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NONCOMMUTATIVE HARDY–LORENTZ SPACES ASSOCIATED WITH SEMIFINITE SUBDIAGONAL ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a maximal subdiagonal algebra of semifinite von Neumann algebra $\mathcal{M}$. For $0 < p \leq \infty$, we define the noncommutative Hardy–Lorentz spaces $H^{p,\omega}(\mathcal{A})$, and give some properties of these spaces. We obtain that the Herglotz maps are bounded linear maps from $\Lambda^p_\omega(\mathcal{M})$ into $\Lambda^p_\omega(\mathcal{M})$, and with this result we characterize the dual spaces of $H^{p,\omega}(\mathcal{A})$ for $1 < p < \infty$. We also present the Hartman–Wintner spectral inclusion theorem of Toeplitz operators and Pisier's interpolation theorem for this case.

1. Introduction

Let $\mathbb{T}$ be the unit circle of a complex plane equipped with a normalized Lebesgue measure dm. We denote by $H^p(\mathbb{T})$ the usual Hardy spaces on $\mathbb{T}$, the space of functions on the unit circle which are in $L^p(\mathbb{T})$ with respect to the Lebesgue measure and whose negative Fourier coefficients vanish. These spaces have played an important role in modern analysis and prediction theory.

In the setting of operator algebraists, a noncommutative version of H_p spaces was given by Arveson. In 1967, Arveson [3] introduced the concept of maximal subdiagonal algebras $\mathcal{A}$ of a von Neumann algebra $\mathcal{M}$, unifying analytic function spaces and nonself-adjoint operator algebras. In the case that $\mathcal{M}$ has a finite trace, $H^p(\mathcal{A})$ may be defined to be the closure of $\mathcal{A}$ in the noncommutative L_p space $L^p(\mathcal{M})$. Subsequently, Arveson's pioneering work has been extended to different cases by several authors. For example, Marsalli and West [18] obtained a

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series of results including a Riesz factorization theorem, a Herglotz transform, and the dual relations between $H^p(\mathcal{A})$ and $H^q(\mathcal{A})$, and Bekjan and Xu [6] presented the more general form of a Riesz and Szegö type factorization for these spaces. Recently, the noncommutative H_p spaces have been developed by Blecher, Bekjan, Labuschagne, Xu, and their coauthors in a series of papers. On the other hand, it is proved by Ji [13] that a semifinite subdiagonal algebra $\mathcal{A}$ is automatically maximal. This eventually led to the definition of noncommutative Hardy spaces associated with semifinite subdiagonal algebras by Bekjan [4]. In this setting, Bekjan [4] obtain that the conjugation and Herglotz maps are bounded linear maps from $L^p(\mathcal{M})$ into $L^p(\mathcal{M})$ for $1 < p < \infty$, the dual relations between $H^p(\mathcal{A})$ and $H^q(\mathcal{A})$, and Pisier's interpolation theorem and so on.

In this article we introduce the noncommutative Hardy–Lorentz spaces associated with semifinite subdiagonal algebras. If $\omega \equiv 1$, the noncommutative Hardy–Lorentz spaces $H^{p,\omega}(\mathcal{A})$ correspond to the noncommutative Hardy spaces $H^p(\mathcal{A})$. By adapting the techniques in [4], [19], and [18], we obtain some properties of these spaces, which include some theorems about Herglotz maps, dual spaces, and Toeplitz operators, and Pisier's interpolation theorem for this case.

2. Preliminaries

Let (Ω, Σ, ν) be a complete σ-finite measure space, and let $L_0(\Omega)$ be the space of all classes of ν-measurable functions defined on Ω. Let $f \in L_0(\Omega)$. Recall that the distribution function of f is defined as

$$d_f(s) = \nu\big(\{t \in \Omega : |f(t)| > s\}\big), \quad s > 0,$$

and its nonincreasing rearrangement is defined as

$$f^*(t) = \inf\{s > 0 : d_f(s) \le t\}, \quad t > 0.$$

It will be sometimes convenient to write $L^p = L^p(0,\infty), 0 < p \le \infty$, and $L_0 = L_0(0,\infty)$ for brevity. When ω is a nonnegative, locally integrable function on $(0,\infty)$ and not identically zero, we say that ω is a weight. For a given weight ω, let $W(t) = \int_0^t \omega(s)\,ds < \infty, t > 0$. We write $W \in \Delta_2$ if $W(2t) \le CW(t), t \ge 0$, holds for some constant $C > 0$.

Let $0 < p \le \infty$. If $(\Omega, \Sigma, \nu) = (\mathbb{R}, \Sigma, \omega(t)\,dt)$, we write $L^p(\omega)$ instead of $L^p(\Omega)$. If the measure $d\nu(t) = \omega(t)\,dt$, we will write d_f^ω and f_ω^* instead of d_f and f^*, respectively.

Let ω be a weight function and let $0 < p < \infty$. We write $\omega \in \mathcal{B}_p$ if there exists some constant C such that

$$\int_r^\infty \frac{\omega(t)}{t^p}\,dt \le C\frac{1}{r^p}\int_0^r \omega(t)\,dt, \quad \forall r > 0,$$

and we write $\omega \in \mathcal{B}_\infty^*$ if there exists some constant C such that

$$\int_0^r \Big[\frac{1}{t}\int_0^t \omega(s)\,ds\Big]\,dt \le C\int_0^r \omega(t)\,dt, \quad \forall r > 0.$$

See [2], [1], and [7] for more information on these weight functions.

For $f \in L_0$ and $0 < p < \infty$, let

$$\|f\|_{\Lambda^p_\omega} = \Big(\int_0^\infty f^*(t)^p \omega(t)\,dt\Big)^{\frac{1}{p}} < \infty.$$

We define the Lorentz space Λ^p_ω as the set of all $f \in L_0$ such that $\|f\|_{\Lambda^p_\omega} < \infty$; that is, $\Lambda^p_\omega = \{f \in L_0 : f^* \in L^p(\omega)\}$. Similarly, we define the Lorentz space Λ^∞_ω as the set of all $f \in L_0$ such that $f^* \in L^\infty(\omega)$. If $W \in \Delta_2$, then $\Lambda^\infty_\omega = L^\infty$.

Let $0 < p < \infty$. The space Γ^p_ω is then defined as the set of all $f \in L_0$ such that

$$\|f\|_{\Gamma^p_\omega} = \Big(\int_0^\infty f^{**}(t)^p \omega(t)\,dt\Big)^{\frac{1}{p}} < \infty,$$

where $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$. For any $0 < p < \infty$, it is well known that Λ^p_ω is a quasi-Banach space if and only if $W(t) \in \Delta_2$. For $0 < p < \infty$, we know that $\Gamma^p_\omega = \Lambda^p_\omega$ if and only if $\omega \in \mathcal{B}_p$. For further results about these spaces, the reader is referred to [2], [7], [14], and [15].

In what follows, we will keep all previous notation throughout the paper, and ω *will always denote a weight function on* $(0,\infty)$ *with* $\omega \notin L_1$ *and* $W \in \Delta_2$.

For $0 < s < \infty$, we define the dilation operator D_s on Λ^p_ω by

$$(D_s f)(t) = f\Big(\frac{t}{s}\Big), \quad 0 < s < \infty, 0 \le t < \infty.$$

Define the lower Boyd indices $\alpha_{\Lambda^p_\omega}$ and the upper Boyd indices $\beta_{\Lambda^p_\omega}$ of Λ^p_ω by

$$\alpha_{\Lambda^p_\omega} = \lim_{s\to\infty} \frac{\log s}{\log \|D_s\|} \qquad \text{and} \qquad \beta_{\Lambda^p_\omega} = \lim_{s\to 0^+} \frac{\log s}{\log \|D_s\|}.$$

It follows from [1] that

$$\alpha_{\Lambda^p_\omega} = \lim_{t\to\infty} \frac{\log t}{\log \overline{W}^{\frac{1}{p}}(t)} \qquad \text{and} \qquad \beta_{\Lambda^p_\omega} = \lim_{t\to 0^+} \frac{\log t}{\log \overline{W}^{\frac{1}{p}}(t)},$$

where $\overline{W}(t) = \sup_{s>0} \frac{W(st)}{W(s)}, t > 0$. It is clear that $0 \le \alpha_{\Lambda^p_\omega} \le \beta_{\Lambda^p_\omega} \le \infty$ and $\alpha_{\Lambda^{rp}_\omega} = r\alpha_{\Lambda^p_\omega}, \beta_{\Lambda^{rp}_\omega} = r\beta_{\Lambda^p_\omega}, r > 0$. If Λ^p_ω is a Banach function space, then $1 \le \alpha_{\Lambda^p_\omega} \le \beta_{\Lambda^p_\omega} \le \infty$. For further results about Boyd indices of quasi-Banach spaces, the reader is referred to [1], [8], and [16].

Let $\mathcal{M}$ be a semifinite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a normal semifinite faithful trace τ. (We refer the reader to [21] and [26] for noncommutative integration.) We denote by $\mathcal{P}(\mathcal{M})$ the complete lattice of all projections in $\mathcal{M}$. For every $x \in \mathcal{M}$, there is a unique polar decomposition $x = u|x|$, where $|x| \in \mathcal{M}^+$ and u is a partial isometry operator. Let $r(x) = u^*u$, and let $l(x) = uu^*$. We call $r(x)$ and $l(x)$ the right and left supports of x, respectively. If x is self-adjoint, then $r(x) = l(x)$. This common projection is then said to be the support of x and denoted by $s(x)$. Let $S(\mathcal{M})^+ = \{x \in \mathcal{M}^+ : \tau(s(x)) < \infty\}$, and let $S(\mathcal{M})$ be the linear span of $S(\mathcal{M})^+$. The closed densely defined linear operator x in $\mathcal{H}$ with domain $D(x)$ is said to be *affiliated* with $\mathcal{M}$ if and only if $u^*xu = x$ for all unitary operators u which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If x is affiliated with $\mathcal{M}$, then we define its distribution function by $\lambda_t(x) = \tau(e_{(t,\infty)}(|x|))$, where $e_{(t,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (t,∞). The decreasing rearrangement of x is defined by $\mu_t(x) = \inf\{s > 0 : \lambda_s(x) \le t\}$. We will denote simply by $\lambda(x)$ and $\mu(x)$ the functions $t \to \lambda_t(x)$ and $t \to \mu_t(x)$, respectively. We call x *τ-measurable* if $\lambda_s(x) < \infty$ for some $s > 0$. The set of all τ-measurable operators will be denoted by $L_{00}(\mathcal{M})$. The set $L_{00}(\mathcal{M})$ is a $*$-algebra with the sum and product being the respective closure of the algebraic sum and product.

The measure topology in $L_{00}(\mathcal{M})$ is the vector space topology defined via the neighborhood base $\{V(\varepsilon,\delta) : \varepsilon, \delta > 0\}$, where

$$V(\varepsilon,\delta) = \big\{x \in L_{00}(\mathcal{M}) : \tau\big(e_{(\varepsilon,\infty)}\big(|x|\big)\big) \le \delta\big\}$$

and $e_{(\varepsilon,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (ε,∞). With respect to the measure topology, $L_{00}(\mathcal{M})$ is a complete topological $*$-algebra. For $0 < p < \infty$, let

$$L^p(\mathcal{M}) = \big\{x \in L_{00}(\mathcal{M}) : \|x\|_p := \tau\big(|x|^p\big)^{\frac{1}{p}} < \infty\big\}.$$

Then $(L^p(\mathcal{M}); \|\cdot\|_p)$ is a Banach (or quasi-Banach for $p < 1$) space. As usual, we put $L^\infty(\mathcal{M}) = \mathcal{M}$, and denote by $\|\cdot\|_\infty$ $(= \|\cdot\|)$ the usual operator norm.

Let $x \in L_{00}(\mathcal{M})$, and let $0 < p \le \infty$. We define

$$\|x\|_{p,\omega} := \|x\|_{\Lambda^p_\omega(\mathcal{M})} = \big\|\mu(x)\big\|_{\Lambda^p_\omega}. \tag{2.1}$$

The noncommutative Lorentz space $\Lambda^p_\omega(\mathcal{M})$ is defined as the space of all $x \in L_{00}(\mathcal{M})$ such that $\|x\|_{p,\omega} < \infty$. If $\omega \equiv 1$, then the noncommutative Lorentz space $\Lambda^p_\omega(\mathcal{M})$ is the usual noncommutative L_p space $L^p(\mathcal{M})$. It is well known that

$$\|x\|_{p,\omega} = \Big(\int_0^\infty pt^{p-1} W\big(\lambda_t(x)\big)\,dt\Big)^{\frac{1}{p}}.$$

Let $0 < p < \infty$. The space $\Gamma^p_\omega(\mathcal{M})$ is defined as the set of all $x \in L_{00}(\mathcal{M})$ such that $\|x\|_{\Gamma^p_\omega(\mathcal{M})} = \|\mu(x)\|_{\Gamma^p_\omega} < \infty$. For $0 < p < \infty$, we infer from Theorem 4 of [24] that $\Lambda^p_\omega(\mathcal{M})$ and $\Gamma^p_\omega(\mathcal{M})$ are quasi-Banach spaces.

We should introduce the Köthe dual spaces generalizing the definition that can be found in [7] in the context of classical Lorentz space $\Lambda^p_\omega, 0 < p \le \infty$. We define the Köthe dual space of $\Lambda^p_\omega(\mathcal{M})$ by

$$\Lambda^p_\omega(\mathcal{M})^\times = \big\{x \in L_{00}(\mathcal{M}) : \|x\|_{\Lambda^p_\omega(\mathcal{M})^\times} = \sup_{\|y\|_{p,\omega} \le 1} \tau\big(|xy|\big) < \infty\big\}.$$

If $x \in \Lambda^p_\omega(\mathcal{M})^\times$, then it is clear that

$$\|x\|_{\Lambda^p_\omega(\mathcal{M})^\times} = \sup_{\|y\|_{p,\omega} \le 1} \|xy\|_1 = \sup\big\{\big|\tau(xy)\big| : \|y\|_{p,\omega} \le 1\big\}.$$

Proposition 2.1.

(1) *Let $0 < p < \infty$. For $x \in \Lambda^p_\omega(\mathcal{M})^\times$, we have*

$$\|x\|_{\Lambda^p_\omega(\mathcal{M})^\times} = \big\|\mu_t(x)\big\|_{(\Lambda^p_\omega)^\times}.$$

Moreover, $(\Lambda^p_\omega)^\times(\mathcal{M}) = \Lambda^p_\omega(\mathcal{M})^\times$ is a noncommutative Banach function space.

