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ON THE EXISTENCE OF UNIVERSAL SERIES BY THE GENERALIZED WALSH SYSTEM

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Abstract. In this paper, we prove the following: let $\omega(t)$ be a continuous function with $\omega(+0)=0$ and increasing in $[0,\infty)$. Then there exists a series of the form

$$\sum_{k=1}^{\infty} c_k\psi_k(x) \quad \text{with} \sum_{k=1}^{\infty} c_k^2\omega(|c_k|)<\infty$$

with the following property: for each $\varepsilon>0$ a weight function $\mu(x), 0<\mu(x)\leq 1, |\{x\in[0,1):\mu(x)\neq 1\}|<\varepsilon$ can be constructed so that the series is universal in the weighted space $L^1_\mu[0,1)$ both with respect to rearrangements and subseries.

1. Introduction and preliminaries

The first case of a universality was observed by Fekete [7] in 1914. He showed that there exists a (formal) real power series

$$\sum_{n=1}^{\infty} a_n x^n, \quad x\in[-1,1],$$

that not only diverges at every point $x\neq 0$ but does so in the worst possible way. Indeed, to every continuous function $g(x)$ on $[-1,1]$ with $g(0)=0$ there exists an increasing sequence $\{n_k\}$ of positive integers such that $S_{n_k}(x)$ converges to $g(x)$ uniformly as $k\to\infty$.

Fekete's example of a universal power (or Taylor) series exhibits two aspects of universality that are generally present. Apart from the first aspect of maximal

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divergence, we have as a second aspect the existence of a single object, which, via a usually countable process, allows us to approximate a maximal class of objects. This suggested the name of universality.

Next, we need some definitions.

Definition 1.1. A functional series

$$\sum_{k=1}^{\infty} f_k(x), \quad f_k(x) \in L^1_\mu[0,1) \tag{1.1}$$

is said to be universal in weighted spaces $L^1_\mu[0,1)$ with respect to rearrangements if for any function $f(x) \in L^1_\mu[0,1)$ the members of (1.1) can be rearranged so that the obtained series $\sum_{k=1}^{\infty} f_{\sigma(k)}(x)$ converges to the function $f(x)$ in the metric $L^1_\mu[0,1)$; that is,

$$\lim_{n\to\infty} \int_0^1 \Big|\sum_{k=1}^{n} f_{\sigma(k)}(x) - f(x)\Big| \cdot \mu(x)\, dx = 0.$$

Definition 1.2. The series (1.1) is said to be universal in weighted spaces $L^1_\mu[0,1)$ in the usual sense if for any function $f(x) \in L^1_\mu[0,1)$ there exists a growing sequence of natural numbers n_k such that the sequence of partial sums of order n_k of the series (1.1) converges to the function $f(x)$ in the metric $L^1_\mu[0,1)$.

Definition 1.3. The series (1.1) is said to be universal in weighted spaces $L^1_\mu[0,1)$ concerning subseries if for any function $f(x) \in L^1_\mu[0,1)$ it is possible to choose a partial series $\sum_{k=1}^{\infty} f_{n_k}(x)$ from (1.1) which converges to the $f(x)$ in the metric $L^1_\mu[0,1)$.

The aforementioned definitions are given not in the most general form and only in the generality in which they will be applied in the present paper.

Here, we consider a question on existence of series by the trigonometric system universal in weighted $L^1_\mu[0,1)$ spaces with respect to rearrangements and subseries.

Note that many papers are devoted to the question on existence of various types of universal series in the sense of convergence almost everywhere and on a measure (see [2], [3], [6], [7], [9], [11]–[13], [15]–[17]). Here we will give those results that are directly related to the theorems proved in this paper.

The first usual universal in the sense of convergence almost everywhere trigonometric series was constructed by D. E. Menshov [12] and V. Ya. Kozlov [11]. They constructed the series of the form

$$\frac{1}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \tag{1.2}$$

such that for any measurable on $[0,2\pi]$ function $f(x)$ there exists an increasing sequence of natural numbers n_k such that the series (1.2) having the sequence of partial sums of order n_k converges to $f(x)$ almost everywhere on $[0,2\pi]$. Note here that, in this result, when $f(x) \in L^1[0,2\pi]$, it is impossible to replace convergence almost everywhere by convergence in the metric $L^1[0,2\pi]$.

This result was extended by A. A. Talalyan [15] to arbitrary orthonormal complete systems. He also established that if $\{\phi_n(x)\}_{n=1}^{\infty}$—the normalized basis of space $L^p[0,1], p>1$—then there exists a series of the form

$$\sum_{k=1}^{\infty} a_k \phi_k(x), \quad a_k \to 0, \tag{1.3}$$

which has the following property: for any measurable function $f(x)$ the members of series (1.3) can be rearranged so that the rearranged series converges on a measure on $[0,1]$ to $f(x)$ (see [16]).

W. Orlicz [13] observed the fact that there exist functional series that are universal with respect to rearrangements in the sense of almost everywhere convergence in the class of almost everywhere finite measurable functions.

