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COVER-STRICT TOPOLOGIES, IDEALS, AND QUOTIENTS FOR SOME SPACES OF VECTOR-VALUED FUNCTIONS

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Abstract. Let X be a completely regular Hausdorff space, let $\mathcal{D}$ be a cover of X, and let $\pi : \mathcal{E} \to X$ be a bundle of Banach spaces (algebras). Let $\Gamma(\pi)$ be the space of sections of π, and let $\Gamma_b(\pi, \mathcal{D})$ be the subspace of $\Gamma(\pi)$ consisting of sections which are bounded on each $D \in \mathcal{D}$. We study the subspace (ideal) and quotient structures of some spaces of vector-valued functions which arise from endowing $\Gamma_b(\pi, \mathcal{D})$ with the cover-strict topology.

1. Introduction

The present article investigates the ideal and quotient structures of certain algebras of vector-valued functions. By using the theory of bundles of topological vector spaces, our results extend to more general algebras many of the results to be found in [1] and [3] regarding the structure of some ideals and quotients of $C(X)$, where X is a completely regular Hausdorff space.

We will be concerned with certain subspaces and quotients of $\Gamma(\pi)$, the space of sections of the bundle of Banach spaces (i.e., Banach bundle) $\pi : \mathcal{E} \to X$, and in particular, we will investigate such structures when $\pi : \mathcal{E} \to X$ is a bundle of Banach algebras (i.e., Banach algebra bundle). (For details of the development of such bundles, and of bundles of topological vector spaces in general, we refer the reader to [5]; further elaboration can be found in [10], [12], [6], and [7].) The essentials are the following (they can be found, e.g., in [7]), but we repeat them here for convenience. Our paper is also related to the *theory of approximation*

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in upper semicontinuous function spaces (see, e.g., [15] and [9] and, even earlier, [13], along with their references).

We will, unless otherwise noted, let X be a completely regular Hausdorff space. The scalar space, either $\mathbb{R}$ or $\mathbb{C}$, will be denoted by $\mathbb{K}$. As usual, a $\mathbb{K}$-valued function f on X is said to vanish at infinity if for each $\varepsilon > 0$ there exists compact $K \subset X$ such that $|f(x)| < \varepsilon$ whenever $x \notin K$. We denote by $S_0^+(X)$ the set of nonnegative upper semicontinuous functions (weights) on X which vanish at infinity. If g is any function defined on X, and if $C \subset X$, we let g_C be the restriction of g to C. If G is a collection of functions on X, then $G_C = \{g_C : g \in G\}$.

Consider now the following situation. Let $\{E_x : x \in X\}$ be a collection of Banach spaces over $\mathbb{K}$, indexed by X, let the total space $\mathcal{E} = \bigcup^\bullet \{E_x : x \in X\}$ be their disjoint union, and let $\pi : \mathcal{E} \to X$ be the natural projection. As usual, $C_b(X)$ will denote the space of $\mathbb{K}$-valued bounded and continuous functions on X. We let $\mathcal{S}$ be a vector space of choice functions $\sigma : X \to \mathcal{E}$ (i.e., $\sigma(x) \in E_x$ for each $x \in X$) such that the following conditions hold:

- (C1) for each $x \in X$, $\phi_x(\mathcal{S}) = \{\sigma(x) : \sigma \in \mathcal{S}\} = E_x$ (in this case, $\mathcal{S}$ is said to be *full*; ϕ_x is the evaluation map at x);
- (C2) $\mathcal{S}$ is a $C_b(X)$-module;
- (C3) for each $\sigma \in \mathcal{S}$, the numerical function $x \mapsto \|\sigma(x)\|$ is upper semicontinuous and bounded on X;
- (C4) $\mathcal{S}$ is closed in the supremum-norm topology, $\|\sigma\| = \sup_{x\in X} \|\sigma(x)\|$; and
- (C5) for each $x \in X$, the relative topology on $E_x \subset \mathcal{E}$ is its norm topology.

Under these conditions, there is a topology on $\mathcal{E}$ (the bundle topology) which makes $\mathcal{S}$ a subspace of the space $\Gamma(\pi)$ of all sections ("section" here is equivalent to "continuous choice function") $\tau : X \to \mathcal{E}$. In this bundle topology, a neighborhood of $z \in E_x \subset \mathcal{E}$ is given by tubes of the form

$$T = T(U, z, \varepsilon) = \big\{z' \in \mathcal{E} : \pi(z') \in U \text{ and } \big\|\sigma\big(\pi(z')\big) - z'\big\| < \varepsilon\big\},$$

where $U \subset X$ is a neighborhood of x, $\sigma \in \Gamma(\pi)$ with $\sigma(\pi(z)) = z$, and $\varepsilon > 0$. Especially, if $\mathcal{S}$ satisfies these conditions, it is a subspace of $\Gamma_b(\pi)$, the space of bounded sections of the bundle $\pi : \mathcal{E} \to X$ (or π, if there can be no confusion). Then the addition map from $\mathcal{E} \vee \mathcal{E}$ to $\mathcal{E}$, $(z, z') \mapsto z + z'$, is continuous, where $\mathcal{E} \vee \mathcal{E} = \{(z, z') \in \mathcal{E} \times \mathcal{E} : \pi(z) = \pi(z')\}$ is the fibered product of $\mathcal{E}$ with itself, and the multiplication map $C_b(X) \times \Gamma_b(\pi) \to \Gamma(\pi)$, $(f, \sigma) \mapsto f\sigma$, is jointly continuous when both $C_b(X)$ and $\Gamma_b(\pi)$ are given their supremum-norms, with $\Gamma_b(\pi)$ then being a Banach space.

If each fiber E_x is a Banach algebra and if $\mathcal{S}$ is an algebra, then π is a bundle of Banach algebras. In this case, multiplication from $\mathcal{E} \vee \mathcal{E}$ to $\mathcal{E}$, $(z, z') \mapsto zz'$, is also continuous, $\Gamma_b(\pi)$ is a Banach algebra, and $\Gamma(\pi)$ is an algebra (evidently, commutative if and only if each fiber E_x is commutative). Call $\pi : \mathcal{E} \to X$ a *line bundle* if each fiber $E_x = \mathbb{K}$.

The intuitive notion in effect here is that if $\pi : \mathcal{E} \to X$ is a Banach bundle and if $\sigma \in \Gamma(\pi)$, then we can think of $\sigma(x)$ as moving continuously through the various spaces E_x as x moves continuously through X. We also note here that, as in [6] and [7], many of the following results or their analogues would also hold in the context of bundles of locally convex vector spaces (resp., locally convex algebras). We have chosen to restrict ourselves to Banach bundles as a ground case in order to keep notational complexity to a minimum.

2. The cover-strict topology on a section space

The subspaces (subalgebras) of $C(X)$ whose ideal and quotient structures are studied in [1] and [3] have topologies that are determined by certain seminorms. Noting that $C(X)$ is (to within topological and algebraic isomorphism) the section space of the trivial bundle $\pi_1 : \mathcal{E}_1 = \bigcup^\bullet_{x\in X} \mathbb{K} \to X$, where $\mathcal{E}_1$ is homeomorphic to $\mathbb{K} \times X$ in its product topology, we adapt the $C(X)$-situation from those papers to the case of certain subspaces of $\Gamma(\pi)$ for the Banach bundle $\pi : \mathcal{E} \to X$.

Lemma 2.1. *Let $D \subset X$, and let $v \in S_0^+(D)$. Let $v' : X \to \mathbb{R}$ be the extension of v defined by $v'(x) = v(x)$ if $x \in D$, and by $v'(x) = 0$ if $x \notin D$. Then $v' \in S_0^+(X)$.*

Proof. Let $\varepsilon > 0$ be given. Then there exists $K \subset D$, compact in the relative topology of D, such that $v(x) = v'(x) < \varepsilon$ for $x \in D \setminus K$. But since K is compact in D, it is also compact (and closed) in X, and it is then evident that $v'(x) < \varepsilon$ for $x \in X \setminus K$. Thus, v' disappears at infinity on X. Now, given that $\varepsilon > 0$, choose $K \subset D$ as above. If $x \in X \setminus K$, then $X \setminus K$ is an open neighborhood of x such that $v'(y) < \varepsilon$ for each $y \in X \setminus K$. If $x \in D$ with $v(x) = v'(x) < \varepsilon$, then there is an open neighborhood $x \in U \subset X$ such that if $y \in U \cap D$, then $v(y) = v'(y) < \varepsilon$. On the other hand, if $y \in U \cap (X \setminus D)$, then $v'(y) = 0 < \varepsilon$, so that in any event $v'(y) < \varepsilon$ if $y \in U$. Thus, $v' \in S_0^+(X)$. ☐

Let $\pi : \mathcal{E} \to X$ be a bundle of Banach spaces, and let $\mathcal{D}$ be a cover of X. Set

$$\Gamma_b(\pi, \mathcal{D}) = \{\sigma \in \Gamma(\pi) : \sigma \text{ is bounded on each } D \in \mathcal{D}\}.$$

(Hence, for $\sigma \in \Gamma_b(\pi)$, $\sigma_D : D \to \mathcal{E}_D$ is continuous, where $\mathcal{E}_D = \bigcup^\bullet\{E_x : x \in D\}$.)