(2) *Let* $\{e_i\}_{i\in\Lambda}$ *be an increasing family of projections in* $\mathcal{M}$ *and* $0<p<\infty$. *If* e_i *converges to* 1 *in the strong operator topology, then*

$$\|xe_i - x\|_{p,\omega} \to 0, \qquad \|e_i x - x\|_{p,\omega} \to 0, \quad \forall x \in \Lambda^p_\omega(\mathcal{M}).$$

(3) *If* $\omega \in \mathcal{B}_p \cap \mathcal{B}^*_\infty$ *and* $1<p<\infty$, *then*

$$\Lambda^p_\omega(\mathcal{M})^* = \Lambda^p_\omega(\mathcal{M})^\times = (\Lambda^p_\omega)^\times(\mathcal{M}) = \Gamma^q_{\widetilde{\omega}}(\mathcal{M}),$$

where $\widetilde{\omega}(t) = t^q W(t)^{-q}\omega(t), t>0, \frac{1}{p}+\frac{1}{q}=1$.

(4) *Let* $\{e_i\}_{i\in\Lambda}$ *be an increasing family of projections in* $\mathcal{M}$ *and* $1<p<\infty$. *If* e_i *converges to* 1 *in the strong operator topology and* $\omega \in \mathcal{B}_p\cap\mathcal{B}^*_\infty$, *then*

$$\|xe_i - x\|_{\Gamma^q_{\widetilde{\omega}}(\mathcal{M})} \to 0, \qquad \|e_i x - x\|_{\Gamma^q_{\widetilde{\omega}}(\mathcal{M})} \to 0, \quad \forall x \in \Gamma^q_{\widetilde{\omega}}(\mathcal{M}),$$

where $\widetilde{\omega}(t) = t^q W(t)^{-q}\omega(t), t>0, \frac{1}{p}+\frac{1}{q}=1$.

Proof. Since $\omega \in \mathcal{B}_p \cap \mathcal{B}^*_\infty$, it follows from Corollary 2.3 and Proposition 2.6 of [1] that $1<\alpha_{\Lambda^p_\omega} \le \beta_{\Lambda^p_\omega} < \infty$. Then Proposition 2.3.3 and Theorem 2.3.4 of [7] and Theorem 3.7 of [8] mean that $\Lambda^p_\omega(\mathcal{M})$ is an interpolation space for the couple $(L^1(\mathcal{M}),\mathcal{M})$. Therefore, by slightly modifying the proof of Proposition 2.4 and Proposition 2.6 in [11], we can prove (1)–(3) and omit the details.

(4) Since $\omega\in\mathcal{B}_p$, applying Proposition 1.7 and Corollary 1.12 of [15] gives $\beta_{\Gamma^q_{\widetilde{\omega}}}<\infty$. According to $\omega\in\mathcal{B}_p\cap\mathcal{B}^*_\infty$, we obtain $\Gamma^q_{\widetilde{\omega}} = (\Lambda^p_\omega)^\times$. Then we see that $\Gamma^q_{\widetilde{\omega}}$ is an Banach space and $1\le \alpha_{\Gamma^q_{\widetilde{\omega}}}$. By Theorem 3.2 of [8], there exist r_1, r_2 with $0< r_1 < \alpha_{\Gamma^q_{\widetilde{\omega}}} \le \beta_{\Gamma^q_{\widetilde{\omega}}} < r_2 < \infty$ such that $\Gamma^q_{\widetilde{\omega}}$ is an interpolation space for the couple (L^{r_1}, L^{r_2}). It follows that $d_f(s)<\infty$ holds for every $s>0$ and $f\in\Gamma^q_{\widetilde{\omega}}$. Therefore, any sequence (f_i) in $\Gamma^q_{\widetilde{\omega}}$ with $f_i\downarrow 0$ satisfying $f_i^{**}(t) = \frac{1}{t}\int_0^t f_i^*(s)\,ds \downarrow 0$, and so $\|f_i\|_{\Gamma^q_{\widetilde{\omega}}} \downarrow 0$. The proof can be done similarly to (2). The details are omitted. □

Remark 2.2.

(1) Let $1<p<\infty$ and $\omega\in\mathcal{B}_p\cap\mathcal{B}^*_\infty$. It follows from Theorem 2.4.9 and Corollary 2.4.23 of [7] that $(\Lambda^p_\omega)^*$ separate points if and only if $(\Lambda^p_\omega)^\times \neq \{0\}$ if and only if $\int_0^1 (\frac{t}{W(t)})^{\frac{1}{p-1}}\,dt < \infty$. If $\int_0^1 (\frac{t}{W(t)})^{\frac{1}{p-1}}\,dt < \infty$, by Proposition 2.1(3), then we deduce that $\Lambda^p_\omega(\mathcal{M})^*$ separates points. Since we deal further with dual space $(\Lambda^p_\omega)^*$ which separates points, we will assume in what follows that $\int_0^1 (\frac{t}{W(t)})^{\frac{1}{p-1}}\,dt < \infty$.

(2) Let $0<p<\infty$. Then $S(\mathcal{M})$ is dense in $\Lambda^p_\omega(\mathcal{M})$. Indeed, $S(\mathcal{M}) \subseteq \Lambda^p_\omega(\mathcal{M})$ is clear. Now let $x\in\Lambda^p_\omega(\mathcal{M})$. Then $\lim_{t\to\infty}\lambda_t(x) = 0$ and $\lim_{t\to\infty}\mu_t(x)=0$. We write $x_n = xe_{[0,n]}(|x|)$. It follows from Proposition 3.2 of [9] that $\mu_t(x)<\infty$ for all $t>0$. Therefore,

$$\mu_t(x-x_n) = \mu_t\big(xe_{(n,\infty)}\big(|x|\big)\big) \le \mu_t(x)\chi_{[0,\lambda_n(x)]}(t) \to 0, \quad n\to\infty$$

and $\mu_t(x-x_n)\le\mu_t(x)$. The dominated convergence theorem shows that

$$\|x-x_n\|_{\Lambda^p_\omega(\mathcal{M})} \to 0, \quad n\to\infty.$$

Thus $\mathcal{M} \cap \Lambda_\omega^p(\mathcal{M})$ is dense in $\Lambda_\omega^p(\mathcal{M})$. Let $y \in \Lambda_\omega^p(\mathcal{M})$. We write $e_n = e_{(\frac{1}{n},\infty)}(|y|)$ and $y_n = ye_n$. According to Lemma 2.5 and Proposition 3.2 of [9], we obtain $y_n \in S(\mathcal{M}) \subseteq \mathcal{M} \cap \Lambda_\omega^p(\mathcal{M})$,

$$\mu_t(y - y_n) = \mu_t\big(ye_{[0,\frac{1}{n}]}(|y|)\big) \leq \frac{1}{n}\mu_t\big(e_{[0,\frac{1}{n}]}(|y|)\big) \leq \frac{1}{n} \to 0, \quad n \to \infty,$$

and $\mu_t(y-y_n) = \mu_t(ye_n^\perp) \leq \mu_t(y)$. By the dominated convergence theorem, we deduce that $\|y-y_n\|_{\Lambda_\omega^p} \to 0, n \to \infty$, and so $S(\mathcal{M})$ is dense in $\Lambda_\omega^p(\mathcal{M})$.

Proposition 2.3. *For $t \in L_{00}(\mathcal{M})$ and $x \in \Lambda_\omega^p(\mathcal{M})$, let $L_t(x) = tx$, and let $R_t(x) = xt$.*

(1) *The operator L_t on $\Lambda_\omega^p(\mathcal{M})$ is bounded if and only if $t \in \mathcal{M}$. Moreover, $\|L_t\| = \|t\|$.*
(2) *The operator R_t on $\Lambda_\omega^p(\mathcal{M})$ is bounded if and only if $t \in \mathcal{M}$. Moreover, $\|R_t\| = \|t\|$.*
(3) *Let $\mathcal{R} = \{R_t : t \in \mathcal{M}\}$, and let $\mathcal{L} = \{L_t : t \in \mathcal{M}\}$. Then $\mathcal{R}$ and $\mathcal{L}$ are subalgebras of $\mathcal{B}(\Lambda_\omega^p(\mathcal{M}))$, the algebra of all bounded linear operators on $\Lambda_\omega^p(\mathcal{M})$.*
(4) *For $\varepsilon > 0$ and $0 < p < \infty$, we set $e_\varepsilon = e_{(\varepsilon,\infty)}(|t|)$ and $\Lambda_l^p(e_\varepsilon) = \{e_\varepsilon x : x \in \Lambda_\omega^p(\mathcal{M})\}$. Then $\Lambda_l^p(e_\varepsilon)$ is a closed subspace of $\Lambda_\omega^p(\mathcal{M})$.*

Proof. (1) If $t \in \mathcal{M}$, then it is clear that $\|L_t\| \leq \|t\|$. Conversely, for $t \in L_{00}(\mathcal{M})$, we write $e_n = e_{(n,\infty)}(|t|), n = 1, 2, \ldots$. By Proposition 21 of [26], there exists $n_0 \in \mathbb{N}^+$ such that $\tau(e_{n_0}) < \infty$, and so $\tau(e_n) < \infty$ for all $n \geq n_0$. Now, we suppose that t is an unbounded operator. Then the projection e_n has positive trace for infinitely many $n \in \mathbb{N}^+$. Without loss of generality we suppose that $0 < \tau(e_n) < \infty$ for all $n \in \mathbb{N}^+$. Since $\tau(e_n) \neq 0$ and $ne_n \leq |t|e_n$, we have

$$n\|e_n\|_{p,\omega} = \Big(\int_0^\infty \mu_s(ne_n)^p\omega(s)\,ds\Big)^{\frac{1}{p}} \leq \Big(\int_0^\infty \mu_s\big(|t|e_n\big)^p\omega(s)\,ds\Big)^{\frac{1}{p}} = \|L_te_n\|_{p,\omega},$$

a contradiction. Thus $t \in \mathcal{M}$. Moreover, if $t \in \mathcal{M}$, then, arguing as before,

$$C\big\|e_{(C,\infty)}\big(|t|\big)\big\|_{p,\omega} \leq \big\|L_te_{(C,\infty)}\big(|t|\big)\big\|_{p,\omega}$$

holds for all $0 < C < \|t\|$. Thus, $C < \|L_t\|$ holds for all $C \in (0, \|t\|)$. Hence $\|t\| \leq \|L_t\|$. This completes the proof. Similarly, (2) holds. (3) and (4) are clear. □

Proposition 2.4. *Let $\mathcal{M}$ be a finite von Neumann algebra. Put $\mathcal{R} = \{R_t : t \in \mathcal{M}\}$, and put $\mathcal{L} = \{L_t : t \in \mathcal{M}\}$. Then $\mathcal{R}$ and $\mathcal{L}$ are subalgebras of $\mathcal{B}(\Lambda_\omega^p(\mathcal{M}))$ which are each other's commutants.*

Proof. First it is clear that $\mathcal{R} \subseteq \mathcal{L}'$. Conversely, for any $y \in \mathcal{M}$, we deduce that

$$\|y\|_{p,\omega} = \Big(\int_0^{\tau(1)} \mu_t(y)^p\omega(t)\,dt\Big)^{\frac{1}{p}} \leq \|y\|W\big(\tau(1)\big)^{\frac{1}{p}}.$$

Since $\mathcal{M}$ is a finite von Neumann algebra, then $\|y\|_{p,\omega} \leq \|y\|W(\tau(1))^{\frac{1}{p}} < \infty$. This means that $\mathcal{M} \subseteq \Lambda_\omega^p(\mathcal{M})$. Let $T \in \mathcal{B}(\Lambda_\omega^p(\mathcal{M}))$ with $TL_t = L_tT$ for all $L_t \in \mathcal{L}$. If $y \in \mathcal{M}$, then we have

$$T(y) = T(L_y1) = L_yT(1) = yT(1). \tag{2.2}$$

Next we show that $T(1) \in \mathcal{M}$. If $T(1) \in \Lambda_\omega^p(\mathcal{M})$ and $T(1) \notin \mathcal{M}$, then the projection $e_n = e_{(n,(n+1)]}(|T(1)|)$ has positive trace for infinitely many $n \in \mathbb{N}^+$. Without loss of generality we suppose that $0 < \tau(e_n) < \infty$ for all $n \in \mathbb{N}^+$. Let $y_n = \frac{e_n}{\|e_n\|_{p,\omega}} U^*$, where $T(1) = U|T(1)|$ is the polar decomposition of $T(1)$. Then

$$e_n\big|T(1)\big| = e_{(n,(n+1)]}\big(\big|T(1)\big|\big)\big|T(1)\big| \geq n e_n.$$

By (2.2), we have that

$$\big\|T(y_n)\big\|_{p,\omega} = \big\|y_n T(1)\big\|_{p,\omega} = \frac{1}{\|e_n\|_{p,\omega}}\big\|e_n\big|T(1)\big|\big\|_{p,\omega} \geq n$$

holds for all $n \in \mathbb{N}^+$. Therefore T is a unbounded operator, a contradiction. Now, let $u_0 = T(1)$. By (2.2), we obtain $T(y) = R_{u_0}(y), y \in \mathcal{M}$. By Remark 2.2(2) and the fact that $T, R_{u_0} \in \mathcal{B}(\Lambda_\omega^p(\mathcal{M}))$, we deduce that $T(y) = R_{u_0}(y)$ holds for all $y \in \Lambda_\omega^p(\mathcal{M})$. Thus $T \in \mathcal{R}$; that is, $\mathcal{R} \supseteq \mathcal{L}'$. Therefore, $\mathcal{R} = \mathcal{L}'$. Similarly, $\mathcal{R}' = \mathcal{L}$. □

Theorem 2.5. *Let $0 < p < \infty$ and let $t = t^* \in \mathcal{M}$. Then L_t is compact if and only if $\Lambda_l^p(e_\varepsilon) := \{e_\varepsilon x : x \in \Lambda_\omega^p(\mathcal{M})\}$ is finite-dimensional for all $\varepsilon > 0$, where $e_\varepsilon = e_{(\varepsilon,\infty)}(|t|)$.*

Proof. First we assume that $L_t : \Lambda_\omega^p(\mathcal{M}) \to \Lambda_\omega^p(\mathcal{M})$ is compact, and we consider the bounded operator $i_{e_\varepsilon} : \Lambda_\omega^p(\mathcal{M}) \to \Lambda_l^p(e_\varepsilon)$ given by $i_{e_\varepsilon} x = e_\varepsilon x$. Clearly, i_{e_ε} is continuous, and hence the composition $i_{e_\varepsilon} \circ L_t : \Lambda_\omega^p(\mathcal{M}) \to \Lambda_l^p(e_\varepsilon)$ is a compact operator. Therefore, $i_{e_\varepsilon} \circ L_t|_{\Lambda_l^p(e_\varepsilon)}$ is a compact operator.

