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NONCONVEX PROXIMAL NORMAL STRUCTURE IN CONVEX METRIC SPACES

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To Professor Jamshid Moori on the occasion of his 70th birthday

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Abstract. Given that A and B are two nonempty subsets of the convex metric space $(X, d, \mathcal{W})$, a mapping $T : A \cup B \to A \cup B$ is noncyclic relatively nonexpansive, provided that $T(A) \subseteq A$, $T(B) \subseteq B$, and $d(Tx, Ty) \leq d(x, y)$ for all $(x, y) \in A \times B$. A point $(p, q) \in A \times B$ is called a *best proximity pair* for the mapping T if $p = Tp$, $q = Tq$, and $d(p, q) = \operatorname{dist}(A, B)$. In this work, we study the existence of best proximity pairs for noncyclic relatively nonexpansive mappings by using the notion of nonconvex proximal normal structure. In this way, we generalize a main result of Eldred, Kirk, and Veeramani. We also establish a common best proximity pair theorem for a commuting family of noncyclic relatively nonexpansive mappings in the setting of convex metric spaces, and as an application we conclude a common fixed-point theorem.

1. Introduction

Let (A, B) be a pair of subsets of a metric space (X, d). A mapping $T\colon A \cup B \to A \cup B$ is said to be *noncyclic* provided that $T(A) \subseteq A$ and $T(B) \subseteq B$. A point $(p, q) \in A \times B$ is called a *best proximity pair* if $p = Tp$, $q = Tq$, and $d(p, q) = \operatorname{dist}(A, B)$ is satisfied, where $\operatorname{dist}(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}$. Eldred, Kirk, and Veeramani [4] proved the existence of a best proximity pair for a relatively nonexpansive mapping using the geometric notion of *proximal normal*

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structure. They also obtained the following theorem in uniformly convex Banach spaces.

Theorem 1.1 (see [4, Corollary 2.2])**.** *Let A and B be two nonempty, bounded, closed, and convex subsets of a uniformly convex Banach space X, and let $T : A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping; that is, T is noncyclic and $\|Tx - Ty\| \leq \|x - y\|$ for all $(x, y) \in A \times B$. Then T has a best proximity pair.*

We mention that Theorem 1.1 is based on the fact that every nonempty, bounded, closed, and convex pair of subsets of a uniformly convex Banach space X has a proximal normal structure, and so the result follows from Theorem 2.2 of [4] (see also [6] for a different approach to the same problem).

This paper is organized as follows: in Section 2, we recall some definitions, concepts, and previous results that we will need. In Section 3, we introduce a geometric notion of a nonconvex proximal normal structure and use it to investigate the existence of best proximity pairs for noncyclic relatively nonexpansive mappings in the setting of convex metric spaces. In this way, we obtain a generalization of Theorem 1.1. Finally, in Section 4, we establish a common best proximity pair theorem for a commuting family of noncyclic relatively nonexpansive mappings.

2. Preliminaries

Throughout this paper, we shall say that a pair (A, B) of subsets of a metric space (X, d) satisfies a property if both A and B satisfy that property. For example, (A, B) is closed if and only if both A and B are closed; $(A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C, \text{and} B \subseteq D$. We shall also adopt the notation

$$\begin{aligned} \delta_x(A) &= \sup\{d(x, y)\colon y \in A\} \quad \text{for all } x \in X, \\ \delta(A, B) &= \sup\{\delta_x(B) : x \in A\}, \\ \operatorname{diam}(A) &= \delta(A, A), \\ \mathcal{D}(x, A) &= \operatorname{dist}(\{x\}, A), \quad \text{for all } x \in X. \end{aligned}$$

Given that (A, B) is a pair of nonempty subsets of a metric space, then its proximal pair is the pair (A_0, B_0) given by

$$\begin{aligned} A_0 &= \{x \in A\colon d(x, y') = \operatorname{dist}(A, B) \text{ for some } y' \in B\}, \\ B_0 &= \{y \in B\colon d(x', y) = \operatorname{dist}(A, B) \text{ for some } x' \in A\}. \end{aligned}$$

Proximal pairs may be empty, but, in particular, if (A, B) is a nonempty and compact pair in a metric space (X, d), then (A_0, B_0) is nonempty. A pair of sets (A, B) is said to be *proximinal* if $A = A_0$ and $B = B_0$.

In [10], Takahashi introduced the notion of convexity in metric spaces as follows.

Definition 2.1. Let (X, d) be a metric space, and let $I := [0, 1]$. A mapping $\mathcal{W} : X \times X \times I \to X$ is said to be a *convex structure* on X provided that, for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, \mathcal{W}(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y).$$

A metric space (X, d) together with a convex structure $\mathcal{W}$ is called a *convex metric space*, which is denoted by $(X, d, \mathcal{W})$. A Banach space and each of its convex subsets are convex metric spaces. But a Frechet space is not necessarily a convex metric space. The other examples of convex metric spaces which are not imbedded in any Banach space can be found in [10].

Definition 2.2. A subset K of a convex metric space $(X, d, \mathcal{W})$ is said to be a *convex set*, provided that $\mathcal{W}(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in I$.