It is also useful to note that even Riemann proved that every convergent numerical series which is not absolutely convergent is universal with respect to rearrangements in the class of all real numbers.

Let $\mu(x)$ be measurable on a $[0,2\pi]$ function with $0<\mu(x)\leq 1, x\in[0,2\pi]$, and let $L^1_\mu[0,2\pi]$ be a space of measurable functions $f(x), x\in[0,2\pi]$ with

$$\int_0^{2\pi} \bigl|f(x)\bigr|\mu(x)\,dx < \infty.$$

M. G. Grigorian [9] constructed a series of the form

$$\sum_{k=-\infty}^{\infty} C_k e^{ikx} \quad \text{with} \quad \sum_{k=-\infty}^{\infty} |C_k|^q < \infty, \forall q>2,$$

which is universal in $L^1_\mu[0,2\pi]$ concerning partial series for some weight function $\mu(x), 0<\mu(x)\leq 1, x\in[0,2\pi]$.

In [6] it is proved that, for any given sequence of natural numbers $\{\lambda_m\}_{m=1}^{\infty}$ with $\lambda_m \nearrow \infty$, there exists a series by a trigonometric system of the form

$$\sum_{k=1}^{\infty} C_k e^{ikx}, \quad C_{-k} = \overline{C}_k, \tag{1.4}$$

with

$$\Bigl|\sum_{k=1}^{m} C_k e^{ikx}\Bigr| \leq \lambda_m, \quad x\in[0,2\pi], m=1,2,\dots,$$

so that, for each $\varepsilon>0$, a weight function $\mu(x)$,

$$0<\mu(x)\leq 1, \quad \bigl|\bigl\{x\in[0,2\pi] : \mu(x)\neq 1\bigr\}\bigr| < \varepsilon,$$

can be constructed so that the series (1.4) is universal in the weighted space $L^1_\mu[0,2\pi]$ with respect simultaneously to rearrangements, as well as to subseries.

Let us denote the generalized Walsh system of order a by Ψ_a (see Definition 2.2 below).

In this paper, we prove the following results.

Theorem 1.4. *Let $\omega(t)$ be a continuous function with $\omega(+0)=0$ and increasing in $[0,\infty)$. Then there exists a series of the form*

$$\sum_{k=1}^{\infty} c_k\psi_k(x) \quad \text{with} \quad \sum_{k=1}^{\infty} c_k^2\omega\big(|c_k|\big) < \infty \tag{1.5}$$

with the following property: for each $\varepsilon > 0$ a weight function $\mu(x), 0 < \mu(x) \leq 1, |\{x \in [0,1) : \mu(x) \neq 1\}| < \varepsilon$ can be constructed so that the series (1.5) is universal in the weighted space $L^1_\mu[0,1)$ with respect to both rearrangements and subseries.

Remark 1.5. Theorem 1.4 for trigonometric and classical Walsh systems was proved in [2] and [3].

2. Basic lemmas

Now, we present the definitions of generalized Rademacher and Walsh systems.

Let a denote a fixed integer, $a \geq 2$, and put $\omega_a = e^{\frac{2\pi i}{a}}$. Now we will give the definitions of generalized Rademacher and Walsh systems [1].

Definition 2.1. The Rademacher system of order a is defined by

$$\varphi_0(x) = \omega_a^k \quad \text{if } x \in \Big[\frac{k}{a}, \frac{k+1}{a}\Big), k = 0, 1, \ldots, a-1, x \in [0,1),$$

and, for $n \geq 0$,

$$\varphi_n(x+1) = \varphi_n(x) = \varphi_0(a^n x).$$

Definition 2.2. The generalized Walsh system of order a is defined by

$$\psi_0(x) = 1,$$

and if $n = \alpha_1 a^{n_1} + \cdots + \alpha_s a^{n_s}$ where $n_1 > \cdots > n_s$, then

$$\psi_n(x) = \varphi_{n_1}^{\alpha_1}(x) \cdot \ldots \cdot \varphi_{n_s}^{\alpha_s}(x).$$

Let us denote the generalized Walsh system of order a by Ψ_a. Note that Ψ_2 is the classical Walsh system. The basic properties of the generalized Walsh system of order a were obtained by H. E. Chrestenson, R. Paley, J. Fine, W. Young, C. Watari, N. Vilenkin, and others (see [1], [8], [14], [18]–[20]). Next, we present some properties of the Ψ_a system.

Property 1. Each nth Rademacher function has period $\frac{1}{a^n}$ and

$$\varphi_n(x) = \text{const} \in \Omega_a = \{1, \omega_a, \omega_a^2, \ldots, \omega_a^{a-1}\} \tag{2.1}$$

if $x \in \Delta_{n+1}^{(k)} = [\frac{k}{a^{n+1}}, \frac{k+1}{a^{n+1}})$, $k = 0, \ldots, a^{n+1}-1$, $n = 1, 2, \ldots$.