We claim that $\operatorname{Ran}(i_{e_\varepsilon} \circ L_t|_{\Lambda_l^p(e_\varepsilon)}) = \Lambda_l^p(e_\varepsilon)$. Indeed, if $y \in \operatorname{Ran}(i_{e_\varepsilon} \circ L_t|_{\Lambda_l^p(e_\varepsilon)})$, then there exists $x \in \Lambda_\omega^p(\mathcal{M})$ such that $y = e_\varepsilon t x \in \Lambda_l^p(e_\varepsilon)$; that is, $\operatorname{Ran}(i_{e_\varepsilon} \circ L_t|_{\Lambda_l^p(e_\varepsilon)}) \subseteq \Lambda_l^p(e_\varepsilon)$. Conversely, if $y \in \Lambda_l^p(e_\varepsilon)$, then there exists $x \in \Lambda_\omega^p(\mathcal{M})$ such that $y = e_\varepsilon x$. Let $t = u|t|$ be the polar decomposition of t, and let $|t| = \int_0^\infty \lambda\, de_\lambda$ be the spectral decomposition of $|t|$. We put $g(\lambda) = \frac{1}{\lambda}\chi_{(\varepsilon,\infty)}$. It is clear that $g(|t|) \in \mathcal{M}$. Let $z = e_\varepsilon g(|t|)u^* x$. Then $e_\varepsilon z = z$,

$$\|z\|_{p,\omega} = \big\|e_\varepsilon g\big(|t|\big)u^* x\big\|_{p,\omega} \leq \big\|g\big(|t|\big)\big\|\|x\|_{p,\omega} < \infty.$$

Thus

$$z = e_\varepsilon g\big(|t|\big)u^* x \in \Lambda_l^p(e_\varepsilon) \subseteq \Lambda_\omega^p(\mathcal{M}).$$

Therefore,

$$\begin{aligned} i_{e_\varepsilon} \circ L_t(z) &= e_\varepsilon t e_\varepsilon g\big(|t|\big)u^* x = e_\varepsilon t g\big(|t|\big)u^* x e_\varepsilon \\ &= e_\varepsilon u e_\varepsilon u^* x = e_\varepsilon x = y; \end{aligned}$$

that is, $\Lambda_l^p(e_\varepsilon) \subseteq \operatorname{Ran}(i_{e_\varepsilon} \circ L_t|_{\Lambda_l^p(e_\varepsilon)})$. Hence the operator $i_{e_\varepsilon} \circ L_t|_{\Lambda_l^p(e_\varepsilon)} : \Lambda_l^p(e_\varepsilon) \to \Lambda_l^p(e_\varepsilon)$ is compact and surjective. Then the result follows directly from classical theory (see III.1.12 in [10]).

Conversely, suppose that for any $n \in \mathbb{N}^+$ we have that $\Lambda_l^p(e_{\frac{1}{n}})$ is finite-dimensional. Set $t_n = e_{\frac{1}{n}} t$, where $e_{\frac{1}{n}} = e_{(\frac{1}{n},\infty)}(|t|)$. Then $\operatorname{Ran}(L_{t_n}) \subseteq \Lambda_l^p(e_{\frac{1}{n}})$.

Since $\Lambda_l^p(e_{\frac{1}{n}})$ is finite-dimensional, L_{t_n} is a finite rank operator. Moreover,

$$\|L_t x - L_{t_n} x\|_{p,\omega} = \left\|(t-t_n)x\right\|_{p,\omega} = \left\|(1-e_{\frac{1}{n}})tx\right\|_{p,\omega} \le \frac{1}{n}\|x\|_{p,\omega}.$$

Then L_{t_n} converges to L_t uniformly, which implies that L_t is compact. □

Theorem 2.6. *Let $t \in \mathcal{M}$, and let $\mathcal{M}$ have no minimal projection. Then the operator $L_t : \Lambda_\omega^p(\mathcal{M}) \to \Lambda_\omega^p(\mathcal{M})$ has finite-dimensional range if and only if $t = 0$.*

Proof. Suppose that $t \neq 0$ and the operator $L_t : \Lambda_\omega^p(\mathcal{M}) \to \Lambda_\omega^p(\mathcal{M})$ has finite-dimensional range. Then there exists $\varepsilon_0 > 0$ such that $\tau(e_{(\varepsilon_0,\infty)}(|t|)) > 0$. Since $\mathcal{M}$ has no minimal projection, then there exists $\{e_n\}_{n=1}^\infty \subseteq \mathcal{P}(\mathcal{M})$ with $\tau(e_{(\varepsilon_0,\infty)}(|t|)) > \tau(e_n) > 0$ such that $e_n e_m = 0$ for any $m \neq n$. Thus, for each $n \in \mathbb{N}^+$, we can define the operator $0 \neq t_n = te_n$, which clearly belongs to $\mathrm{Ran}(L_t)$. On the other hand, the sequence $\{t_n\}$ is linearly independent, and hence $\dim(\mathrm{Ran}(L_t)) = \infty$, and so $t = 0$. The converse is trivial. □

Theorem 2.7. *Let $t = t^* \in \mathcal{M}$, and let $\mathcal{M}$ have no minimal projection. Then the operator $L_t : \Lambda_\omega^p(\mathcal{M}) \to \Lambda_\omega^p(\mathcal{M})$ is compact if and only if $t = 0$.*

Proof. Suppose that L_t is compact. According to Theorem 2.5, we obtain that $\Lambda_l^p(e_{\frac{1}{n}})$ is finite-dimensional for all $n \in \mathbb{N}^+$, where $e_{\frac{1}{n}} = e_{(\frac{1}{n},\infty)}(|t|)$. Since $\Lambda_l^p(e_{\frac{1}{n}}) = \mathrm{Ran}(L_{e_{\frac{1}{n}}})$, by Theorem 2.6, we have $e_{\frac{1}{n}} = 0$, $n \in \mathbb{N}^+$. Therefore, $t = 0$. The converse is trivial. □

3. Noncommutative hardy–lorentz spaces

In this paper, we will assume that $\mathcal{D}$ is a von Neumann subalgebra of $\mathcal{M}$ such that the restriction of τ to $\mathcal{D}$ is still semifinite. Let Φ be the (unique) normal faithful conditional expectation of $\mathcal{M}$ with respect to $\mathcal{D}$ which leaves τ invariant. For a subset K of $L_{00}(\mathcal{M})$, $J(K)$ will denote the set of all Hilbert-adjoints of elements of K.

A w^* closed subalgebra $\mathcal{A}$ of $\mathcal{M}$ is called a *subdiagonal algebra* of $\mathcal{M}$ with respect to Φ (or $\mathcal{D}$) if

(1) $\mathcal{A} + J(\mathcal{A})$ is w^* dense in $\mathcal{M}$;
(2) Φ is multiplicative on $\mathcal{A}$; that is, $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in \mathcal{A}$;
(3) $\mathcal{A} \cap J(\mathcal{A}) = \mathcal{D}$.

$\mathcal{D}$ is then called the *diagonal* of $\mathcal{A}$.

We say that $\mathcal{A}$ is a maximal subdiagonal algebra in $\mathcal{M}$ with respect to Φ in the case that $\mathcal{A}$ is not properly contained in any other subalgebra of $\mathcal{M}$ which is subdiagonal with respect to Φ. We refer to [18] and [6] for noncommutative Hardy spaces associated with finite subdiagonal algebras. It is proved by Ji [13] that a semifinite subdiagonal algebra $\mathcal{A}$ is automatically maximal.

If K is a subset of $\Lambda_\omega^p(\mathcal{M})$ (resp., $L^p(\mathcal{M})$), $[K]_{p,\omega}$ (resp., $[K]_p$) will denote the closed linear span of K in $\Lambda_\omega^p(\mathcal{M})$ (resp., $L^p(\mathcal{M})$) (with respect to the w^* topology in the case of $p=\infty$).

Definition 3.1. Let $\mathcal{M}$ be a semifinite von Neumann algebra and let $0<p\leq\infty$. We define noncommutative Hardy–Lorentz spaces by $H^{p,\omega}(\mathcal{A})=[\mathcal{A}\cap\Lambda_\omega^p(\mathcal{M})]_{p,\omega}$ and $H_0^{p,\omega}(\mathcal{A})=[\mathcal{A}_0\cap\Lambda_\omega^p(\mathcal{M})]_{p,\omega}$.

For $0<p<\infty$, we define

$$H^{\Gamma_\omega^p}(\mathcal{A})=\big[\mathcal{A}\cap\Gamma_\omega^p(\mathcal{M})\big]_{\Gamma_\omega^p(\mathcal{M})}$$

and

$$H_0^{\Gamma_\omega^p}(\mathcal{A})=\big[\mathcal{A}_0\cap\Gamma_\omega^p(\mathcal{M})\big]_{\Gamma_\omega^p(\mathcal{M})}.$$

Remark 3.2.

(1) Let $\mathcal{M}$ be a finite von Neumann algebra. Then

$$\mathcal{M}=\Lambda_\omega^\infty(\mathcal{M})\subseteq\Lambda_\omega^p(\mathcal{M}),\quad 0<p\leq\infty,$$

and so $H^{p,\omega}(\mathcal{A})=[\mathcal{A}]_{p,\omega}$ and $H_0^{p,\omega}(\mathcal{A})=[\mathcal{A}_0]_{p,\omega}$.

(2) Let $\mathcal{M}$ be a finite von Neumann algebra and let $1\leq p\leq\infty$. It follows from [22, Section 3] that

$$H^p(\mathcal{A})=[\mathcal{A}]_p=\big\{x\in L^p(\mathcal{M}):\tau(xy)=0\text{ for all }y\in\mathcal{A}_0\big\}$$

and

$$H_0^p(\mathcal{A})=[\mathcal{A}_0]_p=\big\{x\in L^p(\mathcal{M}):\tau(xy)=0\text{ for all }y\in\mathcal{A}\big\}.$$

Subsequently, Bekjan and Xu [6, Proposition 3.3] have shown that $H^q(\mathcal{A})=H^p(\mathcal{A})\cap L^q(\mathcal{M})$ and $H_0^q(\mathcal{A})=H_0^p(\mathcal{A})\cap L^q(\mathcal{M})$, where $0<p<q\leq\infty$.

(3) Let $0<p\leq q\leq\infty$. Since $\mathcal{M}$ is a finite von Neumann algebra, then $\Lambda_\omega^q(\mathcal{M})\subseteq\Lambda_\omega^p(\mathcal{M})$.

(4) If $\omega\equiv 1$, then $H^{p,\omega}(\mathcal{A})=H^p(\mathcal{A})=[\mathcal{A}\cap L^p(\mathcal{M})]_p$ and $H_0^{p,\omega}(\mathcal{A})=H_0^p(\mathcal{A})=[\mathcal{A}_0\cap L^p(\mathcal{M})]_p$.

Proposition 3.3. *Let $0<p<\infty$, and let $\omega\in\mathcal{B}_q$ for some $1\leq q<\infty$. Then there exists some constant $C>0$ such that $\|\Phi(x)\|_{p,\omega}\leq C\|x\|_{p,\omega}$ holds for all $x\in\mathcal{A}\cap\Lambda_\omega^p(\mathcal{M})$. Consequently, Φ extends to a bounded projection from $H_\omega^p(\mathcal{A})$ onto $\Lambda_\omega^p(\mathcal{D})$. The extension will still be denoted by Φ.*

Proof. First we assume that $\tau(1)<\infty$. Without loss of generality, we assume that $\mathcal{M}$ has no minimal projections. Let $\mathcal{N}$ be any commutative von Neumann subalgebra of $\mathcal{M}$ containing the spectral projection of $|\Phi(x)|$. Combining the fact that $\|\Phi(x)\|_r\leq\|x\|_r, 0<r<\infty$ (see Proposition 3.1 of [4]) with $\mathcal{N}\subseteq\mathcal{D}$, we deduce

$$\begin{aligned}\int_0^t\mu_s\big(\Phi(x)\big)^r\,ds&=\sup\big\{\tau\big(\big(e\big|\Phi(x)\big|e\big)^r\big);e\in\mathcal{P}(\mathcal{M}),\tau(e)\leq t\big\}\\&\leq\sup\big\{\tau\big(\big(\big|\Phi(x)e\big|\big)^r\big);e\in\mathcal{P}(D),\tau(e)\leq t\big\}\end{aligned}$$

$$
\begin{aligned}
&= \sup\{\tau\big(\big(\big|\Phi(xe)\big|\big)^r\big); e \in \mathcal{P}(\mathcal{D}), \tau(e) \le t\} \\
&\le \sup\{\tau\big(|xe|^r\big); e \in \mathcal{P}(\mathcal{D}), \tau(e) \le t\} \\
&\le \int_0^t \mu_s(x)^r\, ds;
\end{aligned}
$$

that is,

$$
\int_0^t \mu_s\big(\Phi(x)\big)^r\, ds \le \int_0^t \mu_s(x)^r\, ds. \tag{3.1}
$$

Let $r > 0$ with $q := \frac{p}{r} \ge 1$. By Theorem 1.7 of [2], we have

$$
\int_0^\infty \Big(\frac{1}{t}\int_0^t \mu_s(x)^r\, ds\Big)^{\frac{p}{r}} \omega(t)\, dt \le C \int_0^\infty \mu_t(x)^p \omega(t)\, dt, \tag{3.2}
$$

where $C > 0$ is a constant. Thus

$$
\big\|\Phi(x)\big\|_{p,\omega} \le \Big(\int_0^\infty \Big(\frac{1}{t}\int_0^t \mu_s(x)^r\, ds\Big)^{\frac{p}{r}} \omega(t)\, dt\Big)^{\frac{1}{p}} \le C\|x\|_{p,\omega}.
$$

If $\mathcal{M}$ has minimal projections, then we replace $\mathcal{M}$ by $\mathcal{M}\overline{\otimes} L^\infty[0,1]$ and $\mathcal{A}$ by $\mathcal{A}\overline{\otimes} L^\infty[0,1]$. Then $\mathcal{A}\overline{\otimes} L^\infty[0,1]$ is a finite subdiagonal subalgebra of $\mathcal{M}\overline{\otimes} L^\infty[0,1]$ with respect to $\Phi\otimes 1$ (see Lemma 3.1 of [5]). By the trivial fact $\mu_t(x\otimes 1) = \mu_t(x)$ and the argument of above, we have $\|\Phi(x)\|_{p,\omega} \le C\|x\|_{p,\omega}$, where $C > 0$ is a constant.

