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SPACEABILITY IN NORM-ATTAINING SETS

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ABSTRACT. We study the existence of infinite-dimensional vector spaces in the sets of norm-attaining operators, multilinear forms, and polynomials. Our main result is that, for every set of permutations P of the set $\{1,\dots,n\}$, there exists a closed infinite-dimensional Banach subspace of the space of n-linear forms on ℓ_1 such that, for all nonzero elements B of such a subspace, the Arens extension associated to the permutation σ of B is norm-attaining if and only if σ is an element of P. We also study the structure of the set of norm-attaining n-linear forms on c_0.

1. Introduction

The Bishop–Phelps theorem [11, p. 97] states that the set of norm-attaining forms on a real or complex Banach space is norm-dense in the set of continuous linear forms. Naturally, we can ask what is the structure of the set of norm-attaining multilinear maps and when is this set an infinite-dimensional vector space or does it contain a large vector space. In this paper, we will look more closely to the existence of nonclosed and closed infinite-dimensional subspaces of the set of norm-attaining mappings.

In recent years, many examples of functions satisfying certain properties or pathologies have been studied. The interest of finding large structures formed by these functions has yielded to the goal of defining concepts such as *lineable* and *spaceable*.

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Definition 1.1. A subset M of a topological vector space E is lineable (respectively, spaceable) in E if $M \cup \{0\}$ contains an infinite-dimensional space (respectively, infinite-dimensional closed space).

The terms *lineable* and *spaceable* were introduced by Gurariy and Quarta [16] and by Aron, Gurariy, and Seoane-Sepúlveda [7]. Since then, many authors have shown their interest in studying the structure and linear properties of several mathematical sets such as the set of continuous nowhere differentiable functions in $C[0,1]$, continuous functions with nonconvergent Fourier series, or the structure of the set of differentiable nowhere monotone functions, among others. For a comprehensive background about lineability and spaceability, we recommend the survey [10] and the very recent book [4].

The concepts of lineability and spaceability of norm-attaining sets have been deeply studied during the last decade. For instance, Bandyopadhyay and Godefroy [9] proved among other results that, if the unit ball of the dual of a Banach space X is w^*-sequentially compact, then the set of norm-attaining functionals contains a norm closed separable subspace M if and only if the dual M^* of M is the canonical quotient of X. More recently, Pellegrino and Teixeira [20] and Botelho et al. [12] investigated lineability properties of the sets of norm-attaining operators for spaces containing an isometric copy of ℓ_q, $1 \leq q < \infty$.

Here we will focus on the study of lineability and spaceability properties of the set of norm-attaining operators, multilinear forms, and polynomials. It was recently proved that there exist compact operators that cannot be approximated by norm-attaining operators. In the second section of this paper, we investigate spaceability properties of this set of operators. The third section is devoted to the study of lineability and spaceability in the set of norm-attaining multilinear forms with special emphasis on the spaces c_0 and ℓ_1. We have two main results for these spaces. First, the set of norm-attaining n-linear forms on c_0 is lineable but not spaceable for every natural number n. Second, given a subset P of permutations of the first n natural numbers, then the set of n-linear forms on ℓ_1 satisfying that only the Arens extensions associated to elements of P are norm-attaining is spaceable. Finally, in the last section, we will study spaceability properties of the set of polynomials whose adjoint is norm-attaining.

1.1. **Notation.** Here we are considering real Banach spaces. For a Banach space X, we will denote by B_X and S_X the closed unit ball and the unit sphere of X, respectively. As usual, the dual of a Banach space X will be denoted by X^*, and the bidual of X will be denoted by X^{**}. Given two Banach spaces X and Y, $\mathcal{L}(X;Y)$ will denote the set of linear bounded operators from X to Y, $\mathcal{L}(^nX;Y)$ will be the set of n-linear continuous mappings from X to Y, and $\mathcal{P}(^nX;Y)$ will stand for the set of n-homogeneous continuous polynomials from X to Y. For the case of $Y = \mathbb{R}$, we will denote by $\mathcal{L}(^nX)$ the set of n-linear continuous forms on X, and for the case $n = 1$, for convenience, we will denote by $\mathcal{L}(X)$ the dual space of X.

The norm of an operator $T \in \mathcal{L}(X;Y)$ is defined by $\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\}$, the norm of a multilinear mapping $A \in \mathcal{L}(^nX;Y)$ is defined by

$\|A\| = \sup\{\|A(x_1, \ldots, x_n)\| : \|x_i\| \leq 1 \text{ for } i = 1, \ldots, n\}$, and the norm of a polynomial $P \in \mathcal{P}(^nX; Y)$ is defined by $\|P\| = \{\|P(x)\| : \|x\| \leq 1\}$. We consider that an operator, a multilinear mapping, or a polynomial is norm-attaining if the supremum defining the norm is in fact a maximum. The set of norm-attaining linear bounded operators, n-linear continuous mappings, and n-homogeneous continuous polynomials will be denoted, respectively, by $NA(\mathcal{L}(X; Y))$, $NA(\mathcal{L}(^nX; Y))$, and $NA(\mathcal{P}(^nX; Y))$. By Σ_n, we will denote the set of possible permutations over the first n natural numbers. Several times we will need to use the c_0 sum of Banach spaces. Given a sequence of Banach spaces $\{X_n\}_{n=1}^{\infty}$, we will denote the space obtained by the c_0 sum of the spaces X_n by

$$X = \bigoplus_{c_0} X_n = \big\{\{x_n\}_{n=1}^{\infty} : x_n \in X_n \text{ for all } n \in \mathbb{N}, \lim_{n\to\infty} \|x_n\| = 0\big\},$$

and we will endow this space with the supremum norm; that is, if $x = \{x_n\}_{n=1}^{\infty} \in X$, then the norm of x is $\|x\| = \max_{n\in\mathbb{N}} \|x_n\|$.

2. Operators

In this section, we study the existence of vector spaces in the sets of norm-attaining operators between Banach spaces. Before we start our study, let us recall some known properties of the sets of norm-attaining functionals on Banach spaces. It is known that, for the Banach space c_0, the set of norm-attaining linear forms consists of just the elements of ℓ_1 with finite support. Since this set does not contain an infinite-dimensional closed space, c_0 is an example of a Banach space such that the set of norm-attaining linear forms is lineable but not spaceable. The same situation happens for the set of n-linear forms on c_0, as we will see in Proposition 3.9.

Nevertheless, in most of the spaces, the set of norm-attaining linear forms is spaceable. For instance, if a Banach space X has a predual, then this predual can be naturally embedded in the set of norm-attaining linear forms of the space. This gives us an infinite-dimensional Banach subspace of the set of norm-attaining linear forms on X.

Proposition 2.1. *For every infinite-dimensional Banach space X with predual X_*, the set of norm-attaining linear forms on X is spaceable.*

At the end of their paper, Bishop and Phelps [11, p. 98] raised the question of extending their results to operators between Banach spaces. Nowadays it is known that, in general, we do not have a Bishop–Phelps theorem for operators; that is, in general the set of norm-attaining operators between two Banach spaces X and Y is not dense in $\mathcal{L}(X; Y)$. The first one who answered this question in the negative was Lindenstrauss [18], who gave an example of a Banach space X such that the identity mapping from X to X with an equivalent renorming cannot be approximated by norm-attaining operators.

As in the Lindenstrauss example, during many years all known cases of operators which cannot be approximated by norm-attaining operators were noncompact operators. The question of whether there exists a compact operator between Banach spaces that cannot be approximated by norm-attaining operators was asked first by Diestel and Uhl in 1976 [14, problem 2, page 6] and three years later by Johnson and Wolfe [17, question 2, page 17]. Recently, Martín [19] answered this question in the negative, making use of the space constructed by Enflo to solve the approximation problem in the negative.

Theorem 2.2 ([19, Theorem 1])**.** *There exist compact linear operators between Banach spaces which cannot be approximated by norm-attaining operators.*

Now we show not only that there are many compact operators that cannot be approximated by norm-attaining operators, but also that there exists two Banach spaces such that the set of compact operators that cannot be approximated by norm-attaining operators contains a large structure.