It is also easily verified that

$$\big(\varphi_n(x)\big)^k = \big(\varphi_n(x)\big)^m, \quad \forall n, k \in \mathcal{N}, \text{where } m = k \,(\text{mod } a). \tag{2.2}$$

Property 2. It is clear that, for any integer n, the Walsh function $\psi_n(x)$ consists of a finite product of Rademacher functions and accepts values from Ω_a.

Property 3. Let $\omega_a = e^{\frac{2\pi i}{a}}$. Then for any natural number m we have

$$\sum_{k=0}^{a-1} \omega_a^{k\cdot m} = \begin{cases} a, & \text{if } m \equiv 0 \,(\text{mod } a), \\ 0, & \text{if } m \neq 0 \,(\text{mod } a). \end{cases} \tag{2.3}$$

Property 4. The generalized Walsh system Ψ_a, $a \geq 2$, is a complete orthonormal system in $L^2[0,1)$ and a basis in $L^p[0,1)$, $p > 1$ (see [14]).

Property 5. From Definition 2.2 we have

$$\psi_i(x) \cdot \psi_j(a^s x) = \psi_{j\cdot a^s + i}(x), \quad \text{where } 0 \leq i, j < a^s, \tag{2.4}$$

and, particularly,

$$\psi_{a^k + j}(x) = \varphi_k(x) \cdot \psi_j(x), \quad \text{if } 0 \leq j \leq a^k - 1. \tag{2.5}$$

Now, for any $m = 1, 2, \ldots$ and $1 \leq k \leq a^m$, we put $\Delta_m^{(k)} = [\frac{k-1}{a^m}, \frac{k}{a^m})$ and consider the following function,

$$I_m^{(k)}(x) = \begin{cases} 1, & \text{if } x \in [0,1) \setminus \Delta_m^{(k)}, \\ 1 - a^m, & \text{if } x \in \Delta_m^{(k)}, \end{cases} \tag{2.6}$$

and we periodically extend these functions on R^1 with period 1.

By $\chi_E(x)$ we denote the characteristic function of the set E; that is,

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases} \tag{2.7}$$

Then, clearly,

$$I_m^{(k)}(x) = \psi_0(x) - a^m \cdot \chi_{\Delta_m^{(k)}}(x), \tag{2.8}$$

and for the natural numbers $m \geq 1$ and $1 \leq i \leq a^m$,

$$a_i(\chi_{\Delta_m^{(k)}}) = \int_0^1 \chi_{\Delta_m^{(k)}}(x) \cdot \overline{\psi_i}(x)\, dx = \mathcal{A} \cdot \frac{1}{a^m}, \quad 0 \leq i < a^m, \tag{2.9}$$

$$b_i(I_m^{(k)}) = \int_0^1 I_m^{(k)}(x) \overline{\psi_i}(x)\, dx = \begin{cases} 0, & \text{if } i = 0 \text{ and } i \geq a^k, \\ -\mathcal{A}, & \text{if } 1 \leq i < a^k, \end{cases} \tag{2.10}$$

where $\mathcal{A} = \text{const} \in \Omega_a$ and $|\mathcal{A}| = 1$.

Hence,

$$\chi_{\Delta_m^{(k)}}(x) = \sum_{i=0}^{a^k - 1} a_i(\chi_{\Delta_m^{(k)}}) \psi_i(x), \tag{2.11}$$

$$I_m^{(k)}(x) = \sum_{i=1}^{a^k - 1} b_i(I_m^{(k)}) \psi_i(x). \tag{2.12}$$

Lemma 2.3. *For any numbers $\gamma \neq 0$, $N_0 > 1$, $\varepsilon \in (0,1)$, and any interval of order a $\Delta = \Delta_m^{(k)} = [\frac{k-1}{a^m}, \frac{k}{a^m})$, $i = 1, \dots, a^m$, there exist a measurable set $E \subset \Delta$ and a polynomial $P(x)$ in the Ψ_a system of the form*

$$P(x) = \sum_{k=N_0}^{N} c_k \psi_k(x)$$

which satisfy the following conditions:

(1) $$|E| > (1-\varepsilon) \cdot |\Delta|;$$

(2) $$P(x) = \begin{cases} \gamma, & \text{if } x \in E, \\ 0, & \text{if } x \notin \Delta; \end{cases}$$

(3) $$\Big[\sum_{k=N_0}^{N} c_k^2\Big]^{\frac{1}{2}} < a \cdot |\gamma| \cdot \sqrt{\frac{|\Delta|}{\varepsilon}}.$$

Proof. We choose natural numbers ν_0 and s so that

$$\nu_0 = \Big[\log_a \frac{1}{\varepsilon}\Big] + 1; \qquad s = [\log_a N_0] + m. \tag{2.13}$$