In the general case when τ is semifinite, we can choose an increasing family of $\{e_i\}_{i\in I}$ of τ-finite projections in $\mathcal{D}$ such that $e_i \to 1$ strongly, where 1 is the identity of $\mathcal{M}$ (see Theorem 2.5.6 in [23]). It follows that $e_i x e_i \to x$ strongly. Therefore, by normality of Φ, we obtain that $\Phi(e_i x e_i) \to \Phi(x)$. On the other hand, by the argument of above,

$$
\big\|\Phi(e_i x e_i) - \Phi(e_j x e_j)\big\|_{p,\omega} \le C\|e_i x e_i - e_j x e_j\|_{p,\omega}, \quad i, j \in I,
$$

where $C > 0$ is a constant. From Proposition 2.1 we obtain $\|e_i x e_i - x\|_{p,\omega} \to 0$, and hence $\|\Phi(e_i x e_i) - \Phi(x)\|_{p,\omega} \to 0$. This implies that

$$
\big\|\Phi(x)\big\|_{p,\omega} \le \lim_i \big\|\Phi(e_i x e_i)\big\|_{p,\omega} \le C \lim_i \|e_i x e_i\|_{p,\omega} = C \lim_i \|x\|_{p,\omega},
$$

where $C > 0$ is a constant. □

Let e be a projection in $\mathcal{D}$. We write $\mathcal{M}_e = e\mathcal{M}e, \mathcal{A}_e = e\mathcal{A}e, \mathcal{D}_e = e\mathcal{D}e$, and let $\Phi_e(x)$ be the restriction of Φ to $\mathcal{M}_e$. From Lemma 3.1 of [4], we have $(\mathcal{A}_e)_0 = e\mathcal{A}_0 e$ and $\mathcal{A}_e$ is a subdiagonal algebra of $\mathcal{M}_e$ with respect to Φ_e and with diagonal $\mathcal{D}_e$.

Since $\mathcal{M}$ is semifinite, we can choose an increasing family of $\{e_i\}_{i\in I}$ of τ-finite projections in $\mathcal{D}$ such that $e_i \to 1$ strongly, where 1 is the identity of $\mathcal{M}$ (see Theorem 2.5.6 in [23]). Throughout the family $\{e_i\}_{i\in I}$ will be used to indicate this net.

Proposition 3.4. *Let $0 < p < \infty$, and let e be a projection in $\mathcal{D}$ with $\tau(e) < \infty$. Then*

(1) $\Lambda^p_\omega(\mathcal{M}_e) = e\Lambda^p_\omega(\mathcal{M})e$,
(2) $H_0^{p,\omega}(\mathcal{A}_e) = eH_0^{p,\omega}(\mathcal{A})e, H^{p,\omega}(\mathcal{A}_e) = eH^{p,\omega}(\mathcal{A})e$.

Proof. (1) Notice that the set $\mathcal{M}_e$ is the reduction of $\mathcal{M}$ by e. The trace τ_e on $\mathcal{M}_e$ is defined by $\tau_e(\cdot) = \tau(e \cdot e)$. We denote the decreasing rearrangement of $x \in L_{00}(\mathcal{M}_e)$ as $\mu_t^e(x)$. Let $x \in L_{00}(\mathcal{M}_e) \subseteq L_{00}(\mathcal{M})$, and let $\mathcal{N}$ be any von Neumann subalgebra of $\mathcal{M}_e$ containing the spectral projections of $|x|$. Since $x \in L_{00}(\mathcal{M}_e)$, we have $s(|x|) \leq e$ and $x(1-e) = 0$. According to Remark 2.3.1 of [9], we obtain

$$\mu_t^e(x) = \inf\big\{\|xE\| : E \in \mathcal{P}(\mathcal{N}), \tau(e-E) \leq t\big\}$$

and

$$\mu_t(x) = \inf\big\{\|xE\| : E \in \mathcal{P}(\mathcal{N}), \tau(1-E) \leq t\big\}.$$

If $E \in \mathcal{P}(\mathcal{N})$ with $\tau(e-E) \leq t$, then there exists $E + (1-e)$ such that $\|xE\| = \|x(E+(1-e))\|$ and $\tau(1-(E+(1-e))) = \tau(e-E) \leq t$. On the other hand, if $E \in \mathcal{P}(\mathcal{N})$ with $\tau(1-E) \leq t$, then we have $\tau(e-E) \leq \tau(1-E) \leq t$. Therefore, $\mu_t(x) = \mu_t^e(x), 0 < t \leq \tau(e)$ and $\mu_t(x) = \mu_t^e(x) = 0, t > \tau(e)$.

Let $x \in \Lambda_\omega^p(\mathcal{M}_e)$. Then there exists $\{x_n\}_{n=1}^\infty \subseteq \mathcal{M}_e$ such that

$$\|x_n - x\|_{\Lambda_\omega^p(\mathcal{M}_e)} \to 0, \quad n \to \infty.$$

By the argument above, we have $x_n = ex_ne, x = exe$ and

$$\mu_t^e(x_n - x) = \mu_t(x_n - x) = 0, \quad t > \tau(e)$$

and

$$\mu_t(x - x_n) = \mu_t^e(x - x_n), \quad 0 < t \leq \tau(e).$$

Therefore,

$$\begin{aligned}\|x_n - x\|_{\Lambda_\omega^p(\mathcal{M})} &= \Big(\int_0^\infty \mu_t(x_n - x)^p \omega(t)\,dt\Big)^{\frac{1}{p}} \\ &= \Big(\int_0^\infty \mu_t^e(x_n - x)^p \omega(t)\,dt\Big)^{\frac{1}{p}} \\ &= \|x_n - x\|_{\Lambda_\omega^p(\mathcal{M}_e)} \to 0, \quad n \to \infty.\end{aligned}$$

Hence $x = exe \in e\Lambda_\omega^p(\mathcal{M})e$. Conversely, if $x \in \Lambda_\omega^p(\mathcal{M})$, then there exists $\{x_n\}_{n=1}^\infty \subseteq \mathcal{M}$ such that $\|x_n - x\|_{\Lambda_\omega^p(\mathcal{M})} \to 0, n \to \infty$. Thus $\|ex_ne - exe\|_{\Lambda_\omega^p(\mathcal{M}_e)} \to 0, n \to \infty$. This implies that $exe \in \Lambda_\omega^p(\mathcal{M}_e)$.

(2) Since $\mathcal{A}_e \cap \Lambda_\omega^p(\mathcal{M}_e) = e\mathcal{A}e \cap e\Lambda_\omega^p(\mathcal{M})e \subseteq eH^{p,\omega}(\mathcal{A})e$, we have $eH^{p,\omega}(\mathcal{A})e \supseteq H^{p,\omega}(\mathcal{A}_e)$. Conversely, if $x \in H^{p,\omega}(\mathcal{A})$, then there exists $\{x_n\}_{n=1}^\infty \subseteq \mathcal{A}$ such that $x_n \to x$ in norm in $H^{p,\omega}(\mathcal{A})$. This tells us that $ex_ne \to exe$ in norm in $H^{p,\omega}(\mathcal{A}_e)$. Thus $exe \in H^{p,\omega}(\mathcal{A}_e)$. The proof of the other containment is similar. □

Proposition 3.5. *Let $0 < p < \infty$. Then $H^{p,\omega}(\mathcal{A})$ is the $\|\cdot\|_{p,\omega}$-closure of $\bigcup_{i\in I} \mathcal{A}_{e_i}$ in $\Lambda_\omega^p(\mathcal{M})$, and $H_0^{p,\omega}(\mathcal{A})$ is the $\|\cdot\|_{p,\omega}$-closure of $\bigcup_{i\in I} (\mathcal{A}_{e_i})_0$ in $\Lambda_\omega^p(\mathcal{M})$.*

Proof. Given $\varepsilon > 0$ and $x \in H^{p,\omega}(\mathcal{A})$, then there exists $a \in \mathcal{A} \cap \Lambda_\omega^p(\mathcal{M})$ such that $\|x - a\|_{p,\omega} < \varepsilon$. By Proposition 2.1(2), we have $\|ae_i - a\|_{p,\omega} \to 0, \|e_ia - a\|_{p,\omega} \to 0$; thus, $\|e_iae_i - a\|_{p,\omega} \to 0$ and $e_iae_i \in \mathcal{A}_{e_i}$. Therefore, there exists i_0 such that $\|e_{i_0}ae_{i_0} - a\|_{p,\omega} < \varepsilon$. It follows that $\|e_{i_0}ae_{i_0} - x\|_{p,\omega} < 2\varepsilon$. We may similarly prove the result about $H_0^{p,\omega}(\mathcal{A})$. □

Theorem 3.6. *Let $\omega \in \mathcal{B}_p \cap \mathcal{B}_\infty^*, 1 < p < \infty$. Then*

$$H^{p,\omega}(\mathcal{A}) = \{x \in \Lambda_\omega^p(\mathcal{M}) : \tau(xy) = 0, \forall y \in \mathcal{A}_0 \cap \Gamma_{\widetilde{\omega}}^q(\mathcal{M})\},$$
$$H_0^{p,\omega}(\mathcal{A}) = \{x \in \Lambda_\omega^p(\mathcal{M}) : \tau(xy) = 0, \forall y \in \mathcal{A} \cap \Gamma_{\widetilde{\omega}}^q(\mathcal{M})\},$$

where $\widetilde{\omega}(t) = t^q W(t)^{-q}\omega(t), t > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We only prove the case $H^{p,\omega}(\mathcal{A})$, the other case being similar. Since $\omega \in \mathcal{B}_p \cap \mathcal{B}_\infty^*$, then Corollary 2.3 and Proposition 2.6 of [1] imply that $1 < \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} < \infty$. From Proposition 3.1 of [11], we obtain

$$H^{p,\omega}(e_i\mathcal{A}e_i) = \{x \in \Lambda_\omega^p(e_i\mathcal{M}e_i) : \tau(xy) = 0, \forall y \in e_i\mathcal{A}_0e_i\}, \quad i \in I.$$

Let $x \in \{x \in \Lambda_\omega^p(\mathcal{M}) : \tau(xy) = 0, \forall y \in \mathcal{A}_0 \cap \Gamma_{\widetilde{\omega}}^q(\mathcal{M})\}$. For any $y \in \mathcal{A}_0 \cap \Gamma_{\widetilde{\omega}}^q(\mathcal{M})$, we have $e_iye_i \in (\mathcal{A}_{e_i})_0 \subseteq \mathcal{A}_0 \cap \Gamma_{\widetilde{\omega}}^q(\mathcal{M})$, and so $\tau(e_iye_ix) = \tau(e_iye_ixe_i) = 0$. Thus $e_ixe_i \in H^{p,\omega}(\mathcal{A}_{e_i}) = e_iH^{p,\omega}(\mathcal{A})e_i$ for all $i \in I$. By Proposition 2.1(2) and (4), we have $\|e_ixe_i - x\|_{p,\omega} \to 0$ and $\|e_iye_i - y\|_{\Gamma_{\widetilde{\omega}}^q(\mathcal{M})} \to 0$. Therefore, $x \in H^{p,\omega}(\mathcal{A})$. Conversely, let $x \in H^{p,\omega}(\mathcal{A})$, and let $y \in \mathcal{A}_0 \cap \Gamma_{\widetilde{\omega}}^q(\mathcal{M})$. Then there exists $\{x_n\} \subseteq \mathcal{A} \cap \Lambda_\omega^p(\mathcal{M})$ such that $x_n \to x$ in $\Lambda_\omega^p(\mathcal{M})$. Hence

$$\begin{aligned}\tau(yx) &= \lim_{n\to\infty} \tau(yx_n) = \lim_{n\to\infty} \tau\big(\Phi(yx_n)\big) \\ &= \lim_{n\to\infty} \tau\big(\Phi(y)\Phi(x_n)\big) = 0.\end{aligned}$$

This implies the desired result. □

Let A_0 and A_1 be two quasi-Banach spaces. Then their sum is defined by $A_0 + A_1 = \{x_0 + x_1 : x_k \in A_k, k = 0, 1\}$ with the quasinorm

$$\|x\|_{A_0+A_1} = \inf\{\|x_0\|_{A_0} + \|x_1\|_{A_1} : x = x_0 + x_1, x_i \in A_i, i = 0, 1\}.$$

It is easy to check that $A_0 + A_1$ is again a quasi-Banach space (and a Banach space if A_0 and A_1 are). For all $x \in A_0 + A_1$ and for all $t > 0$, we let

$$K_t(x; A_0, A_1) = \inf\{\|x_0\|_{A_0} + t\|x_1\|_{A_1} : x = x_0 + x_1, x_i \in A_i, i = 0, 1\}.$$

Proposition 3.7. *There is a constant $C > 0$ such that, for all $x \in H^1(\mathcal{A}) + \mathcal{A}$ and all $t > 0$,*

$$K_t\big(x; H^1(\mathcal{A}), \mathcal{A}\big) \leq CK_t\big(x; L^1(\mathcal{M}), \mathcal{M}\big).$$

Proof. By Proposition 6.1 of [4], there exists a constant $C > 0$ such that

$$\begin{aligned}K_t\big(x; H^1(\mathcal{A}), \mathcal{A}\big) &\leq K_t\big(x; H^1(\mathcal{A}_{e_i}), \mathcal{A}_{e_i}\big) \\ &\leq CK_t\big(x; L^1(\mathcal{M}_{e_i}), \mathcal{M}_{e_i}\big),\end{aligned}$$

where C is independent of e_i. On the other hand, for all $\varepsilon > 0$, there exist $x_1 \in L^1(\mathcal{M}), x_2 \in \mathcal{M}$ such that $x = x_1 + x_2$ and

$$\|x_1\|_{L^1(\mathcal{M})} + t\|x_2\| \leq K_t\big(x, L^1(\mathcal{M}), \mathcal{M}\big) + \varepsilon.$$

Since $e_ixe_i \in L^1(\mathcal{M}_{e_i}) + \mathcal{M}_{e_i}$ and $e_ix_1e_i \in L^1(\mathcal{M}_{e_i}), e_ix_2e_i \in \mathcal{M}_{e_i}$, then

$$K_t\big(x; H^1(\mathcal{A}), \mathcal{A}\big) \leq K_t\big(x; L^1(\mathcal{M}_{e_i}), \mathcal{M}_{e_i}\big) \leq \|e_ix_1e_i\|_{L^1(\mathcal{M})} + t\|e_ix_2e_i\|.$$

This implies that $K_t(x; H^1(\mathcal{A}), \mathcal{A}) \leq K_t(x; L^1(\mathcal{M}), \mathcal{M})$. □