Note that for any cover $\mathcal{D}$, we have

$$\Gamma_b(\pi) \subset \Gamma_b(\pi, \mathcal{D}) \subset \Gamma(\pi).$$

For each $\sigma \in \Gamma_b(\pi, \mathcal{D})$, $D \in \mathcal{D}$, and $v_D \in S_0^+(D)$ the numerical function $x \mapsto v'_D(x)\|\sigma(x)\|$ is in $S_0^+(X)$. Set

$$p_{D,v'_D}(\sigma) = \sup_{x\in X}\{v'_D(x)\|\sigma(x)\|\} = \sup_{x\in D}\{v_D(x)\|\sigma(x)\|\} = \sup_{x\in X} p^x_{D,v'_D}(\sigma) < \infty,$$

where we denote $p^x_{D,v'_D}(\sigma) = v'_D(x)\|\sigma(x)\|$. Then, as $D \in \mathcal{D}$ and $v_D \in S_0^+(D)$ vary, the p_{D,v'_D} form a collection of seminorms on $\Gamma_b(\pi, \mathcal{D})$ satisfying conditions analogous to (C1)–(C4) above:

- (C1′) for each $x \in X$, we have $\phi_x(\Gamma_b(\pi, \mathcal{D})) = \{\sigma(x) : \sigma \in \mathcal{S}\} = E_x$;
- (C2′) $\Gamma_b(\pi, \mathcal{D})$ is a $C_b(X)$-module;
- (C3′) for each $\sigma \in \Gamma_b(\pi, \mathcal{D})$, $D \in \mathcal{D}$, and $v_D \in S_0^+(D)$ the numerical function $x \mapsto v'_D(x)\|\sigma(x)\|$ is in $S_0^+(X)$;
- (C4′) $\Gamma_b(\pi, \mathcal{D})$ is closed in the seminorm topology generated by the p_{D,v'_D}, as $D \in \mathcal{D}$ and $v_D \in S_0^+(D)$ vary; and

(C5′) the topology induced on E_x by the p^x_{D,v'_D} ($D \in \mathcal{D}$, $v_D \in S_0^+(D)$) is equivalent to the original norm topology.

The seminorms p_{D,v'_D} generate a locally convex topology on $\Gamma_b(\pi, \mathcal{D})$ as $D \in \mathcal{D}$, $v_D \in S_0^+(D)$, and $\varepsilon > 0$ vary and, should π be a bundle of Banach algebras, then that topology is a locally multiplicatively convex topology on $\Gamma_b(\pi, \mathcal{D})$ (see [7] for a similar situation and the calculation). In any event, sets of the form

$$N(\sigma, p_{D,v'_D}, \varepsilon) = \{\tau \in \Gamma_b(\pi, \mathcal{D}) : p_{D,v'_D}(\sigma - \tau) < \varepsilon\}$$

then form a subbasis around $\sigma \in \Gamma_b(\pi, \mathcal{D})$. It can also be easily checked that the multiplication $(f, \sigma) \mapsto f\sigma$ is jointly continuous from $C_b(X) \times \Gamma_b(\pi, \mathcal{D})$ to $\Gamma_b(\pi, \mathcal{D})$, so that $\Gamma_b(\pi, \mathcal{D})$ is a topological $C_b(X)$-module.

The seminorms p_{D,v'_D} also determine a topology on the fibered space $\mathcal{E}$. Here, a subbasic neighborhood around $z \in E_x \subset \mathcal{E}$ is given by tubes of the form

$$T(U, \sigma, p_{D,v'_D}, \varepsilon) = \{z' \in \mathcal{E} : p^{\pi(z')}_{D,v_{D'}}\big(\sigma(\pi(z')) - z'\big) < \varepsilon \text{ and } \pi(z') \in U \cap D\},$$

where U is an X-open neighborhood of $x \in D \in \mathcal{D}, v_D \in S_0^+(D), \varepsilon > 0$, and $\sigma \in \Gamma_b(\pi, \mathcal{D})$ is any element such that $\sigma(x) = z$.

Checking the claims about (C1′)–(C5′), we see that (C1′) is satisfied because $\Gamma(\pi)$ is full; (C2′) is self-evident; and (C5′) obtains because the $p^x_{D,v'_D}(\sigma)$ are simply scalar multiples of $\|\sigma(x)\|$. Claim (C3′) follows because $x \mapsto \|\sigma(x)\|$ is upper semicontinuous on X and bounded on each $D \in \mathcal{D}$. As for (C4′), let $\sigma \in \Gamma(\pi)$ be the limit in the topology determined by $\mathfrak{W} = \{p_{D,v'_D} : D \in \mathcal{D}, v_D \in S_0^+(D)\}$ of a net $(\sigma_\lambda) \subset \Gamma_b(\pi, \mathcal{D})$, and let $D \in \mathcal{D}$. For each $x \in D, \chi_x \in S_0^+(D)$, where χ_x is the characteristic function of the singleton set $\{x\}$. Given that $\varepsilon > 0$, then, we see that eventually $\sigma_\lambda \in N(\sigma, D, \chi_x, \varepsilon)$, and equivalently that $\|\sigma_\lambda(x) - \sigma(x)\| < \varepsilon$, so that (σ_λ) converges pointwise on D. Since (σ_λ) does converge, it is eventually bounded in the topology determined by $\mathfrak{W}$, so that there exists m such that, for our given $D \in \mathcal{D}$ and arbitrary $v \in S_0^+(D)$, we eventually have $p_{D,v'_D}(\sigma_\lambda) \leq m$. Setting $v_D = \chi_x$ for $x \in D$, this forces $\|\sigma_\lambda(x)\| \leq m$ for all $x \in D$ if λ is large enough. But σ is the pointwise limit of uniformly bounded functions on D, and hence σ is itself bounded on D. Thus, $\sigma \in \Gamma_b(\pi, \mathcal{D})$.

We regard the topology on $\Gamma_b(\pi, \mathcal{D})$ determined by $\mathfrak{W}$ the cover-strict topology determined by $\mathcal{D}$, and we denote it by $\mathfrak{t}_{\beta,\mathcal{D}}$. If $\Gamma(\pi)$ "is" $C(X)$ in the sense mentioned above, then $\mathfrak{t}_{\beta,\mathcal{D}}$ is the $\mathcal{D}$-strict topology on $C(X, \mathcal{D})$ as defined in [1] and [3]; we have thus modified that definition so as to extend our possibilities to space of vector-valued functions more general than $C(X)$.

A few examples may help to clarify the situation. Let $\pi : \mathcal{E} \to X$ be a bundle of Banach spaces.

Example 2.2.

(2.2.1) If $\mathcal{D} = \{X\}$, then $\mathfrak{t}_{\beta,\mathcal{D}} = \mathfrak{t}_\beta$, the strict topology on $\Gamma_b(\pi)$. This situation was studied in [6] and [7].

(2.2.2) If $\mathcal{D}$ is the collection of singleton subsets of X, then $\mathfrak{t}_{\beta,\mathcal{D}} = \mathfrak{t}_p$, the topology of pointwise convergence.

(2.2.3) If $\mathcal{D}$ is the collection $\mathfrak{K}$ of compact subsets of X, then $\mathfrak{t}_{\beta,\mathcal{D}} = \mathfrak{t}_\kappa$, the topology of compact convergence on $\Gamma(\pi)$.

Denoting by $\mathfrak{t}_u$ the uniform topology on $\Gamma_b(\pi)$ generated by the seminorms $p(\sigma) = \sup_{x\in X}\{\|\sigma(x)\|\}$, we see that on $\Gamma_b(\pi)$ we have $\mathfrak{t}_p \prec \mathfrak{t}_{\beta,\mathcal{D}} \prec \mathfrak{t}_\beta \prec \mathfrak{t}_u$, where $t_1 \prec t_2$ denotes that convergence with respect to t_2 implies convergence with respect to t_1.

It now follows from [5, Proposition 5.11] that there is a bundle $\rho : \mathcal{F} \to X$ of topological vector spaces F_x such that $\Gamma_b(\pi,\mathcal{D})$ is algebraically and topologically isomorphic to a closed subspace of $\Gamma(\rho)$, and such that $F_x \simeq E_x$ for all $x \in X$. Moreover, if $K \subset D \in \mathcal{D}$ is D-compact (and hence X-compact), we have $[\Gamma_b(\pi,\mathcal{D})]_K = \Gamma(\rho_K)$, where $\rho_K : \mathcal{F}_K = \bigcup^\bullet\{F_x : x \in K\} \to K$ is the restriction bundle of ρ to K. Thus every section of the restriction bundle ρ_K can be regarded as the restriction to K of an element of $\Gamma_b(\pi,\mathcal{D})$ (see [5, Theorem 5.9]). We use this identification of $\Gamma_b(\pi,\mathcal{D})$ with a space of sections in the following.

We first provide some completeness results for $\Gamma_b(\pi,\mathcal{D})$ in its cover-strict topology.

Definition 2.3. Let X be as usually given, and let $\mathcal{D}$ be a cover of X. Say that $\mathcal{D}$ is sufficiently open if, given $x \in X$, there exists $D \in \mathcal{D}$ and an X-neighborhood U of x such that $x \in U \subset D$. Call $\mathcal{D}$ sufficiently locally compact if, given $x \in X$, there exists $D \in \mathcal{D}$ and an X-neighborhood U of x such that U is D-compact (and hence X-compact) and $x \in U \subset D$. Especially, open covers are sufficiently open, and locally compact covers are sufficiently locally compact.

