka's inequality for all $x, y, z \in X$. Then $\overline{f}$ satisfies Hlawka's inequality as well.*

Proof. Fix $x, y, z \in X$. If any of the values $\overline{f}(x+y+z)$, $\overline{f}(x)$, $\overline{f}(y)$, $\overline{f}(z)$ is equal to $+\infty$, then the desired inequality for $\overline{f}$ holds. Therefore, we will assume that they are all finite. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ be such that

$$\alpha > \overline{f}(x + y + z), \qquad \beta > \overline{f}(x), \qquad \gamma > \overline{f}(y), \qquad \delta > \overline{f}(z).$$

Therefore, there exist open sets U_{x+y+z}, U_x, U_y, U_z which are neighborhoods of $x + y + z$, x, y, and z, respectively, such that

$$\alpha > \sup_{t \in U_{x+y+z}} f(t), \qquad \beta > \sup_{t \in U_x} f(t), \qquad \gamma > \sup_{t \in U_y} f(t), \qquad \delta > \sup_{t \in U_z} f(t).$$

By the continuity of group operations and by the continuity of division by 2, we derive that there exist open sets U_{x+y}, U_{y+z}, U_{x+z} which are neighborhoods of $x + y$, $y + z$, and $x + z$, respectively, such that

$$\begin{aligned} U_{x+y} + U_{y+z} + U_{x+z} &\subseteq 2U_{x+y+z}, & U_{x+y} - U_{y+z} + U_{x+z} &\subseteq 2U_x, \\ U_{x+y} + U_{y+z} - U_{x+z} &\subseteq 2U_y, & -U_{x+y} + U_{y+z} + U_{x+z} &\subseteq 2U_z. \end{aligned}$$

Fix arbitrary $s \in U_{x+y}$, $t \in U_{y+z}$, $q \in U_{x+z}$. Let $u = \frac{1}{2}(s-t+q)$, $v = \frac{1}{2}(s+t-q)$, and $w = \frac{1}{2}(-s+t+q)$. Then $u \in U_x$, $v \in U_y$, $w \in U_z$, and $u+v+w = \frac{1}{2}(s+t+q) \in U_{x+y+z}$. Therefore, we have

$$\begin{aligned}\alpha + \beta + \gamma + \delta &> f(u+v+w) + f(u) + f(v) + f(w) \\ &\geq f(u+v) + f(v+w) + f(u+w) = f(s) + f(t) + f(q).\end{aligned}$$

Thus,

$$\begin{aligned}\alpha + \beta + \gamma + \delta &\geq \sup_{s \in U_{x+y}} f(s) + \sup_{t \in U_{y+z}} f(t) + \sup_{q \in U_{x+z}} f(q) \\ &\geq \overline{f}(x+y) + \overline{f}(y+z) + \overline{f}(x+z).\end{aligned}$$

The choice of α, β, γ, δ finally leads us to the desired inequality:

$$\overline{f}(x+y+z) + \overline{f}(x) + \overline{f}(y) + \overline{f}(z) \geq \overline{f}(x+y) + \overline{f}(y+z) + \overline{f}(x+z). \qquad \square$$

In our next results, we will prove two more estimates which will be utilized to obtain bounds for the difference $\overline{f}(x) - \underline{f}(x)$.

Theorem 3. *Let X be a topological abelian group, and let $f\colon X \to \mathbb{R}$ satisfy Hlawka's inequality for all $x, y, z \in X$. Then*

$$\underline{f}(x+y) + \underline{f}(x+z) - \overline{f}(x) - \overline{f}(y) \leq \underline{f}(x+y+z) + \underline{f}(z) - \overline{f}(y+z)$$

for all $x, y, z \in X$.

Proof. Fix $x, y, z \in X$. We can assume that the left-hand-side of the estimate to be proved is not equal to $-\infty$. Then, take $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha < \underline{f}(x+y), \qquad \beta < \underline{f}(x+z), \qquad \gamma > \overline{f}(x), \qquad \delta > \overline{f}(y).$$

Consequently, there exist open sets U_{x+y}, U_{x+z}, U_x, U_y which are neighborhoods of $x+y$, $x+z$, x, and y, respectively, such that

$$\alpha < \inf_{t \in U_{x+y}} f(t), \qquad \beta < \inf_{t \in U_{x+z}} f(t), \qquad \gamma > \sup_{t \in U_x} f(t), \qquad \delta > \sup_{t \in U_y} f(t).$$

By the continuity of group operations, there exist open sets U_{x+y+z}, U_z, U_{y+z} which are neighborhoods of $x+y+z$, z, and $y+z$, respectively, such that

$$\begin{aligned} U_{x+y+z} - U_z &\subseteq U_{x+y}, &\qquad U_{x+y+z} - U_{y+z} &\subseteq U_x, \\ U_{x+y+z} + U_z - U_{y+z} &\subseteq U_{x+z}, &\qquad U_{y+z} - U_z &\subseteq U_y. \end{aligned}$$

