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WANDERING SUBSPACES AND FUSION FRAME GENERATORS FOR UNITARY SYSTEMS

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ABSTRACT. This work is inspired by the study of wandering vectors and frame vectors for unitary systems. We investigate the structure and properties of complete wandering subspaces for unitary systems, and, in particular, we consider the unitary systems with a structure similar to wavelet systems. Given a unitary system with a complete wandering subspace, a necessary and sufficient condition for a closed subspace to be a Parseval fusion frame generator is obtained. Moreover, we study the dilation property for Parseval fusion frame generators for unitary groups.

1. INTRODUCTION

In operator theory, wandering vectors and wandering subspaces have been studied for unitary systems and isometry systems (see [8], [14], [21]). Wavelet theory entails the study of wandering vectors for unitary systems. Dai and Larson [8] showed that orthogonal wavelets can be viewed as wandering vectors for dilation-translation unitary systems. The connection between multiresolution analysis and the concept of wandering subspaces of unitary operators in Hilbert spaces was given by Goodman, Lee, and Tang [13]. We know that multiresolution analysis plays an important role in wavelet theory. Indeed, the classical construction of wavelets arises from multiresolution analysis. With the development of wavelets, many different aspects of the wavelet theory have been studied. They are useful

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in many areas of mathematics and theoretical physics (see [2], [3]) and also in practical applications such as image and signal processing. One purpose of this paper is to investigate the properties of wandering subspaces for unitary systems and especially, with a structure similar to wavelet systems.

As a generalization of Riesz bases in Hilbert spaces, frames allow representations of vectors which are not necessarily unique. This property makes them very useful in many fields of applications such as signal and image processing (see [4]) and wavelet and frequency analysis (see [9], [10]). However, in some applications, when we deal with a huge amount of data it is often beneficial to subdivide a large frame system into smaller subsystems and locally combine data vectors. In other words, sometimes we need to construct global frames from smaller local ones. This leads to the concept of fusion frames (frames of subspaces; see [5], [6]), which are an extension to frames. Fusion frames are also very useful. They can provide an extensive framework not only to model sensor networks but also to improve robustness or develop efficient and feasible information processing algorithms (see [6], [22]). Moreover, as we know, many useful frames with a special structure, such as Gabor frames and wavelet frames (see [11], [15], [16]), play an essential role in both theory and applications. Motivated by Gabor analysis, one often considers unitary systems, group-like unitary systems, or projective unitary representations for a countable group. So, another purpose of this paper is to study fusion frames with the structure of unitary systems.

We now review an important example for unitary systems. Let T and D be the operators on the Hilbert space $L^2(\mathbb{R})$ defined by

$$(Tf)(t) = f(t-1), \qquad (Df)(t) = \sqrt{2}f(2t), \quad \text{for } f \in L^2(\mathbb{R}).$$

They are unitary operators and, in fact, bilateral shifts of infinite multiplicity, with wandering subspaces $L^2([0,1])$ and $L^2([-2,-1]\cup[1,2])$, respectively, considered as subspaces of $L^2(\mathbb{R})$. They are not commutative, and we have $TD = DT^2$. Hence,

$$\mathcal{U}_{D,T} = \{D^nT^l : n, l \in \mathbb{Z}\}$$

is an example of a countable unitary system which consists of noncommuting unitary operators, and it does not form a group (see [7], [9]). Usually, $\mathcal{U}_{D,T}$ is called a *wavelet system*. It is well known that the group generated by $\{D, T\}$ is $\{D^nT_\beta : n \in \mathbb{Z}, \beta \in \mathcal{D}\}$, where $\mathcal{D}$ denotes the set of dyadic rational numbers, and for real number β, T_β denotes the translation unitary operator $(T_\beta f)(t) = f(t-\beta)$.

The paper is organized as follows. In Section 2, we first investigate the properties of the local commutants for a set of operators at a subspace of vectors. Then for a unital semigroup of unitary operators, we present a structure characterization of all complete wandering subspaces. More properties of complete wandering subspaces for a special class of unitary systems with a structure similar to wavelet systems are obtained. Section 3 is devoted to the study of fusion frames with the structure of unitary systems. We introduce the concept of fusion frame generators, and, for a unitary system with a complete wandering subspace, we give a necessary and sufficient condition for a closed subspace to be a Parseval fusion frame generator. Moreover, we study the dilation property for Parseval fusion frame generators for unitary groups. We remark that our results partly extend those in [8] and [16] for complete wandering vectors and complete frame vectors for unitary systems.

Throughout this paper, H denotes a complex separable Hilbert space, and $B(H)$ represents the algebra of all bounded linear operators on H. For subsets $W \subseteq H$ and $\mathcal{R} \subseteq B(H)$, let $[W]$ be the closure of the linear span of W, let $w^*(\mathcal{R})$ be the von Neumann algebra generated by $\mathcal{R}$, and let $\mathbb{U}(\mathcal{R})$ be the set of all unitary operators in $\mathcal{R}$. If W is a closed subspace of H, then we write π_W for the orthogonal projection onto W unless otherwise specified.

2. Local commutants and wandering subspaces

Following Dai and Larson [8], a *unitary system* is a subset of unitary operators acting on H which contains the identity operator I. For a unitary system $\mathcal{U}$, a closed subspace W of H is called a *wandering subspace* for $\mathcal{U}$ if UW and VW are orthogonal for all $U, V \in \mathcal{U}$ with $U \neq V$. A wandering subspace W is called *complete* if $\operatorname{span}\{UW : U \in \mathcal{U}\}$ is dense in H. The set of all complete wandering subspaces for $\mathcal{U}$ is denoted by $\mathcal{S}(\mathcal{U})$. It is easy to see that if $\{e_i : i \in \mathbb{I}\}$ is an orthonormal basis for W, then W is a complete wandering subspace for $\mathcal{U}$ if and only if $\{Ue_i : U \in \mathcal{U}, i \in \mathbb{I}\}$ is an orthonormal basis for H. Furthermore, let $\mathcal{R} \subseteq B(H)$ be a set, and let $W \subseteq H$ be a subspace. Call W *cyclic* for $\mathcal{R}$ if $[\mathcal{R}W] = H$ and *separating* for $\mathcal{R}$ if $AW = \{0\}$ implies $A = 0$. The *local commutant* of $\mathcal{R}$ at W is defined by

$$C_W(\mathcal{R}) = \big\{T \in B(H) : (TR - RT)W = \{0\} \text{ for } R \in \mathcal{R}\big\}.$$

It should be mentioned that $(TR - RT)W = \{0\}$ implies $TRW = RTW$, but the reverse implication is not true. The notation $\mathcal{R}'$ will denote the usual commutant of $\mathcal{R}$; that is,

$$\mathcal{R}' = \big\{T \in B(H) : TR = RT \text{ for } R \in \mathcal{R}\big\}.$$

Clearly, $C_W(\mathcal{R})$ contains $\mathcal{R}'$ and is a strongly closed subspace of $B(H)$.