Proposition 2.3. *Let $\{K_\alpha\}_{\alpha \in A}$ be a family of convex subsets of X. Then $\bigcap_{\alpha \in A} K_\alpha$ is also a convex subset of X.*

Definition 2.4. A convex metric space $(X, d, \mathcal{W})$ is said to have *property (C)* if every bounded decreasing net of nonempty, closed, and convex subsets of X has a nonempty intersection.

For example, every weakly compact convex subset of a Banach space has property (C).

Let A be a nonempty subset of a convex metric space $(X, d, \mathcal{W})$. We denote by $\mathcal{B}(x; r)$ the closed ball with center $x \in X$ and radius $r > 0$. Also, the *closed and convex hull* of a set A will be denoted by $\overline{\text{con}}(A)$ and defined as below:

$$\overline{\text{con}}(A) := \bigcap \{C : C \text{ is a closed and convex subset of } X \text{ such that } C \supseteq A\}.$$

The following lemmas will be used in our coming discussion.

Lemma 2.5 ([8, Lemma 4.1])**.** *Let A be a nonempty subset of a convex metric space $(X, d, \mathcal{W})$. Then*

$$\delta_x(A) = \delta_x\big(\overline{\text{con}}(A)\big), \quad \forall x \in X.$$

Lemma 2.6 ([1, Lemma 3.7])**.** *Let (K_1, K_2) be a nonempty pair of subsets of a convex metric space $(X, d, \mathcal{W})$. Then $\delta(K_1, K_2) = \delta(\overline{\text{con}}(K_1), \overline{\text{con}}(K_2))$.*

Definition 2.7 ([7, Definition 2.13]). Let (A, B) be a nonempty pair of subsets of a metric space (X, d). We say that the pair (A, B) is a *proximal compactness pair*, provided that every net $(\{x_\alpha\}, \{y_\alpha\})$ of $A \times B$ satisfying the condition that $d(x_\alpha, y_\alpha) \to \text{dist}(A, B)$ has a convergent subnet in $A \times B$.

It is clear that if (A, B) is a compact pair in a metric space (X, d), then (A, B) is a proximal compactness pair.

Definition 2.8. A Banach space X is said to be

(i) *uniformly convex* if there exists a strictly increasing function $\delta : (0, 2] \to [0, 1]$ such that the following implication holds for all $x, y, p \in X$, $R > 0$ and $r \in [0, 2R]$:

$$\begin{cases} \|x - p\| \leq R, \\ \|y - p\| \leq R, \\ \|x - y\| \geq r \end{cases} \Rightarrow \left\| \frac{x + y}{2} - p \right\| \leq \Big(1 - \delta\Big(\frac{r}{R}\Big)\Big) R;$$

(ii) *strictly convex* if the following implication holds for all $x, y, p \in X$ and $R > 0$:

$$\begin{cases} \|x-p\| \le R, \\ \|y-p\| \le R, \\ x \ne y \end{cases} \Rightarrow \left\| \frac{x+y}{2} - p \right\| < R.$$

At the end of this section, we recall the notion of proximal normal structure, which is an extension of normal structure (see [2]).

Definition 2.9 ([4, Definition 1.2]). A convex pair (A, B) in a Banach space X is said to have *proximal normal structure (PNS)* if for any bounded, closed, and convex proximal pair $(K_1, K_2) \subseteq (A, B)$, for which $\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B)$ and $\delta(K_1, K_2) > \operatorname{dist}(K_1, K_2)$, there exists $(x_1, x_2) \in K_1 \times K_2$ such that

$$\max\big\{\delta_{x_1}(K_2), \delta_{x_2}(K_1)\big\} < \delta(K_1, K_2).$$

Note that if in the above definition $A = B$, then (A, B) has PNS if and only if the set A has a normal structure in the sense of Brodskiĭ and Mil′man [2].

It was announced in [4] that every nonempty, bounded, closed, and convex pair of subsets of a uniformly convex Banach space X has PNS (see [4, Proposition 2.1]). The following theorem guarantees the existence of a best proximity pair for noncyclic relatively nonexpansive mappings.

Theorem 2.10 ([4, Theorem 2.2])**.** *Let (A, B) be a nonempty, weakly compact, and convex pair in a strictly convex Banach space X. Let $T : A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping. If the pair (A, B) has PNS, then T has a best proximity pair.*

Remark 2.11. If in Theorem 2.10 $A = B$, then we do not need the condition of strict convexity of the Banach space X. In this case, we give Kirk's [9] fixed-point theorem. Let A be a nonempty, weakly compact, and convex subset of a Banach space X, and let $T : A \to A$ be a nonexpansive self-mapping; that is,

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in A.$$

If A has the normal structure, then T has a fixed point.

3. Existence results of best proximity pairs

We begin our main results of this section with the following notions.

Definition 3.1. A convex metric space $(X, d, \mathcal{W})$ is said to be a *strictly convex metric space*, provided that for every $r > 0$, x_1, x_2 and $y \in X$ with $d(x_1, y) \le r$, $d(x_2, y) \le r$, and $x_1 \ne x_2$ we have

$$d\Big(\mathcal{W}\Big(x_1, x_2, \frac{1}{2}\Big), y\Big) < r.$$

It is clear that every strictly convex Banach space is a strictly convex metric space.