Proposition 2.4. *Suppose that $\mathcal{D}$ is a sufficiently locally compact cover of X. Let $\pi : \mathcal{E} \to X$ be a Banach bundle. Then $\Gamma_b(\pi,\mathcal{D})$ is $\mathfrak{t}_{\beta,\mathcal{D}}$-complete.*

Proof. Let (σ_λ) be a $\mathfrak{t}_{\beta,\mathcal{D}}$-Cauchy net in $\Gamma_b(\pi,\mathcal{D})$. Then, given $\varepsilon > 0, D \in \mathcal{D}$, and $v_D \in S_0^+(D)$, there exists λ_0 such that if $\lambda, \lambda' \geq \lambda_0$ then $p_{D,v'_D}(\sigma_\lambda - \sigma_{\lambda'}) < \varepsilon$. In particular, for $x \in D$ and $v_D = \chi_x$, the characteristic function of $\{x\}$, it follows that $\chi_x(x)\|\sigma_\lambda(x) - \sigma_{\lambda'}(x)\| = \|\sigma_\lambda(x) - \sigma_{\lambda'}(x)\| < \varepsilon$ eventually, so that $(\sigma_\lambda(x))$ is Cauchy in E_x, and hence convergent in E_x, for each $x \in D$. Define σ to be the pointwise limit of the $\sigma_\lambda(x)$ for $x \in D$. We first claim that σ is bounded on D. If not, then there exists $v_D \in S_0^+(D)$ and a sequence $(x_n) \subset D$ such that $v'_D(x_n)\|\sigma(x_n)\| = p^{x_n}_{D,v'_D}(\sigma(x_n)) > 2n$. On the other hand, there exists λ_0 such that if $\lambda, \lambda' \geq \lambda_0$, then $p_{D,v'_D}(\sigma_\lambda - \sigma_{\lambda'}) < n$; in particular, $p^{x_n}_{D,v'_D}(\sigma_\lambda(x_n) - \sigma_{\lambda_0}(x_n)) < n$ for all n. Then $\lim_\lambda p^{x_n}_{D,v'_D}(\sigma_\lambda(x_n) - \sigma_{\lambda_0}(x_n)) = p^{x_n}_{D,v'_D}(\sigma(x_n) - \sigma_{\lambda_0}(x_n)) \leq n$. Thus, $p^{x_n}_{D,v'_D}(\sigma_{\lambda_0}(x_n)) > n$, which is a contradiction because $\sigma_{\lambda_0} \in \Gamma_b(\pi,\mathcal{D})$ and hence is bounded on D.

Thus, the net (σ_λ) has a pointwise limit σ defined on all of X, and σ is bounded on each $D \in \mathcal{D}$. We claim that σ is continuous on X. Let $x \in X$. Since $\mathcal{D}$ is sufficiently locally compact, there exist $D \in \mathcal{D}$ and X-compact neighborhood U of x such that $x \in U \subset D$. By hypothesis, there exists a D-compact neighborhood U of x such that $x \in U \subset D$. Then $\chi_U \in S_0^+(X)$, and so there exists λ_0 such that if $\lambda, \lambda' \geq \lambda_0$, then

$$p_{D,\chi_U}(\sigma_\lambda - \sigma_{\lambda'}) = \sup_{x\in D}\big\|\sigma_\lambda(x) - \sigma_{\lambda'}(x)\big\| = \sup_{x\in U}\big\|\sigma_\lambda(x) - \sigma_{\lambda'}(x)\big\| < \varepsilon;$$

in other words, (σ_λ) is norm-uniformly Cauchy on U, and thus converges uniformly to σ_U, the restriction of σ to U. But by [5, Theorem 5.9], σ_U is a section in $\Gamma(\pi_U)$, so that σ_U and hence σ are both continuous at x. □

Proposition 2.5. *Suppose that X is first countable, and let $\mathcal{D}$ be a sufficiently open cover of X. Let $\pi : \mathcal{E} \to X$ be a Banach bundle. Then $\Gamma_b(\pi, \mathcal{D})$ is $\mathfrak{t}_{\beta,\mathcal{D}}$-complete.*

Proof. Again, let (σ_λ) be a $\mathfrak{t}_{\beta,\mathcal{D}}$-Cauchy sequence in $\Gamma_b(\pi, \mathcal{D})$. As above, the pointwise limit σ is bounded on each $D \in \mathcal{D}$. We claim that σ is continuous on all of X. If not, suppose that σ is discontinuous at $x \in X$, and choose $D \in \mathcal{D}$ and an X-open U such that $x \in U \subset D$. From the definition of continuity, then, there exist $\varepsilon > 0$, $v_D \in S_0^+(D)$, and $\tau \in \Gamma_b(\pi, \mathcal{D})$ with $\tau(x) = \sigma(x)$, and a sequence of distinct points $(x_n) \subset U \subset D$ such that $x_n \to x$ but such that for no n do we have $\sigma(x_n) \in T = T(U, \tau, v'_D, \varepsilon) = \{z' \in \mathcal{E} : \pi(z') \in U$ and $v'_D(x)\|\sigma(x) - \tau(x)\| = p^{x_n}_{D,v'_D}(\sigma(x_n) - \tau(x_n)) < \varepsilon\}$. That is, $p^{x_n}_{D,v'_D}(\sigma(x_n) - \tau(x_n)) \geq \varepsilon$ for all n, but $x_n \to x$.

Now, let $a = \chi_B$, the characteristic function of the compact set $B = \{x_n : n \in \mathbb{N}\} \cup \{x\} \subset D$. Then $a \in S_0^+(D)$. Since (σ_λ) is $\mathfrak{t}_{\beta,\mathcal{D}}$-Cauchy, it follows that there exists λ_0 such that for $\lambda, \lambda' \geq \lambda_0$ we have $p_{D,a'}(\sigma_\lambda - \sigma_{\lambda'}) < \varepsilon/2$, and hence that $\sup_{y\in B}\{a(y)\|\sigma_\lambda(y) - \sigma_{\lambda_0}(y)\| = \sup_{y\in B}\{\|\sigma_\lambda(y) - \sigma_{\lambda_0}(y)\| < \varepsilon/2$. Passing to the limit in λ, we have $\sup_{y\in B}\{\|\sigma(y) - \sigma_{\lambda_0}(y)\| \leq \varepsilon/2$. Thus, $p^{x_n}_{D,a'}(\sigma_{\lambda_0}(x_n) - \tau(x_n)) > \varepsilon/2$ for all n. But $\sigma_{\lambda_0} \in \Gamma_b(\pi, \mathcal{D}) \subset \Gamma(\pi)$, and $x_n \to x$, so that $\sigma_{\lambda_0}(x_n) \in T(U, \tau, v'_D, \varepsilon/2)$ eventually; this is a contradiction, since $p^{x_n}_{D,a'}(\sigma_{\lambda_0}(x_n) - \tau(x_n)) > \varepsilon/2$ implies that $\sigma_{\lambda_0}(x_n) \notin T(U, \tau, v'_D, \varepsilon/2)$. □

We can also obtain a Stone–Weierstrass result for $\Gamma_b(\pi, \mathcal{D})$.

Theorem 2.6. *Let $\pi : \mathcal{E} \to X$ be as generally given, and let $M \subset \Gamma_b(\pi, \mathcal{D})$ be a fiberwise dense $C_b(X)$-module (i.e., $M_x = \{\sigma(x) : \sigma \in \Gamma_b(\pi)\}$ is dense in E_x for each $x \in X$). Then M is $\mathfrak{t}_{\beta,\mathcal{D}}$-dense in $\Gamma_b(\pi, \mathcal{D})$.*

Proof. Let $\varepsilon > 0$, and suppose that $D_k \in \mathcal{D}$, $v_k \in S_0^+(D_k)$ for $k = 1, \ldots, n$. Then

$$N = \big\{\sigma \in \Gamma_b(\pi, \mathcal{D}) : p_{D_k,v'_k}(\sigma) = \sup_{x\in D_k} v_k(x)\big\|\sigma(x)\big\| = \sup_{x\in X} v'_k(x)\big\|\sigma(x)\big\| < \varepsilon \text{ for } k = 1, \ldots, n\big\}$$

is a $\mathfrak{t}_{\beta,\mathcal{D}}$-basic neighborhood of $0 \in \Gamma_b(\pi, \mathcal{D})$. Let $\tau \in \Gamma_b(\pi, \mathcal{D})$; the goal is to find $\sigma \in M$ such that $\sigma - \tau \in N$.