Theorem 2.3. *There exists two Banach spaces X and Y such that the set of compact linear operators from X to Y which cannot be approximated by norm-attaining operators is spaceable.*

Proof. By Theorem 2.2, there exist two Banach spaces X and Z and a compact linear operator $T \in \mathcal{L}(X;Z)$ of norm 1 such that T cannot be approximated by norm-attaining operators. Let $Y = \bigoplus_{c_0} Z$, and consider the operators $T_m : X \to Y$ defined by $T_m(x) = (0, \ldots, 0, T(x), 0 \ldots)$, where $T(x)$ is in the mth coordinate of Y.

Let us consider the vector space defined by

$$\mathcal{S} := \Big\{ G = \sum_{m\in\mathbb{N}} \lambda_m T_m : \{\lambda_m\}_{m\in\mathbb{N}} \in c_0 \Big\}.$$

Note that $\mathcal{S}$ is a closed infinite-dimensional Banach space, and for all $G \in \mathcal{S}$, $\|G\| = \sup_{m\in\mathbb{N}} |\lambda_m|$.

To see that the operators $G = \sum_{m\in\mathbb{N}} \lambda_m T_m$ are compact, we can check that, for every sequence $\{x_n\}_{n=1}^{\infty}$ in the unit ball of X, the sequence $\{G(x_n)\}_{n=1}^{\infty}$ has a subsequence convergent to some point in Y. Since T is a compact operator, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{x_n\}_{n=1}^{\infty}$ such that $\{T(x_{n_k})\}_{k=1}^{\infty}$ is convergent to a point z in Z. Let y be the sequence $\{\lambda_m z\}_{m=1}^{\infty}$. Clearly, y is an element of Y.

Since $T(x_{n_k})$ is convergent to z when k goes to infinity, we have

$$\big\|G(x_{n_k}) - y\big\| = \sup_{m\in\mathbb{N}} \big\|\lambda_m T(x_{n_k}) - \lambda_m z\big\| \leq \sup_{m\in\mathbb{N}} |\lambda_m| \big\|T(x_{n_k}) - z\big\| \overset{k\to\infty}{\longrightarrow} 0.$$

Hence G is a compact operator.

To finish, we only need to see that any operator G in $\mathcal{S}$ cannot be approximated by operators that attain their norm.

Since T cannot be approximated by norm-attaining operators, there exists a positive number ϵ such that, if $V \in \mathcal{L}(X;Z)$ is an operator with $\|T - V\| < \epsilon$, then V is not norm-attaining. Let us fix this $\epsilon > 0$ and a nonzero operator G in $\mathcal{S}$. Without loss of generality, we can assume that G has norm 1. We will prove that, if W is an operator in $\mathcal{L}(X;Y)$ of norm 1 such that $\|G - W\| < \epsilon$, then W is not norm-attaining. Let us assume that this is not the case and that W is a norm-attaining operator of norm 1 in $\mathcal{L}(X;Y)$ with $\|G - W\| < \epsilon$. Since

W is norm-attaining, then there exists a point $x_0 \in X$ of norm 1 such that $\|W(x_0)\| = \|W\| = 1$. Let us denote by $\{y_m\}_{m=1}^{\infty} = W(x_0)$. By the norm of Y, there exists some natural number m_0 such that $\|y_{m_0}\| = 1$. We can consider the operator $V \in \mathcal{L}(X;Z)$ defined by $V := \pi_{m_0} \circ W$, where π_{m_0} is the projection of the m_0th coordinate from the space Y into the space Z. Then V is a norm-attaining operator in $\mathcal{L}(X;Z)$ since $\|V(x_0)\| = \|(\pi_{m_0} \circ W)(x_0)\| = \|y_{m_0}\| = 1 = \|W\| \geq \|V\|$. But $\|T - V\| \leq \|G - W\| < \epsilon$, contradicting the fact that there does not exist a norm-attaining operator V in $\mathcal{L}(X;Z)$ with $\|T - V\| < \epsilon$.

Therefore, every operator W in $\mathcal{L}(X;Y)$ with $\|G - W\| < \epsilon$ is not norm-attaining. $\square$

As a consequence of this result, we get that the set of compact operators from X to Y that do not attain their norm is spaceable.

Corollary 2.4. *There exist two Banach spaces X and Y such that the set of compact linear operators from X to Y that do not attain their norm is spaceable.*

Remark 2.5. It is worth mentioning that Martín [19, Theorem 8] gave a version of Theorem 2.2 where the domain and range spaces of the operators that cannot be approximated by norm-attaining operators coincide. The same idea used by Martín to obtain this result can be used here to modify the proof of Theorem 2.3 and Corollary 2.4, obtaining versions where the range space Y is the same as the domain space X.

In the same paper (see [18]) in which he provided the first example where the Bishop–Phelps theorem fails for operators, Lindenstrauss gave a related result that involves the second adjoint of linear operators, starting a new research direction. Let us denote by $T^* : Y^* \to X^*$ the adjoint of the operator $T : X \to Y$ defined by $T^*(y^*)(x) = y^*(T(x))$ for all $x \in X$, $y^* \in Y^*$. Lindenstrauss proved that, for any two Banach spaces X and Y, the set of norm-attaining operators from X into Y whose second adjoints attain their norms is dense in $\mathcal{L}(X;Y)$. Ten years later, Zizler [21] improved this result, showing that using only one adjoint is enough; that is, for any two Banach spaces X and Y, the set of operators from X to Y such that its adjoint is norm-attaining is norm-dense in $\mathcal{L}(X;Y)$. Following this idea, we study spaceability of the set of operators whose adjoint is norm-attaining.

In Section 4 we will prove that every compact polynomial has a norm-attaining transpose. Since every compact operator from X to Y is a compact polynomial of degree 1 from X to Y and the adjoint of the operator coincides with the transpose of the polynomial in this case, as a particular case Theorem 4.1 implies the following result.

Proposition 2.6. *If X and Y are Banach spaces, then every compact operator from X to Y satisfies that its adjoint is norm-attaining.*

As a consequence of this result and using the fact that the class of compact operators between two Banach spaces is a closed infinite-dimensional vector space, we obtain the following result.

Corollary 2.7. *Given two Banach spaces X and Y, the set of operators from X to Y whose adjoint is norm-attaining is spaceable.*

3. Multilinear forms

Since there is no general version of the Bishop–Phelps result for operators, due to the duality between operators and bilinear forms, we cannot expect a general Bishop–Phelps theorem for bilinear forms. Therefore, there is no general Bishop–Phelps theorem for multilinear forms. However, we can still study the existence of subspaces of the set of norm-attaining multilinear forms.

For instance, we can get a generalization of Proposition 2.1 in the following way.

Proposition 3.1. *For every infinite-dimensional Banach space X with predual X_* and every natural number n, the set of norm-attaining n-linear forms on X is spaceable.*

Proof. Since X has predual, by Proposition 2.1, $NA(\mathcal{L}(X))$ is spaceable. Hence there exists an infinite-dimensional Banach space $Y \subset NA(\mathcal{L}(X))$.

Let f be a nonzero norm-attaining linear form of X^*. Then, for every norm-attaining linear form g of X^*, we have that the n-linear form $g*f*\cdots*f$ defined by $(g*f*\cdots*f)(x_1,\ldots,x_n) = g(x_1)\prod_{j=2}^n f(x_i)$ is a norm-attaining n-linear form of $\mathcal{L}(^nX)$. In particular, the space

$$\mathcal{S} := \{g*f*\cdots*f : g \in Y\}$$

is an infinite-dimensional closed subspace of $NA(\mathcal{L}(^nX))$. □

However, if the space does not have a predual, then Proposition 3.1 does not need to be true in general, as we will show in Theorem 3.9 using the space c_0. However, before we give the proof of Theorem 3.9, we will study the behavior of the norm-attaining multilinear form on c_0.