Definition 3.2. Let (A, B) be a pair of nonempty subsets of a convex metric space $(X, d, \mathcal{W})$, and let $T : A \cup B \to A \cup B$ be a noncyclic mapping. The pair (A, B) is said to be a *T-regular pair* provided that

$$\Big(\mathcal{W}\Big(x, Tx, \frac{1}{2}\Big), \mathcal{W}\Big(y, Ty, \frac{1}{2}\Big)\Big) \in A \times B, \quad \forall (x, y) \in A \times B.$$

Note that if $A = B$, then the set A is said to be a *T-regular set* (see [11] for Banach spaces). We also note that every convex pair of subsets of X is a T-regular pair for any noncyclic mapping T defined on the union of the considered pair. The next example shows that the reverse implication does not hold.

Example 3.3. Consider $X = \mathbb{R}$ with the usual metric and usual convex structure $\mathcal{W}(x, y, \lambda) = \lambda x + (1 - \lambda)y$. Let $A = \mathbb{Q} \cap [-1, 0]$, and let $B = \mathbb{Q}^c \cap [1, 2]$. Define $T : A \cup B \to A \cup B$ with

$$T(x) = \begin{cases} x, & \text{if } x \in A, \\ \sqrt{x}, & \text{if } x \in B. \end{cases}$$

Then T is noncyclic and it is easy to see that the pair (A, B) is a T-regular pair; that is, $(\frac{x+Tx}{2}, \frac{y+Ty}{2}) \in A \times B$ for all $(x, y) \in A \times B$.

Definition 3.4. Let (A, B) be a nonempty, bounded, and closed pair in a convex metric space $(X, d, \mathcal{W})$, and let $T : A \cup B \to A \cup B$ be a noncyclic mapping. (A, B) is said to be a *T-regular reflexive pair* provided that (A, B) is T-regular and any descending chain consisting of nonempty, closed, and T-regular pairs which are subsets of (A, B) have a nonempty intersection.

Note that if (A, B) is a nonempty and compact pair in $(X, d, \mathcal{W})$, then (A, B) is T-regular reflexive.

Let us illustrate this notion with the following example.

Example 3.5. Consider the nonreflexive Banach space $X = l^1$ with the canonical basis $\{e_n\}$ and the usual convex structure. Let $A = B = \{xe_n : -1 \le x \le 1, n \in \mathbb{N}\}$, and define $T : A \to A$ with $Tx = -x$. Note that A is nonconvex and also is not weakly compact. Besides, A is T-regular reflexive. In fact, for each $x \in A$ we have $\mathcal{W}(x, Tx, \frac{1}{2}) = \frac{x+Tx}{2} = 0 \in A$; that is, (A, B) is a T-regular pair. Moreover, if $\{(E_j, F_j)\}$ is a descending chain of nonempty, closed, and T-regular pairs which are subsets of (A, B), then $(0, 0) \in (\bigcap_j E_j, \bigcap_j F_j)$. Therefore, (A, B) is a T-regular reflexive pair.

The following lemma will be used in the sequel.

Lemma 3.6. *Let (A, B) be a nonempty, bounded, and closed pair in a convex metric space $(X, d, \mathcal{W})$, and let $T : A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping such that (A, B) is a T-regular reflexive pair. Then (A_0, B_0) is a nonempty, closed, and T-regular pair.*

Proof. For all $n \in \mathbb{N}$ put

$$A_n := \Big\{x \in A : \mathcal{D}(x, B) \le \operatorname{dist}(A, B) + \frac{1}{n}\Big\},$$
$$B_n := \Big\{y \in B : \mathcal{D}(y, A) \le \operatorname{dist}(A, B) + \frac{1}{n}\Big\}.$$

Thus, (A_n, B_n) is a nonempty, bounded, and closed pair for each $n \in \mathbb{N}$. Let $x \in A_n$. Then $\mathcal{D}(x, B) \leq \operatorname{dist}(A, B) + \frac{1}{n}$. We now have

$$\begin{aligned}\mathcal{D}(Tx, B) \leq \mathcal{D}(Tx, TB) &= \inf_{y \in B} d(Tx, Ty)\\ &\leq \inf_{y \in B} d(x, y) = \mathcal{D}(x, B) \leq \operatorname{dist}(A, B) + \frac{1}{n};\end{aligned}$$

that is, $Tx \in A_n$. Therefore, $T(A_n) \subseteq A_n$. Similarly, we can see that $T(B_n) \subseteq B_n$, and so T is noncyclic on $A_n \cup B_n$ for all $n \in \mathbb{N}$. Besides, (A_n, B_n) is a T-regular pair; indeed, if $x \in A_n$, then for each $y \in B$ we have

$$d\Big(\mathcal{W}\Big(x, Tx, \frac{1}{2}\Big), y\Big) \leq \frac{1}{2} d(x, y) + \frac{1}{2} d(Tx, y),$$

which implies that

$$\mathcal{D}\Big(\mathcal{W}\Big(x, Tx, \frac{1}{2}\Big), B\Big) \leq \frac{1}{2}\mathcal{D}(x, B) + \frac{1}{2}\mathcal{D}(Tx, B) \leq \operatorname{dist}(A, B) + \frac{1}{n}.$$