For each $k = 1, \ldots, n$ there is a D_k-compact set K_k, which is hence X-compact, such that $v'_k(x)\|\tau(x)\| < \varepsilon$ if $x \in X \setminus K_k$. Letting $K = K_1 \cup \cdots \cup K_n$, we then have $v'_k(x)\|\tau(x)\| < \varepsilon$ for each $k = 1, \ldots, n$ whenever $x \notin K$. Since K is compact, we can apply the Stone–Weierstrass theorem of [5, Corollary 4.3] and the isometric isomorphism $\Gamma(\pi_K) \simeq [\Gamma(\pi)]_K$ to M_K in order to assert the existence of $\sigma \in \Gamma(\pi)$ such that

$$\sup_{x\in K}\big\|\sigma_K(x) - \tau_K(x)\big\| = \sup_{x\in K}\big\|\sigma(x) - \tau(x)\big\| < \varepsilon' := \varepsilon/\big(2\max\big\{\|v_k\|\big\}\big).$$

Now, choose an open neighborhood U of K such that $\|\sigma(y) - \tau(y)\| < \varepsilon'$ if $y \in U$; that X is completely regular then allows us to choose a continuous $f : X \to [0,1]$ such that $f(K) = 1$ and $f(X \setminus U) = 0$. Then $f\sigma \in M$, and we have

(1) $v_k'(y)\|\sigma(y) - \tau(y)\| = v_k'(y)\|(f\sigma)(y) - \tau(y)\| < \varepsilon$ for all $y \in K$;

(2) if $y \in U \setminus K$, then

$$\begin{aligned} v_k'(y)\big\|(f\sigma)(y) - \tau(y)\big\| &= v_k'(y)\big\|(f\sigma)(y) - (f\tau)(y)\big\| + v_k'(y)(1-f)(y)\big\|\tau(y)\big\| \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon; \end{aligned}$$

and

(3) if $y \notin U$ and $k = 1, \ldots, n$, then

$$\begin{aligned} v_k'(y)\big\|(f\sigma)(y) - \tau(y)\big\| &\leq v_k'(y)\big\|(f\sigma)(y) - (f\tau)(y)\big\| + v_k'(y)(1-f)(y)\big\|\tau(y)\big\| \\ &= v_k'(y)(1-f)(y)\big\|\tau(y)\big\| \\ &\leq v_k'(y)\big\|\tau(y)\big\| \\ &< \varepsilon, \end{aligned}$$

so that $f\sigma - \tau \in N$. □

The reader may wish to compare these bundle-oriented Stone–Weierstrass and completeness results with those of [2], [1], and [3], which are obtained for certain subspaces of $C(X)$.

3. Ideal structure in $\Gamma_b(\pi, \mathcal{D})$

In this section, unless otherwise specified, we suppose that $\pi : \mathcal{E} \to X$ is a bundle of Banach algebras, and we examine the ideal structure of $\Gamma_b(\pi, \mathcal{D})$ in the cover-strict topology $\mathfrak{t}_{\beta,\mathcal{D}}$. Because of the ubiquity of compact sets in the definition of $\mathfrak{t}_{\beta,\mathcal{D}}$, it turns out that this ideal structure is similar to that of $\Gamma_b(\pi)$ in its β (strict) topology, which was studied in [7].

Lemma 3.1. *Let $\pi : \mathcal{E} \to X$ be a bundle of Banach algebras, and let $J \subset \Gamma(\pi)$ be an ideal which is also a $C(X)$-module. Let $K \subset X$ be compact. Then $J_K = \{\sigma_K : \sigma \in J\}$ is an ideal in $[\Gamma(\pi)]_K$, which is also a $C(K)$-module.*

Proof. Evidently J_K is an ideal. Now, let $f \in C(K), \sigma \in [\Gamma(\pi)]_K$. Then there exist $f^* \in C_b(X) \subset C(X)$ and $\sigma^* \in J$ such that $f = (f^*)_K$ and $\sigma = (\sigma^*)_K$. We therefore have $f\sigma = (f^*)_K(\sigma^*)_K = (f^*\sigma^*)_K \in [\Gamma(\pi)]_K$, since J is a $C(X)$-module. □

(We note that the hypotheses of this lemma, and other results below, beg the question as to when an ideal might be a submodule. We address this later in the present section.)

We first establish a relationship between $\mathfrak{t}_{\beta,\mathcal{D}}$-closed subspaces in $\Gamma_b(\pi, \mathcal{D})$ and closed subspaces in the fibers E_x.

Proposition 3.2. *Let $\pi : \mathcal{E} \to X$ be a bundle of Banach spaces, and let $J \subset \Gamma_b(\pi, \mathcal{D})$ be a $\mathfrak{t}_{\beta,\mathcal{D}}$-closed subspace which is also a $C_b(X)$-module. Set $(\overline{J_x})^x = \{\sigma \in \Gamma_b(\pi, \mathcal{D}) \colon \sigma \in \overline{J_x}\}$. Then $(\overline{J_x})^x$ is a $\mathfrak{t}_{\beta,\mathcal{D}}$-closed submodule in $\Gamma_b(\pi, \mathcal{D})$, and $J = \bigcap_{x \in X} (\overline{J_x})^x$.*

Proof. Let $x \in X$. Noting that $\overline{J_x}$ is a subspace in E_x, it is then easy to see that $(\overline{J_x})^x$ is a submodule of $\Gamma_b(\pi, \mathcal{D})$. Now, suppose that $\sigma \in \Gamma_b(\pi, \mathcal{D})$ is in the $\mathfrak{t}_{\beta,\mathcal{D}}$-closure of $(\overline{J_x})^x$. Then for each $D \in \mathcal{D}$ (with $x \in D$), $v_D \in S_0^+(D)$, and $\varepsilon > 0$, there exists $\tau \in (\overline{J_x})^x$ such that $p_{D,v'_D}(\sigma - \tau) < \varepsilon$. In particular, if $x \in D$, then we have $p^x_{D,v'_D}(\sigma - \tau) = v'(x)\|\sigma(x) - \tau(x)\| < \varepsilon$, and since the p^x_{D,v'_D} generate the (norm) topology on E_x, this gives us $\sigma(x) \in \overline{J_x}$ or $\sigma \in (\overline{J_x})^x$. If $x \notin D$, then $v'_D(x) = 0$, so that $p^x_{D,v'_D}(\sigma - \tau) = 0 < \varepsilon$.

Set $J' = \bigcap_{x \in X} (\overline{J_x})^x$. Evidently, $J \subset (\overline{J_x})^x$ for each $x \in X$, so that $J \subset J'$. It is evident that J' is a $\mathfrak{t}_{\beta,\mathcal{D}}$-closed submodule of $\Gamma_b(\pi, \mathcal{D})$; we claim that $J' \subset J$.

Let $\sigma \in J'$, and let N be a $\mathfrak{t}_{\beta,\mathcal{D}}$-basic neighborhood of σ. We will show that $J \cap N \neq \varnothing$, so that $\sigma \in \overline{J} = J$. Given $\sigma \in J'$, therefore, let $\varepsilon > 0$, $D_k \in \mathcal{D}$, and $v_k = v_{D_k} \in S_0^+(D_k)$ for $k = 1, \dots, n$. We may take N to be of the form $N = \{\tau \in \Gamma_b(\pi, \mathcal{D}) : p_{D_k, v'_k}(\sigma - \tau) < \varepsilon,\ k = 1, \dots, n\}$.

As in the proof of the Stone–Weierstrass result above (Theorem 2.6), there exists a compact $K \subset X$ such that $v_k(y)\|\sigma(y)\| < \varepsilon$ whenever $y \in X \setminus K$ for each $k = 1, \dots, n$. For each $x \in K$, since $\sigma(x) \in \overline{J_x}$, there exists $\tau_x \in J$ such that $v'_k(x)\|\sigma(x) - \tau_x(x)\| < \varepsilon' = \varepsilon/2$, for $k = 1, \dots, n$. And, by the upper semicontinuity of the function $y \mapsto v'_k(y)\|\sigma(y) - \tau_x(y)\|$, there exists for each $x \in K$ a neighborhood U_x of x such that $v'_k(y)\|\sigma(y) - \tau_x(y)\| < \varepsilon'$ for each $k = 1, \dots, n$ and $y \in K$. Choose x_j, $j = 1, \dots, m$ such that the $U_j = U_{x_j}$, $j = 1, \dots, m$ form a finite subcover of compact K. From [4, Lemma 1], we can choose continuous functions $f_j : X \to [0, 1]$, $j = 1, \dots, m$ such that the following hold:

(1) $\sum_j f_j(y) = 1$ for $y \in K$;
(2) f_j is supported on U_j for each $j = 1, \dots, m$; and
(3) $\sum_j f_j(y) \leq 1$ for each $y \in X$.

Then, as in the proof of Proposition 7 of [7], we set $\tau = \sum_j f_j \tau_{x_j}$. Noting that J is a $C_b(X)$-submodule of $\Gamma_b(\pi, \mathcal{D})$, we also have $\tau \in J$. For $y \in X$, we have the following three possibilities.