Lemma 3.2. *If f is a linear form on c_0 that attains its norm on a point $x_0 \in B_{c_0}$, then $f(x) = f(x_0)$ for all $x \in c_0$ such that $x(j) = x_0(j)$ whenever $|x_0(j)| = 1$.*

Proof. Let k be such that $|x_0(k)| < 1$, and take $0 < \delta < 1 - |x_0(k)|$. Then

$$\big|f(x_0)\big| \le 1/2\big(\big|f(x_0 - \delta e_k)\big| + \big|f(x_0 + \delta e_k)\big|\big) \le 1/2\big(\|f\| + \|f\|\big) = \|f\| = \big|f(x_0)\big|.$$

Hence $|f(x_0)| = |f(x_0 + \delta e_k)|$, and so $f(e_k) = 0$.

Let us now consider $x \in c_0$ such that $x(j) = x_0(j)$ whenever $|x_0(j)| = 1$. Define $z := x - x_0$. Note that $z(j) = 0$ whenever $|x_0(j)| = 1$. Then

$$f(z) = \sum_{j=1}^{\infty} z(j)f(e_j) = \sum_{\{k:|x_0(k)|<1\}} z(k)f(e_k) = 0;$$

that is, $f(x) = f(x_0)$. □

Lemma 3.3. *If A is an n-linear form on c_0 that attains its norm on a point $(x_1,\ldots,x_n) \in B_{c_0} \times \cdots \times B_{c_0}$, then $A(y_1,\ldots,y_n) = A(x_1,\ldots,x_n)$ for all elements $(y_1,\ldots,y_n)$ in $B_{c_0} \times \cdots \times B_{c_0}$ such that, for every $i \in \{1,\ldots,n\}$, $y_i(j) = x_i(j)$ whenever $|x_i(j)| = 1$.*

Proof. Let $(y_1, \dots, y_n) \in B_{c_0} \times \cdots \times B_{c_0}$ be such that, for every $i \in \{1, \dots, n\}$, $y_i(j) = x_i(j)$ whenever $|x_i(j)| = 1$. Decompose

$$\begin{aligned} &A(y_1, \dots, y_n) - A(x_1, \dots, x_n) \\ &\quad = A(y_1 - x_1, y_2, \dots, y_n) + A(x_1, y_2 - x_2, y_3, \dots, y_n) + \cdots \\ &\qquad + A(x_1, \dots, x_{n-1}, y_n - x_n). \end{aligned} \tag{3.1}$$

Define $f_n(x) := A(x_1, \dots, x_{n-1}, x)$. Clearly, f_n is a linear form that has the same norm as A, and it is attained at x_n. By Lemma 3.2, it follows that $f_n(x_n) = f_n(y_n)$; that is, $A(x_1, \dots, x_{n-1}, y_n - x_n) = 0$.

Define the linear form $f_{n-1}(x) := A(x_1, \dots, x_{n-2}, x, y_n)$. Since

$$f_{n-1}(x_{n-1}) = A(x_1, \dots, x_{n-2}, x_{n-1}, y_n) = f_n(y_n) = f_n(x_n) = \|A\|,$$

the form f_{n-1} has the same norm as A and it is attained at x_{n-1}. By Lemma 3.2, it follows that $f_{n-1}(x_{n-1}) = f_{n-1}(y_{n-1})$; that is, $A(x_1, \dots, x_{n-2}, x_{n-1} - y_{n-1}, y_n) = 0$.

Clearly, the last step of this iterative process will be as follows. Define $f_1(x) := A(x, y_2, \dots, y_n)$. Since

$$f_1(x_1) = A(x_1, y_2, \dots, y_n) = f_2(y_2) = f_2(x_2) = \cdots = f_n(y_n) = f_n(x_n) = \|A\|,$$

the linear form f_1 has the same norm as A, and it is attained at x_1. By Lemma 3.2, it follows that $f_1(x_1) = f_1(y_1)$; that is, $A(x_1 - y_1, y_2, \dots, y_n) = 0$. Equality (3.1) gives the result. □

Let us denote by $\mathrm{Ext}_{n,m} = \{(x_1, \dots, x_n) \in c_0 \times \cdots \times c_0 : x_i(j) = \pm 1 \text{ for } j \leq m \text{ and } x_i(j) = 0 \text{ for } j > m, 1 \leq i \leq n\}$. Note that c_0 does not have extreme points and that the set $\{x \in c_0 : x(j) = \pm 1 \text{ for } j \leq m, x(j) = 0 \text{ for } j > m\}$ is the set of extremal points of the finite-dimensional space $(\mathbb{R}^m, \|\cdot\|_\infty)$ when considering the natural embedding of the space $\mathbb{R}^m$ in c_0.

Corollary 3.4. *Every norm-attaining n-linear form on c_0 attains its norm at a point of $\mathrm{Ext}_{n,m}$ for some m (that depends on the n-linear form).*

Proof. Let A be an n-linear form on c_0 that attains its norm at $(x_1, \dots, x_n) \in B_{c_0} \times \cdots \times B_{c_0}$. For each $i \in \{1, \dots, n\}$, consider $m_i := \min\{j : |x_i(k)| < 1 \text{ for all } k \geq j\}$, and take $m := \max\{m_i : i = 1, \dots, n\}$. For each $i \in \{1, \dots, n\}$, define $y_i(j) := 1$ whenever $x_i(j) > 0$ and $j \leq m$, define $y_i(j) := -1$ whenever $x_i(j) < 0$ and $j \leq m$, and define $y_i(j) = 0$ for all $j > m$. By Lemma 3.3, we conclude that $A(y_1, \dots, y_n) = A(x_1, \dots, x_n) = \|A\|$. □

Corollary 3.5. *For every natural number n,*

$$\begin{aligned} NA\big(\mathcal{L}(^n c_0)\big) \subseteq \big\{&A \in \mathcal{L}(^n c_0) : \exists m \in \mathbb{N} \textit{ such that } A(e_{k_1}, \dots, e_{k_n}) = 0 \\ &\textit{if } k_1, \dots, k_n > m\big\}. \end{aligned}$$

Proof. Given a norm-attaining n-linear form A on c_0, by Corollary 3.4 there exist a natural number m and an n-tuple $(x_1, \dots, x_n) \in \mathrm{Ext}_{n,m}$ where A is norm-attaining.

Then, by Corollary 3.4 again, for every $k > m$ we have

$$\begin{aligned}A(x_1, \ldots, x_n) &= A(x_1, \ldots, x_{i-1}, x_i + e_k, x_{i+1}, \ldots, x_n)\\ &= A(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)\\ &\quad + A(x_1, \ldots, x_{i-1}, e_k, x_{i+1}, \ldots, x_n),\end{aligned}$$

and so $A(x_1, \ldots, x_{i-1}, e_k, x_{i+1}, \ldots, x_n) = 0$ if $k > m$.

Now, given $k_1, k_2 > m$, we have that

$$\begin{aligned}A(x_1, \ldots, x_n) &= A(x_1, \ldots, x_{i-1}, x_i + e_{k_1}, x_{i+1}, \ldots, x_{r-1}, x_r + e_{k_2}, x_{r+1}, \ldots, x_n)\\ &= A(x_1, \ldots, x_n) + A(x_1, \ldots, e_{k_1}, \ldots, x_r, \ldots, x_n)\\ &\quad + A(x_1, \ldots, x_i, \ldots, e_{k_2}, \ldots, x_n) + A(x_1, \ldots, e_{k_1}, \ldots, e_{k_2}, \ldots, x_n)\\ &= A(x_1, \ldots, x_n) + A(x_1, \ldots, e_{k_1}, \ldots, e_{k_2}, \ldots, x_n),\end{aligned}$$

and so $A(x_1, \ldots, e_{k_1}, \ldots, e_{k_2}, \ldots, x_n) = 0$ if $k_1, k_2 > m$. After n steps we see that $A(e_{k_1}, \ldots, e_{k_n}) = 0$ for all $k_1, \ldots, k_n > m$, and so the conclusion holds. ☐

Remark 3.6. For $n \geq 2$, the inclusion in Corollary 3.5 is strict. Indeed, if we denote $\varphi(x) = \sum_{j=1}^{\infty} \frac{x(j)}{2^j}$, then the continuous n-linear form $A : c_0^n \to \mathbb{R}$ defined by

$$A(x_1, \ldots, x_n) = x_1(1) \prod_{k=2}^{n} \varphi(x_k)$$

satisfies $A(e_{k_1}, \ldots, e_{k_n}) = 0$ if $k_1 \geq 2$, but A does not attain its norm.