Fix arbitrarily $s \in U_{x+y+z}$, $t \in U_z$, and $q \in U_{y+z}$, and apply Hlawka's inequality with the substitution $x = s - q$, $y = q - t$, and $z = t$ to get

$$f(s-t) + f(s+t-q) - f(s-q) - f(q-t) \leq f(s) + f(t) - f(q).$$

Note that $s - t \in U_{x+y}$, $s + t - q \in U_{x+z}$, $s - q \in U_x$, and $q - t \in U_y$. Therefore,

$$\alpha + \beta - \gamma - \delta < f(s) + f(t) - f(q).$$

Thus, since s, t, q were arbitrary and independent elements of respective neighborhoods, we obtain

$$\begin{aligned}\alpha+\beta-\gamma-\delta &\le \inf_{s\in U_{x+y+z}} f(s)+\inf_{t\in U_z} f(t)-\sup_{q\in U_{y+z}} f(q)\\ &\le \underline{f}(x+y+z)+\underline{f}(z)-\overline{f}(y+z). \qquad \square\end{aligned}$$

Theorem 4. *Let X be a topological abelian group, and let $f\colon X\to\mathbb{R}$ satisfy Hlawka's inequality for all $x,y,z\in X$. Then*

$$\overline{f}(x+y)-\underline{f}(x)-\underline{f}(z)\le \overline{f}(y)+\overline{f}(x+y+z)-\underline{f}(x+z)-\underline{f}(y+z)$$

for all $x,y,z\in X$.

Proof. Fix $x,y,z\in X$. We can assume that the right-hand side of the estimate to be proved is not equal to $+\infty$. Then, take $\alpha,\beta,\gamma,\delta\in\mathbb{R}$ such that

$$\alpha>\overline{f}(y),\qquad \beta>\overline{f}(x+y+z),\qquad \gamma<\underline{f}(x+z),\qquad \delta<\underline{f}(y+z).$$

Consequently, there exist open sets U_y, U_{x+y+z}, U_{x+z}, U_{y+z} which are neighborhoods of y, $x+y+z$, $x+z$, and $y+z$, respectively, such that

$$\alpha>\sup_{t\in U_y} f(t),\qquad \beta>\sup_{t\in U_{x+y+z}} f(t),\qquad \gamma<\inf_{t\in U_{x+z}} f(t),\qquad \delta<\inf_{t\in U_{y+z}} f(t).$$

By the continuity of group operations, there exist open sets U_{x+y}, U_x, U_z which are neighborhoods of $x+y$, x, and z, respectively, such that

$$\begin{gathered}U_{x+y}-U_x\subseteq U_y,\qquad U_{x+y}+U_z\subseteq U_{x+y+z},\\ U_x+U_z\subseteq U_{x+z},\qquad U_{x+y}-U_x+U_z\subseteq U_{y+z}.\end{gathered}$$

Fix arbitrarily $u\in U_{x+y}$, $v\in U_x$, and $w\in U_z$, and apply Hlawka's inequality with the substitution $x=u-v$, $y=v$, and $z=w$ to obtain

$$f(u)-f(v)-f(w)\le f(u-v)+f(u+w)-f(v-w)-f(u-v+w).$$

Note that $u-v\in U_y$, $u+w\in U_{x+y+z}$, $v+w\in U_{x+z}$, and $u-v+w\in U_{y+z}$. Therefore,

$$f(u)-f(v)-f(w)<\alpha+\beta-\gamma-\delta.$$

Thus, since u, v, w were arbitrary and independent elements of respective neighborhoods, we get

$$\begin{aligned}\sup_{u\in U_{x+y}} f(u)-\inf_{v\in U_x} f(v)-\inf_{w\in U_z} f(w) &\le \overline{f}(x+y)-\underline{f}(x)-\underline{f}(z)\\ &<\alpha+\beta-\gamma-\delta. \qquad \square\end{aligned}$$

Let us introduce a function $\omega\colon X\to\mathbb{R}\cup\{+\infty\}$ by $\omega=\overline{f}-\underline{f}$. Then, ω is a nonnegative upper-semicontinuous mapping which vanishes precisely at the points of continuity of f. Corollaries 2 and 3 say that, under an additional assumption that X is a topological abelian group uniquely divisible by 2 which is generated by any neighborhood of zero, if ω is finite at a point, then ω is finite everywhere; moreover, if $\omega(0)=0$, then $\omega=0$ on X. Using our Theorems 3 and 4, we are able to avoid, at least partially, this additional assumption upon X.

If we substitute $x = 0$ and $y = x$ in Theorem 3 and $y = 0$ in Theorem 4, then we get

$$\begin{aligned} -\omega(x) + \underline{f}(z) - \overline{f}(0) &\le \underline{f}(z) - \omega(x+z), \\ \omega(x) - \underline{f}(z) &\le \overline{f}(0) + \omega(x+z) - \underline{f}(z), \end{aligned}$$

for all $x, z \in X$. Therefore, if additionally $\underline{f}(z) > -\infty$ (or equivalently, f is locally bounded below at z), then

$$\begin{aligned} -\omega(x) - \overline{f}(0) &\le -\omega(x+z), \\ \omega(x) &\le \overline{f}(0) + \omega(x+z). \end{aligned}$$

Let us list some consequences of Theorems 3 and 4. First, we show that the assumptions of unique division by 2 in Corollary 2 can be omitted.