(1) If $y \in K$, then for any $k = 1, \dots, n$ we have

$$\begin{aligned} v'_k(y)\big\|\sigma(y) - \tau(y)\big\| &= v'_k(y)\Big\|\sum_j f_j(y)\big(\sigma(y) - \tau_{x_j}(y)\big)\Big\| \\ &\leq v'_k(y) \sum_{j \text{ s.t. } y \in U_j} f_j(y)\big\|\sigma(y) - \tau_{x_j}(y)\big\| \\ &< \varepsilon' < \varepsilon. \end{aligned}$$

(2) If $y \in \bigcup_{j=1}^m U_j \setminus K$, then for any $k = 1, \dots, n$ we have

$$\begin{aligned} \eta &= v'_k(y)\big\|\sigma(y) - \tau(y)\big\| \\ &= v'_k(y)\Big\|\Big[\sum_j f_j(y)\big(\sigma(y) - \tau_{x_j}(y)\big)\Big] - \Big(1 - \sum_j f_j(y)\Big)\sigma(y)\Big\| \end{aligned}$$

$$\begin{aligned}&\le v_k'(y)\sum_j f_j(y)\big\|\sigma(y)-\tau_{x_j}(y)\big\| + v_k'(y)\Big(1-\sum_j f_j(y)\Big)\big\|\sigma(y)\big\|\\ &< \varepsilon'+\varepsilon'=\varepsilon\end{aligned}$$

(because $y \in U_j$ for some j, but $y \notin K$, and $0 \le \sum_j f_j(y) \le 1$).

(3) If $y \in X \setminus \bigcup_{j=1}^m U_j$, then for each $k = 1, \dots, n$ we have

$$v_k'(y)\big\|\sigma(y)-\tau(y)\big\| = v_k'(y)\Big\|\sum_j f_j(y)\big(\sigma(y)-\tau_{x_j}(y)\big)\Big\| = 0,$$

because y is not in the support of any of the f_j.

Thus, for all $y \in X$, and for each $k = 1, \dots, n$, we have $v_k'(y)\|\sigma(y) - \tau(y)\| < \varepsilon$, so that $\tau \in N$, which, remember, is a $\mathfrak{t}_{\beta,\mathcal{D}}$-basic neighborhood of σ. Hence $\sigma \in \overline{J}$, so that $J' \subset \overline{J} = J$. □

Corollary 3.3. *Let $\pi : \mathcal{E} \to X$ be a bundle of Banach algebras, and let $J \subset \Gamma_b(\pi,\mathcal{D})$ be a $\mathfrak{t}_{\beta,\mathcal{D}}$-closed ideal which is also a $C_b(X)$-module. Set $(\overline{J_x})^x = \{\sigma \in \Gamma_b(\pi,\mathcal{D})\colon \sigma \in \overline{J_x}\}$. Then $(\overline{J_x})^x$ is a $\mathfrak{t}_{\beta,\mathcal{D}}$-closed ideal in $\Gamma_b(\pi,\mathcal{D})$, and $J = \bigcap_{x\in X}(\overline{J_x})^x$.*

Corollary 3.4. *Let $\pi : \mathcal{E} \to X$ be a bundle of commutative Banach algebras, and let $\mathcal{D}$ be a cover of X. Suppose that $J \subset \Gamma_b(\pi,\mathcal{D})$ is a $\mathfrak{t}_{\beta,\mathcal{D}}$-closed proper ideal which is also a $C_b(X)$-submodule. Then there exists $x \in X$ such that $\overline{J_x} = \overline{\phi_x(J)}$ is a closed proper ideal in E_x.*

Proof. Suppose that $\overline{J_x} = E_x$ for all $x \in X$. Then $(\overline{J_x})^x = (E_x)^x = \Gamma_b(\pi,\mathcal{D})$ for each $x \in X$, and so $J = \bigcap_{x\in X}(\overline{J_x})^x = \Gamma_b(\pi,\mathcal{D})$. Hence, if $J \subset \Gamma_b(\pi,\mathcal{D})$ is to be a closed proper ideal, there must be some $x \in X$ such that $\overline{J_x} \subset E_x$ is a closed proper ideal. □

Proposition 3.5. *Suppose that $H : \Gamma_b(\pi,\mathcal{D}) \to \mathbb{K}$ is a $\mathfrak{t}_{\beta,\mathcal{D}}$-continuous nontrivial multiplicative homomorphism, and set $J = \ker H$. Then there exists $x \in X$ such that $\overline{J_x}$ is a proper ideal in E_x.*

Proof. By the preceding, it suffices to show that J is a $C_b(X)$-module. If not, there exists $\sigma \in J$ and $f \in C_b(X)$ such that $f\sigma \notin J$. But $f(f\sigma)\sigma = f^2\sigma^2 \in J$, so that $H(f^2\sigma^2) = H((f\sigma)^2) = [H(f\sigma)]^2 = 0$, a contradiction to $f\sigma \notin J$. □

Proposition 3.6. *Suppose that $\pi : \mathcal{E} \to X$ is a bundle of commutative Banach algebras with maximal ideal spaces $\Delta(E_x)$. Let $H \in \Delta = \Delta(\Gamma_b(\pi,\mathcal{D}))$, the space of nontrivial $\mathfrak{t}_{\beta,\mathcal{D}}$-continuous homomorphisms from $\Gamma_b(\pi,\mathcal{D})$ to $\mathbb{K}$. Then there exist unique $x \in X$ and $h \in \Delta(E_x)$ such that $H = h \circ \phi_x$.*

Proof. Let $H \in \Delta$, and let $J = \ker H$. Choose $x \in \overline{J_x}$ such that $\overline{J_x}$ is a closed proper ideal of E_x. Then $\frac{E_x}{\overline{J_x}} \neq 0$. Now, $\phi_x : \Gamma_b(\pi,\mathcal{D}) \to E_x$ maps J into J_x, so that there is a unique linear map $\phi_x' : \frac{\Gamma_b(\pi,\mathcal{D})}{J} \to \frac{E_x}{\overline{J_x}}$ which makes this diagram commute:

$$\begin{array}{ccc}\Gamma_b(\pi,\mathcal{D}) & \overset{\phi_x}{\longrightarrow} & E_x\\ q\downarrow & & \downarrow q_x\\ \frac{\Gamma_b(\pi,\mathcal{D})}{J} & \underset{\phi_x'}{\longrightarrow} & \frac{E_x}{\overline{J_x}}\end{array}$$

(Here, q and q_x are the quotient maps.) Since $\phi_x : \Gamma_b(\pi, \mathcal{D}) \to E_x$ is surjective so is ϕ'_x. But $\frac{\Gamma_b(\pi,\mathcal{D})}{J}$ is 1-dimensional because J is the kernel of a multiplicative homomorphism. Since ϕ'_x is surjective, this forces $\frac{E_x}{J_x}$ to be 1-dimensional, in particular making $\overline{J_x}$ a closed maximal ideal in the topological algebra E_x, and hence the kernel of some $h \in \Delta(E_x)$; that is, $\overline{J_x} = \ker h$. Then $h \circ \phi_x : \Gamma_b(\pi, \mathcal{D}) \to \mathbb{K}$ is a continuous nontrivial algebra homomorphism (because evaluation is continuous; see [5, Corollary 2.7]). If $\sigma \in J$, then $\sigma(x) \in J_x \subset \overline{J_x}$, so $(h \circ \phi_x)(\sigma) = 0$, and thus $J = \ker H \subset \ker(h \circ \phi_x)$. But $\ker H$ and $\ker(h \circ \phi_x)$ are both closed maximal ideals, and so $\ker H = \ker(h \circ \phi_x)$. Finally, since H and $h \circ \phi_x$ are both multiplicative, we have $H = h \circ \phi_x$. □

Note that these are the same results, and proofs, as in [7, Propositions 3, 4].

Corollary 3.7. *Under the situation as described in Proposition 3.6, we may identify $\Delta(\Gamma_b(\pi, \mathcal{D}))$ as a point set with the disjoint union $\bigcup^\bullet \{\Delta(E_x) : x \in X\}$.*

Proof. From the preceding results, the map $\mu : \Delta(\Gamma_b(\pi, \mathcal{D})) \to \bigcup^\bullet \{\Delta(E_x) : x \in X\}$, given by $\mu(H) = h \in \Delta(E_x)$ if $H = h \circ \phi_x$, is a bijection. □

We now show that we have a spectral synthesis-like property in $\Gamma_b(\pi, \mathcal{D})$.

Theorem 3.8. *Let $J \subset \Gamma_b(\pi, \mathcal{D})$ be a $\mathfrak{t}_{\beta,\mathcal{D}}$-closed ideal which is also a $C_b(X)$-module. If E_x satisfies spectral synthesis for each $x \in X$, then J is the intersection of all $\mathfrak{t}_{\beta,\mathcal{D}}$-closed maximal ideals in $\Gamma_b(\pi, \mathcal{D})$ which contain it.*

Proof. We can essentially repeat the proof of [7, Proposition 9]. To begin, set $P = \{x \in X : \overline{J_x}$ is a proper ideal in $E_x\}$. Note in general that for $H \in \Delta = \Delta(\Gamma_b(\pi, \mathcal{D}))$, with $H = h \circ \phi_x$ for $h \in \Delta(E_x)$ and $x \in P$, we have the following: $J \subset \ker H$, if and only if $J_x \subset \ker h$, if and only if $\overline{J_x} \subset \ker h$, if and only if $(\overline{J_x})^x \subset \ker(h \circ \phi_x)$. But when $x \in P$, then $\overline{J_x} \subset E_x$ is a proper ideal, and so we have $\overline{J_x} = \bigcap\{\ker(h \circ \phi_x) : h \in \Delta(E_x)$ and $\overline{J_x} \subset \ker h\}$. Hence, for $x \in P$ we have

$$\begin{aligned}\overline{J_x} &= \bigcap\big\{\ker(h \circ \phi_x) : h \in \Delta(E_x) \text{ and } \overline{J_x} \subset \ker h\big\}\\ &= \bigcap\big\{\ker H : H = h \circ \phi_x \in \Delta\big(\Gamma_b(\pi, \mathcal{D})\big) \text{ and } \overline{J_x} \subset \ker h\big\}.\end{aligned}$$

Finally,

$$\begin{aligned}J &= \bigcap_{x\in X} (\overline{J_x})^x = \bigcap_{x\in P} (\overline{J_x})^x\\ &= \bigcap_{x\in P} \big\{\ker H : H = h \circ \phi_x \in \Delta\big(\Gamma_b(\pi, \mathcal{D})\big) \text{ and } \overline{J_x} \subset \ker h\big\}\\ &= \bigcap_{x\in X} \big\{\ker H : H = h \circ \phi_x \in \Delta\big(\Gamma_b(\pi, \mathcal{D})\big) \text{ and } \overline{J_x} \subset \ker h\big\}.\end{aligned}$$

□

The converse to the preceding is also true.