On the other hand, notice that, for the linear case, the set of linear forms on c_0 that are norm-attaining are the elements of ℓ_1 with finite support. However, for $n \geq 2$, there exist n-linear forms that are norm-attaining but do not have finite support. For instance, the n-linear form

$$A(x_1, \ldots, x_n) = \Big(\Big(\frac{x_1(1) + x_1(2)}{2}\Big)x_2(1) + \Big(\frac{x_1(1) - x_1(2)}{2}\Big) \sum_{k=2}^{\infty} \frac{x_2(k)}{2^k}\Big) \prod_{j=3}^{n} x_j(1)$$

has norm 1, and satisfies $A(e_1 + e_2, e_1, e_1, \ldots, e_1) = 1$, but if k is bigger than 1, then $A(e_1 - e_2, e_k, e_1, \ldots, e_1) = 2^{-k}$.

Lemma 3.7. *Let us fix a natural number m. If $\mathcal{S}$ is a vector space of norm-attaining n-linear forms on c_0 such that, for all A in $\mathcal{S}$, A is norm-attaining at an n-tuple $(x_1, \ldots, x_n) \in \mathrm{Ext}_{n,m}$, then the dimension of $\mathcal{S}$ is at most m^n.*

Proof. We prove this by contradiction. Let $\mathcal{S}$ be the aforementioned vector space, and assume that $\mathcal{S}$ has dimension bigger than m^n. Then we can find a set $\{A_j\}_{j=1}^{m^n+1}$ of n-linear forms in $\mathcal{S}$ that are linearly independent.

Now we construct a new set of m^n n-linear forms in $\mathcal{S}$ in the following way. If every n-linear form of the set $\{A_j\}_{j=1}^{m^n+1}$ is zero at the n-tuple $(e_1, \ldots, e_1)$, then define $B_j = A_j$ for $j = 1, \ldots, m^n$. If this is not the case, then there exists an n-linear form in the set $\{A_j\}_{j=1}^{m^n+1}$ that is not zero at the n-tuple $(e_1, \ldots, e_1)$. Without loss of generality, we can assume that $A_{m^n+1}(e_1, \ldots, e_1) \neq 0$. Then we define the n-linear forms $B_j = A_j - \frac{A_j(e_1,\ldots,e_1)}{A_{m^n+1}(e_1,\ldots,e_1)} A_{m^n+1}$ for $j = 1, \ldots, m^n$. By construction, $B_j(e_1, \ldots, e_1) = 0$ for $j = 1, \ldots, m^n$.

By repeating the same process for all the vectors $(e_{j_1}, \dots, e_{j_n})$, $1 \le j_1, \dots, j_n \le m$, and using the fact that $\{A_j\}_{j=1}^{m^n+1}$ is a set of linearly independent n-linear forms, we can find a nonzero n-linear form A with $A(e_{j_1}, \dots, e_{j_n}) = 0$ for all $1 \le j_1, \dots, j_n \le m$. Then, since $\mathcal{S}$ is a vector space, the n-linear form A is in $\mathcal{S}$. Therefore,

$$\begin{aligned} 0 \neq \|A\| &= \max\big\{A(x_1, \dots, x_n) : (x_1, \dots, x_n) \in \mathrm{Ext}_{n,m}\big\} \\ &= \max\Big\{ \sum_{1 \le j_1, \dots, j_n \le m} A(e_{j_1}, \dots, e_{j_n}) \prod_{i=1}^{n} x_i(j_i) : (x_1, \dots, x_n) \in \mathrm{Ext}_{n,m} \Big\} = 0, \end{aligned}$$

which is a contradiction. Hence $\mathcal{S}$ has dimension at most m^n. □

Lemma 3.8. *Let $\mathcal{S}$ be an infinite-dimensional vector space of norm-attaining n-linear forms on c_0. Let A be a norm-attaining n-linear form in $\mathcal{S}$, and let ϵ be a positive number. If A is norm-attaining at an n-tuple in $\mathrm{Ext}_{n,m}$, then there exist an n-linear form B in $\mathcal{S}$ and a natural number $\tilde{m} > m$ such that $A + \epsilon B$ is norm-attaining at an n-tuple in $\mathrm{Ext}_{n,\tilde{m}}$, but $A + \epsilon B$ is not norm-attaining at any n-tuple in $\mathrm{Ext}_{n,j}$ for $j < \tilde{m}$. Even more, B can be chosen such that $\|A\| < \|A + \epsilon B\| \le \|A\| + \epsilon$.*

Proof. Fix a positive number ϵ. Then, for every n-linear form B in $\mathcal{S}$ of norm 1, we have that $A \pm \epsilon B$ are norm-attaining n-linear forms of $\mathcal{S}$. Since $\mathcal{S}$ has infinite dimensions, if $\mathcal{H}$ is an algebraic complement in $\mathcal{S}$ of $\mathrm{span}(A)$, then $\mathcal{H}$ is infinite-dimensional too. Hence, if $\{B_i : i \in I\}$ is a Hamel basis of $\mathcal{H}$ with $\|B_i\| = 1$ for every $i \in I$, then $\{A + \epsilon B_i : i \in I\}$ is an infinite set of linearly independent elements of $\mathcal{S}$. Thus, by Lemma 3.7, there exists one B in $\mathcal{S}$ with $\|B\| = 1$ and such that $A \pm \epsilon B$ are not norm-attaining at any n-tuple in $\mathrm{Ext}_{n,m}$. Also, as a consequence of Corollary 3.4, $A \pm \epsilon B$ are not norm-attaining at any n-tuple in $\mathrm{Ext}_{n,j}$ for $j \le m$.

By hypothesis, A is norm-attaining at an n-tuple $(y_1, \dots, y_n) \in \mathrm{Ext}_{n,m}$. Then, since $A \pm \epsilon B$ is not norm-attaining at any n-tuple of $\mathrm{Ext}_{n,m}$, we have that $A \pm \epsilon B$ is not norm-attaining at $(y_1, \dots, y_n)$. Since we are working on the real case, we have that $|(A + \epsilon B)(y_1, \dots, y_n)| \ge |A(y_1, \dots, y_n)|$ or $|(A - \epsilon B)(y_1, \dots, y_n)| \ge |A(y_1, \dots, y_n)|$. Without loss of generality, we assume that $|(A+\epsilon B)(y_1, \dots, y_n)| \ge |A(y_1, \dots, y_n)|$. Then

$$\|A\| + \epsilon \ge \|A + \epsilon B\| > \big|(A + \epsilon B)(y_1, \dots, y_n)\big| \ge \big|A(y_1, \dots, y_n)\big| = \|A\|.$$

Since $A+\epsilon B$ is norm-attaining, by Corollary 3.4, there exists a positive number k such that $A+\epsilon B$ is norm-attaining at an n-tuple in $\mathrm{Ext}_{n,k}$. Let $\tilde{m}$ be the smallest natural number such that $A + \epsilon B$ is norm-attaining at an n-tuple in $\mathrm{Ext}_{n,\tilde{m}}$. By construction, $\tilde{m}$ needs to be bigger than m, and $A + \epsilon B$ is not norm-attaining at any n-tuple in $\mathrm{Ext}_{n,j}$ for j smaller than $\tilde{m}$. □

Now we are ready to present our main result about norm-attaining n-linear forms on c_0.