Proposition 3.8. *Let $\omega \in \mathcal{B}_p \cap \mathcal{B}_\infty^* 1 < p < \infty$, and let $1 \leq r < \alpha_{\Lambda_\omega^p}$. Then*

$$\big(H^r(\mathcal{A}) + \mathcal{A}\big) \cap \Lambda_\omega^p(\mathcal{M}) = H^{p,\omega}(\mathcal{A}),$$
$$\big(H_0^r(\mathcal{A}) + \mathcal{A}_0\big) \cap \Lambda_\omega^p(\mathcal{M}) = H_0^{p,\omega}(\mathcal{A}).$$

Proof. We only verify the first equation, the other case being similar. Let $x \in H^{p,\omega}(\mathcal{A}) \subseteq \Lambda_\omega^p(\mathcal{M})$. By Proposition 3.1 of [11], we have

$$e_i x e_i \in H^{p,\omega}(\mathcal{A}_{e_i}) = H^r(\mathcal{A}_{e_i}) \cap \Lambda_\omega^p(\mathcal{M}_{e_i}), \quad i \in I.$$

Thus $e_i x e_i \in H^r(\mathcal{A}_{e_i}) + \mathcal{A}_{e_i}$ and $e_i x e_i \in \Lambda_\omega^p(\mathcal{M}_{e_i})$. By Proposition 2.1, we have $\|x - e_i x e_i\|_{p,\omega} \to 0$. Since $\omega \in \mathcal{B}_p \cap \mathcal{B}_\infty^*$, it follows from Corollary 2.3 and Proposition 2.6 of [1] that $1 < \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} < \infty$. Therefore, Theorem 3.7 of [8] means that $\Lambda_\omega^p(\mathcal{M})$ is an interpolation space for the couple $(L^r(\mathcal{M}), \mathcal{M}), 1 \leq r < \alpha_{\Lambda_\omega^p}$. By Lemma 6.5(iii) of [4] and Proposition 3.7, there exist two constants $C_0, C_1 > 0$ such that

$$\|x - e_i x e_i\|_{H^r(\mathcal{A})+\mathcal{A}} \leq C_0 \|x - e_i x e_i\|_{L^r(\mathcal{M})+\mathcal{M}} \leq C_0 C_1 \|x - e_i x e_i\|_{p,\omega} \to 0.$$

Thus $x \in (H^r(\mathcal{A}) + \mathcal{A}) \cap \Lambda_\omega^p(\mathcal{M})$. Conversely, if $x \in (H^r(\mathcal{A}) + \mathcal{A}) \cap \Lambda_\omega^p(\mathcal{M})$, then Proposition 3.1 of [11] means that

$$e_i x e_i \in \big(H^r(\mathcal{A}_{e_i}) + \mathcal{A}_{e_i}\big) \cap \Lambda_\omega^p(\mathcal{M}_{e_i}) = H^r(\mathcal{A}_{e_i}) \cap \Lambda_\omega^p(\mathcal{M}_{e_i}) = H^{p,\omega}(\mathcal{A}_{e_i}).$$

Therefore, $e_i x e_i \in H^{p,\omega}(\mathcal{A})$. It follows that $x \in H^{p,\omega}(\mathcal{A})$. $\square$

Remark 3.9.

(1) Let $\mathcal{M}$ be a finite von Neumann algebra, and let $\omega \in \mathcal{B}_p \cap \mathcal{B}_\infty^*, 1 < p < \infty$. Then

$$H^{p,\omega}(\mathcal{A}) = \big\{x \in \Lambda_\omega^p(\mathcal{M}) : \tau(xy) = 0, \forall y \in \mathcal{A}_0\big\},$$
$$H_0^{p,\omega}(\mathcal{A}) = \big\{x \in \Lambda_\omega^p(\mathcal{M}) : \tau(xy) = 0, \forall y \in \mathcal{A}\big\}.$$

Moreover, if $\omega \in \mathcal{B}_p \cap \mathcal{B}_\infty^*$ and $1 \leq r < \alpha_{\Lambda_\omega^p}$, then

$$H^r(\mathcal{A}) \cap \Lambda_\omega^p(\mathcal{M}) = H^{p,\omega}(\mathcal{A}),$$
$$H_0^r(\mathcal{A}) \cap \Lambda_\omega^p(\mathcal{M}) = H_0^{p,\omega}(\mathcal{A}).$$

(2) Let $\omega \equiv 1, 1 < p < \infty$. Then

$$H^p(\mathcal{A}) = \big\{x \in L^p(\mathcal{M}) : \tau(xy) = 0, \forall y \in \mathcal{A}_0 \cap L^q(\mathcal{M})\big\},$$
$$H_0^p(\mathcal{A}) = \big\{x \in L^p(\mathcal{M}) : \tau(xy) = 0, \forall y \in \mathcal{A} \cap L^q(\mathcal{M})\big\}.$$

Moreover, if $1 \leq r < p$, then

$$\big(H^r(\mathcal{A}) + \mathcal{A}\big) \cap L^p(\mathcal{M}) = H^p(\mathcal{A}),$$
$$\big(H_0^r(\mathcal{A}) + \mathcal{A}\big) \cap L^p(\mathcal{M}) = H_0^p(\mathcal{A}).$$

Lemma 3.10. *Let $0 < p, p_0, p_1 < \infty$ with $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$ and $\omega \in \mathcal{B}_{\min\{1,p\}}$. Then $\Lambda_\omega^p(\mathcal{M}) = \Lambda_\omega^{p_0}(\mathcal{M}) \odot \Lambda_\omega^{p_1}(\mathcal{M})$ and $\Lambda_\omega^{p_0}(\mathcal{M}) \odot \Lambda_\omega^{p_1}(\mathcal{M})$ is a quasi-Banach space with the quasinorm*

$$\|x\|_0 = \inf\big\{\|x_0\|_{p_0,\omega}\|x_1\|_{p_1,\omega} : x = x_0 x_1, x_i \in \Lambda_\omega^{p_i}(\mathcal{M}), i = 0, 1\big\},$$

where $\Lambda_\omega^{p_0}(\mathcal{M}) \odot \Lambda_\omega^{p_1}(\mathcal{M} = \{x : x = x_0x_1, x_i \in \Lambda_\omega^{p_i}(\mathcal{M}), i = 0,1\})$. *Moreover,* $\|\cdot\|_{p,\omega}$ *is equivalent to* $\|\cdot\|_0$.

Proof. Since $p_i \geq p$, it follows from Corollary 2.3 of [1] that $\omega \in \mathcal{B}_{p_i}, i = 0,1$. Therefore, by Theorem 2 of [15], we have $\Lambda_\omega^p(\mathcal{M})$ is $\frac{p}{2}$-convex and $\Lambda_\omega^{p_i}(\mathcal{M})$ is $\frac{p_i}{2}$-convex, $i = 0,1$. Therefore, by slightly modifying the proof of Lemma 2.1 and Theorem 2.5 in [5], we can prove that $(\Lambda_\omega^{p_0}(\mathcal{M})\odot\Lambda_\omega^{p_1}(\mathcal{M}), \|\cdot\|_0)$ is a quasi-Banach space. For $x \in \Lambda_\omega^p(\mathcal{M})$, we write $x = u|x| = u|x|^{\frac{p}{p_0}}|x|^{\frac{p}{p_1}}$. Thus $u|x|^{\frac{p}{p_0}} \in \Lambda_\omega^{p_0}(\mathcal{M})$, $|x|^{\frac{p}{p_1}} \in \Lambda_\omega^{p_1}(\mathcal{M})$, and $\|u|x|^{\frac{p}{p_0}}\|_{p_0,\omega}\||x|^{\frac{p}{p_1}}\|_{p_1,\omega} \leq \|x\|_{p,\omega}$. This implies that $\|x\|_0 \leq \|x\|_{p,\omega}$ and $\Lambda_\omega^p(\mathcal{M}) \subseteq \Lambda_\omega^{p_0}(\mathcal{M}) \odot \Lambda_\omega^{p_1}(\mathcal{M})$. Since $\omega \in \mathcal{B}_p$, then $\Lambda_\omega^p(\mathcal{M}) = \Gamma_\omega^p(\mathcal{M})$. For $y \in \Lambda_\omega^{p_0}(\mathcal{M})$ and $z \in \Lambda_\omega^{p_1}(\mathcal{M})$, there exist two constants $C_0, C_1 > 0$ such that

$$
\begin{aligned}
\|yz\|_{p,\omega} &\leq C_0\Big(\int_0^\infty \Big(\frac{1}{t}\int_0^t \mu_s(yz)\,ds\Big)^p \omega(t)\,dt\Big)^{\frac{1}{p}} \\
&\leq C_0\Big(\int_0^\infty \Big(\frac{1}{t}\int_0^t \mu_s(y)\mu_s(z)\,ds\Big)^p \omega(t)\,dt\Big)^{\frac{1}{p}} \\
&\leq C_0C_1\Big(\int_0^\infty \big(\mu_t(y)\mu_t(z)\big)^p \omega(t)\,dt\Big)^{\frac{1}{p}} \\
&= C_0C_1\Big(\int_0^\infty \mu_t(y)^p\omega(t)^{\frac{p}{p_0}}\mu_t(z)^p\omega(t)^{\frac{p}{p_1}}\,dt\Big)^{\frac{1}{p}} \\
&\leq C_0C_1\|y\|_{p_0,\omega}\|z\|_{p_1,\omega}.
\end{aligned}
$$

This completes the proof. $\square$

Proposition 3.11. *Let* $\mathcal{M}$ *be a finite von Neumann algebra,* $0 < p, p_0, p_1 < \infty$ *with* $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$ *and* $\omega \in \mathcal{B}_{\min\{1,p\}}$. *Then for* $x \in H^{p,\omega}(\mathcal{A})$ *and* $\varepsilon > 0$ *there exist* $x_0 \in H^{p_0,\omega}(\mathcal{A})$ *and* $x_1 \in H^{p_1,\omega}(\mathcal{A})$ *such that* $x = x_0x_1$ *and* $\|x_0\|_{p_0,\omega}\|x_1\|_{p_1,\omega} \leq \|x\|_0 + \varepsilon \leq \|x\|_{p,\omega} + \varepsilon$. *Consequently,*

$$\|x\|_0 = \inf\big\{\|x_0\|_{p_0,\omega}\|x_1\|_{p_1,\omega} : x = x_0x_1, x_i \in H^{\omega,p_i}(\mathcal{A}), i = 0,1\big\},$$

and $\|\cdot\|_{p,\omega}$ *is equivalent to* $\|\cdot\|_0$.

Proof. The proof can be done similarly to Theorem 4.8 in [5] by using Lemma 3.10. The details are omitted. $\square$

4. The conjugation and herglotz map

Let $u \in \mathrm{Re}(\mathcal{A} \cap \Lambda_\omega^p(\mathcal{M}))$. Then $u = \mathrm{Re}\, x$ for some $x \in \mathcal{A} \cap \Lambda_\omega^p(\mathcal{M})$. We write $a = x - \frac{1}{2}\Phi(x - x^*)$, and so $a \in \mathcal{A} \cap \Lambda_\omega^p(\mathcal{M}), u = \mathrm{Re}\, a$, $\Phi(\mathrm{Im}\, a) = 0$. Therefore, there exists $\widetilde{u} = \mathrm{Im}\, a \in \mathrm{Re}(\mathcal{A} \cap \Lambda_\omega^p(\mathcal{M}))$ such that $a = u + i\widetilde{u} \in \mathcal{A} \cap \Lambda_\omega^p(\mathcal{M})$ and $\Phi(\widetilde{u}) = \Phi(\mathrm{Im}\, a) = 0$. By a similar discussion as [4], we have that such an element of $\mathrm{Re}(\mathcal{A}\cap\Lambda_\omega^p(\mathcal{M}))$ is unique. Thus we can define $\widetilde{u} = \mathrm{Im}\, a$, where $a \in \mathcal{M}$ is the unique element of $\mathcal{M}$ with $a = \mathrm{Re}\, a$ and $\Phi(\mathrm{Im}\, a) = 0$. It is obvious that $\sim : x \mapsto \widetilde{x}$ is a real linear. We shall call $\widetilde{u}$ the *conjugate* of u.

Lemma 4.1. *Let* $e \in \mathcal{D}$, *and let* $\omega \in \mathcal{B}_\infty^* \cap \mathcal{B}_p, 1 < p < \infty$. *If* $u \in \mathrm{Re}(\mathcal{A}\cap\Lambda_\omega^p(\mathcal{M}))$, *then* $\widetilde{eue} = e\widetilde{u}e$.

Proof. Let $u \in \mathrm{Re}(\mathcal{A} \cap \Lambda^p_\omega(\mathcal{M}))$. For $u = \mathrm{Re}(x)$, $x \in \mathcal{A} \cap \Lambda^p_\omega(\mathcal{M})$, we have $eue = \mathrm{Re}(exe)$, $exe \in \mathcal{A}_e \cap \Lambda^p_\omega(\mathcal{M}_e)$. Then $eue \in \mathrm{Re}(\mathcal{A}_e \cap \Lambda^p_\omega(\mathcal{M}_e))$. On the other hand, for $a = x - \frac{1}{2}\Phi(x - x^*)$,

$$eae = exe - e\frac{1}{2}\Phi(x - x^*)e = exe - \frac{1}{2}\Phi(exe - ex^*e), \qquad \Phi\big(\mathrm{Im}(eae)\big) = 0.$$

Thus $\widetilde{eue} = \mathrm{Im}(eae) = e\,\mathrm{Im}(a)e = e\widetilde{u}e$. $\square$

Proposition 4.2. *Let $1 < p, q < \infty$ with $\frac{1}{q} + \frac{1}{p} = 1$ and $\omega \in \mathcal{B}_p \cap \mathcal{B}^*_\infty$. Then*

$$\big[\mathrm{Re}\big(\mathcal{A} \cap \Lambda^p_\omega(\mathcal{M})\big)\big]_{p,\omega} = \big[\mathcal{M}^{sa} \cap \Lambda^p_\omega(\mathcal{M})\big]_{p,\omega} = \Lambda^p_\omega(\mathcal{M})^{sa}.$$

Moreover,

$$\big[\mathrm{Re}\big(\mathcal{A} \cap \Gamma^q_{\widetilde{\omega}}(\mathcal{M})\big)\big]_{\Gamma^q_{\widetilde{\omega}}(\mathcal{M})} = \big[\mathcal{M}^{sa} \cap \Gamma^q_{\widetilde{\omega}}(\mathcal{M})\big]_{\Gamma^q_{\widetilde{\omega}}(\mathcal{M})} = \Gamma^q_{\widetilde{\omega}}(\mathcal{M})^{sa},$$

where $\widetilde{\omega}(t) = t^q W(t)^{-q}\omega(t), t > 0$.