Proposition 3.9. *Suppose that for each $\mathfrak{t}_{\beta,\mathcal{D}}$-closed proper ideal and $C(X)$-submodule J of $\Gamma_b(\pi, \mathcal{D})$, J is the intersection of all $\mathfrak{t}_{\beta,\mathcal{D}}$-closed maximal ideals which contain it. Then for each $x \in X$, E_x satisfies spectral synthesis.*

Proof. Fix $x \in X$, and let $I \subset E_x$ be a (norm-)closed proper ideal. Then $I^x = \{\sigma \in \Gamma_b(\pi, \mathcal{D}) : \sigma(x) \in I\}$ is evidently a proper ideal in $\Gamma_b(\pi, \mathcal{D})$, which from the proof of Proposition 3.2 above is also $\mathfrak{t}_{\beta,\mathcal{D}}$-closed; it is also easily checked that I^x is a $C(X)$-submodule of $\Gamma_b(\pi, \mathcal{D})$. Observe that if $x, y \in X$ with $x \neq y$, then $(I^x)_y = \phi_y(I_x) = E_y$; for if $t \in E_y$ and if $\sigma \in \Gamma_b(\pi, \mathcal{D})$ is such that $\sigma(y) = t$, then for any $f \in C_b(X)$ such that $f(y) = 1$ and $f(x) = 0$ we have $(f\sigma)(y) = t$ and $(f\sigma)(x) = 0 \in E_x$, so that $f\sigma \in I^x$.

Hence,

$$\begin{aligned} I^x &= \bigcap_{y \in X} \big\{\ker(h \circ \phi_y) : h \in \Delta(E_y) \text{ and } I^x \subset \ker(h \circ \phi_y)\big\} \\ &= \bigcap \big\{\ker(h \circ \phi_x) : h \in \Delta(E_x) \text{ and } I^x \subset \ker(h \circ \phi_x)\big\}, \end{aligned}$$

so that

$$I = (I^x)_x = \bigcap \big\{\ker h : h \in \Delta(E_x) \text{ and } I \subset \ker h\big\}. \qquad \square$$

The results and their proofs above are analogous to results in [7], which addresses the case of section spaces with their strict topologies. Unlike in [1], we are unable because of our more general situation to assume a Stone–Weierstrass property to arrive at a short proof identifying the closed maximal ideals. Note also that by [1, Theorem 4.4] for each $\mathfrak{t}_{\beta,\mathcal{D}}$-closed ideal $I \subset C_b(X, \mathcal{D})$ there is a corresponding closed zero set $Z(I) \subset X$ such that $Z(I) = \{x \in X : f(x) = 0 \text{ for all } f \in I\}$, and the converse also holds. This correspondence does not hold once we leave the situation of 1-dimensional fibers and continuous $\mathbb{K}$-valued functions. Moreover, such a hope is vain even in the situation where the original bundle $\pi : \mathcal{E} \to X$ is a continuously normed (i.e., $x \mapsto \|\sigma(x)\|$ is continuous for each $\sigma \in \Gamma(\pi)$) bundle of Banach algebras. This is so even though it is easy to check in this continuously normed situation that for any closed $C \subset X$ the set $I = \{\sigma \in \Gamma_b(\pi) : \sigma_C = 0\}$ is a norm-closed ideal, and conversely that for each norm-closed ideal I, the set $C = \{x \in X : \sigma(x) = 0 \text{ for all } \sigma \in I\}$ is closed. (See Proposition 4.3 and Corollary 4.4.)

Example 3.10.

(3.10.1) Let $X = [0, 1]$, and consider $C(X, \mathbb{K}^2)$, identified as the section space of the trivial bundle $\pi_2 : \mathcal{E}_2 = \bigcup^{\bullet}_{x \in X} \mathbb{K}^2 \to X$, where $\mathcal{E}_2$ is homeomorphic to $\mathbb{K}^2 \times X$ in its product topology. Writing $\sigma(x) = (\sigma_1(x), \sigma_2(x))$, we have $J = \{\sigma \in C(X, \mathbb{K}^2) : \sigma_1(x) = 0\}$ and $I = \{\sigma \in C(X, \mathbb{K}^2) : \sigma_2(x) = 0\}$ as distinct closed ideals and $C(X)$-submodules of $C(X, \mathbb{K}^2)$, but $Z(I) = Z(J) = \varnothing$.

(3.10.2) Let $X = [0, 1]$, and let $\pi_0 : \mathcal{E}_0 = \bigcup^{\bullet}_{x \in X} \mathbb{K} \to X$ be the "spiky" bundle, whose section space $\Gamma(\pi_0)$ can be identified with $c_0(X)$ (see [10]). Let $M = \{f \in \Gamma(\pi) : f(x) = 0 \text{ for } x \text{ irrational}\} \subset \Gamma(\pi)$; evidently, M is a closed $C(X)$-submodule and ideal in $\Gamma(\pi)$. If we endow $\Gamma(\pi)$ with the $\mathfrak{t}_{\beta,\mathcal{D}}$ topology arising from $\mathcal{D} = \{X\}$, then $\mathfrak{t}_{\beta,\mathcal{D}} = \mathfrak{t}_\beta$, the strict topology on $\Gamma(\pi)$. But from [6, Corollary 4], when X is compact, $\mathfrak{t}_\beta = \mathfrak{t}_u$, the (norm) uniform topology on $\Gamma(\pi)$. Then $Z(M) = X \setminus \mathbb{Q}$.

In the preceding, a common premise is that we deal with subspaces (ideals) of $\Gamma_b(\pi, \mathcal{D})$ which are $C_b(X)$-modules. When is this premise guaranteed to obtain? We provide a sufficient condition in the case of bundles of Banach algebras. The key definition and the results are couched in terms of left approximate identities, but the right- and two-sided versions may be easily imagined.

Proposition 3.11. *Let $\pi : \mathcal{E} \to X$ be a bundle of Banach algebras, and suppose that $J \subset \Gamma_b(\pi, \mathcal{D})$ is a $\mathfrak{t}_{\beta,\mathcal{D}}$-closed ideal. Suppose that $\Gamma_b(\pi, \mathcal{D})$ has a left approximate identity (σ_λ) in the $\mathfrak{t}_{\beta,\mathcal{D}}$ topology. Then J is also a $C_b(X)$-module.*

Proof. Let $\tau \in J$ and $f \in C_b(X)$. Noting that $\sigma_\lambda \tau \in J$ and that $f\sigma_\lambda \in \Gamma_b(\pi, \mathcal{D})$, it follows that in the $\mathfrak{t}_{\beta,\mathcal{D}}$-topology we have $f\tau = \lim_\lambda f(\sigma_\lambda \tau) = \lim_\lambda (f\sigma_\lambda)\tau \in J$. $\square$

Then when is $\Gamma_b(\pi, \mathcal{D})$ guaranteed to have a left approximate identity for the $\mathfrak{t}_{\beta,\mathcal{D}}$ topology?

Definition 3.12 (see [8, Definition 2.1]). Suppose that $\{A_x : x \in X\}$ is a collection of Banach algebras. Say that $\{A_x : x \in X\}$ has uniformly bounded left approximate identities if (1) for each $x \in X$, A_x has a bounded left approximate identity $\{a_{\lambda_x} : \lambda_x \in \Lambda_x\}$ with $\|a_{\lambda_x}\| \le m_x < \infty$ for all $\lambda_x \in \Lambda_x$, and (2) $m = \sup_{x \in X} m_x < \infty$.

Proposition 3.13. *Let $\pi : \mathcal{E} \to X$ be a bundle of Banach algebras, and let $\mathcal{D}$ be a cover of X. Suppose also that for each $D \in \mathcal{D}$, the fibers $E_x (x \in D)$ have uniformly bounded left approximate identities. Then $\Gamma_b(\pi, \mathcal{D})$ has a left approximate identity.*

Proof. Let $\mathfrak{F} \subset \Gamma_b(\pi, \mathcal{D})$ be a finite set, let $\varepsilon > 0$, and let $D_k \in \mathcal{D}$, $v_k \in S_0^+(D_k)$, for $k = 1, \dots, n$. Then $N = \{\tau \in \Gamma_b(\pi, \mathcal{D}) : \rho_{D_k v_k'}(\tau) < \varepsilon \text{ for } k = 1, \dots, n\}$ is a basic $\mathfrak{t}_{\beta,\mathcal{D}}$-neighborhood of $0 \in \Gamma_b(\pi, \mathcal{D})$.