Theorem 3.9. *For every natural number n, the set of norm-attaining n-linear forms on c_0 is lineable but not spaceable.*

Proof. Let us denote by $\mathbb{R}^{n_0}$ the real n_0-dimensional space equipped with the supremum norm. Notice that the space $(\mathbb{R}^{n_0}\times\cdots\times\mathbb{R}^{n_0}, \|\cdot\|_\infty)$ is finite-dimensional; hence its unit ball is compact. Therefore, every element of $\mathcal{L}({}^n\mathbb{R}^{n_0})$ is norm-attaining. Hence every element A of $\mathcal{L}({}^n c_0)$ with all but a finite number of coordinates zero is norm-attaining. Therefore, $NA(\mathcal{L}({}^n c_0))$ is lineable.

Now, let us show that $NA(\mathcal{L}({}^n c_0))$ is not spaceable. Assume this is not the case. Then there exists an infinite-dimensional closed subspace $\mathcal{S}$ of $NA(\mathcal{L}({}^n c_0))$.

By Lemma 3.8, we can find a natural number m_1 bigger than 1 and an n-linear form A_1 in $\mathcal{S}$ such that A_1 is norm-attaining at an n-tuple of Ext_{n,m_1}, but A_1 is not norm-attaining at any n-tuple of $\mathrm{Ext}_{n,r}$ for $r < m_1$. Consider

$$\epsilon_1 = \|A_1\| - \max_{(x_1,\ldots,x_n)\in\mathrm{Ext}_{n,m_1-1}} \big|A_1(x_1,\ldots,x_n)\big| > 0.$$

By Lemma 3.8 again, we can find a natural number m_2 bigger than m_1 and an n-linear form B in $\mathcal{S}$ such that $A_2 = A_1 + \frac{\epsilon_1}{3}B$ is norm-attaining at an n-tuple in Ext_{n,m_2}, but it is not norm-attaining at any n-tuple in $\mathrm{Ext}_{n,k}$ for $k \le m_1$ and $\|A_2\| > \|A_1\|$. Let

$$\delta_2 = \|A_2\| - \max_{(x_1,\ldots,x_n)\in\mathrm{Ext}_{n,m_1}} \big|A_2(x_1,\ldots,x_n)\big| > 0,$$

and let $\epsilon_2 = \min\{\frac{\epsilon_1}{3}, \delta_2\}$.

Using this idea and proceeding by induction, we are going to construct three sequences as follows. If we have the first j elements $\{A_k\}_{k=1}^j$, $\{m_k\}_{k=1}^j$, and $\{\epsilon_k\}_{k=1}^j$, then, by Lemma 3.8, we can find an n-linear form B in $\mathcal{S}$ and a natural number m_{j+1} bigger than m_j such that $A_{j+1} = A_j + \frac{\epsilon_j}{3}B$ is norm-attaining at an n-tuple in $\mathrm{Ext}_{n,m_{j+1}}$, but it is not norm-attaining at any n-tuple in $\mathrm{Ext}_{n,k}$ for $k \le m_j$ and $\|A_{j+1}\| > \|A_j\|$. Let

$$\delta_{j+1} = \|A_{j+1}\| - \max_{(x_1,\ldots,x_n)\in\mathrm{Ext}_{n,m_j}} \big|A_{j+1}(x_1,\ldots,x_n)\big| > 0,$$

and let $\epsilon_{j+1} = \min\{\frac{\epsilon_1}{3^j}, \frac{\epsilon_2}{3^{j-1}}, \ldots, \frac{\epsilon_j}{3}, \delta_{j+1}\}$.

Then, since the sequence $\{\epsilon_j\}_{j=1}^\infty$ is convergent to zero, we have that $\{A_j\}_{j=1}^\infty$ is a Cauchy sequence with $\|A_r - A_j\| \le \frac{\epsilon_j}{3} + \frac{\epsilon_j}{3^2} + \cdots + \frac{\epsilon_j}{3^{k-j}} < \frac{\epsilon_j}{2}$ for all $r, j \in \mathbb{N}$ with $r > j$. Let A be the limit of the sequence $\{A_j\}_{j=1}^\infty$. Then, since the sequence $\{\|A_j\|\}_{j=1}^\infty$ is strictly increasing, $\|A\| > \|A_j\|$ for all $j \in \mathbb{N}$, and since $\mathcal{S}$ is closed, A is an element of $\mathcal{S}$. By Corollary 3.4, there exist a natural number m and an n-tuple $(x_1,\ldots,x_n) \in \mathrm{Ext}_{n,m}$ where A is norm-attaining. Also by Corollary 3.4, for any m_j bigger than or equal to m, A is norm-attaining at an n-tuple in Ext_{n,m_j}. Let us fix m_j bigger than m. Then A_{j+1} is not norm-attaining at $(x_1,\ldots,x_n)$, and hence

$$\begin{aligned}
\|A\| &= \big|A(x_1,\ldots,x_n)\big| \\
&\le \big|A_{j+1}(x_1,\ldots,x_n)\big| \\
&\qquad + \big|(A - A_{j+1})(x_1,\ldots,x_n)\big| \\
&\le \big|A_{j+1}(x_1,\ldots,x_n)\big| + \epsilon_{j+1}/2 \\
&\le \|A_{j+1}\| - \delta_{j+1} + \epsilon_{j+1}/2
\end{aligned}$$

$$\leq \|A_{j+1}\| - \epsilon_{j+1} + \epsilon_{j+1}/2$$
$$< \|A_{j+1}\| < \|A\|,$$

which is a contradiction. Hence $\mathcal{S}$ cannot be closed. □

3.1. **Multilinear Arens extensions on ℓ_1.** In 1951, Arens [3] found a natural way to extend a continuous bilinear mapping between Banach spaces. This procedure was generalized by Aron and Berner [5] to arbitrary multilinear mappings, and was simplified by Davie and Gamelin [13] by using Goldstine's theorem and limits in the *weak-star topology* denoted by $w(X^{**}, X^*)$. In this method, the Arens extension of an n-linear form $A \in \mathcal{L}(X_1, \ldots, X_n)$ at a point $(x_1^{**}, \ldots, x_n^{**})$ of $X_1^{**} \times \cdots \times X_n^{**}$ associated to a permutation $\sigma \in \Sigma_n$ is defined by

$$A_\sigma(x_1^{**}, \ldots, x_n^{**}) = \lim_{d_{\sigma(1)}} \cdots \lim_{d_{\sigma(n)}} A(x_{d_1}, \ldots, x_{d_n}),$$

where $\{x_{d_i}\}_{d_i}$ is a bounded net in X_i ($\|x_{d_i}\| \leq \|x_i^{**}\|$ for all d_i) $w(X^*, X)$ convergent to $x_i^{**} \in X_i^{**}$ for $i = 1, \ldots, n$. The mapping A_σ is called an *Arens extension* of A, and there are $n!$ Arens extensions that may be different from each other.

Motivated by Lindenstrauss's result and using the Arens extensions to the second duals, Acosta [1] proved a Lindenstrauss-type result for bilinear forms. Afterward, in [6] the denseness of bilinear forms whose Arens extensions to the biduals attain their norms at the same point was established. The generalization of Lindenstrauss' result to n-linear vector-valued mappings was finally obtained in [2] in its strongest form, where the authors showed that, for any n Banach spaces, the set of n-linear forms such that all of their Arens extensions are norm-attaining is norm-dense.

It is important to remark that in the generalization of Lindenstrauss's result all the extensions attain their norm at the same point. However, there exist n-linear forms such that only some of its extensions are norm-attaining and the others are not. The first example of these multilinear forms was given in 2003 by Aron et al. [6], where the authors provided a bilinear form such that only one of its two Arens extensions attains its norm. This example was generalized to n-linear forms and subsets of Σ_n by Falcó et al. [15]. The authors obtained an n-linear form on ℓ_1 such that only the extensions associated to an arbitrary but fixed subset of Σ_n are norm-attaining.