Proposition 2.1. *Let $\mathcal{R} \subseteq B(H)$ be a set, and let $W \subseteq H$ be a cyclic subspace for $\mathcal{R}$. Then the following hold:*

(1) *The subspace W is separating for $C_W(\mathcal{R})$.*
(2) *If $\mathcal{R}$ is a semigroup, then $C_W(\mathcal{R}) = \mathcal{R}'$.*
(3) *If $A \in C_W(\mathcal{R})$ with dense range, then AW is a cyclic subspace for $\mathcal{R}$.*
(4) *Suppose that W is also a separating subspace for $\mathcal{R}$. If $R_1, R_2 \in \mathcal{R}$ with $R_1R_2, R_2R_1 \in \mathcal{R}$ and $R_1R_2 \neq R_2R_1$, then neither R_1 nor R_2 is in $C_W(\mathcal{R})$.*
(5) *Let $\mathcal{R} = \mathcal{R}_1\mathcal{R}_2$, where $\mathcal{R}_1$ is a semigroup. Then $C_W(\mathcal{R}) \subseteq \mathcal{R}_1'$.*
(6) *If $T \in C_W(\mathcal{R})$ is invertible, then $C_{TW}(\mathcal{R}) = C_W(\mathcal{R})T^{-1}$.*
(7) *For any $A \in \mathcal{R}'$ and $B \in C_W(\mathcal{R})$, we have $AB \in C_W(\mathcal{R})$.*
(8) *Let $\mathcal{R}$ be a semigroup, and let a set $\widetilde{\mathcal{R}} \subseteq B(H)$ such that $\mathcal{R} \subseteq \widetilde{\mathcal{R}}$, $\mathcal{R}' = \widetilde{\mathcal{R}}'$. Then $C_W(\widetilde{\mathcal{R}}) = C_W(\mathcal{R}) = \mathcal{R}'$.*

Proof. (1) If $A \in C_W(\mathcal{R})$ and $AW = \{0\}$, then for all $R \in \mathcal{R}$, we have $ARW = RAW = \{0\}$. Hence, $A[\mathcal{R}W] = AH = \{0\}$, implying $A = 0$. So, W is separating for $C_W(\mathcal{R})$.

(2) It is sufficient to show $C_W(\mathcal{R}) \subseteq \mathcal{R}'$. Suppose that $A \in C_W(\mathcal{R})$. Then for any $R, T \in \mathcal{R}$, we have $RT \in \mathcal{R}$, and so

$$AR(Tx) = A(RT)x = (RT)Ax = R(TA)x = RA(Tx)$$

for any $x \in W$. Since $[\mathcal{R}W] = H$, it follows that $AR = RA$; that is, $A \in \mathcal{R}'$.

(3) Since $A \in C_W(\mathcal{R})$, for any $R \in \mathcal{R}$ and $x \in W$, we have $ARx = RAx$. Also, $[\mathcal{R}W] = H$ and A has dense range, so $[\mathcal{R}AW] = [AH] = H$. Hence, AW is a cyclic subspace for $\mathcal{R}$.

(4) Suppose on the contrary that $R_1 \in C_W(\mathcal{R})$. Then $(R_1R_2 - R_2R_1)W = \{0\}$ because of $R_2 \in \mathcal{R}$. Since $R_1R_2, R_2R_1 \in \mathcal{R}$ and W is a separating subspace for $\mathcal{R}$, we obtain $R_1R_2 = R_2R_1$, which contradicts the assumption $R_1R_2 \neq R_2R_1$. So, $R_1 \notin C_W(\mathcal{R})$. Similarly, we can get that $R_2 \notin C_W(\mathcal{R})$.

(5) We have $\mathcal{R}_1\mathcal{R} \subseteq \mathcal{R}$, clearly. Let $A \in C_W(\mathcal{R})$ and $B \in \mathcal{R}_1$. Then for any $R \in \mathcal{R}$ and $x \in W$, we have $ARx = RAx$, and, moreover,

$$A(BR)x = (BR)Ax = B(RA)x = B(AR)x$$

because $BR \in \mathcal{R}$. That is, $(AB)Rx = (BA)Rx$ for all $R \in \mathcal{R}$, $x \in W$. Since $[\mathcal{R}W] = H$, it follows that $AB = BA$, and so $A \in \mathcal{R}_1'$. Hence, $C_W(\mathcal{R}) \subseteq \mathcal{R}_1'$.

(6) By definition, we have

$$\begin{aligned} C_{TW}(\mathcal{R}) &= \big\{A \in B(H) : (AR - RA)TW = \{0\} \text{ for } R \in \mathcal{R}\big\} \\ &= \big\{A \in B(H) : (ART - RAT)W = \{0\} \text{ for } R \in \mathcal{R}\big\} \\ &= \big\{A \in B(H) : (ATR - RAT)W = \{0\} \text{ for } R \in \mathcal{R}\big\} \\ &= \big\{A \in B(H) : AT \in C_W(\mathcal{R})\big\} \\ &= C_W(\mathcal{R})T^{-1}, \end{aligned}$$

where the third equality follows from the fact that $T \in C_W(\mathcal{R})$.

(7) For any $A \in \mathcal{R}', B \in C_W(\mathcal{R})$ and $R \in \mathcal{R}$, we have

$$(AB)Rx = A(BR)x = A(RB)x = (RA)Bx = R(AB)x$$

for all $x \in W$. That is, $(ABR - RAB)W = \{0\}$. Hence, $AB \in C_W(\mathcal{R})$.

(8) By the assumptions and the statement (2), we have $C_W(\mathcal{R}) = \mathcal{R}' = \widetilde{\mathcal{R}}'$. But, it is easy to see that $\widetilde{\mathcal{R}}' \subseteq C_W(\widetilde{\mathcal{R}}) \subseteq C_W(\mathcal{R})$. So $C_W(\widetilde{\mathcal{R}}) = C_W(\mathcal{R}) = \mathcal{R}'$. $\square$

Denote $C_1(H)$ by the space of trace-class operators and by $\mathrm{tr}(\cdot)$ the trace of a trace-class operator. It is well known that $B(H)$ can be identified with the dual of $C_1(H)$ via the pairing $(T, S) = \mathrm{tr}(TS)$ for $S \in B(H)$, $T \in C_1(H)$. For a subspace $\mathcal{R}$ of $B(H)$, call $\mathcal{R}$ *reflexive* if

$$\mathcal{R} = \big\{T \in B(H) : Tx \in [\mathcal{R}x] \text{ for all } x \in H\big\},$$

and n-*reflexive* if the n-fold ampliation $\mathcal{R}^{(n)} := \{T^{(n)} : T \in \mathcal{R}\}$ is a reflexive subspace of $B(H^{(n)})$. A famous result tells us that a weakly closed subspace of $B(H)$ is n-reflexive if and only if the preannihilator $\mathcal{R}_\perp$ in $C_1(H)$ is a trace-class norm $\|\cdot\|_1$-closed linear span of operators of rank at most n (see, e.g., [19]).

Proposition 2.2. *Let $\mathcal{R} \subseteq B(H)$, and let W be a subspace of H. Then*

$$\big(C_W(\mathcal{R})\big)_\perp = \overline{\operatorname{span}}^{\|\cdot\|_1}\big\{[R, x \otimes y] : R \in \mathcal{R}, x \in W, y \in H\big\},$$

where $x \otimes y$ denotes the rank-one operator defined by $(x \otimes y)z = \langle z, y\rangle x$ for $z \in H$, and $[R, x \otimes y] = R(x \otimes y) - (x \otimes y)R$. Hence, $C_W(\mathcal{R})$ is 2-reflexive.