Proof. First we assume that τ is finite. The second equality is apparent, and hence we concern ourselves only with the first. Since $\mathrm{Re}(\mathcal{A} \cap \Lambda^p_\omega(\mathcal{M})) \subseteq \mathcal{M}^{sa} \cap \Lambda^p_\omega(\mathcal{M})$, we have

$$\big[\mathrm{Re}\big(\mathcal{A} \cap \Lambda^p_\omega(\mathcal{M})\big)\big]_{p,\omega} \subseteq \big[\mathcal{M}^{sa} \cap \Lambda^p_\omega(\mathcal{M})\big]_{p,\omega}.$$

Similarly,

$$\big[\mathrm{Re}\big(\mathcal{A} \cap \Gamma^q_{\widetilde{\omega}}(\mathcal{M})\big)\big]_{\Gamma^q_{\widetilde{\omega}}(\mathcal{M})} \subseteq \big[\mathcal{M}^{sa} \cap \Gamma^q_{\widetilde{\omega}}(\mathcal{M})\big]_{\Gamma^q_{\widetilde{\omega}}(\mathcal{M})}.$$

Conversely, suppose that $x \in \mathcal{M}^{sa}$. (a) From Proposition 5.3 of [18], we obtain $x \in [\mathrm{Re}(\mathcal{A})]_p$, $1 \le p < \infty$. Since $1 < p < \infty$ and $\omega \in \mathcal{B}^*_\infty \cap \mathcal{B}_p$, then Corollary 2.3 and Proposition 2.6 of [1] imply that $1 < \alpha_{\Lambda^p_\omega} \le \beta_{\Lambda^p_\omega} < \infty$, and so there exist $r_1, r_2 > 0$ with $1 < r_1 < \alpha_{\Lambda^p_\omega} \le \beta_{\Lambda^p_\omega} < r_2 < \infty$. It follows from Theorem 3.7 of [8] that $\Lambda^p_\omega(\mathcal{M})$ is an interpolation space for the couple $(L^{r_1}(\mathcal{M}), L^{r_2}(\mathcal{M}))$. Thus $L^{r_1}(\mathcal{M}) \supseteq \Lambda^p_\omega(\mathcal{M}) \supseteq L^{r_2}(\mathcal{M})$. This implies that $x \in [\mathrm{Re}(\mathcal{A})]_{r_2} \subseteq [\mathrm{Re}(\mathcal{A})]_{p,\omega}$.

(b) By (a) and Proposition 2.1, we have $\Lambda^p_\omega(\mathcal{M}) \subseteq L^1(\mathcal{M})$ and $\Lambda^p_\omega(\mathcal{M})^* = \Lambda^p_\omega(\mathcal{M})^\times = \Gamma^q_{\widetilde{\omega}}(\mathcal{M})$. A similar discussion to the proof of Proposition 5.3 in [18] shows that $x \in [\mathrm{Re}(\mathcal{A})]_{\Gamma^q_{\widetilde{\omega}}(\mathcal{M})}$.

Now let τ be semifinite. Since $\tau(e_i) < \infty$, we have

$$[\mathrm{Re}\,\mathcal{A}_{e_i}]_{p,\omega} = [\mathcal{M}^{sa}_{e_i}]_{p,\omega} = \Lambda^p_\omega(\mathcal{M}_{e_i})^{sa} = e_i\Lambda^p_\omega(\mathcal{M})^{sa}e_i$$

and

$$[\mathrm{Re}\,\mathcal{A}_{e_i}]_{\Gamma^q_{\widetilde{\omega}}(\mathcal{M})} = [\mathcal{M}^{sa}_{e_i}]_{\Gamma^q_{\widetilde{\omega}}(\mathcal{M})} = \Gamma^q_{\widetilde{\omega}}(\mathcal{M}_{e_i})^{sa} = e_i\Gamma^q_{\widetilde{\omega}}(\mathcal{M})^{sa}e_i.$$

This implies the desired result. $\square$

We say a Lorentz space Λ^r_ω has an *order continuous norm* if for every net (f_i) in Λ^r_ω such that $f_i \downarrow 0$ we have $\|f_i\|_{r,\omega} \downarrow 0$. Let $0 < r < \infty$. It follows from Proposition 2.3.3 and Theorem 2.3.4 of [7] that the norm on Λ^r_ω is order continuous. Thus Theorem 3.2 of [8] implies that Λ^r_ω is an interpolation space for the couple (L^p, L^q), whence $0 < p < \alpha_{\Lambda^r_\omega} \le \beta_{\Lambda^r_\omega} < q \le \infty$, and so

$$L^p(\mathcal{M}) \cap L^q(\mathcal{M}) \hookrightarrow \Lambda^r_\omega(\mathcal{M}) \hookrightarrow L^q(\mathcal{M}) + L_p(\mathcal{M}),$$

where "$\hookrightarrow$" denotes a continuous embedding. If in addition $p \geq 1$, then, by Theorem 3.2 and Theorem 3.7 of [8], we have that $\Lambda_\omega^r(\mathcal{M})$ is an interpolation space for the couple $(L^p(\mathcal{M}), L^q(\mathcal{M}))$.

Let $1 < p < \infty$, and let $\omega \in \mathcal{B}_p \cap \mathcal{B}_\infty^*$. From Corollary 2.3 and Proposition 2.6 of [1], we obtain $1 < \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} < \infty$, and so there exist $r_1, r_2 > 0$ such that

$$1 < r_1 < \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} < r_2 < \infty.$$

According to Theorem 3.2 and Theorem 3.7 of [8], we obtain that $\Lambda_\omega^p(\mathcal{M})$ is an interpolation space for the couple $(L^{r_1}(\mathcal{M}), L^{r_2}(\mathcal{M}))$, and so $\Lambda_\omega^p(\mathcal{M}) \subseteq L^{r_1}(\mathcal{M}) + L^{r_2}(\mathcal{M})$. Then, by Lemma 4.3 of [4], $\sim : \Lambda_\omega^p(\mathcal{M})^{sa} \to \Lambda_\omega^p(\mathcal{M})^{sa}$ is well defined. Now consider the standard complexification $\overline{\sim}$ of $\sim$. Since $\overline{\sim}$ is bounded on $L^{r_i}(\mathcal{M})$, $i = 1, 2$ (cf. [4]), we can conclude that $\overline{\sim} : \Lambda_\omega^p(\mathcal{M}) \to \Lambda_\omega^p(\mathcal{M})$ is well defined, and it is a bounded linear operator.

By restriction we get that $\overline{\sim} : \Lambda_\omega^p(\mathcal{M})^{sa} \to \Lambda_\omega^p(\mathcal{M})^{sa}$ is bounded. We define the Herglotz map $H : \mathrm{Re}(\mathcal{A}) \cap \Lambda_\omega^p(\mathcal{M}) \to \mathcal{A} \cap \Lambda_\omega^p(\mathcal{M})$ by $H(x) = x + i\widetilde{x}$. It is clear that H is a bounded real linear operator. From the above discussion, we obtain the following result.

Theorem 4.3. *Let $1 < p < \infty$, and let $\omega \in \mathcal{B}_\infty^* \cap \mathcal{B}_p$. The real linear maps*

$$\begin{aligned} \sim &: \mathrm{Re}\big(\mathcal{A} \cap \Lambda_\omega^p(\mathcal{M})\big) \to \mathrm{Re}\big(\mathcal{A} \cap \Lambda_\omega^p(\mathcal{M})\big), \\ H &: \mathrm{Re}\big(\mathcal{A} \cap \Lambda_\omega^p(\mathcal{M})\big) \to \mathcal{A} \cap \Lambda_\omega^p(\mathcal{M}) \end{aligned}$$

extend to real linear maps

$$\sim : \Lambda_\omega^p(\mathcal{M})^{sa} \to \Lambda_\omega^p(\mathcal{M})^{sa}, \qquad H : \Lambda_\omega^p(\mathcal{M})^{sa} \to H^{p,\omega}(\mathcal{A}).$$

If $x \in \Lambda_\omega^p(\mathcal{M})^{sa}$, then $H(x) = x + i\widetilde{x} \in H^{p,\omega}(\mathcal{A})$ and $\Phi(\widetilde{x}) = 0$. Both $\sim$ and H are bounded.

Corollary 4.4. *Let $1 < p < \infty$, and let $\omega \in \mathcal{B}_\infty^* \cap \mathcal{B}_p$. Then*

$$\mathrm{Re}\, H^{p,\omega}(\mathcal{A}) = \big[\mathrm{Re}\big(\mathcal{A} \cap \Lambda_\omega^p(\mathcal{M})\big)\big]_{p,\omega} = \Lambda_\omega^p(\mathcal{M})^{sa}.$$

Proof. The second equality has already been established. Let $x \in H^{p,\omega}(\mathcal{A})$. Take $\{x_n\} \subseteq \mathcal{A} \cap \Lambda_\omega^p(\mathcal{M})$ such that $\|x_n - x\|_{p,\omega} \to 0, n \to \infty$. Then $\|\mathrm{Re}\, x_n - \mathrm{Re}\, x\|_{p,\omega} \to 0, n \to \infty$. Thus $\mathrm{Re}\, H^{p,\omega}(\mathcal{A}) \subseteq [\mathrm{Re}(\mathcal{A} \cap \Lambda_\omega^p(\mathcal{M}))]_{p,\omega}$. Conversely, we only need to show that $\mathrm{Re}\, H^{p,\omega}(\mathcal{A})$ is closed. Let $\{x_n\} \subseteq H^{p,\omega}(\mathcal{A})$ with $\|\mathrm{Re}\, x_n - y\|_{p,\omega} \to 0$, $n \to \infty$. By continuity of adjunction, we have $y \in \Lambda_\omega^p(\mathcal{M})^{sa}$. From Theorem 4.3, we see that $\|H(\mathrm{Re}\, x_n) - H(y)\|_{p,\omega} \to 0, n \to \infty$ and $H(\mathrm{Re}\, x_n), H(y) \in H^{p,\omega}(\mathcal{A})$. This implies that $y = \mathrm{Re}(y + i\widetilde{y}) = \mathrm{Re}\, H(y) \in \mathrm{Re}\, H^{p,\omega}(\mathcal{A})$. ∎

Proposition 4.5. *Let $x \in H_\omega^p(\mathcal{M})$, and let $\omega \in \mathcal{B}_\infty^* \cap \mathcal{B}_p, 1 < p < \infty$. Then*

(1) $\overline{\widetilde{\mathrm{Re}\, x}} = (1 - \Phi)(\mathrm{Im}\, x)$,
(2) $\overline{\widetilde{\mathrm{Im}\, x}} = -(1 - \Phi)(\mathrm{Re}\, x)$,
(3) $\overline{\widetilde{x}} = -i(1 - \Phi)(x)$,
(4) $\overline{\widetilde{\overline{\widetilde{x}}}} = -(1 - \Phi)(x)$,
(5) $\overline{\widetilde{x^*}} = \overline{\widetilde{x}}^* = i(1 - \Phi)(x^*)$.

Proof. The proof is similar to the proof of Lemma 4.4 in [4]. □

Proposition 4.6. *Let $\omega \in \mathcal{B}_\infty^* \cap \mathcal{B}_p$, and let $1 < q, p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\tau(\overline{\widetilde{x}}y) = -\tau(x\overline{\widetilde{y}}), \quad x \in \Lambda_\omega^p(\mathcal{M}), y \in \Gamma_{\widetilde{\omega}}^q(\mathcal{M}),$$

where $\widetilde{\omega}(t) = t^q W(t)^{-q}\omega(t), t > 0$.

Proof. Let $x, y \in \mathrm{Re}(\mathcal{A})$. Since $e_ixe_i, e_iye_i \in \mathrm{Re}(\mathcal{A}_{e_i})$, it follows from Lemma 5.1 of [18] and Lemma 4.1 that

$$\tau(e_ixe_i\overline{\widetilde{y}}) = \tau(e_ixe_i\overline{\widetilde{e_iye_i}}) = -\tau(\overline{\widetilde{e_ixe_i}}e_iye_i) = -\tau(\overline{\widetilde{x}}e_iye_i) \tag{4.1}$$

for all $i \in I$. Note that $[\mathrm{Re}\,\mathcal{A}_{e_i}]_{p,\omega} = \mathrm{Re}\,H^{p,\omega}(\mathcal{A}_{e_i})$ and $[\mathrm{Re}\,\mathcal{A}_{e_i}]_{\Gamma_{\widetilde{\omega}}^p} = \mathrm{Re}\,H^{\Gamma_{\widetilde{\omega}}^p}(\mathcal{A}_{e_i})$. It is clear that $e_ixe_iye_i \in L^1(\mathcal{M}_{e_i})$ for $e_ixe_i \in \Lambda_\omega^p(\mathcal{M}_{e_i}), e_iye_i \in \Gamma_{\widetilde{\omega}}^q(\mathcal{M}_{e_i})$. From the continuity of $\sim$ and the $L^1(\mathcal{M}_{e_i})$-continuity of τ, we infer that the equation (4.1) holds for $x \in \mathrm{Re}\,H^{p,\omega}(\mathcal{A}), y \in \mathrm{Re}\,H^{\Gamma_{\widetilde{\omega}}^q}(\mathcal{A})$. Then the result now follows by breaking x and y into their real and imaginary parts and applying the result already established in the obvious manner on the four pairwise products; that is, $\tau(e_ixe_i\overline{\widetilde{y}}) = -\tau(\overline{\widetilde{x}}e_iye_i)$ holds for $x \in \Lambda_\omega^p(\mathcal{M}), y \in \Gamma_{\widetilde{\omega}}^q(\mathcal{M})$. Take limits to obtain the desired result. □

Using the same method of Theorem 6.2 in [18], we obtain the following result.

Theorem 4.7. *Let $1 < p < \infty$, $\omega \in \mathcal{B}_\infty^* \cap \mathcal{B}_p$. Then*

$$\Lambda_\omega^p(\mathcal{M}) = H_0^{p,\omega}(\mathcal{A}) \oplus \Lambda_\omega^p(\mathcal{D}) \oplus J\big(H_0^{p,\omega}(\mathcal{A})\big).$$

The relevant projections are $x \mapsto \frac{1}{2}[x+i\overline{\widetilde{x}}-\Phi(x)]$; $x \mapsto \Phi(x)$; $x \mapsto \frac{1}{2}[x-i\overline{\widetilde{x}}-\Phi(x)]$.