Corollary 4. *Let X be a topological abelian group, and let $f\colon X \to \mathbb{R}$ satisfy Hlawka's inequality for all $x, y, z \in X$. Then*

$$\big|\omega(x) - \omega(x+z)\big| \le \overline{f}(0)$$

for all $x, z \in X$, for which $\underline{f}(z)$ and $\omega(x+z)$ are finite.

Corollary 5. *Let X be a topological abelian group, and let $f\colon X \to \mathbb{R}$ satisfy Hlawka's inequality for all $x, y, z \in X$. If $\omega(0) < +\infty$, then the set $\{x \in X : \omega(x) < +\infty\}$ is a subgroup of X.*

Proof. Fix $y, z \in \{x \in X : \omega(x) < +\infty\}$. Then in particular $\underline{f}(z) > -\infty$. By Corollary 4 applied with x replaced by $y - z$, we see that ω is finite at $y - z$. □

Corollary 6. *Let X be a topological abelian group which is generated by any neighborhood of zero, and let $f\colon X \to \mathbb{R}$ satisfy Hlawka's inequality for all $x, y, z \in X$. If $\omega(0) < +\infty$, then ω is finite on X.*

Proof. Since ω is upper-semicontinuous, then there exists a neighborhood U of zero such that ω is bounded at U. By our assumption, U generates group X, and by Corollary 5, the set $\{x \in X : \omega(x) < +\infty\}$ is a group and it contains U. Consequently, ω is finite at every point. □

Remark 3. It can happen that $\omega(0) = \infty$, but $\overline{f}(0) = 0$. It is clear that $f(0) \ge 0$ (to see this, it is enough to put $x = y = z$ in Hlawka's inequality). Therefore, $\overline{f}(0) = 0$ implies that $f(0) = 0$.

Corollary 7. *Let X be a topological abelian group, and let $f\colon X \to \mathbb{R}$ satisfy Hlawka's inequality for all $x, y, z \in X$. If $\overline{f}(0) = 0$, then every point $z \in X$ at which $\underline{f}(z) > -\infty$ is a period of ω.*

Proof. Fix $x, z \in X$, and assume that $\underline{f}(z) > -\infty$. If $\omega(x+z)$ is finite, then by Theorem 4 function, $\omega(x)$ is finite, and by Corollary 4 we have $\omega(x) = \omega(x+z)$ for every $x \in X$. If, on the other hand $\omega(x+z) = +\infty$, then by Theorem 3 we get $\omega(x) = +\infty$. □

Remark 4. If $a\colon \mathbb{R} \to \mathbb{R}$ is a discontinuous additive function, then $f = -a^2$ satisfies all assumptions of the above corollary, but $\underline{f} = -\infty$ on $\mathbb{R}$. Note that for this function, we have $\omega = +\infty$ on $\mathbb{R}$.

Corollary 8. *Let X be a topological abelian group, and let $f\colon X \to \mathbb{R}$ satisfy Hlawka's inequality for all $x, y, z \in X$. If $\overline{f}(0) = 0$ and $\underline{f}(0) > -\infty$, then ω is an even function.*

Proof. Apply Theorem 3 with $y = 0$ and $z = -x$ to obtain (after some reductions) $-\omega(x) \leq -\omega(-x)$ which, after replacing x by $-x$, implies that ω is even. □

Corollary 9. *Let X be a topological abelian group, and let $f\colon X \to \mathbb{R}$ satisfy Hlawka's inequality for all $x, y, z \in X$. If $\overline{f}(0) = 0$ and f is continuous at a point, then f is continuous at zero.*

Proof. If f is continuous at some $z \in X$, then $\omega(z) = 0$, and by Corollary 4 also $\omega(x) = \omega(x + z)$ for all $x \in X$. Therefore, $\omega(0) = \omega(z) = 0$. □

Next, we state an analogue of Corollary 3 for arbitrary topological abelian groups.

Corollary 10. *Let X be a topological abelian group, and let $f\colon X \to \mathbb{R}$ satisfy Hlawka's inequality for all $x, y, z \in X$. If f is continuous at zero and $f(0) = 0$, then f is continuous at every point at which $\underline{f}$ is finite.*

Proof. Assume that $\underline{f}(z) > -\infty$ for some $z \in X$, and apply Theorem 3 with $x = y = 0$ to get

$$0 = -\omega(0) - \overline{f}(0) \leq -\omega(z) \leq 0;$$

that is, f is continuous at z. □

On joining the last corollary with Corollary 6, we get a direct generalization of Corollary 3.

Corollary 11. *Let X be a topological abelian group which is generated by any neighborhood of zero, and let $f\colon X \to \mathbb{R}$ satisfy Hlawka's inequality for all $x, y, z \in X$. If f is continuous at zero and $f(0) = 0$, then f is continuous on X.*

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