Proof. For any $A \in B(H)$, we have

$$\begin{aligned}
\operatorname{tr}\big(A[R, x \otimes y]\big) &= \operatorname{tr}\big(A(Rx \otimes y - x \otimes R^*y)\big) \\
&= \operatorname{tr}(ARx \otimes y) - \operatorname{tr}(Ax \otimes R^*y) \\
&= \langle ARx, y\rangle - \langle Ax, R^*y\rangle \\
&= \langle ARx, y\rangle - \langle RAx, y\rangle \\
&= \big\langle (AR - RA)x, y\big\rangle.
\end{aligned}$$

This implies that $A \in C_W(\mathcal{R})$ if and only if A is annihilated by all trace-class operators of the form $[R, x \otimes y]$ for $R \in \mathcal{R}$ and $x \in W, y \in H$. So, $(C_W(\mathcal{R}))_\perp = \overline{\operatorname{span}}^{\|\cdot\|_1}\{[R, x \otimes y] : R \in \mathcal{R}, x \in W, y \in H\}$. $\square$

We now want to characterize the set of all complete wandering subspaces for a unital semigroup of unitaries. For this, we need two lemmas.

Lemma 2.3. *Suppose that $\mathcal{U}$ is a unitary system on H and $W \in \mathcal{S}(\mathcal{U})$.*

(1) *If $\Omega \in \mathcal{S}(\mathcal{U})$ with $\dim \Omega = \dim W$, then there exists a unitary operator $T \in C_W(\mathcal{U})$ such that $\Omega = TW$.*
(2) *If T is a unitary operator in $C_W(\mathcal{U})$, then $TW \in \mathcal{S}(\mathcal{U})$.*

Proof. (1) Suppose $\Omega \in \mathcal{S}(\mathcal{U})$ such that $\dim \Omega = \dim W$. Let $\{e_i\}_{i\in\mathbb{I}}$ and $\{f_i\}_{i\in\mathbb{I}}$ be orthonormal bases for W and Ω, respectively, where $\mathbb{I}$ is an index set with cardinal number $\dim W$. Then both $\{Ue_i : i \in \mathbb{I}, U \in \mathcal{U}\}$ and $\{Uf_i : i \in \mathbb{I}, U \in \mathcal{U}\}$ are orthonormal bases for H. So, we can define a unitary operator T on H by $TUe_i = Uf_i$ for all $i \in \mathbb{I}$, $U \in \mathcal{U}$. Then clearly $TW = \Omega$. Since $I \in \mathcal{U}$, for any $U \in \mathcal{U}$, we have $TUe_i = Uf_i = UTe_i$ for all $i \in \mathbb{I}$, and hence, $T \in C_W(\mathcal{U})$.

(2) Let T be a unitary operator in $C_W(\mathcal{U})$. We first prove that $UTW \perp VTW$ for all $U, V \in \mathcal{U}$ with $U \neq V$. In fact, for any $U, V \in \mathcal{U}$ with $U \neq V$ and $x, y \in W$, since $W \in \mathcal{S}(\mathcal{U})$, we have

$$\langle UTx, VTy\rangle = \langle TUx, TVy\rangle = \langle Ux, Vy\rangle = 0.$$

So, $UTW \perp VTW$ for all $U \neq V$. Second, we verify that $\overline{\operatorname{span}}\{UTW : U \in \mathcal{U}\} = H$. Let $y \perp \operatorname{span}\{UTW : U \in \mathcal{U}\}$. Then for all $U \in \mathcal{U}$, $x \in W$, we have

$$\langle T^{-1}y, Ux\rangle = \langle y, TUx\rangle = \langle y, UTx\rangle = 0.$$

Since $\overline{\operatorname{span}}\{UW : U \in \mathcal{U}\} = H$, it follows that $T^{-1}y = 0$, and so $y = 0$. Thus, $\overline{\operatorname{span}}\{UTW : U \in \mathcal{U}\} = H$. So $TW \in \mathcal{S}(\mathcal{U})$ as required. $\square$

Lemma 2.4 ([17, Corollary 1.2])**.** *Let $\mathcal{U}$ be a unitary group on H, and let W be a complete wandering subspace for $\mathcal{U}$. Then every complete wandering subspace for $\mathcal{U}$ has the same dimension as W.*

The following theorem is one main result in this section and plays a key role in the rest of the paper.

Theorem 2.5. *Let $\mathcal{U}$ be a unital semigroup of unitaries in $B(H)$, and suppose that there exists an element $W \in \mathcal{S}(\mathcal{U})$. Then*

$$\mathcal{S}(\mathcal{U}) = \big\{TW : T \in \mathbb{U}(\mathcal{U}')\big\}.$$

Proof. We claim that $\mathcal{U}$ is, in fact, a group. Otherwise, there exists $A \in \mathcal{U}$ such that $A^{-1} \notin \mathcal{U}$. For every $B \in \mathcal{U}$, since $\mathcal{U}$ is a semigroup, we have $AB \in \mathcal{U}$ and $AB \neq I$. Fix a nonzero vector $x \in W$. Then for all $y \in W$, one has $\langle A^{-1}x, By\rangle = \langle x, ABy\rangle = 0$. This implies that $A^{-1}x \perp BW$ for all $B \in \mathcal{U}$, contradicting the fact $[\mathcal{U}W] = H$. So, $\mathcal{U}$ is a group.

Suppose that $\Omega \in \mathcal{S}(\mathcal{U})$. By Lemma 2.4, we obtain $\dim\Omega = \dim W$. Then the result follows immediately from Lemma 2.3. $\square$

The following result shows the commutativity of local commutants.

Proposition 2.6. *Let $\mathcal{U}$ be a unitary system on H, and suppose that $C_W(\mathcal{U})$ is abelian for some $W \in \mathcal{S}(\mathcal{U})$. Then $C_\Omega(\mathcal{U})$ is abelian for all $\Omega \in \mathcal{S}(\mathcal{U})$ with $\dim\Omega = \dim W$.*

Proof. Suppose that $C_W(\mathcal{U})$ is abelian for some $W \in \mathcal{S}(\mathcal{U})$ and $\Omega \in \mathcal{S}(\mathcal{U})$ with $\dim\Omega = \dim W$. By Lemma 2.3(1), we know that there exists $T \in \mathbb{U}(C_\Omega(\mathcal{U}))$ such that $W = T\Omega$. Then $C_W(\mathcal{U}) = C_{T\Omega}(\mathcal{U}) = C_\Omega(\mathcal{U})T^*$ by Proposition 2.1(6). Clearly, $T^* \in C_W(\mathcal{U})$ and $T^* \in (C_W(\mathcal{U}))'$. Since T is normal, we have $T \in (C_W(\mathcal{U}))'$ by the Fuglede–Putnam theorem. This implies that $C_\Omega(\mathcal{U}) = C_W(\mathcal{U})T$ is abelian. $\square$

In the remainder of this section, unless otherwise specified, we always suppose that $\mathcal{U}$ is a unitary system in $B(H)$ and that $\mathcal{U}$ contains a subset $\mathcal{U}_0$ which is a group such that $\mathcal{U}\mathcal{U}_0 = \mathcal{U}$. Since this is just the case of the wavelet system $\mathcal{U}_{D,T} = \{D^nT^l : n, l \in \mathbb{Z}\}$ for $L^2(\mathbb{R})$, where $\mathcal{U}_0 = \{T^l : l \in \mathbb{Z}\}$, we call $\mathcal{U}$ a *wavelet-like unitary system*.