Theorem 3.10 ([15, Theorem 13])**.** *Given a subset $P \subseteq \Sigma_n$, there exists an n-linear form $A(P) \in \mathcal{L}(^n\ell_1)$ with $\|A(P)\| = 1$ such that $A(P)_\sigma$ is norm-attaining if and only if $\sigma \in P$.*

To be more specific, the n-linear form they used was

$$A(P)(e_{k_1}, e_{k_2}, \ldots, e_{k_n}) = \begin{cases} (\frac{k_1}{k_1+1})^n & \text{if } k_1 = k_2 = \cdots = k_n, \\ 0 & \text{otherwise,} \end{cases}$$

if P is the empty set, and

$$A(P)(e_{k_1}, e_{k_2}, \ldots, e_{k_n}) = \begin{cases} \prod_{i=1}^n \frac{k_i}{k_i+1} & \text{if } \exists \sigma \in P, k_{\sigma(1)} \leq \cdots \leq k_{\sigma(n)}, \\ 0 & \text{otherwise,} \end{cases}$$

if P is not empty.

This asymmetry between the Arens extensions reveals the importance of the stronger condition of attaining their norms simultaneously in the result of Acosta et al. [6].

Here we will prove that, for a fixed subset P of Σ_n, if we denote by

$$NA_P\big(\mathcal{L}(^n\ell_1)\big) = \big\{A \in \mathcal{L}(^n\ell_1) : A_\sigma \text{ is norm-attaining if and only if } \sigma \in P\big\},$$

then $NA_P(\mathcal{L}(^n\ell_1))$ is spaceable. To prove the spaceability of these multilinear forms, in Theorem 3.12 we will need to make use of the following lemma.

Lemma 3.11. *For every natural number n, consider n sequences of nonnegative numbers $\{x_i(t)\}_{t=1}^\infty$, $i = 1, \ldots, n$ with $\sum_{t=1}^\infty x_i(t) \le 1$. If*

$$\sum_{t=1}^\infty x_1(t) \cdots x_n(t) > \delta$$

for some $1 > \delta > 3/4$, then there exists only one natural number m_0 such that $x_1(m_0), \ldots, x_n(m_0) > \delta$.

Proof. First, we will prove that there exists m_1 such that $x_1(m_1) > \delta$. Assume that this is not the case. Then $x_1(t) \le \delta$ for all $t \in \mathbb{N}$. Therefore,

$$\begin{aligned}
\delta &< \sum_{t=1}^\infty x_1(t) \cdots x_n(t) \\
&\le \delta \sum_{t=1}^\infty x_2(t) \cdots x_n(t) \\
&\le \delta \sum_{t_2,\ldots,t_n=1}^\infty x_2(t_2) \cdots x_n(t_n) \\
&= \delta\Big(\sum_{t_2=1}^\infty x_2(t_2)\Big) \cdots \Big(\sum_{t_n=1}^\infty x_n(t_n)\Big) \\
&\le \delta,
\end{aligned}$$

which is a contradiction. Therefore, there exists m_1 with $x_1(m_1) > \delta$. Note that, since $\sum_{t=1}^\infty x_1(t) \le 1$, m_1 is unique.

We can repeat the same argument to see that, for $i = 1, \ldots, n$, there exists only one m_i such that $x_i(m_i) > \delta$. It only remains to see that $m_1 = \cdots = m_n$ and take $m_0 = m_1$. Assume that $m_i \ne m_j$ for some $1 \le i, j \le n$, $i \ne j$. Then

$$\begin{aligned}
\delta < \sum_{t=1}^\infty x_1(t) \cdots x_n(t) &\le \sum_{t=1}^\infty x_i(t) x_j(t) \\
&= x_i(m_i)x_j(m_i) + x_i(m_j)x_j(m_j) + \sum_{\substack{t\in\mathbb{N},\\ t\ne m_i,m_j}} x_i(t)x_j(t) \\
&\le x_i(m_i)1/4 + 1/4x_j(m_j) + \sum_{\substack{t_i,t_j\in\mathbb{N},\\ t_i,t_j\ne m_i,m_j}} x_i(t_i)x_j(t_j)
\end{aligned}$$

$$\leq 1/2 + \Big(\sum_{\substack{t_i \in \mathbb{N}, \\ t_i \neq m_i, m_j}} x_i(t_i) \Big) \Big(\sum_{\substack{t_j \in \mathbb{N}, \\ t_j \neq m_i, m_j}} x_j(t_j) \Big)$$
$$< 1/2 + 1/16 < 3/4 < \delta,$$

which is a contradiction. Therefore, considering $m_0 = m_1$, we have the result and m_0 is unique. □

Now we present one of our main results of this paper.

Theorem 3.12. *For every set $P \subseteq \Sigma_n$, the set $NA_P(\mathcal{L}(^n\ell_1))$ is spaceable.*

Proof. Let us fix a set $P \subseteq \Sigma_n$, and consider a disjoint partition of the natural numbers into an infinite number of infinite sets $\{\mathbb{N}_m\}_{m=1}^{\infty}$; that is, $\bigcup_m \mathbb{N}_m = \mathbb{N}$ and $\mathbb{N}_m \cap \mathbb{N}_{m'} = \emptyset$ iff $m \neq m'$ with $\mathbb{N}_m$ being infinite for $m = 1, 2, \ldots$.

The sets $\mathbb{N}_m$ are naturally ordered by the order defined on the natural numbers. Therefore, we can assume that $\mathbb{N}_m = \{(m,t)\}_{t=1}^{\infty}$ with $(m,t) < (m,k)$ if and only if $t < k$.

Let

$$A(m)(e_{k_1}, \ldots, e_{k_n}) = \begin{cases} A(P)(e_{t_1}, \ldots, e_{t_n}) & \text{if } k_i = (m, t_i) \in \mathbb{N}_m, i = 1, \ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

where $A(P)$ is the n-linear form of Theorem 3.10.

Let $\mathcal{S}$ be the vector space defined by

$$\mathcal{S} := \Big\{ B = \sum_{m \in \mathbb{N}} \lambda_m A(m) : \lambda_m \in \mathbb{R}, \lim_{m \to \infty} \lambda_m = 0 \Big\}.$$

For every B in $\mathcal{S}$, we have

$$\begin{aligned} \|B\| &= \sup_{x_1, \ldots, x_n \in B_{\ell_1}} \big| B(x_1, \ldots, x_n) \big| \\ &= \sup_{k_1, \ldots, k_n} \big| B(e_{k_1}, \ldots, e_{k_n}) \big| \\ &= \sup_{m \in \mathbb{N}} \sup_{k_1, \ldots, k_n \in \mathbb{N}} \big| \lambda_m A(m)(e_{k_1}, \ldots, e_{k_n}) \big| \\ &= \sup_{m \in \mathbb{N}} |\lambda_m| \big\| A(P) \big\| = \sup_{m \in \mathbb{N}} |\lambda_m| \\ &= \max_{m \in \mathbb{N}} |\lambda_m|. \end{aligned}$$

Now, we prove that, for all B in $\mathcal{S} \setminus \{0\}$, B_σ is norm-attaining if and only if $\sigma \in P$. Let us fix $B \in \mathcal{S} \setminus \{0\}$ of norm 1.

First, we prove that B_σ is norm-attaining for $\sigma \in P$. Let us fix σ in P. Since B has norm 1, there exists m_0 with $\|B\| = |\lambda_{m_0}| = 1$. For simplicity, we will assume that $\lambda_{m_0} = 1$. Then, since $\sigma \in P$,

$$\lim_{(m_0, k_{\sigma(1)}) \to \infty} \cdots \lim_{(m_0, k_{\sigma(n)}) \to \infty} B(e_{(m_0, k_1)}, \ldots, e_{(m_0, k_n)})$$
$$= \lim_{k_{\sigma(1)} \to \infty} \cdots \lim_{k_{\sigma(n)} \to \infty} A(P)(e_{k_1}, \ldots, e_{k_n}) = 1.$$

Hence, considering a weak-star cluster point x^{**} of the sequence $\{e_{(m_0,k)}\}_{k=1}^{\infty}$, we obtain

$$B_\sigma(x^{**},\dots,x^{**})=1.$$

Thus B_σ is norm-attaining.