Hence, $\mathcal{W}(x, Tx, \frac{1}{2}) \in A_n$. By a similar way, $\mathcal{W}(y, Ty, \frac{1}{2}) \in B_n$ for any $y \in B$. Thereby, (A_n, B_n) is a T-regular pair for all $n \in \mathbb{N}$. Note that $A_0 = \bigcap_{n \geq 1} A_n$, $B_0 = \bigcap_{n \geq 1} B_n$. Since (A, B) is a T-regular reflexive pair, (A_0, B_0) is a nonempty and also closed pair. On the other hand, if $(x, y) \in A_0 \times B_0$, then $(x, y) \in A_n \times B_n$ for all $n \in \mathbb{N}$, and since (A_n, B_n) is a T-regular pair, $(\mathcal{W}(x, Tx, \frac{1}{2}), \mathcal{W}(y, Ty, \frac{1}{2})) \in A_n \times B_n$ for all $n \in \mathbb{N}$, which implies that $(\mathcal{W}(x, Tx, \frac{1}{2}), \mathcal{W}(y, Ty, \frac{1}{2})) \in A_0 \times B_0$; that is, (A_0, B_0) is a T-regular pair. ☐

Next we shall state the first main result of this section.

Theorem 3.7. *Let (A, B) be a nonempty, bounded, and closed pair in a strictly convex metric space $(X, d, \mathcal{W})$, and let $T : A \cup B \to A \cup B$ be a generalized noncyclic contraction; that is, T is noncyclic and*

$$d(Tx, Ty) \leq r \max\big\{d(x, y), d(x, Ty), d(y, Tx)\big\} + (1 - r)\operatorname{dist}(A, B),$$

for some $r \in [0, 1)$ and for all $(x, y) \in A \times B$. If (A, B) is a T-regular reflexive pair, then T has a best proximity pair.

Proof. Let Σ denote the collection of all nonempty, bounded, and closed pairs $(E, F) \subseteq (A, B)$ such that T is noncyclic on $E \cup F$ and (E, F) is a T-regular pair. By Lemma 3.6, (A_0, B_0) is a nonempty, closed, and T-regular pair, and so $(A_0, B_0) \in \Sigma \neq \emptyset$. Also, Σ is partially ordered with respect to reverse inclusion; that is, $(E_1, F_1) \leq (E_2, F_2) \Leftrightarrow (E_2, F_2) \subseteq (E_1, F_1)$. Let $\{(E_j, F_j)\}_j$ be a descending chain in Σ and $E = \bigcap_j E_j$, $F = \bigcap_j F_j$. Since (A, B) is T-regular reflexive, (E, F) is a nonempty pair. Moreover, (E, F) is closed, and it is easy to see that T is noncyclic on $E \cup F$. Now, suppose $(x, y) \in E \times F$. Thus, $(x, y) \in E_j \times F_j$ for all j. By the fact that each pair (E_j, F_j) is T-regular, we obtain $(\mathcal{W}(x, Tx, \frac{1}{2}), \mathcal{W}(y, Ty, \frac{1}{2})) \in E_j \times F_j$ for any j, and so $(\mathcal{W}(x, Tx, \frac{1}{2}), \mathcal{W}(y, Ty, \frac{1}{2})) \in E \times F$; that is, (E, F) is T-regular, which implies that $(E, F) \in \Sigma$. It now follows from Zorn's lemma that Σ has a minimal element, say (K_1, K_2). Let

$$L_1 := \overline{\operatorname{con}}\big(T(K_1)\big) \cap K_1 \qquad \text{and} \qquad L_2 := \overline{\operatorname{con}}\big(T(K_2)\big) \cap K_2.$$

Then (L_1, L_2) is a closed pair. Also, T is noncyclic on $L_1 \cup L_2$. Besides, if $(x, y) \in L_1 \times L_2$, then $(Tx, Ty) \in \overline{\text{con}}(T(K_1)) \times \overline{\text{con}}(T(K_2))$. Since both the pairs $(\overline{\text{con}}(T(K_1)), \overline{\text{con}}(T(K_2)))$ and (K_1, K_2) are T-regular, we have

$$\Big(\mathcal{W}\Big(x, Tx, \frac{1}{2}\Big), \mathcal{W}\Big(y, Ty, \frac{1}{2}\Big)\Big) \in L_1 \times L_2,$$

and so (L_1, L_2) is T-regular. Now, from the minimality of (K_1, K_2) we obtain $L_1 = K_1$ and $L_2 = K_2$. Therefore, $(K_1, K_2) \subseteq (\overline{\text{con}}(T(K_1)), \overline{\text{con}}(T(K_2)))$; hence,

$$\big(\overline{\text{con}}(K_1), \overline{\text{con}}(K_2)\big) \subseteq \big(\overline{\text{con}}\big(T(K_1)\big), \overline{\text{con}}\big(T(K_2)\big) \subseteq \big(\overline{\text{con}}(K_1), \overline{\text{con}}(K_2)\big),$$

which concludes that $\overline{\text{con}}(K_1) = \overline{\text{con}}(T(K_1))$ and $\overline{\text{con}}(K_2) = \overline{\text{con}}(T(K_2))$. Let $x \in K_1$ be an arbitrary element. For all $y \in K_2$ we have

$$\begin{aligned} d(Tx, Ty) &\le r \max\big\{d(x, y), d(x, Ty), d(Tx, y)\big\} + (1 - r) \operatorname{dist}(A, B) \\ &\le r\delta(K_1, K_2) + (1 - r) \operatorname{dist}(A, B). \end{aligned}$$