Lemma 2.7. *Let $W \in \mathcal{S}(\mathcal{U})$, and let $U \in \mathcal{U}_0$. Then $UW \in \mathcal{S}(\mathcal{U})$, and there exists a unique unitary operator T_U in $C_W(\mathcal{U})$ such that $T_Ux = Ux$ for all $x \in W$.*

Proof. Clearly, $UW \in \mathcal{S}(\mathcal{U})$ and $\dim W = \dim UW$. Let $\{e_i\}_{i\in\mathbb{I}}$ be an orthonormal basis for W. Similar to the proof of Lemma 2.3(1), define an operator T_U on H by $Ve_i \mapsto VUe_i$ for all $i \in \mathbb{I}, V \in \mathcal{U}$. Then T_U is in $C_W(\mathcal{U})$ and unitary and satisfies $T_Ux = Ux$ for all $x \in W$. Suppose that there are two operators T_1, T_2 in $C_W(\mathcal{U})$ such that $T_1x = T_2x = Ux$ for all $x \in W$. Then

$$T_1Vx = VT_1x = VUx = VT_2x = T_2Vx$$

for all $V \in \mathcal{U}$, $x \in W$. The uniqueness of the operator T_U follows from $[\mathcal{U}W] = H$. $\square$

By this lemma, for a given $W \in \mathcal{S}(\mathcal{U})$, we get a well-defined map

$$K_W : \mathcal{U}_0 \to \mathbb{U}\big(C_W(\mathcal{U})\big), \qquad U \mapsto T_U.$$

Note that $\mathcal{U}_0$ will not usually be contained in $C_W(\mathcal{U})$.

Theorem 2.8. *Let $W \in \mathcal{S}(\mathcal{U})$. Then $K_W(\mathcal{U}_0)$ is a group, and K_W is a group anti-isomorphism. Moreover, if we regard the elements in $\mathcal{S}(\mathcal{U})$ as orthogonal projections, then the set $\mathcal{U}_0 W$ is contained in a connected subset of $\mathcal{S}(\mathcal{U})$ in the norm topology.*

Proof. For any $U_1, U_2 \in \mathcal{U}_0$, $T \in \mathcal{U}$, and $x \in W$, we have

$$\begin{aligned} K_W(U_2)K_W(U_1)Tx &= K_W(U_2)TK_W(U_1)x = K_W(U_2)TU_1x \\ &= TU_1K_W(U_2)x = TU_1U_2x \\ &= TK_W(U_1U_2)x = K_W(U_1U_2)Tx. \end{aligned}$$

Since $[\mathcal{U}W] = H$, we know $K_W(U_2)K_W(U_1) = K_W(U_1U_2)$ and $K_W(I) = I$ by taking $U_1 = U_2 = I$. Then clearly $K_W(\mathcal{U}_0)$ is a group, and K_W is an antihomomorphism. Moreover, if $U_1, U_2 \in \mathcal{U}_0$ are different, then $U_1W \neq U_2W$ as $U_1W \perp U_2W$. This implies $K_W(U_1) \neq K_W(U_2)$, and hence, K_W is one-to-one.

Observe that the closure of the span$\{K_W(\mathcal{U}_0)\}$ in the strong operator topology is the von Neumann algebra $w^*(K_W(\mathcal{U}_0))$ and is contained in $C_W(\mathcal{U})$. Define a map

$$\mathbb{U}\big(w^*\big(K_W(\mathcal{U}_0)\big)\big) \to B(H), \qquad U \mapsto \pi_{UW}.$$

Since U is a unitary operator, we know that $\pi_{UW} = U\pi_W U^*$. Then it is easy to see that the map $U \mapsto \pi_{UW}$ is norm continuous. Recalling that Theorem 2.5 and that the unitary group of a von Neumann algebra is norm connected (see [18]), we can get that $\{\pi_{UW} : U \in \mathbb{U}(w^*(K_W(\mathcal{U}_0)))\}$ is norm connected in $\{\pi_\Omega : \Omega \in \mathcal{S}(\mathcal{U})\}$. For $U \in \mathcal{U}_0, x \in W$, since $Ux = T_Ux = K_W(U)x$, we have $UW = K_W(U)W$, and hence,

$$\{\pi_{UW} : U \in \mathcal{U}_0\} \subseteq \big\{\pi_{UW} : U \in \mathbb{U}\big(w^*\big(K_W(\mathcal{U}_0)\big)\big)\big\},$$

as required. $\square$

In the case that $\mathcal{U}_0$ is abelian, the domain of the map K_W can be enlarged as follows.

Theorem 2.9. *Let $\mathcal{U}$ be a waveletlike unitary system such that $\mathcal{U}_0$ is abelian.*

(1) *If $U \in \mathbb{U}(w^*(\mathcal{U}_0))$, then $U\mathcal{S}(\mathcal{U}) \subseteq \mathcal{S}(\mathcal{U})$.*
(2) *For $W \in \mathcal{S}(\mathcal{U})$, the map K_W extends to a homomorphism from $\mathbb{U}(w^*(\mathcal{U}_0))$ into $\mathbb{U}(C_W(\mathcal{U}))$.*

Proof. (1) Let $U \in \mathbb{U}(w^*(\mathcal{U}_0))$, $W \in \mathcal{S}(\mathcal{U})$, and write $\Omega = UW$. To show $\Omega \in \mathcal{S}(\mathcal{U})$, denote

$$E_W = [\mathcal{U}_0 W] = \big[w^*(\mathcal{U}_0)W\big].$$

Then clearly $UE_W \subseteq E_W$ and $U^*E_W \subseteq E_W$, from which we have $UE_W = E_W$. Let $T \in \mathcal{U}$, but $T \notin \mathcal{U}_0$. Then $TS \notin \mathcal{U}_0$ for all $S \in \mathcal{U}_0$, and so $TS_1W \perp S_2W$ for all $S_1, S_2 \in \mathcal{U}_0$, because $W \in \mathcal{S}(\mathcal{U})$. This yields $TE_W \perp E_W$. More generally, if $T_1, T_2 \in \mathcal{U}$ such that $T_1\mathcal{U}_0 \neq T_2\mathcal{U}_0$, we have $T_1\mathcal{U}_0 \cap T_2\mathcal{U}_0 = \emptyset$. Hence, $T_1U_1W \perp T_2U_2W$ for all $U_1, U_2 \in \mathcal{U}_0$. Thus, $T_1E_W \perp T_2E_W$, and then $T_1\Omega \perp T_2\Omega$ as $\Omega = UW \subseteq E_W$. On the other hand, suppose that $T_1, T_2 \in \mathcal{U}$ such that $T_1 \neq T_2$ but $T_1\mathcal{U}_0 = T_2\mathcal{U}_0$. Then there exists some $U_1 \in \mathcal{U}_0$ such that $U_1 \neq I$, $T_1U_1 = T_2$. Noting that $\mathcal{U}_0$ is abelian, so is $w^*(\mathcal{U}_0)$ and $UU_1 = U_1U$. It follows from $U_1W \perp W$ that $\langle U_1Ux, Uy\rangle = \langle UU_1x, Uy\rangle = \langle U_1x, y\rangle = 0$ for all $x, y \in W$. This yields $U_1\Omega \perp \Omega$, and then $T_1\Omega \perp T_2\Omega (= T_1U_1\Omega)$. Also,

$$[\mathcal{U}_0\Omega] = [\mathcal{U}_0UW] = \big[w^*(\mathcal{U}_0)UW\big] = \big[w^*(\mathcal{U}_0)W\big] = E_W.$$

Thus, $[\mathcal{U}\Omega] = [\mathcal{U}\mathcal{U}_0\Omega] = [\mathcal{U}E_W] \supseteq [\mathcal{U}W] = H$. So $\Omega \in \mathcal{S}(\mathcal{U})$.