Proposition 4.8. *Let $1 < p < \infty$, let $\frac{1}{p} + \frac{1}{q} = 1$, and let $\omega \in \mathcal{B}_\infty^* \cap \mathcal{B}_p$. If $\widetilde{\omega}(t) = t^q W(t)^{-q}\omega(t), t > 0$, then*

(1) *the real linear maps*

$$\sim\, : \mathrm{Re}\big(\mathcal{A} \cap \Gamma_{\widetilde{\omega}}^q(\mathcal{M})\big) \to \mathrm{Re}\big(\mathcal{A} \cap \Gamma_{\widetilde{\omega}}^q(\mathcal{M})\big),$$
$$H : \mathrm{Re}\big(\mathcal{A} \cap \Gamma_{\widetilde{\omega}}^q(\mathcal{M})\big) \to \mathcal{A} \cap \Gamma_{\widetilde{\omega}}^q(\mathcal{M})$$

extend to real linear maps

$$\sim\, : \Gamma_{\widetilde{\omega}}^q(\mathcal{M})^{sa} \to \Gamma_{\widetilde{\omega}}^q(\mathcal{M})^{sa}, \qquad H : \Gamma_{\widetilde{\omega}}^q(\mathcal{M})^{sa} \to H^{\Gamma_{\widetilde{\omega}}^q}(\mathcal{A}).$$

If $x \in \Gamma_{\widetilde{\omega}}^q(\mathcal{M})^{sa}$, then $H(x) = x + i\widetilde{x} \in H^{\Gamma_{\widetilde{\omega}}^q}(\mathcal{A})$ and $\Phi(\widetilde{x}) = 0$. Both $\sim$ and H are bounded.

(2) $\Gamma_{\widetilde{\omega}}^q(\mathcal{M}) = H_0^{\Gamma_{\widetilde{\omega}}^q} \oplus \Gamma_{\widetilde{\omega}}^q(\mathcal{D}) \oplus J(H_0^{\Gamma_{\widetilde{\omega}}^q})$. *The relevant projections are $x \mapsto \frac{1}{2}[x + i\overline{\widetilde{x}} - \Phi(x)]$; $x \mapsto \Phi(x)$; $x \mapsto \frac{1}{2}[x - i\overline{\widetilde{x}} - \Phi(x)]$.*

Proof. Let $q > 1$, and let $\frac{1}{p} + \frac{1}{q} = 1$. The fact $\Lambda_\omega^p(\mathcal{M})^* = \Gamma_{\widetilde{\omega}}^q(\mathcal{M})$ (Proposition 2.1) and Proposition 4.6 imply that the adjoint of $\overline{\sim} : \Lambda_\omega^p(\mathcal{M}) \to \Lambda_\omega^p(\mathcal{M})$ is $-\overline{\sim} : \Gamma_{\widetilde{\omega}}^q(\mathcal{M}) \to \Gamma_{\widetilde{\omega}}^q(\mathcal{M})$. Therefore, (1) follows immediately from Theorem 4.3. A similar discussion to the proof of Theorem 4.7 shows that

$$\Gamma_{\widetilde{\omega}}^q(\mathcal{M}) = H_0^{\Gamma_{\widetilde{\omega}}^q} \oplus \Gamma_{\widetilde{\omega}}^q(\mathcal{D}) \oplus J(H_0^{\Gamma_{\widetilde{\omega}}^q}).$$ □

Corollary 4.9. *Let $1 < p < \infty$, let $\frac{1}{p} + \frac{1}{q} = 1$, and let $\omega \in \mathcal{B}^*_\infty \cap \mathcal{B}_p$. Then*

$$H^{p,\omega}(\mathcal{A})^* = H^{\Gamma^q_{\widetilde{\omega}}}(\mathcal{M})$$

with equivalent quasinorms, where $\widetilde{\omega}(t) = t^q W(t)^{-q}\omega(t), t > 0$.

Proof. Proposition 4.8 shows that $\Gamma^q_{\widetilde{\omega}}(\mathcal{M})/H_0^{\Gamma^q_{\widetilde{\omega}}}(\mathcal{A}) = J(H^{\Gamma^q_{\widetilde{\omega}}}(\mathcal{A}))$ as vector spaces. By Theorem 4.7, we deduce that $H_0^{\Gamma^q_{\widetilde{\omega}}}(\mathcal{A})$ is the annihilator of $H^{p,\omega}(\mathcal{A})$ in $\Gamma^q_{\widetilde{\omega}}(\mathcal{M})$. This implies that $H^{p,\omega}(\mathcal{A})^* = H^{\Gamma^q_{\widetilde{\omega}}}(\mathcal{M})$ with equivalent quasinorms. □

5. Interpolation

Let us recall the definitions of the K_t functionals which are fundamental in the real interpolation method. Let A_0, A_1 be a compatible couple of Banach (or quasi-Banach) spaces. This just means that A_0, A_1 are continuously included into a larger topological vector space so that we can consider unambiguously the sets $A_0 + A_1$ and $A_0 \cap A_1$. For all $x \in A_0 + A_1$ and for all $t > 0$, we let

$$K_t(x; A_0, A_1) = \inf\big\{\|x_0\|_{A_0} + t\|x_1\|_{A_1} : x = x_0 + x_1, x_i \in A_i, i = 0, 1\big\}.$$

Recall that the real interpolation space $(A_0, A_1)_{\theta,p}$ is defined as the space of all $x \in A_0 + A_1$ such that $\|x\|_{\theta,p} < \infty$, where

$$\|x\|_{\theta,p} = \Big(\int_0^\infty \big(t^{-\theta}K_t(x, A_0, A_1)\big)^p \frac{dt}{t}\Big)^{\frac{1}{p}}.$$

Let $\Psi_{\omega,p}(f) = (\int_0^\infty t^{-p}|f(t)|^p\omega(t)\,dt)^{\frac{1}{p}}, 0 < p < \infty$. The real interpolation space $(A_0, A_1)_{\Psi_{\omega,p}}$ is defined as the space of all $x \in A_0 + A_1$ such that $\Psi_{\omega,p}(K_t(x, A_0, A_1)) < \infty$. Let $A_0 = L^1(\mathcal{M}), A_1 = \mathcal{M}$. By Corollary 2.3 of [21] (or the discussion following Theorem 4.4 of [9]), we have

$$\begin{aligned}\Psi_{\omega,p}\big(K_t\big(x; L^1(\mathcal{M}), \mathcal{M}\big)\big) &= \Big(\int_0^\infty t^{-p}K_t\big(x; L^1(\mathcal{M}), \mathcal{M}\big)^p\omega(t)\,dt\Big)^{\frac{1}{p}} \\ &= \Big(\int_0^\infty \Big(\frac{1}{t}\int_0^t \mu_s(x)\,ds\Big)^p\omega(t)\,dt\Big)^{\frac{1}{p}} = \|x\|_{\Gamma^p_\omega(\mathcal{M})}.\end{aligned}$$

Therefore, $\Gamma^p_\omega(\mathcal{M}) = (L^1(\mathcal{M}), \mathcal{M})_{\Psi_{\omega,p}}$.

Proposition 5.1. *Let $0 < p < \infty$, let ω be a weighted function, and let $x \in L_0(\mathcal{M})$. Then there exists a constant $C > 0$ such that*

$$K_t\big(\mu(x); \Lambda^p_\omega, \Lambda^\infty_\omega\big) \le K_t\big(x; \Lambda^p_\omega(\mathcal{M}), \mathcal{M}\big) \le CK_t\big(\mu(x); \Lambda^p_\omega, \Lambda^\infty_\omega\big).$$

Proof. For $x \in \Lambda^p_\omega(\mathcal{M}) + \mathcal{M}$, we write $x = x_0 + x_1$, where $x_0 \in \Lambda^p_\omega(\mathcal{M}), x_1 \in \mathcal{M}$. It follows that $\mu_t(x) \le \mu_{t-\varepsilon}(x_0) + \mu_\varepsilon(x_1)$. Letting $\varepsilon \to 0$, we have $\mu_t(x) \le \mu_t(x_0) + \|\mu_t(x_1)\|_{\Lambda^\infty_\omega}, t > 0$. Thus

$$\begin{aligned}K_t\big(\mu(x); \Lambda^p_\omega, \Lambda^\infty_\omega\big) &\le \big\|\mu_t(x_0)\big\|_{\Lambda^p_\omega} + t\big\|\mu_t(x_1)\big\|_{\Lambda^\infty_\omega} \\ &= \|x_0\|_{\Lambda^p_\omega(\mathcal{M})} + t\|x_1\|_\infty,\end{aligned}$$

and so

$$K_t\big(x; \Lambda^p_\omega(\mathcal{M}), \mathcal{M}\big) \ge K_t\big(\mu(x); \Lambda^p_\omega, \Lambda^\infty_\omega\big).$$

Conversely, if $0 < x \in L_{00}(\mathcal{M})$ and $t > 0$, then we take $a = (\mu(x))^*_\omega(t^p)$. We write $x_0 = (x-a)e_{(a,\infty)}(|x|)$ and $x_1 = x - x_0 = xe_{[0,a]}(|x|) + ae_{(a,\infty)}(|x|)$. Therefore,

$$\mu(x_0)^*_\omega(s) = \big(\mu(x)^*_\omega(s) - a\big)\chi_{(0,t^p)}(s)$$

and $\mu(x_1) \leq a$. Since $t \to \mu_t(x)$ is decreasing, we obtain that

$$\int_0^{t^p} a^p \, ds \leq \int_0^{t^p} \big(\mu(x)\big)^*_\omega(s)^p \, ds.$$

Hence there exist two constants $C_0, C_1 > 0$ such that

$$\begin{aligned} K_t\big(x; \Lambda^p_\omega(\mathcal{M}), \mathcal{M}\big) &\leq \|x_0\|_{\Lambda^p_\omega(\mathcal{M})} + t\|x_1\|_\infty \\ &\leq \big\|\mu(x_0)^*_\omega\big\|_p + ta \\ &= \Big(\int_0^{t^p} \big(\mu(x)^*_\omega(s) - a\big)^p \, ds\Big)^{\frac{1}{p}} + \Big(\int_0^{t^p} a^p \, ds\Big)^{\frac{1}{p}} \\ &\leq C_0\Big(\int_0^{t^p} \big(\mu(x)^*_\omega(s)\big)^p \, ds\Big)^{\frac{1}{p}} \\ &\leq C_0 C_1 K_t\big(\mu(x); \Lambda^p_\omega, \Lambda^\infty_\omega\big). \end{aligned}$$

□

Corollary 5.2. *Let $0 < \theta < 1$, let $0 < q \leq \infty$, and let $\frac{1}{r} = \frac{1-\theta}{p}$. Then*

$$\big(\Lambda^p_\omega(\mathcal{M}), \mathcal{M}\big)_{\theta,q} = \Lambda^{r;q}_\omega(\mathcal{M}).$$

Proof. It follows immediately from Theorem 2.6.2 and Theorem 2.6.5 of [7] and Proposition 5.1. □

Proposition 5.3. *Let $0 < p_0 < p < p_1 < \infty$, let $\omega \in \mathcal{B}_{p_0}$, and let $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then*

$$\begin{aligned} \Big(\big(L^1(\mathcal{M}), \mathcal{M}\big)_{\Psi_{\omega,p_0}}, \big(L^1(\mathcal{M}), \mathcal{M}\big)_{\Psi_{\omega,p_1}}\Big)_{\theta,p} &= \big(\Lambda^{p_0}_\omega(\mathcal{M}), \Lambda^{p_1}_\omega(\mathcal{M})\big)_{\theta,p} \\ &= \Lambda^p_\omega(\mathcal{M}) = \big(L^1(\mathcal{M}), \mathcal{M}\big)_{\Psi_{\omega,p}}. \end{aligned}$$

Proof. By Corollary 2.3 of [1] and $\omega \in \mathcal{B}_{p_0}$, we have $\omega \in \mathcal{B}_{p_1}$ and $\omega \in \mathcal{B}_p$. Thus $\Gamma^{p_i}_\omega(\mathcal{M}) = \Lambda^{p_i}_\omega(\mathcal{M}), i = 0, 1$, and $\Gamma^p_\omega(\mathcal{M}) = \Lambda^p_\omega(\mathcal{M})$. Therefore,

$$\big(L^1(\mathcal{M}), \mathcal{M}\big)_{\Psi_{\omega,p_i}} = \Lambda^{p_i}_\omega(\mathcal{M}), \quad i = 0, 1.$$

Then the result follows from Theorem 2.6.5 of [7] and Proposition 5.1. □

Proposition 5.4. *Let $0 < p_0 < p_1 < \infty$, and let $\omega \in \mathcal{B}_{p_0}$. There is a constant $C > 0$ such that, for all $x \in H^{p_0,\omega}(\mathcal{A}) + H^{p_1,\omega}(\mathcal{A})$ and all $t > 0$,*

$$K_t\big(x; H^{p_0,\omega}(\mathcal{A}), H^{p_1,\omega}(\mathcal{A})\big) \leq CK_t\big(x; \Lambda^{p_0}_\omega(\mathcal{M}), \Lambda^{p_1}_\omega(\mathcal{M})\big).$$

Proof. By Proposition 3.7 and Proposition 3.8, we have

$$\big(H^1(\mathcal{A}), \mathcal{A}\big)_{\Psi_{\omega,p_i}} = H^{p_i,\omega}(\mathcal{A}), \quad i = 0, 1.$$

From the proof of Proposition 5.3, we obtain

$$\big(L^1(\mathcal{M}), \mathcal{M}\big)_{\Psi_{\omega,p_i}} = \Lambda^{p_i}_\omega(\mathcal{M}), \quad i = 0, 1.$$

Then the result follows from Theorem 2.2 of [12]. □

Proposition 5.5. *Let $0 < p_0 < p < p_1 < \infty$, and let $\omega \in \mathcal{B}_{p_0}$. Then, for $0 < \theta < 1, \frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$, we have*

$$\left(H^{p_0,\omega}(\mathcal{A}), H^{p_1,\omega}(\mathcal{A})\right)_{\theta,p} = H^{p,\omega}(\mathcal{A}).$$

Proof. It follows immediately from Proposition 5.4 and Proposition 3.8. □

6. Toeplitz operators

Let $\omega \in \mathcal{B}_\infty^* \cap \mathcal{B}_p, 1 < p < \infty$. For $t \in \mathcal{M}$, the (left) Toeplitz operator with symbol t is defined as

$$T_t : H^{p,\omega}(\mathcal{A}) \to H^{p,\omega}(\mathcal{A}) : h \to P(th),$$

where P is the projection from $\Lambda_\omega^p(\mathcal{M})$ onto $H^{p,\omega}(\mathcal{A})$. Then $T_t(h) = (P \circ L_t)(h)$, $h \in H^{p,\omega}(\mathcal{A})$. By Theorem 4.7, we obtain that P is bounded. This implies that T_t is bounded. Similarly, the (left) Toeplitz operator with symbol t on $H^{\Gamma^q_{\widetilde{\omega}}}(\mathcal{A})$ is defined as

$$T_t : H^{\Gamma^q_{\widetilde{\omega}}}(\mathcal{A}) \to H^{\Gamma^q_{\widetilde{\omega}}}(\mathcal{A}) : h \to P_1(th),$$

where P_1 is the projection from $\Gamma^q_{\widetilde{\omega}}(\mathcal{M})$ onto $H^{\Gamma^q_{\widetilde{\omega}}}(\mathcal{A})$ and

$$\widetilde{\omega}(t) = t^q W(t)^{-q}\omega(t), \quad t > 0.$$

From Proposition 4.8, we get that the projection P_1 is bounded. Thus T_t is bounded. Our basic references for Toeplitz operators on $H^2(\mathcal{A})$ in this context are [19] and [17].