Now, if we set $\rho := r\delta(K_1, K_2) + (1 - r) \operatorname{dist}(A, B)$, then we have $T(K_2) \subseteq \mathcal{B}(Tx; \rho)$. Thus, $\overline{\text{con}}(K_2) = \overline{\text{con}}(T(K_2)) \subseteq \mathcal{B}(Tx; \rho)$. It follows from Lemma 2.5 that

$$\delta_{Tx}(K_2) = \delta_{Tx}\big(\overline{\text{con}}(K_2)\big) \le \rho, \quad \forall x \in K_1.$$

By using Lemma 2.6, we obtain

$$\begin{aligned} \delta(K_1, K_2) &= \delta\big(\overline{\text{con}}(K_1), \overline{\text{con}}(K_2)\big) = \delta\big(\overline{\text{con}}\big(T(K_1)\big), \overline{\text{con}}\big(T(K_2)\big)\big) \\ &= \delta\big(T(K_1), T(K_2)\big) \le \delta\big(T(K_1), K_2\big) = \sup_{x \in K_1} \delta_{Tx}(K_2) \le \rho \end{aligned}$$

so that $\delta(K_1, K_2) = \operatorname{dist}(A, B)$, which concludes that $d(p, q) = \operatorname{dist}(A, B)$ for all $(p, q) \in K_1 \times K_2$. On the other hand, both p and q are fixed points of T; indeed, if $p \ne Tp$, then $\mathcal{W}(p, Tp, \frac{1}{2}) \in K_1$ from the T-regularity of (K_1, K_2). We have $d(p, q) = d(Tp, q) = \operatorname{dist}(A, B)$. Since X is a strictly convex metric space, we obtain

$$\operatorname{dist}(A, B) \le d\Big(\mathcal{W}\Big(p, Tp, \frac{1}{2}\Big), q\Big) < \operatorname{dist}(A, B),$$

which is a contradiction, and so $p = Tp$. Similarly, $q = Tq$, and this completes the proof of the theorem. □

The next result is a straightforward consequence of Theorem 3.7.

Corollary 3.8 ([5, Theorem 3.10])**.** *Let (A, B) be a nonempty, weakly compact, and convex pair in a strictly convex Banach space X. Suppose $T : A \cup B \to A \cup B$ is a generalized noncyclic contraction mapping. Then T has a best proximity pair.*

Example 3.9. Let $X := [-1, 1]$, and define a metric d on X by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ |x| + |y|, & \text{if } x \ne y. \end{cases}$$

Define $\mathcal{W} : X \times X \times I \to X$ with

$$\mathcal{W}(x, y, \lambda) = \lambda \min\big\{|x|, |y|\big\}$$

for each $x, y \in X$ and $\lambda \in I$. We show that $\mathcal{W}$ is a convex structure on X. Let $x, y \in X$, and let $\lambda \in I$. We may assume that $|x| \leq |y|$. Then for each $u \in X$ we have

$$\begin{aligned} d\big(u, \mathcal{W}(x, y, \lambda)\big) &= |u| + \lambda \min\big\{|x|, |y|\big\} = |u| + \lambda|x| \leq |u| + |x| \\ &= \lambda\big(|u| + |x|\big) + (1-\lambda)\big(|u| + |x|\big) \\ &\leq \lambda\big(|u| + |x|\big) + (1-\lambda)\big(|u| + |y|\big) \\ &= \lambda d(u, x) + (1-\lambda) d(u, y). \end{aligned}$$

This implies that $(X, d, \mathcal{W})$ is a convex metric space. We now show that the convex metric space $(X, d, \mathcal{W})$ is strictly convex. Suppose $x, y, z \in X$ such that $x \neq y$ and $d(x, z) \leq r$ and $d(y, z) \leq r$ so that $|x| + |z| \leq r$ and $|y| + |z| \leq r$. Therefore,

$$d\Big(\mathcal{W}\Big(x, y, \frac{1}{2}\Big), z\Big) = \frac{1}{2} \min\big\{|x|, |y|\big\} + |z| < |x| + |z| \leq r,$$

and this follows the strict convexity of X. Let $A = \{\frac{1}{n} : n \geq 2\} \cup \{0\}$ and $B = \{\frac{\sqrt{2}}{n} : n \geq 2\} \cup \{0\}$, and define the noncyclic mapping $T : A \cup B \to A \cup B$ with

$$Tx = \begin{cases} x^2, & \text{if } x \in A, \\ \frac{\sqrt{2}}{n^2}, & \text{if } x \in B, x = \frac{\sqrt{2}}{n}. \end{cases}$$

Then (A, B) is a T-regular pair; indeed, if $(\frac{1}{n}, \frac{\sqrt{2}}{m}) \in A \times B$ for some integers $m, n \geq 2$, then we have

$$\Big(\mathcal{W}\Big(\frac{1}{n}, \frac{1}{n^2}, \frac{1}{2}\Big), \mathcal{W}\Big(\frac{\sqrt{2}}{m}, \frac{\sqrt{2}}{m^2}, \frac{1}{2}\Big)\Big) = \Big(\frac{1}{2n^2}, \frac{\sqrt{2}}{2m^2}\Big) \in A \times B.$$