(2) Let $W \in \mathcal{S}(\mathcal{U})$, and let $U \in \mathbb{U}(w^*(\mathcal{U}_0))$. Then we have $UW \in \mathcal{S}(\mathcal{U})$ by the conclusion (1). Similar to Lemma 2.7, we can get a unique unitary operator $T_U \in C_W(\mathcal{U})$ such that $T_Ux = Ux$ for all $x \in W$. Define a map

$$K_W : U\big(w^*(\mathcal{U}_0)\big) \to \mathbb{U}\big(C_W(\mathcal{U})\big), \qquad U \mapsto T_U.$$

Let $U_1, U_2 \in \mathbb{U}(w^*(\mathcal{U}_0))$, $x \in W$, and let $T \in \mathcal{U}$. Note that TU_1 is in the strongly closed linear span of $\mathcal{U}$; thus, similar to the proof of Theorem 2.8, one has

$$\begin{aligned} K_W(U_2)K_W(U_1)Tx &= K_W(U_2)TK_W(U_1)x = K_W(U_2)TU_1x \\ &= TU_1K_W(U_2)x = TU_1U_2x \\ &= TK_W(U_1U_2)x = K_W(U_1U_2)Tx. \end{aligned}$$

This implies that $K_W(U_2)K_W(U_1) = K_W(U_1U_2) = K_W(U_2U_1)$, since $[\mathcal{U}W] = H$ and $\mathcal{U}_0$ is abelian. □

We now give two examples to illustrate some of the results in this section.

Example 2.10. Let $\{e_n\}_{n=-\infty}^{+\infty}$ be an orthonormal basis for a separable Hilbert space H, and let S be the bilateral shift of multiplicity one; that is, $Se_n = e_{n+1}$ for any $n \in \mathbb{Z}$. Let $\mathcal{U} = \{S^{2n} : n \in \mathbb{Z}\}$ be the group generated by S^2, and let $W = \operatorname{span}\{e_0, e_1\}$. Then it is easy to check $W \in \mathcal{S}(\mathcal{U})$. By Proposition 2.1(2) and Theorem 2.5, we have

$$\mathcal{S}(\mathcal{U}) = \big\{TW : T \in \mathbb{U}\big(\{S^2\}'\big)\big\}.$$

More generally, given a positive integer k, let $\mathcal{U}_k = \{S^{kn} : n \in \mathbb{Z}\}$ be the group generated by S^k, and let $W_k = \operatorname{span}\{e_0, e_1, \ldots, e_{k-1}\}$. Then $W_k \in \mathcal{S}(\mathcal{U}_k)$ and

$$\mathcal{S}(\mathcal{U}_k) = \big\{TW_k : T \in \mathbb{U}\big(\{S^k\}'\big)\big\}.$$

Example 2.11. Let D, T be operators on $L^2(\mathbb{R})$ defined in the Introduction. A family of closed subspaces $\{\Omega_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ is said to be a *multiresolution analysis* if it satisfies the following conditions (see [7], [13], [20]):

(i) $\Omega_j \subset \Omega_{j+1}$ for each $j \in \mathbb{Z}$;
(ii) $D(\Omega_j) = \Omega_{j+1}$ and $T(\Omega_0) = \Omega_0$;
(iii) $\overline{\bigcup_j \Omega_j} = L^2(\mathbb{R})$ and $\bigcap_j \Omega_j = \{0\}$;
(iv) there exists a scaling function $\varphi \in \Omega_0$ such that $\{T^k\varphi : k \in \mathbb{Z}\}$ is an orthonormal basis for Ω_0.

For every $j \in \mathbb{Z}$, let W_j be the orthogonal complement of Ω_j in Ω_{j+1}. From [20], we know that there exists $\psi \in W_0$ such that $\{T^k\psi : k \in \mathbb{Z}\}$ is an orthonormal basis for W_0, and so we can obtain an orthonormal basis $\{D^jT^k\psi : j, k \in \mathbb{Z}\}$ for $L^2(\mathbb{R})$. Then W_0 is a complete wandering subspace for the wavelet system $\mathcal{U}_{D,T} = \{D^nT^l : n, l \in \mathbb{Z}\}$ and the unitary group $\mathcal{U}_1 = \{D^n : n \in \mathbb{Z}\}$, respectively. So by Proposition 2.1, we have the following:

(1) $C_{W_0}(\mathcal{U}_1) = \{D\}'$.
(2) $C_{W_0}(\mathcal{U}_{D,T}) \subseteq \{D\}' \cap \{T\}'$.
(3) Let $\widetilde{W} \subseteq H$ and $A \in \mathbb{U}(C_{W_0}(\mathcal{U}_{D,T}))$ such that $AW_0 = \widetilde{W}$. Then $C_{\widetilde{W}}(\mathcal{U}_{D,T}) = C_{W_0}(\mathcal{U}_{D,T})A^*$.

3. Wandering subspaces and fusion frame generators

In this section, we consider fusion frames with the structure of unitary systems. In the case when the unitary system has a complete wandering subspace, we obtain a necessary and sufficient condition for a closed subspace to be a Parseval fusion frame generator. Moreover, we want to study the dilation property for Parseval fusion frames and Parseval fusion frame generators.

Definition 3.1 (see [5], [6]). Let $\mathbb{I}$ be some index set, let $\{W_i\}_{i\in\mathbb{I}}$ be a family of closed subspaces in a Hilbert space H, and let $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights; that is, $v_i > 0$ for all $i \in \mathbb{I}$. Then the family $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is called a *fusion frame* (*frame of subspaces*) for H if there exist constants $0 < C \leq D < \infty$ such that

$$C\|x\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \|\pi_{W_i} x\|^2 \leq D\|x\|^2 \quad \text{for all } x \in H.$$

We call C and D the *fusion frame bounds*, and if we only have the upper bound, then $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is said to be a *Bessel fusion sequence*. The family $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is called a *Parseval fusion frame* provided that $C = D = 1$, and an *orthonormal fusion basis* if $H = \sum_{i\in\mathbb{I}} \oplus W_i$. Moreover, we call a fusion frame with respect to $\{v_i\}_{i\in\mathbb{I}}$ *v-uniform* if $v := v_i = v_j$ for all $i, j \in \mathbb{I}$.

By [5, Proposition 3.23], the family $\{W_i\}_{i\in\mathbb{I}}$ of closed subspaces in H is a 1-uniform Parseval fusion frame if and only if it is an orthonormal fusion basis.