Define the coefficients c_n, a_i, b_j, and the function $P(x)$ in the following way:

$$P(x) = \gamma \cdot \chi_{\Delta_m^{(k)}}(x) \cdot I_{\nu_0}^{(1)}(a^s x), \quad x \in [0,1], \tag{2.14}$$

$$c_n = c_n(P) = \int_0^1 P(x)\overline{\psi_n}(x)\, dx, \quad \forall n \geq 0, \tag{2.15}$$

$$a_i = a_i(\chi_{\Delta_m^{(k)}}), \quad 0 \leq i < a^m, \qquad b_j = b_j(I_{\nu_0}^{(1)}), \quad 1 \leq j < a^{\nu_0}. \tag{2.16}$$

Taking into account (2.1)–(2.3), (2.5)–(2.7), and (2.9)–(2.12) for $P(x)$, we obtain

$$\begin{aligned} P(x) &= \gamma \cdot \sum_{i=0}^{a^m-1} a_i \psi_i(x) \cdot \sum_{j=1}^{a^{\nu_0}-1} b_j \psi_j(a^s x) \\ &= \gamma \cdot \sum_{j=1}^{a^{\nu_0}-1} b_j \cdot \sum_{i=0}^{a^m-1} a_i \psi_{j \cdot a^s + i}(x) = \sum_{k=N_0}^{N} c_k \psi_k(x), \end{aligned} \tag{2.17}$$

where

$$c_k = c_k(P) = \begin{cases} -\mathcal{K} \cdot \frac{\gamma}{a^m} \text{ or } 0, & \text{if } k \in [N_0, N], \\ 0, & \text{if } k \notin [N_0, N], \end{cases} \tag{2.18}$$

$$\mathcal{K} \in \Omega_a, \qquad |\mathcal{K}| = 1, \qquad N = a^{s+\nu_0} + a^m - a^s - 1. \tag{2.19}$$

Set

$$E = \big\{x \in \Delta : P(x) = \gamma\big\}.$$

By (2.7), (2.8), and (2.14), we have

$$|E| = a^{-m}(1-a^{-\nu_0}) > (1-\epsilon)|\Delta|,$$

$$P(x) = \begin{cases} \gamma, & \text{if } x \in E, \\ \gamma(1-a^{\nu_0}), & \text{if } x \in \Delta \setminus E, \\ 0, & \text{if } x \notin \Delta. \end{cases}$$

From relations (2.13), (2.18), and (2.19) we obtain

$$\begin{aligned} \max_{N_0 \le m \le N} \int_0^1 \Big| \sum_{k=N_0}^{m} c_k \psi_k(x) \Big| \, dx &< \Big[\int_0^1 \big| P(x) \big|^2 \, dx \Big]^{\frac{1}{2}} \le \Big[\sum_{k=N_0}^{N} |c_k|^2 \Big]^{\frac{1}{2}} \\ &= |\gamma| \cdot |\Delta| \cdot \sqrt{a^{\nu_0+s} + a^m} = |\gamma| \cdot \sqrt{|\Delta|} \cdot \sqrt{a^{\nu_0}+1} \\ &< |\gamma| \cdot \sqrt{|\Delta|} \cdot \sqrt{\frac{a}{\varepsilon}} < a \cdot |\gamma| \cdot \sqrt{\frac{|\Delta|}{\varepsilon}}. \end{aligned}$$

□

Lemma 2.4. *Let $\omega(t)$ be a continuous function increasing in $[0,\infty)$ and $\omega(+0) = 0$. Then, for any given numbers $0 < \varepsilon < \frac{1}{2}$, $N_0 > 2$, and a step function*

$$f(x) = \sum_{s=1}^{q} \gamma_s \cdot \chi_{\Delta_s}(x), \tag{2.20}$$

where each Δ_s is an interval of the form $\Delta_m^{(i)} = [\frac{i-1}{2^m}, \frac{i}{2^m}]$, $1 \le i \le 2^m$, there exist a measurable set $E \subset [0,1)$ and a polynomial $P(x)$ of the form

$$P(x) = \sum_{k=N_0}^{N} c_k \psi_k(x),$$

which satisfy the following conditions:

(1) $$|E| > 1 - \varepsilon,$$

(2) $$P(x) = f(x), \quad \textit{for } x \in E,$$

(3) $$\sum_{k=N_0}^{N} |c_k|^2 \cdot \omega\big(|c_k|\big) < \varepsilon,$$

(4) $$\max_{N_0 \le M \le N} \int_e \Big| \sum_{k=N_0}^{M} c_k \psi_k(x) \Big| \, dx < \varepsilon + \int_e \big| f(x) \big| \, dx$$

for every measurable subset e of E.