Besides, (A, B) is T-regular reflexive. In fact, for every T-regular pair $(E, F) \subseteq (A, B)$ and $(x, y) \in E \times F$, from the T-regularity of (E, F) we conclude that

$$\Big(\frac{1}{2^{2^k-1} n^{2^k}}, \frac{\sqrt{2}}{2^{2^k-1} m^{2^k}}\Big) \in (E, F), \quad \forall k \in \mathbb{N},$$

which implies that $(0, 0) \in E \times F$ for any T-regular pair $(E, F) \subseteq (A, B)$, and so (A, B) is T-regular reflexive. On the other hand, for $(x, y) \in A \times B$ with $x = \frac{1}{n}$ and $y = \frac{\sqrt{2}}{m}$ for integers $m, n \geq 2$, we have

$$d(Tx, Ty) = \frac{1}{n^2} + \frac{\sqrt{2}}{m^2} \leq \frac{1}{2}\Big(\frac{1}{n} + \frac{\sqrt{2}}{m}\Big) = \frac{1}{2} d(x, y);$$

that is, T is a generalized noncyclic contraction. Thereby, all of the assumptions of Theorem 3.7 hold, and T has a best proximity pair which is the point $(0, 0) \in A \times B$.

Motivated by Definition 2.9, we introduce the notion of *nonconvex proximal normal structure* as below.

Definition 3.10. Let (A, B) be a nonempty pair of subsets of the convex metric space $(X, d, \mathcal{W})$, and let $T : A \cup B \to A \cup B$ be a noncyclic mapping so that (A, B) is a T-regular pair. We say that (A, B) has *nonconvex proximal normal structure (NPNS)* provided that for any bounded, closed, proximinal, and T-regular pair $(K_1, K_2) \subseteq (A, B)$, for which $\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B)$ and $\delta(K_1, K_2) > \operatorname{dist}(K_1, K_2)$, there exists $(x, y) \in K_1 \times K_2$ with its proximal point $(x', y') \in K_1 \times K_2$ (i.e., $d(x, y') = d(x', y) = \operatorname{dist}(K_1, K_2)$) such that

$$\max\{\delta_x(K_2), \delta_{y'}(K_1), \delta_{x'}(K_2), \delta_y(K_1)\} < \delta(K_1, K_2).$$

It is worth noting that NPNS $\not\Rightarrow$ PNS. Let us illustrate this fact with the next example.

Example 3.11. Consider the Banach space $X = l^\infty$ with canonical basis $\{e_n\}_{n\in\mathbb{N}}$, and let $A = \{xe_n : -1 \leq x \leq 1, n \in \mathbb{N}\}$. Note that A is not convex, and so the notion of normal structure for the set A is absurd. Now, define the mapping $T : A \to A$ with $T(xe_n) = -xe_n$. Obviously, A is T-regular. Moreover, A is T-regular reflexive because, for every nonempty and closed subset K of A which is T-regular, we have $p := 0 = \mathcal{W}(\mathbf{x}, T\mathbf{x}, \frac{1}{2}) \in K$ for any $\mathbf{x} \in K$. On the other hand, if $\operatorname{diam}(K) > 0$, by the fact that T is a self-mapping on K, then there exists $xe_n \in K$ for some $x \in [-1, 1]$ and $n \in \mathbb{N}$ so that $Tx = -xe_n \in K$. We now have $\delta_p(K) = \sup_{xe_n\in K} \|xe_n\|$ and

$$\operatorname{diam}(K) \geq 2 \sup_{xe_n\in K} \|xe_n\| = 2\delta_p(K).$$

Thus, $\delta_p(K) \leq \frac{1}{2} \operatorname{diam}(K)$; that is, A has NPNS.

The following theorem, which is an extension of Theorem 1.1, is the second main result of this section.

Theorem 3.12. *Let (A, B) be a nonempty, bounded, and closed pair in a strictly convex metric space $(X, d, \mathcal{W})$, and let $T : A\cup B \to A\cup B$ be a noncyclic relatively nonexpansive mapping such that (A, B) is a T-regular reflexive pair. If (A, B) is a proximal compactness pair and has NPNS, then T has a best proximity pair.*

Proof. Suppose that $\mathfrak{F}$ denotes the collection of all nonempty, closed, and T-regular pairs (E, F) which are subsets of (A, B) and $d(x, y) = \operatorname{dist}(A, B)$ for some $(x, y) \in E\times F$. By Lemma 3.6, (A_0, B_0) is a nonempty, closed, and T-regular pair, and so $(A_0, B_0) \in \mathfrak{F}$. Let $\{(E_j, F_j)\}_j$ be a descending chain in $\mathfrak{F}$. Set $E := \bigcap E_j$ and $F := \bigcap F_j$. A similar argument of Theorem 3.7 implies that (E, F) is a nonempty, bounded, closed, and T-regular pair. Also, if $(x_j, y_j) \in E_j \times F_j$ such that $d(x_j, y_j) = \operatorname{dist}(A, B)$, then, by the fact that (A, B) has proximal compactness, $\{(x_j, y_j)\}$ has a convergent subsequence, say $\{(x_{j_i}, y_{j_i})\}$, for which $x_{j_i} \to x \in A$ and $y_{j_i} \to y \in B$. Thus,