Let $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ be a Bessel fusion sequence for H. The *analysis operator* θ is defined by

$$\theta : H \to \Big(\sum_{i\in\mathbb{I}} \oplus W_i\Big)_{\ell^2} \quad \text{with } \theta(x) = \{v_i \pi_{W_i} x\}_{i\in\mathbb{I}},$$

where

$$\Big(\sum_{i\in\mathbb{I}} \oplus W_i\Big)_{\ell^2} := \Big\{\{x_i\}_{i\in\mathbb{I}} : x_i \in W_i \text{ and } \sum_{i\in\mathbb{I}} \|x_i\|^2 < \infty\Big\}$$

is the usual (external) direct sum of Hilbert spaces. It is easy to see that the adjoint operator θ^* is given by

$$\theta^* : \Big(\sum_{i\in\mathbb{I}} \oplus W_i\Big)_{\ell^2} \to H \quad \text{with } \theta^*(x) = \sum_{i\in\mathbb{I}} v_i x_i,$$

where $x = \{x_i\}_{i\in\mathbb{I}} \in (\sum_{i\in\mathbb{I}} \oplus W_i)_{\ell^2}$. The *frame operator* S is defined by

$$S : H \to H \quad \text{with } Sx = \theta^*\theta(x) = \sum_{i\in\mathbb{I}} v_i^2 \pi_{W_i} x.$$

Clearly, a Bessel fusion sequence $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is a fusion frame if and only if the frame operator is positive and invertible on H.

For our purpose, and motivated by frame vectors for unitary systems (see [16]), we introduce the following concept.

Definition 3.2. Let $\mathcal{U}$ be a unitary system on H. A closed subspace W of H is called a *fusion frame generator* (resp., *Parseval fusion frame generator* and *Bessel fusion sequence generator*) for $\mathcal{U}$ with respect to $\{v_U\}_{U\in\mathcal{U}}$ of weights, if $\{(UW, v_U)\}_{U\in\mathcal{U}}$ is a fusion frame (resp., Parseval fusion frame and Bessel fusion sequence) for H.

Let $\mathcal{R} \subseteq B(H)$ be a set, and let W be a subspace of H. We say that two operators $A, B \in B(H)$ are *linearly dependent* on W if there exists some nonzero constant μ such that $Ax = \mu Bx$ for all $x \in W$. Denote that

$$\begin{aligned} C_W^g(\mathcal{R}) = \big\{&T \in B(H) : TR \text{ and } RT \text{ are linearly dependent on } W \\ &\text{for all } R \in \mathcal{R}\big\}, \\ \mathcal{R}'_g = \big\{&T \in B(H) : TR \text{ and } RT \text{ are linearly dependent on } H \\ &\text{for all } R \in \mathcal{R}\big\}. \end{aligned}$$

We call $C_W^g(\mathcal{R})$ and $\mathcal{R}'_g$ the *generalized local commutant of* $\mathcal{R}$ *at* W and the *generalized commutant of* $\mathcal{R}$, respectively. Clearly, $C_W^g(\mathcal{R})$ contains $\mathcal{R}'_g$, but it is not necessarily a subspace.

Proposition 3.3. *Let $\mathcal{U}$ be a unitary system on H, let W be a fusion frame generator for $\mathcal{U}$ with respect to some weights, and let T be an invertible operator in $C_W^g(\mathcal{U})$. Then TW is a fusion frame generator for $\mathcal{U}$ with respect to the same weights as W.*

Proof. Note that $T \in C_W^g(\mathcal{U})$ implies $TUW = UTW$ for every $U \in \mathcal{U}$. The result is immediate by [12, Theorem 2.4]. □

The following result shows that all Parseval fusion frame generators for a unitary system $\mathcal{U}$ can be characterized in terms of operators in $C_W^g(\mathcal{U})$, where W is a complete wandering subspace for $\mathcal{U}$.

Theorem 3.4. *Let $\mathcal{U}$ be a unitary system on H, let W be a complete wandering subspace for $\mathcal{U}$, and let Ω be a closed subspace of H such that $\dim\Omega = \dim W$. Then Ω is a Parseval fusion frame generator for $\mathcal{U}$ with respect to some family $\{v_U\}_{U\in\mathcal{U}}$ of weights if and only if there are a coisometry $A \in C_W^g(\mathcal{U})$ (i.e., A^* is an isometry) and a nonzero constant μ such that $\Omega = AW$ and the operator μA is isometric on W.*

Proof. Suppose that Ω is a Parseval fusion frame generator for $\mathcal{U}$ with respect to the weights $\{v_U\}_{U\in\mathcal{U}}$. Let $\{e_i\}_{i\in\mathbb{I}}$ and $\{f_i\}_{i\in\mathbb{I}}$ be orthonormal bases for W and Ω, respectively, where $\mathbb{I}$ is an index set with cardinal number $\dim W$ and it can be ∞. Note that for $U \in \mathcal{U}$, $\{Uf_i\}_{i\in\mathbb{I}}$ constitutes an orthonormal basis for $U\Omega$. Then for any $x \in H$, we have

$$\sum_{i\in\mathbb{I}}\sum_{U\in\mathcal{U}} v_U^2 \big|\langle x, Uf_i\rangle\big|^2 = \sum_{i\in\mathbb{I}}\sum_{U\in\mathcal{U}} v_U^2 \big|\langle \pi_{U\Omega}x, Uf_i\rangle\big|^2$$

$$
\begin{aligned}
&= \sum_{U\in\mathcal{U}} v_U^2 \|\pi_{U\Omega} x\|^2 \\
&= \|x\|^2.
\end{aligned}
$$

Taking into account that $\{Ue_i\}_{i\in\mathbb{I},U\in\mathcal{U}}$ is an orthonormal basis for H, we can define a linear isometric operator $B : H \to H$ by

$$
Bx = \sum_{i\in\mathbb{I}} \sum_{U\in\mathcal{U}} v_U \langle x, Uf_i\rangle Ue_i, \quad x \in H.
$$

Obviously, B is of closed range. Denote P by the orthogonal projection onto BH, and let $A = B^*(= B^*P)$. Then for all $x \in W$, $U, V \in \mathcal{U}$ and $i \in \mathbb{I}$, we have

$$
\begin{aligned}
\langle Vx, AUe_i\rangle = \langle BVx, Ue_i\rangle &= \Big\langle \sum_{j\in\mathbb{I}} \sum_{S\in\mathcal{U}} v_S \langle Vx, Sf_j\rangle Se_j, Ue_i \Big\rangle \\
&= v_U \langle Vx, Uf_i\rangle.
\end{aligned}
$$

Recalling $[\mathcal{U}W] = H$, we get $AUe_i = v_U Uf_i$ and, in particular, $Ae_i = v_I f_i$ by choosing $U = I$. It follows that

$$
AUe_i = \frac{v_U}{v_I} UAe_i
$$

for all $i \in \mathbb{I}$, $U \in \mathcal{U}$. Then clearly the coisometry $A \in C_W^g(\mathcal{U})$ and $\Omega = [AW]$. Moreover, for every $x \in W$,

$$
\begin{aligned}
\|Ax\|^2 &= \Big\|\sum_{i\in\mathbb{I}} \langle x, e_i\rangle Ae_i\Big\|^2 = v_I^2 \Big\|\sum_{i\in\mathbb{I}} \langle x, e_i\rangle f_i\Big\|^2 \\
&= v_I^2 \sum_{i\in\mathbb{I}} \big|\langle x, e_i\rangle\big|^2 = v_I^2 \|x\|^2.
\end{aligned}
$$

This implies that the operator $\frac{1}{v_I}A$ is isometric on W. It turns out that AW is closed, and so $\Omega = AW$.