Proof. Let $0 < \varepsilon < 1$ be an arbitrary number. For any positive number η with

$$\eta < \frac{\varepsilon^2}{a^2} \cdot \Big[\int_0^1 f^2(x) \, dx \Big]^{-1}, \tag{2.21}$$

by definition of function $\omega(t)$, there exists a positive number $\delta < \varepsilon$ so that, for any t, $0 < t < \delta$, we have

$$\omega(t) < \omega(\delta) < \eta. \tag{2.22}$$

Without restriction of generality, we assume that

$$0 < a \cdot |\gamma_s| \cdot \sqrt{\frac{|\Delta_s|}{\varepsilon}} < \delta, \quad s = 1, 2, \ldots, q. \tag{2.23}$$

Applying Lemma 2.3 consecutively, we can find a sequence of sets $E_s \subset \Delta_s$ and polynomials

$$P_s(x) = \sum_{k=N_{s-1}}^{N_s - 1} c_k^{(s)} \psi_k(x), \quad s = 1, 2, \ldots, q, \tag{2.24}$$

which, for all $1 \le s \le q$, satisfy the following conditions:

$$|E_s| > (1 - \varepsilon) \cdot |\Delta_s|, \tag{2.25}$$

$$P_s(x) = \begin{cases} \gamma_s, & \text{if } x \in E_s \\ 0, & \text{if } x \notin \Delta_s, \end{cases} \tag{2.26}$$

$$\Big[\sum_{k=N_{s-1}}^{N_s - 1} |c_k^{(s)}|^2 \Big]^{\frac{1}{2}} < a \cdot |\gamma_s| \cdot \sqrt{\frac{|\Delta_s|}{\varepsilon}}. \tag{2.27}$$

We define a set E and a polynomial $P(x)$ as follows:

$$E = \bigcup_{s=1}^{q} E_s, \tag{2.28}$$

$$P(x) = \sum_{k=1}^{q} c_k \psi_k(x) = \sum_{s=1}^{q} \Big[\sum_{k=N_{s-1}}^{N_s - 1} c_k^{(s)} \psi_k(x) \Big], \tag{2.29}$$

where

$$c_k = c_k^{(s)}, \quad \text{for } N_{s-1} \le k < N_s, s = 1, 2, \ldots, q, N = N_q - 1. \tag{2.30}$$

From (2.20), (2.25), (2.26), (2.28), and (2.29) we get

$$|E| > 1 - \varepsilon,$$
$$P(x) = f(x), \quad \text{if } x \in E.$$

Taking relations (2.23), (2.27), and (2.30), for any $k \in [N_0, N]$ we have

$$|c_k| \le \max_{1 \le s \le q} \Big[a \cdot |\gamma_s| \cdot \sqrt{\frac{|\Delta_s|}{\varepsilon}} \Big] < \delta. \tag{2.31}$$

Hence, and from (2.22), it follows that

$$\omega\big(|c_k|\big) < \omega(\delta) < \eta, \quad \forall k \in [N_0, N].$$

Consequently, from (2.21) and (2.27) we get

$$\sum_{k=N_0}^{N} |c_k|^2 \cdot \omega\big(|c_k|\big) < \eta \cdot \sum_{s=1}^{q} \Big[\sum_{k=N_{s-1}}^{N_s - 1} |c_k^{(s)}|^2 \Big] < \eta \cdot \frac{a^2}{\varepsilon} \cdot \Big[\int_0^1 f^2(x)\, dx \Big] < \varepsilon.$$

That is, statements (1)–(3) of Lemma 2.4 are satisfied. Now we will check the fulfillment of statement (4).

For any number M, $N_0 \le M < N$, we can find s_0, $1 \le s_0 \le q$ such that $N_{s_0} < M < N_{s_0+1}$. Then, from (2.29) and (2.30) we have

$$\sum_{k=N_0}^{M} c_k \psi_k(x) = \sum_{s=1}^{s_0} P_s(x) + \sum_{k=N_{s_0}}^{M} c_k \psi_k(x). \tag{2.32}$$

Given the choice of δ and that $P(x) = f(x)$ for $x \in E$, we then obtain, from relations (2.23), (2.27), (2.29), and (2.32) for any measurable set $e \subset E$

$$\begin{aligned} &\int_e \Big| \sum_{k=N_0}^{M} c_k \psi_k(x) \Big| \, dx \\ &\le \int_e \Big| \sum_{s=1}^{s_0} P_s(x) \Big| \, dx + \int_0^1 \Big| \sum_{k=N_{s_0}}^{M} c_k \psi_k(x) \Big| \, dx \\ &< \int_e \big| P(x) \big| \, dx + |\gamma_{s_0+1}| \cdot a \cdot \sqrt{\frac{|\Delta_{s_0+1}|}{\varepsilon}} \\ &< \int_e \big| f(x) \big| \, dx + \varepsilon. \end{aligned}$$