Now we see that, if $\sigma\notin P$, then B_σ is not norm-attaining. Fix $\sigma\notin P$, and assume that B_σ attains its norm. Then there exists $(x_1^{**},\dots,x_n^{**})\in B_{\ell_1^{**}}\times\cdots\times B_{\ell_1^{**}}$ with $B_\sigma(x_1^{**},\dots,x_n^{**})=1$. Let $\{x_{d_i}\}_{d_i\in D_i}$ be nets in B_{ℓ_1} weak-star convergent to x_i^{**} for $i=1,\dots,n$. To simplify the notation, we will assume that σ is the identity permutation denoted by Id. Therefore,

$$\lim_{d_1}\cdots\lim_{d_n}B(x_{d_1},\dots,x_{d_n})=B_{\mathrm{Id}}(x_1^{**},\dots,x_n^{**})=1.$$

Fix $1>\delta>3/4$. Then there exists $\alpha_1\in D_1$ with

$$\lim_{d_2}\cdots\lim_{d_n}B(x_{d_1},x_{d_2},\dots,x_{d_n})>\delta$$

for all $d_1>\alpha_1$. For fixed $d_1>\alpha_1$, there exists $\alpha_{1,2}\in D_2$ with

$$\lim_{d_3}\cdots\lim_{d_n}B(x_{d_1},x_{d_2},x_{d_3},\dots,x_{d_n})>\delta$$

for all $d_2>\alpha_{1,2}$. Fix d_2 with $d_2>\alpha_{1,2}$.

In this way, for fixed $d_1,\dots,d_i$, for $1\le i<n-1$, there exists $\alpha_{1,\dots,i+1}$ with

$$\lim_{d_{i+2}}\cdots\lim_{d_n}B(x_{d_1},x_{d_2},\dots,x_{d_n})>\delta$$

for all $d_{i+1}>\alpha_{1,\dots,i+1}$. Fix d_{i+1} with $d_{i+1}>\alpha_{1,\dots,i+1}$.

After $n-1$ steps, for fixed $d_1,\dots,d_{n-1}$, there exists $\alpha_{1,\dots,n}$ with

$$B(x_{d_1},x_{d_2},\dots,x_{d_n})>\delta$$

for all $d_n>\alpha_{1,\dots,n}$. Fix d_n with $d_n>\alpha_{1,\dots,n}$.

Then

$$B(x_{d_1},\dots,x_{d_n})>\delta.$$

Now, for every $m\in\mathbb{N}$, define $\pi_m:\ell_1\mapsto\ell_1$ by $(\pi_m(x))(t)=x(t)$ if $t\in\mathbb{N}_m$ and $(\pi_m(x))(t)=0$ if $t\notin\mathbb{N}_m$.

Observe that

$$\begin{aligned}\delta<B(x_{d_1},\dots,x_{d_n})&=B\Big(\sum_{m_1\in\mathbb{N}}\pi_{m_1}(x_{d_1}),\dots,\sum_{m_n\in\mathbb{N}}\pi_{m_n}(x_{d_n})\Big)\\&=\sum_{m_1,\dots,m_n\in\mathbb{N}}B\big(\pi_{m_1}(x_{d_1}),\dots,\pi_{m_n}(x_{d_n})\big)\\&=\sum_{m\in\mathbb{N}}B\big(\pi_m(x_{d_1}),\dots,\pi_m(x_{d_n})\big)\\&=\sum_{m\in\mathbb{N}}A(m)\big(\pi_m(x_{d_1}),\dots,\pi_m(x_{d_n})\big)\\&\le\sum_{m\in\mathbb{N}}\big\|\pi_m(x_{d_1})\big\|\cdots\big\|\pi_m(x_{d_n})\big\|.\end{aligned}$$

Therefore, by Lemma 3.11, there exists only one natural number m_0 such that $\|\pi_{m_0}(x_{d_1})\|, \dots, \|\pi_{m_0}(x_{d_n})\| > \delta$.

Now, if we consider another $\tilde{d}_n > \alpha_{1,\dots,n}$, by the same argument, there exists only one natural number $\tilde{m}_0$ with

$$\big\|\pi_{\tilde{m}_0}(x_{d_1})\big\|, \dots, \big\|\pi_{\tilde{m}_0}(x_{\tilde{d}_n})\big\| > \delta.$$

But, since $1 \geq \|x_{d_1}\| = \sum_{m\in\mathbb{N}} \|\pi_m(x_{d_1})\|$, and $\|\pi_{m_0}(x_{d_1})\|, \|\pi_{\tilde{m}_0}(x_{d_1})\| > \delta > 3/4$, we have $\tilde{m}_0 = m_0$.

Therefore, for fixed $d_1, \dots, d_{n-1}$, there exists m_0 with

$$\big\|\pi_{m_0}(x_{d_1})\big\|, \dots, \big\|\pi_{m_0}(x_{d_n})\big\| > \delta$$

for all $d_n \in D_n$ with $d_n > \alpha_{1,\dots,n}$.

Now, fix $d_1, \dots, d_{n-2}$, and consider $\tilde{d}_{n-1} \in D_{n-1}$ with $\tilde{d}_{n-1} > \alpha_{1,\dots,n-1}$. Then there exists $\tilde{\alpha}_{1,\dots,n}$ with

$$B(x_{d_1}, \dots, x_{\tilde{d}_{n-1}}, x_{d_n}) > \delta$$

for all $d_n > \tilde{\alpha}_{1,\dots,n}, \alpha_{1,\dots,n}$.

Fix $d_n > \alpha_{1,\dots,n}, \tilde{\alpha}_{1,\dots,n}$. Arguing as before, we find $\tilde{m}_0$ with

$$\big\|\pi_{\tilde{m}_0}(x_{d_1})\big\|, \dots, \big\|\pi_{\tilde{m}_0}(x_{\tilde{d}_{n-1}})\big\|, \big\|\pi_{\tilde{m}_0}(x_{d_n})\big\| > \delta$$

for all $d_n \in D_n$ with $d_n > \tilde{\alpha}_{1,\dots,n}, \alpha_{1,\dots,n}$. But, as before, we have $1 \geq \|x_{d_n}\| = \sum_{m\in\mathbb{N}} \|\pi_m(x_{d_n})\|$, and $\|\pi_{m_0}(x_{d_n})\|, \|\pi_{\tilde{m}_0}(x_{d_n})\| > \delta > 3/4$, and hence $\tilde{m}_0 = m_0$.

Therefore, if $d_1, \dots, d_{n-2}$ are fixed, then

$$\big\|\pi_{m_0}(x_{d_1})\big\|, \dots, \big\|\pi_{m_0}(x_{d_n})\big\| > \delta$$

for every $d_{n-1} \in D_{n-1}$ with $d_{n-1} > \alpha_{1,\dots,n-1}$ and every $d_n \in D_n$ with $d_n > \alpha_{1,\dots,n}$, where $\alpha_{1,\dots,n-1}$ depends on $d_1, \dots, d_{n-2}$ and $\alpha_{1,\dots,n}$ depends on $d_1, \dots, d_{n-1}$.