$$d(x, y) = \lim_i d(x_{j_i}, y_{j_i}) = \operatorname{dist}(A, B).$$

Therefore, there exists an element $(x, y) \in E \times F$ such that $d(x, y) = \operatorname{dist}(A, B)$ so that every increasing chain in $\mathfrak{F}$ is bounded above with respect to the reverse inclusion relation, and by using Zorn's lemma we obtain a minimal element for $\mathfrak{F}$, say (K_1,K_2). We note that since (K_{1_0},K_{2_0}) is a nonempty, closed, and T-regular subset of (K_1,K_2) and $\mathrm{dist}(K_{1_0},K_{2_0})=\mathrm{dist}(K_1,K_2)(=\mathrm{dist}(A,B))$, we must have $K_1=K_{1_0}$ and $K_2=K_{2_0}$ by the minimality of (K_1,K_2); that is, (K_1,K_2) is a proximinal pair. Moreover, by an equivalent argument of Theorem 3.7, we have $\overline{\mathrm{con}}(K_1)=\overline{\mathrm{con}}(T(K_1))$ and $\overline{\mathrm{con}}(K_2)=\overline{\mathrm{con}}(T(K_2))$. We now consider the two following cases.

Case 1: $\delta(K_1,K_2)=\mathrm{dist}(K_1,K_2)(=\mathrm{dist}(A,B))$.

Then the result follows from the equivalent argument of Theorem 3.7.

Case 2: $\delta(K_1,K_2)>\mathrm{dist}(K_1,K_2)$.

Since (A,B) has NPNS, there exists a point $(p,q)\in K_1\times K_2$ with its proximal point $(p',q')\in K_1\times K_2$ and $\lambda\in(0,1)$ such that

$$\max\big\{\delta_p(K_2),\delta_{q'}(K_1),\delta_q(K_1),\delta_{p'}(K_2)\big\}\le\lambda\delta(K_1,K_2).$$

Put

$$\begin{aligned}H_1&:=\big\{x\in K_1:\delta_x(K_2)\le\lambda\delta(K_1,K_2)\big\},\\ H_2&:=\big\{y\in K_2:\delta_y(K_1)\le\lambda\delta(K_1,K_2)\big\}.\end{aligned}$$

Then $(p,q'),(p',q)\in H_1\times H_2$, and so (H_1,H_2) is a nonempty and closed subset of (K_1,K_2) with $\mathrm{dist}(H_1,H_2)=\mathrm{dist}(K_1,K_2)$. Also, if $x\in H_1$, then, by Lemma 2.5 and the relative nonexpansiveness of T,

$$\begin{aligned}\delta_{Tx}(K_2)&=\delta_{Tx}\big(\overline{\mathrm{con}}(K_2)\big)=\delta_{Tx}\big(\overline{\mathrm{con}}\big(T(K_2)\big)\big)\\ &=\delta_{Tx}\big(T(K_2)\big)\le\delta_x(K_2)\le\lambda\delta(K_1,K_2),\end{aligned}$$

which implies that $Tx\in H_1$. Thus, $T(H_1)\subseteq H_1$. Similarly, $T(H_2)\subseteq H_2$, and so T is noncyclic on $H_1\cup H_2$. Moreover, (H_1,H_2) is a T-regular pair. In fact, if $x\in H_1$, then for any $y\in K_2$ we have

$$\begin{aligned}d\Big(\mathcal{W}\Big(x,Tx,\frac{1}{2}\Big),y\Big)&\le\frac{1}{2}\big[d(x,y)+d(Tx,y)\big]\\ &\le\frac{1}{2}\big[\delta_x(K_2)+\delta_{Tx}(K_2)\big]\le\lambda\delta(K_1,K_2),\end{aligned}$$

which concludes that $\mathcal{W}(x,Tx,\frac{1}{2})\in H_1$. Equivalently, $\mathcal{W}(y,Ty,\frac{1}{2})\in H_2$ for all $y\in H_2$. It now follows from the minimality of (K_1,K_2) that $H_1=K_1$ and $H_2=K_2$. Thereby, $\delta_x(K_2)\le\lambda\delta(K_1,K_2)$ for all $x\in K_1$, which implies that

$$\delta(K_1,K_2)=\sup_{x\in K_1}\delta_x(K_2)\le\lambda\delta(K_1,K_2),$$

which is a contradiction. □

Here, we give an improved version of Theorem 1.1.

Theorem 3.13. *Let (A,B) be a nonempty, bounded, and weakly closed pair in a uniformly convex Banach space X, and let $T:A\cup B\to A\cup B$ be a noncyclic relatively nonexpansive mapping such that (A,B) is a T-regular pair. Then T has a best proximity pair.*