□

3. Proof of main results

Proof. Let $\omega(t)$ be a continuous function, increasing in $[0, \infty)$ and $\omega(+0) = 0$, and let

$$\big\{ f_n(x) \big\}_{n=1}^{\infty} \tag{3.1}$$

be a sequence of all step functions with rational values and rational jump points. Applying Lemma 2.4 consecutively, we can find a sequence of sets $\{E_s\}_{s=1}^{\infty}$ and a sequence of polynomials

$$P_s(x) = \sum_{k=N_{s-1}}^{N_s - 1} c_k^{(s)} \psi_k(x), \tag{3.2}$$

where $1 = N_0 < N_1 < \cdots < N_s < \cdots$, $s = 1, 2, \ldots$, which satisfy the following conditions:

$$|E_s| > 1 - 2^{-2(s+1)}, \quad E_s \subset [0,1], \tag{3.3}$$

$$P_s(x) = f_s(x), \quad x \in E_s, \tag{3.4}$$

$$\sum_{k=N_{s-1}}^{N_s-1} |c_k^{(s)}| \cdot \omega\big(|c_k^{(s)}|\big) < 2^{-2s}, \tag{3.5}$$

$$\max_{N_{s-1}\le p<N_s} \Big[\int_e \Big| \sum_{k=N_{s-1}}^{p} c_k^{(s)}\psi_k(x)\Big|\,dx\Big] < 2^{-2(s+1)} + \int_e \big|f_s(x)\big|\,dx \tag{3.6}$$

for any measurable set $e \subset E$.

Denote

$$\sum_{k=1}^{\infty} c_k\psi_k(x) = \sum_{s=1}^{\infty}\Big[\sum_{k=N_{s-1}}^{N_s-1} c_k^{(s)}\psi_k(x)\Big], \tag{3.7}$$

where $c_k = c_k^{(s)}$, for $N_{s-1} \le k < N_s$, $s = 1, 2, \dots$.

Let ε be an arbitrary positive number, and setting

$$\begin{cases} \Omega_n = \bigcap_{s=n}^{\infty} E_s, \quad n = 1, 2, \dots; \\ E = \Omega_{n_0} = \bigcap_{s=n_0}^{\infty} E_s, \quad n_0 = [\log_{1/2}\varepsilon] + 1; \\ B = \bigcup_{n=n_0}^{\infty} \Omega_n = \Omega_{n_0} \cup (\bigcup_{n=n_0+1}^{\infty} \Omega_n \setminus \Omega_{n-1}). \end{cases} \tag{3.8}$$

It is clear that $|B| = 1$ and $|E| > 1 - \varepsilon$ (see (3.3)).

We define a function $\mu(x)$ in the following way:

$$\mu(x) = \begin{cases} 1, & \text{for } x \in E \cup ([0,1) \setminus B); \\ \mu_n, & \text{for } x \in \Omega_n \setminus \Omega_{n-1}, n \ge n_0 + 1, \end{cases} \tag{3.9}$$

where

$$\begin{cases} \mu_n = [2^{2n} \cdot \prod_{s=1}^{n} h_s]^{-1}, \\ h_s = \|f_s(x)\|_C + \max_{N_{s-1}\le p<N_s} \|\sum_{k=N_{s-1}}^{p} c_k^{(s)}\psi_k(x)\|_C + 1, \end{cases} \tag{3.10}$$

where $\|g(x)\|_C = \max_{x\in[0,1)} |g(x)|$, $g(x)$ is a bounded function on $[0, 1)$.

From (3.5) and (3.8)–(3.10) we obtain the following:

(A) $\mu(x)$ is a measurable function and

$$0 < \mu(x) \le 1, \quad \Big|\Big\{x \in [0,1) : \mu(x) \ne 1\Big\}\Big| < \varepsilon.$$

(B) $\sum_{k=1}^{\infty} |c_k|^2 \cdot \omega(|c_k|) < \infty$.

Hence, we obviously have

$$\lim_{k\to\infty} c_k = 0. \tag{3.11}$$

It follows from (3.8)–(3.10) that, for all $s \ge n_0$ and $p \in [N_{s-1}, N_s)$,

$$\begin{aligned} &\int_{[0,1)\setminus\Omega_s} \Big|\sum_{k=N_{s-1}}^{p} c_k^{(s)}\psi_k(x)\Big|\mu(x)\,dx \\ &= \sum_{n=s+1}^{\infty}\Big[\int_{\Omega_n\setminus\Omega_{n-1}} \Big|\sum_{k=N_{s-1}}^{p} c_k^{(s)}\psi_k(x)\Big|\mu_n\,dx\Big] \\ &\le \sum_{n=s+1}^{\infty} 2^{-2n}\Big[\int_0^1 \Big|\sum_{k=N_{s-1}}^{p} c_k^{(s)}\psi_k(x)\Big|h_s^{-1}\,dx\Big] < \frac{1}{3}2^{-2s}. \end{aligned} \tag{3.12}$$

By (3.4) and (3.8)–(3.10), for all $s \geq n_0$ we have

$$\begin{aligned}
&\int_0^1 \big|P_s(x) - f_s(x)\big|\mu(x)\,dx \\
&\quad = \int_{\Omega_s} \big|P_s(x) - f_s(x)\big|\mu(x)\,dx \\
&\qquad + \int_{[0,1)\setminus\Omega_s} \big|P_s(x) - f_s(x)\big|\mu(x)\,dx \\
&\quad = \sum_{n=s+1}^{\infty} \Big[\int_{\Omega_n\setminus\Omega_{n-1}} \big|P_s(x) - f_s(x)\big|\mu_n\,dx\Big] \\
&\quad \leq \sum_{n=s+1}^{\infty} 2^{-2s} \Big[\int_0^1 \Big(\big|f_s(x)\big| + \Big|\sum_{k=N_{s-1}}^{N_s-1} c_k^{(s)}\psi_k(x)\Big|\Big) h_s^{-1}\,dx\Big] \\
&\quad < \frac{1}{3}2^{-2s} < 2^{-2s}. \qquad (3.13)
\end{aligned}$$