We will do one more case for the sake of completeness. Fix $d_1, \dots, d_{n-3}$, and consider $\tilde{d}_{n-2} \in D_{n-2}$ with $\tilde{d}_{n-2} > \alpha_{1,\dots,n-2}$. Then there exists $\tilde{\alpha}_{1,\dots,n-1}$ with

$$\lim_{d_n} B(x_{d_1}, \dots, x_{\tilde{d}_{n-2}}, x_{\tilde{d}_{n-1}}, x_{d_n}) > \delta$$

for all $\tilde{d}_{n-1} > \tilde{\alpha}_{1,\dots,n-1}, \alpha_{1,\dots,n-1}$. Then, for fixed $\tilde{d}_{n-1}$, there exists $\tilde{\alpha}_{1,\dots,n}$ with

$$B(x_{d_1}, \dots, x_{\tilde{d}_{n-2}}, \dots, x_{\tilde{d}_{n-1}}, \dots, x_{\tilde{d}_n}, \dots, x_{d_n}) > \delta$$

for all $\tilde{d}_n > \tilde{\alpha}_{1,\dots,n}, \alpha_{1,\dots,n}$.

Fix $\tilde{d}_n > \alpha_{1,\dots,n}, \tilde{\alpha}_{1,\dots,n}$. Arguing as before, we find $\tilde{m}_0$ with

$$\big\|\pi_{\tilde{m}_0}(x_{d_1})\big\|, \dots, \big\|\pi_{\tilde{m}_0}(x_{\tilde{d}_{n-2}})\big\|, \big\|\pi_{\tilde{m}_0}(x_{\tilde{d}_{n-1}})\big\|, \big\|\pi_{\tilde{m}_0}(x_{d_n})\big\| > \delta$$

for all $d_n \in D_n$ with $d_n > \tilde{\alpha}_{1,\dots,n}$. But, as before, $1 \geq \|x_{d_n}\| = \sum_{m\in\mathbb{N}} \|\pi_m(x_{d_n})\|$, and $\|\pi_m(x_{d_n})\|, \|\pi_{m_0}(x_{\tilde{d}_n})\| > \delta > 3/4$, and hence $\tilde{m}_0 = m_0$.

Therefore, if $d_1, \dots, d_{n-3}$ are fixed, then

$$\big\|\pi_{m_0}(x_{d_1})\big\|, \dots, \big\|\pi_{m_0}(x_{d_n})\big\| > \delta$$

for every $d_{n-2} \in D_{n-2}$ with $d_{n-2} > \alpha_{1,\dots,n-2}$, every $d_{n-1} \in D_{n-1}$ with $d_{n-1} > \alpha_{1,\dots,n-1}$, and every $d_n \in D_n$ with $d_n > \alpha_{1,\dots,n}$, where $\alpha_{1,\dots,n-2}$ depends on $d_1, \dots, d_{n-3}$, $\alpha_{1,\dots,n-1}$ depends on $d_1, \dots, d_{n-2}$, and $\alpha_{1,\dots,n}$ depends on $d_1, \dots, d_{n-1}$.

The same argument can be repeated to get

$$\big\|\pi_{m_0}(x_{d_1})\big\|, \dots, \big\|\pi_{m_0}(x_{d_n})\big\| > \delta$$

for every $d_1 \in D_1$ with $d_1 > \alpha_1$, $d_2 \in D_2$ with $d_2 > \alpha_{1,2}, \dots, d_n \in D_n$ with $d_n > \alpha_{1,\dots,n}$, where $\alpha_{1,\dots,i}$ depends on $d_1, \dots, d_{i-1}$ for $i = 2, \dots, n$.

Since this holds for every δ with $3/4 < \delta < 1$, we have

$$\lim_{d_i} \big\|\pi_{m_0}(x_{d_i}) - x_{d_i}\big\| = 0,$$

and hence $\{\pi_{m_0}(x_{d_i})\}_{d_i \in D_i}$ weak-star converges to x_i^{**} for $i = 1, \dots, n$.

Now, consider the map $\pi : \ell_1 \mapsto \ell_1$ defined by $(\pi(x))(t) = x(m_0, t)$. Since π is $\|\cdot\|$-$\|\cdot\|$-continuous, π is ω-ω-continuous, and the canonical extension $\widehat{\pi}$ defined from ℓ_1^{**} into ℓ_1^{**} is ω^*-ω^*-continuous. Therefore, as $\{x_{d_i}\}_{d_i \in D_i}$ is ω^*-convergent to x_i^{**}, we have that $\{\pi(x_{d_i})\}_{d_i \in D_i}$ is ω^*-convergent to $\widehat{\pi}(x_i^{**})$, and since π has norm 1, $\widehat{\pi}(x_i^{**}) \in B_{\ell_1^{**}}$.

Then, by using the multilinear form $A(m)$ defined at the beginning of the proof and the multilinear form $A(P)$ of Theorem 3.10, we have

$$\begin{aligned}
A(P)_{\mathrm{Id}}\big(\hat{\pi}(x_1^{**}), \dots, \hat{\pi}(x_n^{**})\big) &= \lim_{d_1} \cdots \lim_{d_n} A(P)\big(\pi(x_{d_1}), \dots, \pi(x_{d_n})\big) \\
&= \lim_{d_1} \cdots \lim_{d_n} A(m_0)(x_{d_1}, \dots, x_{d_n}) \\
&= \lim_{d_1} \cdots \lim_{d_n} B\big(\pi_{m_0}(x_{d_1}), \dots, \pi_{m_0}(x_{d_n})\big) \\
&= B_{\mathrm{Id}}(x_1^{**}, \dots, x_n^{**}) = 1.
\end{aligned}$$

Since $\mathrm{Id} \notin P$, $A(P)_{\mathrm{Id}}$ is not norm-attaining, and this is a contradiction. Therefore, B_{Id} is not norm-attaining. Since $\mathcal{S}$ is an infinite-dimensional Banach space, this concludes the proof. □

4. Polynomials

To finish, we want to show a generalization of Theorem 2.6 and Corollary 2.7. For this, we will use the definition of the transpose of a polynomial introduced by Aron and Schottenloher in [8].

Given a polynomial P of degree n between the Banach spaces X and Y, $P \in \mathcal{P}(^nX; Y)$, the transpose of the polynomial is defined by

$$\begin{aligned}
P^* : Y^* &\longrightarrow \mathcal{P}(^nX), \\
y^* &\rightsquigarrow P^*(y^*) : X \longrightarrow \mathbb{K}, \\
x &\rightsquigarrow y^*\big(P(x)\big).
\end{aligned}$$

Then

$$\|P^*\| = \sup_{y^* \in B_{Y^*}} \big\|P^*(y^*)\big\| = \sup_{y^* \in B_{Y^*}} \sup_{x \in B_X} \big|y^*\big(P(x)\big)\big| = \|P\|.$$

Theorem 4.1. *If X and Y are Banach spaces and n is a natural number, then the transpose of every compact n-homogeneous polynomial from X to Y is norm-attaining.*

Proof. Given a compact polynomial $P \in \mathcal{P}(^nX;Y)$, since P is compact, there exists a point $y_0 \in \overline{P(B_X)}$ such that $\|y_0\| = \|P\|$. Then, as a consequence of the Hahn–Banach theorem, there exists a linear and continuous functional $y_0^* \in Y^*$ of norm 1 with $|y_0^*(y_0)| = \|y_0\|$. Then

$$\begin{aligned} \|P^*\| &= \|P\| = \|y_0\| \\ &= \big|y_0^*(y_0)\big| \\ &\leq \sup_{y\in\overline{P(B_X)}} \big|y_0^*(y)\big| \\ &= \sup_{y\in P(B_X)} \big|y_0^*(y)\big| \\ &= \sup_{x\in B_X} \big|y_0^*\big(P(x)\big)\big| \\ &= \big\|P^*(y_0^*)\big\| \\ &\leq \sup_{y^*\in B_{Y^*}} \big\|\big(P^*(y^*)\big)\big\| = \|P^*\|. \end{aligned}$$

Therefore, $\|P^*\| = \|P^*(y_0^*)\|$. Hence P^* is norm-attaining. □

As a consequence of this result and by using the fact that the class of compact polynomials of degree n between two infinite-dimensional Banach spaces is a closed infinite-dimensional vector space, we obtain the following result.

Corollary 4.2. *Given two infinite-dimensional Banach spaces X and Y, the set of all n-homogeneous continuous polynomials from X to Y whose transpose is norm-attaining is spaceable.*

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