For each $\sigma \in \mathfrak{F}$ we may choose compact $K_\sigma \subset X$ such that for each $k = 1, \dots, n$ we have $v_k'(x)\|\sigma(x)\| < \varepsilon/3$ whenever $x \in X \setminus K_\sigma$. It is then clear that $v_k'(x)\|\sigma(x)\| < \varepsilon/3$ for all $k = 1, \dots, n$ and for all $\sigma \in F$ whenever $x \in X \setminus K$, where $K = \bigcup\{K_\sigma : \sigma \in \mathfrak{F}\}$; note that K is also compact.

Consider the Banach algebra $\Gamma(\pi_K) \simeq [\Gamma(\pi)]_K$ consisting of all restrictions to K of sections in $\Gamma(\pi)$. From [8], because the left approximate identities in $\{E_x : x \in K\}$ are uniformly bounded, $\Gamma(\pi_K)$ itself has a (norm) bounded left approximate identity. Thus, we can choose $\tau \in \Gamma(\pi)$ such that $v_k(x)\|(\tau_K\sigma_K - \sigma_K)(x)\| = v_k(x)\|(\tau\sigma - \sigma)(x)\| < \varepsilon/3$ for each $\sigma \in F$, $k = 1, \dots, n$ and each $x \in K$. From the upper semicontinuity of the seminorms $y \mapsto v_k'(y)\|\sigma(y)\|$, there exists an X-open neighborhood U of K such that $v_k'(x)\|(\tau\sigma - \sigma)(x)\| < \varepsilon/3$ for all $\sigma \in \mathfrak{F}$ and $x \in U$, and we may also choose $f \in C_b(X)$, $f : X \to [0, 1]$ such that $f(K) = 1$ and $f(X \setminus U) = 0$. Let $\tau' = f\tau$. We then check that for each $k = 1, \dots, n$ and for each $\sigma \in \mathfrak{F}$ we have

(1) $v_k(x)\|(\tau'\sigma - \sigma)(x)\| = v_k'(x)\|(\tau'\sigma - \sigma)(x)\| < \varepsilon/3$ if $x \in K$;
(2) $v_k'(x)\|(\tau'\sigma - \sigma)(x)\| = 0$ if $x \in X \setminus U$; and
(3) if $x \in U \setminus K$, then

$$\begin{aligned} v_k'(x)\big\|(\tau'\sigma-\sigma)(x)\big\| &= v_k'(x)\big\|(f\tau\sigma-\sigma)(x)\big\| \\ &= v_k'(x)\big\|\big[(f\tau\sigma-f\sigma)+(1-f)\sigma\big](x)\big\| \\ &\le v_k'(x)\big\|(f\tau\sigma-f\sigma)(x)\big\| + v_k'(x)(1-f)(x)\big\|\sigma(x)\big\| \\ &< 2\varepsilon/3. \end{aligned}$$

Thus, for each $\sigma \in \mathfrak{F}$ and $k = 1, \ldots, n$, we have $\rho_{D_k,v_k}(\tau'\sigma - \sigma) < \varepsilon$; that is, $\tau'\sigma - \sigma \in N$.

Thus, $\Gamma_b(\pi, \mathcal{D})$ under the $\mathfrak{t}_{\beta,\mathcal{D}}$-topology satisfies the conditions of [14, Propositions 3.1, 3.2] and hence has a left approximate identity. □

4. Quotients of $\Gamma_b(\pi, \mathcal{D})$

We now examine the nature of quotients of $\Gamma_b(\pi, \mathcal{D})$. In this section, we will assume that $\pi : \mathcal{E} \to X$ is a bundle of Banach algebras and that $\mathfrak{t}_{\beta,\mathcal{D}}$-closed ideals in $\Gamma_b(\pi, \mathcal{D})$ are also $C_b(X)$-submodules. From the previous section, it is clear that this will certainly be the case if $\Gamma_b(\pi, \mathcal{D})$ has a $\mathfrak{t}_{\beta,\mathcal{D}}$-approximate identity of some sort.

Let $\mathcal{D}$ be a cover of X. Suppose that I is a $\mathfrak{t}_{\beta,\mathcal{D}}$-closed ideal. Then, from [5, Proposition 9.1], we have $I = \{\sigma \in \Gamma_b(\pi, \mathcal{D}) : \sigma(x) \in \overline{I_x} \text{ for all } x \in X\}$ (where, as before, $I_x = \{\sigma(x) : \sigma \in I\}$). We examine the quotient algebra $\Gamma_b(\pi, \mathcal{D})/I$.

Consider the set $\mathcal{G} = \bigcup^{\bullet}\{\frac{E_x}{\overline{I_x}} : x \in X\}$. For $\sigma \in \Gamma_b(\pi, \mathcal{D})$, we define the choice function $\widehat{\sigma} : X \to \mathcal{G}$ by $\widehat{\sigma}(x) = \sigma(x) + \overline{I_x}$; it is clear that for $f \in C(X)$, we have $\widehat{f\sigma} = f \cdot \widehat{\sigma}$. Since $I = \bigcap_{x\in X}(\overline{I_x})^x$, the map $\sigma \mapsto \widehat{\sigma}$ defines an injective $C(X)$-homomorphism from $\frac{\Gamma_b(\pi,\mathcal{D})}{I}$ to the space of choice functions from X to $\mathcal{G}$. It is also clear that, for $D \in \mathcal{D}$ and $v \in S_0^+(D)$, we have

$$p_{D,v_D'}(\sigma) = \sup_{x\in X} v_D'(x)\big\|\sigma(x)\big\| \ge \sup_{x\in X} v_D'(x)\big\|\sigma(x) + \overline{I_x}\big\| \stackrel{\text{def}}{=} \widehat{p}_{D,v_D'}(\widehat{\sigma}),$$

so that the collection of seminorms $\{\widehat{p}_{D,v'} : D \in \mathcal{D}, v \in S_0^+(D)\}$ defines a topology $\widehat{\mathfrak{t}}_{\beta,\mathcal{D}}$ on the image in that space of choice functions of $\frac{\Gamma_b(\pi,\mathcal{D})}{I}$ under $\widehat{\ }$.

Moreover, let $\varepsilon > 0$, $D \in \mathcal{D}$, and let $v_D \in S_0^+(D)$. Suppose that $\widehat{p}^x_{D,v'}(\widehat{\sigma}(x)) = v'(x)\|\sigma(x) + \overline{I_x}\| < \varepsilon$. Then there exists $z \in I_x$ and $\tau \in I$ such that $z = \tau(x)$ and such that

$$v_D'(x)\big\|\sigma(x) + \tau(x)\big\| < \varepsilon.$$

By the upper semicontinuity of the functions involved, there exists a neighborhood U of x such that, if $y \in U$, then

$$v_D'(y)\big\|\sigma(y) + \tau(y)\big\| < \varepsilon.$$

Then clearly

$$v_D'(y)\big\|\sigma(y) + \overline{I_y}\big\| < \varepsilon;$$

that is, the map $y \mapsto v'(y)\|\sigma(y) + \overline{I_y}\|$ is upper semicontinuous for each $\sigma \in \Gamma_b(\pi, \mathcal{D})$. This leads to the following proposition.

Proposition 4.1. *Suppose that $\pi : \mathcal{E} \to X$ is a bundle of Banach algebras, that $\mathcal{D}$ is a cover of X, and that I is a $\mathfrak{t}_{\beta,\mathcal{D}}$-closed ideal of $\Gamma_b(\pi,\mathcal{D})$. Then there is a bundle $\pi_I : \mathcal{G} \to X$ with fibers $G_x = \frac{E_x}{I_x}$ such that for each $\sigma \in \Gamma_b(\pi,\mathcal{D})$ we have $\widehat{\sigma} \in \Gamma_b(\pi_I,\mathcal{D})$. The map $\widehat{\ }\colon \frac{\Gamma_b(\pi,\mathcal{D})}{I} \to \Gamma_b(\pi_I,\mathcal{D})$, $\sigma + I \mapsto [x \mapsto \widehat{\sigma}(x) = \sigma(x) + \overline{I_x}]$ is an injective and continuous $C_b(X)$-homomorphism whose image is $\widehat{\mathfrak{t}}_{\beta,\mathcal{D}}$-dense in $\Gamma_b(\pi_I,\mathcal{D})$.*

Proof. All but the density claim have been noted. For that, we need only observe that by our original assumption, $\Gamma(\pi)$ is full, so that $\Gamma_b(\pi,\mathcal{D})$ is also full. Thus, if $z + \overline{I_x} \in G_x = \frac{E_x}{I_x}$, then we can choose $\sigma \in \Gamma_b(\pi,\mathcal{D})$ such that $\sigma(x) = z$, which yields $\widehat{\sigma}(x) = \sigma(x)+\overline{I_x} = z+\overline{I_x}$. Now, apply Theorem 2.6 (Stone–Weierstrass). □