Taking relations (A), (3.6), and (3.8)–(3.10) into account, for all $p \in [N_{s-1}, N_s)$ and $s \geq n_0 + 1$, we obtain

$$\begin{aligned}
&\int_0^1 \Big|\sum_{k=N_{s-1}}^{p} c_k^{(s)}\psi_k(x)\Big|\mu(x)\,dx \\
&\quad = \int_{\Omega_s} \Big|\sum_{k=N_{s-1}}^{p} c_k^{(s)}\psi_k(x)\Big|\mu(x)\,dx \\
&\qquad + \int_{[0,1)\setminus\Omega_s} \Big|\sum_{k=N_{s-1}}^{p} c_k^{(s)}\psi_k(x)\Big|\mu(x)\,dx \\
&\quad < \sum_{n=n_0+1}^{s} \Big[\int_{\Omega_n\setminus\Omega_{n-1}} \Big|\sum_{k=N_{s-1}}^{p} c_k^{(s)}\psi_k(x)\Big|\,dx\Big] \cdot \mu_n + \frac{1}{3}2^{-2s} \\
&\quad < \sum_{n=n_0+1}^{s} \Big(2^{-2(s+1)} + \int_{\Omega_n\setminus\Omega_{n-1}} \big|f_s(x)\big|\,dx\Big)\mu_n + \frac{1}{3}2^{-2s} \\
&\quad = 2^{-2(s+1)} \cdot \sum_{n=n_0+1}^{s} \mu_n + \int_{\Omega_s} \big|f_s(x)\big|\mu(x)\,dx + \frac{1}{3}2^{-2s} \\
&\quad < \int_0^1 \big|f_s(x)\big|\mu(x)\,dx + 2^{-2s}. \qquad (3.14)
\end{aligned}$$

Let $f(x) \in L^1_\mu[0,1)$ be any function; that is, $\int_0^1 |f(x)|\mu(x)\,dx < \infty$.

It is easy to see that we can choose a function $f_{\nu_1}(x)$ from the sequence (3.1) such that

$$\int_0^1 \big|f(x) - f_{\nu_1}(x)\big|\mu(x)\,dx < 2^{-2}, \quad \nu_1 > n_0 + 1. \qquad (3.15)$$

Hence, we have

$$\int_0^1 \big|f_{\nu_1}(x)\big|\mu(x)\,dx < 2^{-2} + \int_0^1 \big|f(x)\big|\mu(x)\,dx. \tag{3.16}$$

From Definition 2.2 and from relations (A), (3.13), and (3.15) we obtain, with $m_1 = 1$,

$$\begin{aligned}
&\int_0^1 \big|f(x) - \big[P_{\nu_1}(x) + c_{m_1}\psi_{m_1}(x)\big]\big|\mu(x)\,dx \\
&\quad\le \int_0^1 \big|f(x) - f_{\nu_1}(x)\big|\mu(x)\,dx + \int_0^1 \big|f_{\nu_1}(x) - P_{\nu_1}(x)\big|\mu(x)\,dx \\
&\qquad + \int_0^1 \big|c_{m_1}\psi_{m_1}(x)\big|\mu(x)\,dx < 2\cdot 2^{-2} + |c_{m_1}|.
\end{aligned} \tag{3.17}$$

Assume that numbers $\nu_1 < \nu_2 < \cdots < \nu_{q-1}$, $m_1 < m_2 < \cdots < m_{q-1}$ are chosen in such a way that the following condition is satisfied:

$$\begin{aligned}
&\int_0^1 \Big|f(x) - \sum_{s=1}^{j}\big[P_{\nu_s}(x) + c_{m_s}\psi_{m_s}(x)\big]\Big|\mu(x)\,dx \\
&\quad < 2\cdot 2^{-2j} + |c_{m_j}|, \quad 1 \le j \le q-1.
\end{aligned} \tag{3.18}$$

Now, we choose a function $f_{\nu_q}(x)$ from the sequence (3.1) such that

$$\int_0^1 \Big|\Big(f(x) - \sum_{s=1}^{q-1}\big[P_{\nu_s}(x) + c_{m_s}\psi_{m_s}(x)\big]\Big) - f_{n_q}(x)\Big|\mu(x)\,dx < 2^{-2q}, \tag{3.19}$$

where $\nu_q > \nu_{q-1}$, $\nu_q > m_{q-1}$.