Conversely, let A and μ be of the properties described as in the theorem. Write (still) $\{e_i\}_{i\in\mathbb{I}}$ for an orthonormal basis for W, and let $f_i = \mu Ae_i$ for all $i \in \mathbb{I}$. Then $\{f_i\}_{i\in\mathbb{I}}$ is an orthonormal basis for Ω, because $\Omega = AW$ and μA is isometric on W. Noting that $A \in C_W^g(\mathcal{U})$, we have, for each $U \in \mathcal{U}$, a nonzero constant λ_U so that $AUx = \lambda_U UAx$ for all $x \in W$. Denote

$$
v_U = \Big|\frac{\lambda_U}{\mu}\Big|
$$

for all $U \in \mathcal{U}$. Since A^* is an isometry and $\{UW\}_{U\in\mathcal{U}}$ is an orthonormal fusion basis for H, we have, for all $x \in H$,

$$
\begin{aligned}
\|x\|^2 = \|A^*x\|^2 &= \sum_{U\in\mathcal{U}} \|\pi_{UW} A^*x\|^2 \\
&= \sum_{U\in\mathcal{U}} \sum_{i\in\mathbb{I}} \big|\langle \pi_{UW} A^*x, Ue_i\rangle\big|^2 \\
&= \sum_{U\in\mathcal{U}} \sum_{i\in\mathbb{I}} \big|\langle x, AUe_i\rangle\big|^2 = \sum_{U\in\mathcal{U}} \sum_{i\in\mathbb{I}} |\lambda_U|^2 \big|\langle x, UAe_i\rangle\big|^2
\end{aligned}
$$

$$= \sum_{U\in\mathcal{U}}\sum_{i\in\mathbb{I}}\Big|\frac{\lambda_U}{\mu}\Big|^2\big|\langle x, Uf_i\rangle\big|^2 = \sum_{U\in\mathcal{U}}\sum_{i\in\mathbb{I}} v_U^2\big|\langle \pi_{U\Omega}x, Uf_i\rangle\big|^2$$
$$= \sum_{U\in\mathcal{U}} v_U^2\|\pi_{U\Omega}x\|^2.$$

This shows that $\{(U\Omega, v_U)\}_{U\in\mathcal{U}}$ is a Parseval fusion frame for H; that is, Ω is a Parseval fusion frame generator for $\mathcal{U}$ with respect to $\{v_U\}_{U\in\mathcal{U}}$. The proof is complete. □

It is well known that frames have a natural geometric interpretation as sequences of vectors which can be dilated to bases, and a similar dilation property holds true for frame vectors (see [16]). We now consider the generalizations of this dilation property for fusion frames.

Proposition 3.5. *Let $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ be a Parseval fusion frame for H. Then there exist a Hilbert space $K \supseteq H$ and an orthonormal fusion basis $\{N_i\}_{i\in\mathbb{I}}$ for K such that $PN_i = W_i$ for all $i \in \mathbb{I}$, where P is the orthogonal projection from K onto H.*

Proof. Let $\theta : H \to (\sum_{i\in\mathbb{I}} \oplus W_i)_{\ell^2}$ be the analysis operator for $\{(W_i, v_i)\}_{i\in\mathbb{I}}$. Since $\{W_i, v_i\}_{i\in\mathbb{I}}$ is a Parseval fusion frame, we have that θ is an isometry with closed range. Denote the Hilbert space $K = H \oplus \theta(H)^\perp$, and define a linear operator

$$U : K \to \Big(\sum_{i\in\mathbb{I}} \oplus W_i\Big)_{\ell^2} \quad \text{by } x \oplus y \mapsto \theta x + y,$$

where $x \in H$ and $y \in \theta(H)^\perp$. Then clearly U is unitary. Let E_i be the canonical embedding of W_i in $(\sum_{i\in\mathbb{I}} \oplus W_i)_{\ell^2}$, and let $N_i = U^*E_i$. Then $\{E_i\}_{i\in\mathbb{I}}$ is an orthonormal fusion basis for $(\sum_{i\in\mathbb{I}} \oplus W_i)_{\ell^2}$, and hence $\{N_i\}_{i\in\mathbb{I}}$ constitutes an orthonormal fusion basis for K. Denote by P the orthogonal projection from K onto H. It is easily seen that $\theta = U|_H$, $\theta^* = PU^*$ and that $\theta^*(\{\dots, 0, x_i, 0, \dots\}) = v_i x_i$ for $x_i \in W_i$. Then $PN_i = PU^*E_i = \theta^*E_i = W_i$. □

We remark that the above result appeared in [1]. Here, we present a different proof and a smaller dilation space "K." We next study the dilation property for fusion frame generators.

Let $\mathcal{U}$ be a unitary group on H, and let e_U be the element in the Hilbert space $\ell^2(\mathcal{U})$ which takes values 1 at U and zero elsewhere. Then $\{e_U : U \in \mathcal{U}\}$ is an orthonormal basis for $\ell^2(\mathcal{U})$. The *left regular representation* of $\mathcal{U}$ on $\ell^2(\mathcal{U})$ gives the unitary group $\{L_U\}_{U\in\mathcal{U}}$, where we describe the transformation of L_U on each element of the orthonormal basis as follows:

$$L_U e_V = e_{UV}, \quad V \in \mathcal{U}.$$

Theorem 3.6. *Suppose that $\mathcal{U}$ is a unitary group on H and Ω is a Parseval fusion frame generator for $\mathcal{U}$ with respect to the weights $\{v_U\}_{U\in\mathcal{U}}$. Then there exist a Hilbert space $K \supseteq H$, a unitary group $\mathcal{G}$ on K which has a complete wandering subspace W, and a group isomorphism α from $\mathcal{U}$ onto $\mathcal{G}$ such that $U\Omega = P\alpha(U)W$ for all $U \in \mathcal{U}$, where P is the orthogonal projection from K*

onto H. In particular, if Ω is a v-uniform Parseval fusion frame generator for $\mathcal{U}$, then $\mathcal{G}$ and W can be chosen so that H is an invariant subspace of $\mathcal{G}$ and $\mathcal{G}|_H = \mathcal{U}$.