We can actually do a little better if the original bundle $\pi : \mathcal{E} \to X$ is continuously normed. If this is the case, then for $\sigma \in \Gamma(\pi)$, the set $\{x \in X : \sigma(x) = 0\}$ is closed, and hence $Z(I) = \{x \in X : \sigma(x) = 0 \text{ for each } \sigma \in I\}$ is also closed in X. Let $C = Z(I)$. We can then form the cover $\mathcal{D}_C = \mathcal{D} \cap C = \{D \cap C : D \in \mathcal{D}\}$ of C, and the restriction bundle $\pi_C : \mathcal{E}_C = \bigcup^{\bullet}_{x\in C} E_x \to C$; this is a full bundle, and $[\Gamma(\pi)]_C \subset \Gamma(\pi_C)$. We can then consider the space of sections $\Gamma_b(\pi_C,\mathcal{D}_C)$ and topologize it in the usual fashion by using the seminorms $p_{D\cap C,v'_{D\cap C}}$, as $D \in \mathcal{D}$ and $v \in S_0^+(D)$ vary. Moreover, as noted in [3], we have $[S_0^+(X)]_C = S_0^+(C)$; that is, each upper semicontinuous function on C which disappears at infinity (on C) is the restriction to C of such a function defined on all of X. Consider then the bundle $\pi_C : \mathcal{E}_C \to C$, and topologize $\Gamma_b(\pi_C,\mathcal{D}_C)$ using the seminorms $p_{D\cap C,v'_{D\cap C}}$. (Note that we may have only $[\Gamma_b(\pi,\mathcal{D})]_C$ being a subspace of $\Gamma_b(\pi_C,\mathcal{D}_C)$.) Given $I \subset \Gamma_b(\pi,\mathcal{D})$, we then consider the map $T : \frac{\Gamma_b(\pi,\mathcal{D})}{I} \to \Gamma_b(\pi_C,\mathcal{D}_C)$ defined by $T(\sigma + I) = \sigma_C$, where $\frac{\Gamma_b(\pi,\mathcal{D})}{I}$ is given its quotient topology $\widehat{\mathfrak{t}}_{\beta,\mathcal{D}}$ determined by the $\widehat{p}_{D,v'_D}$ (where $\widehat{p}_{D,v'_D}(\sigma + I) = \inf_{\tau\in I} p_{D,v'_D}(\sigma + \tau)$), and where $\Gamma_b(\pi_C,\mathcal{D}_C)$ is given the topology determined by the seminorms $p_{D\cap C,v'_{D\cap C}}$.

Theorem 4.2. *Suppose that $\pi : \mathcal{E} \to X$ is a continuously normed bundle of Banach algebras, $\mathcal{D}$ a cover of X, and I a $\mathfrak{t}_{\beta,\mathcal{D}}$-closed ideal of $\Gamma_b(\pi,\mathcal{D})$. Let $C = Z(I)$. Then the map $T : \frac{\Gamma_b(\pi,\mathcal{D})}{I} \to \Gamma_b(\pi_C,\mathcal{D}_C)$, $\sigma + I \mapsto \sigma_C$, is a topological isomorphism onto its image in $\Gamma_b(\pi_C,\mathcal{D}_C)$.*

Proof. We alter the method of [3, Proposition 3.1] only slightly. Note that T is easily seen to be injective. Then for $\sigma \in \Gamma_b(\pi,\mathcal{D})$, $\tau \in I$, $D \in \mathcal{D}$, and $v \in S_0^+(D)$, we have

$$\begin{aligned} p_{D\cap C,v'_{D\cap C}}(\sigma_C) &= \sup_{x\in D\cap C} v'_{D\cap C}(x)\|\sigma(x)\| \\ &= \sup_{x\in D\cap C} v'_{D\cap C}(x)\|\sigma(x) + \tau(x)\| \\ &\le \sup_{x\in D} v'_D(x)\|\sigma(x) + \tau(x)\| \\ &= p_{D,v'_D}(\sigma + \tau), \end{aligned}$$

since $\tau(x) = 0$ on C. Hence $p_{D\cap C,v'_{D\cap C}}(\sigma) \le \widehat{p}_{D,v'_D}(\sigma)$, and T is continuous.

Now, given $\sigma \in \Gamma_b(\pi, \mathcal{D})$, $D \in \mathcal{D}$, $v \in S_0^+(D)$, and $\varepsilon > 0$, set $K_\varepsilon = \{x \in D : v'_D(x)\|\sigma(x)\| \geq p_{D\cap C, v'_{D\cap C}}(\sigma) + \varepsilon\}$. Then, from the fact that the map $x \mapsto v'_D(x)\|\sigma(x)\|$ is in $S_0^+(D)$, it follows that K_ε is compact (in D and hence in X); we also have $K_\varepsilon \cap C = \varnothing$. Choose $f \in C(X)$, $f : X \to [0,1]$ such that $f(K_\varepsilon) = 1$ and $f(C) = 0$, and set $\tau = -f\sigma$. Then

$$\begin{aligned}\widehat{p}_{D,v'_D}(\sigma + I) &\leq p_{D,v'_D}(\sigma + \tau)\\ &= \sup_{x\in X} v'_D(x)\big\|\sigma(x) - f(x)\sigma(x)\big\|\\ &= \sup_{x\in X\setminus K_\varepsilon} v'_D(x)(1-f)(x)\big\|\sigma(x)\big\|\\ &= \sup_{x\in D\setminus K_\varepsilon} v'_D(x)(1-f)(x)\big\|\sigma(x)\big\|\\ &\leq p_{D\cap C, v'_{D\cap C}}(\sigma_C) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\widehat{p}_{D,v'_D}(\sigma + I) \leq p_{D\cap C, v'_{D\cap C}}(\sigma_C),$$

so that T^{-1} is continuous. □

In our present situation, dealing with bundles of topological vector spaces deriving originally from Banach bundles, we do not obtain the full strength of [3, Proposition 3.1], simply because, as seen in Example 3.10, we do not have the correspondence between closed sets in X and zero sets of ideals I in $\Gamma_b(\pi, \mathcal{D})$. However, if we have continuous line bundles, so that every fiber is $\mathbb{K}$, we do obtain that correspondence between $\mathfrak{t}_{\beta,\mathcal{D}}$-closed submodules in $\Gamma_b(\pi, \mathcal{D})$ and closed sets in X.

Proposition 4.3. *Let $\pi : \mathcal{E} \to X$ be a continuously normed bundle with constant fiber $\mathbb{K}$, and let $\mathcal{D}$ be a cover of X. Then there is a one-to-one correspondence between closed sets in X and $\mathfrak{t}_{\beta,\mathcal{D}}$-closed submodules of $\Gamma_b(\pi, \mathcal{D})$; namely, if $C \subset X$ is closed, then there is a unique $\mathfrak{t}_{\beta,\mathcal{D}}$-closed submodule M of $\Gamma_b(\pi, \mathcal{D})$ such that C is the zero set of M.*

Proof. Evidently, if M is any submodule of $\Gamma_b(\pi, \mathcal{D})$, its zero set $Z(M)$ is closed in X, since the norm function is continuous. Now, let $C \subset X$ be closed, and let $M = \{\sigma \in \Gamma_b(\pi, \mathcal{D}) : \sigma_C = 0\}$. It is clear that M is a $C(X)$-submodule of $\Gamma_b(\pi, \mathcal{D})$; consider its $\mathfrak{t}_{\beta,\mathcal{D}}$-closure $L = \overline{M}$. Then $Z(L) \subset Z(M)$, and since any net $(\sigma_\lambda) \subset M$ which $\mathfrak{t}_{\beta,\mathcal{D}}$-converges to $\sigma \in L$ also converges pointwise, we have $Z(M) \subset Z(L)$. Thus, there is a closed submodule whose zero set is C. Suppose now that M and L are two $\mathfrak{t}_{\beta,\mathcal{D}}$-closed submodules such that $Z(M) = Z(L)$. By Corollary 3.3 above, we have $M = \bigcap_{x\in X}(\overline{M_x})^x$ and $L = \bigcap_{x\in X}(\overline{L_x})^x$. But since $Z(M) = Z(L)$, and since for any $x \in X$ we have either $M_x = \mathbb{K} = L_x$ or $M_x = \{0\} = L_x$, this forces $L = M$. □

Corollary 4.4. *Assume the conditions of Proposition 4.3, and suppose also that $\Gamma(\pi)$ is an algebra. Then there is a one-to-one correspondence between $\mathfrak{t}_{\beta,\mathcal{D}}$-closed ideals of $\Gamma_b(\pi, \mathcal{D})$ and the closed sets of X.*

Proof. Let $I \subset \Gamma_b(\pi, \mathcal{D})$ be a $\mathfrak{t}_{\beta,\mathcal{D}}$-closed ideal. Writing $I = \bigcap_{x \in X} (\overline{I_x})^x$ as in Proposition 3.2, we see that I is the intersection of $\mathfrak{t}_{\beta,\mathcal{D}}$-closed submodules, and hence a $\mathfrak{t}_{\beta,\mathcal{D}}$-closed submodule, with zero set $C = Z(I)$. On the other hand, if $C \subset X$ is closed, then $I = \{\sigma \in \Gamma_b(\pi, \mathcal{D}) : \sigma_C = 0\}$ is a $\Gamma_b(\pi, \mathcal{D})$-submodule, and $C = Z(I)$. □

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