Proposition 6.1. *Let $\omega \in \mathcal{B}_\infty^* \cap \mathcal{B}_p$, and let $1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$. Then*

(1) *$P^2 = P$ and $P \geq 0$;*
(2) *if $x \in \Lambda_\omega^p(\mathcal{M}), y \in \Gamma^q_{\widetilde{\omega}}(\mathcal{M})$, then $(Px, y) = (x, P_1 y)$, where $(x, y) := \tau(xy^*)$ is the duality between $\Lambda_\omega^p(\mathcal{M})$ and $\Gamma^q_{\widetilde{\omega}}(\mathcal{M})$.*

Proof. (1) Let $x \in \Lambda_\omega^p(\mathcal{M})$. It follows from Theorem 4.3 that there exist $x_1 \in H^{p,\omega}(\mathcal{A}), x_2 \in J(H_0^{p,\omega}(\mathcal{A}))$ such that $x = x_1 + x_2$. Then $(Px, x) = (x_1, x_1) + (x_1, x_2) = \tau(x_1 x_1^*) \geq 0$; that is, $P \geq 0$. $P^2 = P$ is clear.

(2) Let $x \in \Lambda_\omega^p(\mathcal{M}), y \in \Gamma^q_{\widetilde{\omega}}(\mathcal{M})$. By Theorem 4.3, there exist

$$x_1 \in H^{p,\omega}(\mathcal{A}), \qquad x_2 \in J\big(H_0^{p,\omega}(\mathcal{A})\big), \qquad y_1 \in H^{\Gamma^q_{\widetilde{\omega}}}(\mathcal{A}), \qquad y_2 \in J\big(H_0^{\Gamma^q_{\widetilde{\omega}}}(\mathcal{A})\big)$$

such that $x = x_1 + x_2$, $y = y_1 + y_2$. For $x_2 \in J(H_0^{p,\omega}(\mathcal{A}))$ and $y_1 \in H^{\Gamma^q_{\widetilde{\omega}}}(\mathcal{A})$, it is clear that there exist $z_2 \in H_0^{p,\omega}(\mathcal{A})$ and $\{y_{1n}\} \subseteq \mathcal{A} \cap \Gamma^q_{\widetilde{\omega}}(\mathcal{M})$ such that $x_2 = z_2^*$ and $y_{1n} \to y_1$ in $H^{\Gamma^q_{\widetilde{\omega}}}(\mathcal{A})$. Thus, by Theorem 3.6, we have

$$(x_2, y_{1n}) = (z_2^*, y_{1n}) = \overline{(y_{1n}, z_2^*)} = \overline{\tau(y_{1n} z_2)} = 0.$$

This implies that $(x_2, y_1) = 0$. Similarly, $(x_1, y_2) = 0$. Therefore, we obtain

$$(Px, y) = (x_1, y_1 + y_2) = (x_1, y_1) + (x_1, y_2) = (x_1, y_1)$$

and

$$(x, P_1 y) = (x_1 + x_2, y_1) = (x_1, y_1) + (x_2, y_1) = (x_1, y_1);$$

that is, $(Px, y) = (x, P_1 y)$. □

Remark 6.2. Let $\omega \in \mathcal{B}_\infty^* \cap \mathcal{B}_p$, and let $1 < p < \infty$.

(1) Given $t \in \mathcal{M}$, it is clear that

$$\|T_t x\|_{p,\omega} = \big\|P(tx)\big\|_{p,\omega} \le \|P\|\|t\|\|x\|_{p,\omega}, \quad x \in H^{p,\omega}(\mathcal{A}).$$

(2) Given $t \in \mathcal{M}$, then $T_t^* = T_{t^*}$. Indeed, let $\widetilde{\omega}(t) = t^q W(t)^{-q}\omega(t)$, let $t > 0$, and let $\frac{1}{p} + \frac{1}{q} = 1$. For $x \in H^{p,\omega}(\mathcal{A}), y \in H^{\Gamma^q_{\widetilde{\omega}}}(\mathcal{A})$, by Proposition 6.1, we have

$$(T_t x, y) = \big(P(tx), y\big) = (tx, P_1 y) = (tx, y).$$

According to Proposition 6.1 and the fact $P(x) = x$, we obtain

$$\begin{aligned}(tx, y) &= \tau(txy^*) = \tau(xy^*t) = (x, t^*y)\\ &= \big(P(x), t^*y\big) = \big(x, P_1(t^*y)\big) = (x, T_{t^*}y),\end{aligned}$$

which implies that $(T_t)^* = T_{t^*}$.

(3) If $s, t \in \mathcal{M}$ and $t \in \mathcal{A}$ or $s^* \in \mathcal{A}$, then we have $T_s T_t = T_{st}$. Indeed, let $x \in H^{p,\omega}(\mathcal{A}), y \in H^{\Gamma^q_{\widetilde{\omega}}}(\mathcal{A})$. Then

$$\begin{aligned}(T_s T_t x, y) &= (PL_s PL_t x, y) = \big(PL_t x, (L_s)^* P_1 y\big)\\ &= \big(P(tx), L_{s^*} y\big) = \big(P(tx), s^* y\big).\end{aligned}$$

If $t \in \mathcal{A}$, then

$$\big(P(tx), s^*y\big) = (tx, s^*y) = (stx, y) = (T_{st}x, y).$$

On the other hand, if $s^* \in \mathcal{A}$, then

$$\big(P(tx), s^*y\big) = \big(tx, P_1(s^*y)\big) = (tx, s^*y) = (T_{st}x, y).$$

In either case we have $(T_s T_t x, y) = (T_{st}x, y)$, as required.

Lemma 6.3. *Let $\mathcal{M}$ be a finite von Neumann algebra. Then $\mathcal{M}^{-1} := \{x \in \mathcal{M} : x^{-1} \in \mathcal{M}\}$ is dense in $\Lambda^p_\omega(\mathcal{M})$, $0 < p \le \infty$.*

Proof. It is clear that $\Lambda^p_\omega(\mathcal{M}) = [\mathcal{M}]_{p,\omega}$. We only need to show that $\mathcal{M}^{-1}$ is dense in $\mathcal{M}$ under the natural norm on $\Lambda^p_\omega(\mathcal{M})$. For every $x \in \mathcal{M}$ and $\varepsilon > 0$, let $x = u|x|$ be the polar decomposition of x. Here u is a partial isometry that can be extended to a unitary in the finite von Neumann algebra $\mathcal{M}$ (see [25], V Proposition 1.38). We will consistently write u for this extension. Put

$$f(t) = \begin{cases} t, & \text{if } t > \frac{\varepsilon}{\|1\|_{p,\omega}},\\ \frac{\varepsilon}{\|1\|_{p,\omega}}, & \text{if } t \le \frac{\varepsilon}{\|1\|_{p,\omega}},\end{cases}$$

$$g(t) = \begin{cases} \frac{1}{t}, & \text{if } t > \frac{\varepsilon}{\|1\|_{p,\omega}},\\ \frac{\|1\|_{p,\omega}}{\varepsilon}, & \text{if } t \le \frac{\varepsilon}{\|1\|_{p,\omega}},\end{cases}$$

and

$$h(t) = \begin{cases} 0, & \text{if } t > \frac{\varepsilon}{\|1\|_{p,\omega}},\\ \frac{\varepsilon}{\|1\|_{p,\omega}} - t, & \text{if } t \le \frac{\varepsilon}{\|1\|_{p,\omega}}.\end{cases}$$

Set $y = uf(|x|)$. Then $y^{-1} = g(|x|)u^*$. This implies that $y \in \mathcal{M}^{-1}$. By the Borel function calculus, we have $0 \leq h(|x|) \leq \frac{\varepsilon}{\|1\|_{p,\omega}}$, and so $\|h(|x|)\|_{p,\omega} \leq \varepsilon$. Thus

$$\begin{aligned}\|x-y\|_{p,\omega}^p &= \big\|u|x| - uf\big(|x|\big)\big\|_{p,\omega}^p \\ &= \int_0^{\tau(1)} \mu_t\big(\big|u|x| - uf\big(|x|\big)\big|\big)^p \omega(t)\,dt \\ &= \int_0^{\tau(1)} \mu_t\big(\big||x| - f\big(|x|\big)\big|\big)^p \omega(t)\,dt \\ &= \big\|h\big(|x|\big)\big\|_{p,\omega}^p \leq \varepsilon. \end{aligned}$$

□

For $x \in \mathcal{B}(\Lambda_\omega^p(\mathcal{M}))$, we will denote the spectral radius and spectrum of the operator x by $r(x)$ and $\sigma(x)$, respectively.

Theorem 6.4 (Hartman–Wintner spectral inclusion)**.** *Let $\omega \in \mathcal{B}_\infty^* \cap \mathcal{B}_p$, $1 < p < \infty$. If $\mathcal{M}$ is a semifinite von Neumann algebra and $t \in \mathcal{M}$, then $\sigma(t) = \sigma(L_t) \subseteq \sigma(T_t)$, and $r(t) = r(L_t) \leq r(T_t)$.*

Proof. Note that $\sigma(L_t) \subseteq \sigma(T_t)$ is equivalent to $\rho(T_t) \subseteq \rho(L_t)$. Since $L_t - \lambda 1 = L_{t-\lambda 1}$, we have $T_t - \lambda 1 = T_{t-\lambda 1}$. To prove the second inclusion, we only need to show that if T_t is bounded below, then so is L_t. Suppose that there exists some constant $C > 0$ such that $\|T_t x\|_{p,\omega} \geq C\|x\|_{p,\omega}$ for all $x \in H^{p,\omega}(\mathcal{A})$. Let $\varepsilon > 0$, and let $0 \neq x \in \Lambda_\omega^p(\mathcal{M})$. We write $x_i = e_i x e_i \in \Lambda_\omega^p(\mathcal{M}_{e_i})$. Let P be the projection from $\Lambda_\omega^p(\mathcal{M})$ onto $H^{p,\omega}(\mathcal{A})$ and $\delta = \frac{\varepsilon}{\frac{C}{\|P\|C_1} + \|t\|}$, where the constant C_1 is taken from the triangle inequality in Lorentz spaces Λ_ω^p. By Lemma 6.3, there exists $y_i \in \mathcal{M}_{e_i}^{-1}$ such that $\|x_i - y_i\|_{p,\omega} \leq \frac{\delta}{\|P\|C_1}$. Applying Proposition 1.2 of [20] to y_i, we can find a unitary $u_i \in \mathcal{M}_{e_i}$, $h_i \in \mathcal{A}_{e_i}$ such that $y_i = h_i u_i$. Moreover,

$$\mu(y_i) = \mu(y_i^* y_i)^{\frac{1}{2}} = \mu(y_i y_i^*)^{\frac{1}{2}} = \mu(h_i u_i u_i^* h_i^*)^{\frac{1}{2}} = \mu(h_i h_i^*)^{\frac{1}{2}} = \mu(h_i),$$

and so

$$\begin{aligned} C\|y_i\|_{p,\omega} = C\|h_i\|_{p,\omega} &\leq \|T_t h_i\|_{p,\omega} \\ &\leq \|P\|\|t h_i\|_{p,\omega} = \|P\|\|t h_i u_i\|_{p,\omega} = \|P\|\|t y_i\|_{p,\omega}. \end{aligned}$$

Thus

$$\begin{aligned} C_1\|L_t x_i\|_{p,\omega} = C_1\|t x_i\|_{p,\omega} &\geq \|t y_i\|_{p,\omega} - C_1\big\|t(x_i - y_i)\big\|_{p,\omega} \\ &\geq \frac{C}{\|P\|}\|y_i\|_{p,\omega} - \|t\|\delta \\ &\geq \frac{C}{\|P\|C_1}\big(\|x_i\|_{p,\omega} - C_1\|x_i - y_i\|_{p,\omega}\big) - \|t\|\delta \\ &\geq \frac{C}{\|P\|C_1}\big(\|x_i\|_{p,\omega} - \delta\big) - \|t\|\delta \\ &= \frac{C}{\|P\|C_1}\|x_i\|_{p,\omega} - \varepsilon, \end{aligned}$$

which implies that $\|L_t x_i\|_{p,\omega} \geq \frac{C}{\|P\|C_1^2}\|x_i\|_{p,\omega}$. Therefore,

$$\begin{aligned} C_1\|L_t x\|_{p,\omega} &\geq \|L_t x_i\|_{p,\omega} - C_1\|P\|\|t\|\|x_i - x\|_{p,\omega} \\ &\geq \frac{C}{\|P\|C_1^2}\|x_i\|_{p,\omega} - C_1\|P\|\|t\|\|x_i - x\|_{p,\omega} \\ &\geq \frac{C}{\|P\|C_1^3}\big(\|x\|_{p,\omega} - C_1\|x_i - x\|_{p,\omega}\big) - C_1\|P\|\|t\|\|x_i - x\|_{p,\omega} \\ &= \frac{C}{\|P\|C_1^3}\|x\|_{p,\omega} - \Big(\frac{C}{\|P\|C_1^2} + C_1\|P\|\|t\|\Big)\|x_i - x\|_{p,\omega}. \end{aligned}$$

By Proposition 2.1, we have $\|x_i - x\|_{p,\omega} \to 0$. Therefore, $\|L_t x\|_{p,\omega} \geq \frac{C}{\|P\|C_1^4}\|x\|_{p,\omega}$. Hence, if T_t is invertible, then L_t is invertible; that is, $\sigma(L_t) \subseteq \sigma(T_t)$. Similarly, it follows from the fact $\|t\| = \|L_t\|$ that $t \in \mathcal{M}^{-1}$ if and only if $L_t \in \mathcal{L}^{-1}$. This implies that $\lambda \in \rho(t)$ if and only if $L_{t-\lambda t}$ is invertible, and hence $\rho(t) = \rho(L_t)$. Then the first equality holds. □

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