Proof. Since Ω is a Parseval fusion frame generator for $\mathcal{U}$ with respect to the weights $\{v_U\}_{U\in\mathcal{U}}$, we can define an isometric operator

$$\theta : H \to \ell^2(\mathcal{U}) \otimes H \quad \text{by } x \mapsto \sum_{U\in\mathcal{U}} e_U \otimes v_U \pi_{U\Omega} x,$$

where the notation $\otimes$ denotes the tensor product. Let the Hilbert space $K = H \oplus \theta(H)^{\perp}$, and define

$$B : K \to \ell^2(\mathcal{U}) \otimes H \quad \text{by } x \oplus y \mapsto \theta x + y,$$

where $x \in H$ and $y \in \theta(H)^{\perp}$. Then B is a unitary operator. Denote $\widetilde{L}_U = B^*(L_U \otimes I)B$ for every $U \in \mathcal{U}$ and $\mathcal{G} = \{\widetilde{L}_U : U \in \mathcal{U}\}$. It is easy to see that $\mathcal{G}$ constitutes a unitary group on K, $B^*(e_U \otimes H)$ is a complete wandering subspace for $\mathcal{G}$ for every $U \in \mathcal{U}$, and the mapping

$$\alpha : \mathcal{U} \to \mathcal{G} \quad \text{by } U \mapsto \widetilde{L}_U$$

is an isomorphism. Put $W = B^*(e_I \otimes H)$, and let P be the orthogonal projection from K onto H. We now want to prove that $P\widetilde{L}_U W = U\Omega$ for all $U \in \mathcal{U}$. In fact, we have $\theta^* = PB^*$, clearly. Then for $U \in \mathcal{U}$, $x, y \in H$, we obtain

$$\begin{aligned}\big\langle P\widetilde{L}_U B^*(e_I \otimes x), y\big\rangle &= \big\langle PB^*(e_U \otimes x), y\big\rangle = \big\langle \theta^*(e_U \otimes x), y\big\rangle \\ &= \langle e_U \otimes x, \theta y\rangle = \Big\langle e_U \otimes x, \sum_{S\in\mathcal{U}} e_S \otimes v_S \pi_{S\Omega} y\Big\rangle \\ &= v_U \langle \pi_{U\Omega} x, y\rangle.\end{aligned}$$

It follows that $P\widetilde{L}_U B^*(e_I \otimes x) = v_U \pi_{U\Omega} x$. Hence, $P\widetilde{L}_U W = U\Omega$, and $PW = \Omega$ by taking $U = I$.

In particular, suppose that Ω is a v-uniform Parseval fusion frame generator for $\mathcal{U}$. Then for $x \in H$, $U \in \mathcal{U}$, one has

$$\begin{aligned}(L_U \otimes U)Bx = (L_U \otimes U)\theta x &= (L_U \otimes U)\Big(\sum_{S\in\mathcal{U}} e_S \otimes v\pi_{S\Omega} x\Big) \\ &= \sum_{S\in\mathcal{U}} e_{US} \otimes vU\pi_{S\Omega} x = \sum_{S\in\mathcal{U}} e_S \otimes vU\pi_{U^{-1}S\Omega} x \\ &= \sum_{S\in\mathcal{U}} e_S \otimes v\pi_{S\Omega} Ux = \theta Ux \\ &= BUx.\end{aligned}$$

So, $B^*(L_U \otimes U)B|_H = U$. Take $\widehat{\mathcal{G}} = \{\widehat{L}_U : U \in \mathcal{U}\}$, where $\widehat{L}_U = B^*(L_U \otimes U)B$. Then $\widehat{\mathcal{G}}$ is a unitary group on K and $\widehat{\mathcal{G}}|_H = \mathcal{U}$. Additionally, we can check that the subspace $W = B^*(e_I \otimes H)$ defined above is still a complete wandering subspace for $\widehat{\mathcal{G}}$, and $U\Omega = P\widehat{L}_U W$ for all $U \in \mathcal{U}$. This completes the proof. $\square$

Let us conclude by considering the equivalence of unitary representations for groups, which is determined by wandering subspaces.

Let $\mathcal{G}$ be a group. A *unitary representation* of $\mathcal{G}$ on a Hilbert space H is a group homomorphism π from $\mathcal{G}$ into the unitary group on H; as usual, write $(\mathcal{G}, \pi, H)$ or simply π for such a representation. If a closed nonzero subspace K of H is invariant under each operator in $\pi(\mathcal{G})$, then the mapping $g \mapsto \pi(g)|_K$ is a unitary representation of $\mathcal{G}$ on K, which is called a *subrepresentation* of π. Let H_1, H_2 be two Hilbert spaces. Call two unitary representations $(\mathcal{G}, \pi_1, H_1)$ and $(\mathcal{G}, \pi_2, H_2)$ *unitarily equivalent* if there is a unitary operator $T : H_1 \to H_2$ such that $T\pi_1(g) = \pi_2(g)T$ for all $g \in \mathcal{G}$.

Proposition 3.7. *Let $\mathcal{G}$ be a group, and let $(\mathcal{G}, \pi_1, H_1)$, $(\mathcal{G}, \pi_2, H_2)$ be two unitary representations such that $\pi_1(\mathcal{G})$ and $\pi_2(\mathcal{G})$ admit complete wandering subspaces W_1 and W_2, respectively. Then the following hold:*

(1) *If $\dim W_1 = \dim W_2$, then π_1, π_2 are unitarily equivalent.*
(2) *If $\dim W_1 < \dim W_2$, then π_1 is equivalent to a subrepresentation of π_2.*

Proof. (1) Suppose that $\dim W_1 = \dim W_2$, and let $\{e_i\}_{i\in\mathbb{I}}$, $\{f_i\}_{i\in\mathbb{I}}$ be orthonormal bases for W_1, W_2, respectively, where $\mathbb{I}$ is an index set with cardinal number $\dim W_1$. Then let $\{\pi_1(g)e_i : i \in \mathbb{I}, g \in \mathcal{G}\}$, $\{\pi_2(g)f_i : i \in \mathbb{I}, g \in \mathcal{G}\}$ be orthonormal bases for H_1, H_2, respectively. Define a unitary operator $T : H_1 \to H_2$ by

$$T\pi_1(g)e_i = \pi_2(g)f_i \quad \text{for all } i \in \mathbb{I}, g \in \mathcal{G}.$$

Then for any $g, h \in \mathcal{G}$ and $i \in \mathbb{I}$, we have

$$T\pi_1(g)\pi_1(h)e_i = T\pi_1(gh)e_i = \pi_2(gh)f_i = \pi_2(g)\pi_2(h)f_i = \pi_2(g)T\pi_1(h)e_i.$$

Hence, $T\pi_1(g) = \pi_2(g)T$, which implies that π_1, π_2 are unitarily equivalent.

(2) Let $m = \dim W_1$ and $n = \dim W_2$. By the hypothesis, we know $m < \infty$. Take an m-dimensional subspace N of W_2. Then N is a wandering subspace (not necessarily complete) for $\pi_2(\mathcal{G})$. Denote

$$K = \overline{\operatorname{span}}\{\pi_2(\mathcal{G})N\}.$$

Then clearly, K is a closed subspace of H_2 and is invariant under every operator in $\pi_2(\mathcal{G})$. Define a new mapping

$$\tilde{\pi}_2 : \mathcal{G} \to B(K) \quad \text{by } g \mapsto \pi_2(g)|_K.$$

We have that $\tilde{\pi}_2$ is a unitary representation of $\mathcal{G}$ on K. Consider the unitary representations $(\mathcal{G}, \pi_1, H_1)$ and $(\mathcal{G}, \tilde{\pi}_2, K)$. By (1), there is a unitary operator $A : H_1 \to K$ such that $A\pi_1(g) = \tilde{\pi}_2(g)A$ for all $g \in \mathcal{G}$. This shows that π_1, $\tilde{\pi}_2$ are unitarily equivalent. $\square$

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