This, with (3.18), implies

$$\begin{aligned}
\int_0^1 \big|f_{\nu_q}(x)\big|\mu(x)\,dx &< 2^{-2q} + 2\cdot 2^{-2(q-1)} + |c_{m_{q-1}}| \\
&= 9\cdot 2^{-2q} + |c_{m_{q-1}}|.
\end{aligned} \tag{3.20}$$

From (3.13), (3.14), and (3.20) we have

$$\int_0^1 \big|f_{\nu_q}(x) - P_{\nu_q}(x)\big|\mu(x)\,dx < 2^{-2\nu_q}, \tag{3.21}$$

$$P_{\nu_q}(x) = \sum_{k=N_{\nu_{q-1}}}^{N_{\nu_q}-1} c_k^{(\nu_q)}\psi_k(x),$$

$$\max_{N_{\nu_{q-1}} \le p < N\nu_q} \int_0^1 \Big|\sum_{k=N_{\nu_{q-1}}}^{p} c_k^{(\nu_q)}\psi_k(x)\Big|\mu(x)\,dx < 10\cdot 2^{-2q} + |c_{m_{q-1}}|. \tag{3.22}$$

Denote

$$m_q = \min\big\{n \in N : n \notin \big\{\big\{\{k\}_{k=N_{\nu_{s-1}}}^{N_{\nu_s}-1}\big\}_{s=1}^{q} \cup \{m_s\}_{s=1}^{q-1}\big\}\big\}. \tag{3.23}$$

Taking into account the relations (A), (3.19), and (3.21), we get

$$\begin{aligned}\int_0^1 \Big| f(x) - \sum_{s=1}^{q} \big[P_{\nu_s}(x) + c_{m_s}\psi_{m_s}(x)\big]\Big| \mu(x)\,dx \\ \leq \int_0^1 \Big|\Big(f(x) - \sum_{s=1}^{q-1}\big[P_{\nu_s}(x) + c_{m_s}\psi_{m_s}(x)\big]\Big) - f_{\nu_q}(x)\Big|\mu(x)\,dx \\ + \int_0^1 \big|f_{\nu_q}(x) - P_{\nu_q}(x)\big|\mu(x)\,dx \\ + \int_0^1 \big|c_{m_q}\psi_{m_q}(x)\big|\mu(x)\,dx < 2\cdot 2^{-2q} + |c_{m_q}|. \end{aligned} \tag{3.24}$$

Thus, by induction, we can choose from series (3.7) a sequence of members

$$c_{m_q}\psi_{m_q}(x), \quad q = 1, 2, \ldots,$$

and a sequence of polynomials

$$P_{\nu_q}(x) = \sum_{k=N_{\nu_{q-1}}}^{N_{\nu_q}-1} c_k^{(\nu_q)}\psi_k(x), \quad N_{n_q-1} > N_{n_{q-1}}, q = 1, 2, \ldots, \tag{3.25}$$

such that conditions (3.22)–(3.24) are satisfied for all $q \geq 1$.

Taking into account the choice of $P_{\nu_q}(x)$ and $c_{m_q}\psi_{m_q}(x)$ (see (3.22) and (3.25)), we conclude that the series

$$\sum_{q=1}^{\infty}\Big[\sum_{k=N_{\nu_{q-1}}}^{N_{\nu_q}-1} c_k^{(\nu_q)}\psi_k(x) + c_{m_q}\psi_q(x)\Big]$$

is obtained from the series (3.7) by rearrangement of members.

It follows from (3.11), (3.21), and (3.24) that this series converges to the function $f(x)$ in the metric $L^1_\mu[0,1)$; that is, the series (3.7) is universal with respect to rearrangements (see Definition 1.1).

On the other hand, it is easy to see that, for any function $f(x) \in L^1_\mu[0,1)$, from the sequence (3.2) one can choose polynomials

$$P_{r_s}(x) = \sum_{k=N_{r_{s-1}}}^{N_{r_s}-1} c_k^{(r_s)}\psi_k(x), \quad r_{s-1} < r_s, s = 1, 2, \ldots,$$

so that the following conditions are satisfied:

$$\int_0^1 \Big| f(x) - \sum_{s=1}^{N} P_{r_s}(x)\Big|\mu(x)\,dx < 2^{-N}, \quad N = 1, 2, \ldots,$$

$$\max_{N_{r_{s-1}} \leq m < Nr_s} \int_0^1 \Big|\sum_{k=N_{r_{s-1}}}^{p} c_k^{(r_s)}\psi_k(x)\Big|\mu(x)\,dx < 2^{-N}, \quad N = 1, 2, \ldots.$$

Hence, it follows that the subseries

$$\sum_{s=1}^{\infty}\Big[\sum_{k=N_{r_{s-1}}}^{N_{r_s}-1} c_k^{(r_s)}\psi_k(x)\Big]$$

of series (3.7) converges to $f(x)$ in the metric of $L^1_\mu[0,1)$. This means that series (3.7) is universal in $L^1_\mu[0,1)$ by subseries (see Definition 1.3). □

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