Proof. Let Γ be the collection of all nonempty, weakly closed, and T-regular pairs $(E,F) \subseteq (A,B)$ such that $\operatorname{dist}(E,F) = \operatorname{dist}(A,B)$. Then $(A_0,B_0) \in \Gamma$. Since (A,B) is weakly compact, every descending chain in Γ has a nonempty intersection, and so, by a similar argument of Theorem 3.7, Γ has a minimal element such as (K_1,K_2), which is a nonempty, weakly closed, and T-regular pair, and we have $\operatorname{dist}(K_1,K_2) = \operatorname{dist}(A,B)$. Suppose that T does not have a best proximity pair. We get a contradiction by showing that (A,B) has NPNS. Let $(H_1,H_2) \subseteq (A,B)$ be a nonempty, closed, proximinal, and T-regular pair for which $\operatorname{dist}(H_1,H_2) = \operatorname{dist}(A,B)$ and $\delta(H_1,H_2) > \operatorname{dist}(H_1,H_2)$. Suppose $(p,q) \in H_1 \times H_2$, and let $(p',q') \in H_1 \times H_2$ be such that $\|p-q'\| = \|p'-q\| = \operatorname{dist}(H_1,H_2)$. Since T is noncyclic relatively nonexpansive, we have

$$\|Tp - Tq'\| \le \|p - q'\| = \operatorname{dist}(H_1,H_2).$$

From the strict convexity of X, we must have $p \neq Tp$ and $q' \neq Tq'$. Also, since (H_1,H_2) is a T-regular pair, $(\frac{p+Tp}{2}, \frac{q'+Tq'}{2}) \in H_1 \times H_2$. We have

$$\operatorname{dist}(A,B) \le \Big\|\frac{p+Tp}{2} - \frac{q'+Tq'}{2}\Big\| \le \frac{1}{2}\big[\|p-q'\| + \|Tp-Tq'\|\big] \le \operatorname{dist}(A,B).$$

Put $R := \delta(H_1,H_2)$ and $r := \min\{\|p-Tp\|, \|q'-Tq'\|\}$. For any $y \in H_2$ we have

$$\begin{cases} \|p-y\| \le R, \\ \|Tp-y\| \le R, \\ \|p-Tp\| \ge r. \end{cases}$$

Uniform convexity of X concludes that $\|\frac{p+Tp}{2} - y\| \le (1-\delta(\frac{r}{R}))R$ for all $y \in H_2$; hence, $\delta_{\frac{p+Tp}{2}}(K_2) \le (1-\delta(\frac{r}{R}))R < R$. Similarly, we can see that $\delta_{\frac{q'+Tq'}{2}}(H_1) < R$. Equivalently, for the pair $(\frac{p'+Tp'}{2}, \frac{q+Tq}{2}) \in H_1 \times H_2$ we have

$$\delta_{\frac{p'+Tp'}{2}}(H_2) < R, \qquad \delta_{\frac{q+Tq}{2}}(H_1) < R.$$

Now, if we set $u := \frac{p+Tp}{2}$, $u' := \frac{p'+Tp'}{2}$ and $v := \frac{q+Tq}{2}$, $v' := \frac{q'+Tq'}{2}$, then (u,v) is a member of $H_1 \times H_2$ with its proximal point $(u',v') \in H_1 \times H_2$, and we have

$$\max\big\{\delta_u(H_2), \delta_v(H_1), \delta_{u'}(H_2), \delta_{v'}(H_1)\big\} < \delta(H_1,H_2).$$

This implies that (A,B) has NPNS and the result follows. □

In a special case, we conclude the following generalization of the *Browder fixed-point theorem* (see [3]).

Corollary 3.14. *Let A be a nonempty, bounded, and closed subset of a uniformly convex Banach space X, and let $T : A \to A$ be a nonexpansive mapping such that A is T-regular. Then T has a fixed point.*

Let us illustrate Theorem 3.13 with the following example.

Example 3.15. Given $k \in [0,1]$, let $A = \{(1+k)e_{2n} : n \in \mathbb{N}\}$ and $B = \{(1+k)e_{2m-1} : m \in \mathbb{N}\}$ be subsets of a uniformly convex Banach space l^2. It is clear that

(A, B) is a bounded, closed, and *nonconvex* pair in X and we have $\operatorname{dist}(A, B) = \sqrt{2}$. Define the noncyclic mapping $T : A \cup B \to A \cup B$ with

$$T\big((1+k)e_{2n}\big) = (1+k^2)e_{2n}, \qquad T\big((1+k)e_{2m-1}\big) = (1+k^3)e_{2m-1}.$$

Then (A, B) is a T-regular pair; indeed, if $(x, y) = ((1+k)e_{2n}, (1+k')e_{2m-1}) \in A \times B$, then

$$\begin{aligned}\frac{x+Tx}{2} &= \frac{(1+k)e_{2n} + (1+k^2)e_{2n}}{2} = \Big[1 + \frac{k+k^2}{2}\Big]e_{2n} \in A,\\ \frac{y+Ty}{2} &= \frac{(1+k')e_{2m-1} + (1+k'^3)e_{2m-1}}{2} = \Big[1 + \frac{k'+k'^3}{2}\Big]e_{2m-1} \in B.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|Tx - Ty\| &= \big\|(1+k^2)e_{2n} - (1+k'^3)e_{2m-1}\big\| = \big[(1+k^2)^2 + (1+k'^3)^2\big]^{\frac{1}{2}}\\ &\le \big[(1+k^2)^2 + (1+k'^2)^2\big]^{\frac{1}{2}} \le \big[(1+k)^2 + (1+k')^2\big]^{\frac{1}{2}} = \|x-y\|;\end{aligned}$$

that is, T is a noncyclic relatively nonexpansive mapping. Thus, Theorem 3.13 guarantees the existence of a best proximity pair for the mapping T. Note that for any $m, n \in \mathbb{N}$, (e_{2n}, e_{2m-1}) is a best proximity pair of T. It is worth noting that the existence of best proximity pairs of T cannot be concluded from Theorem 1.1 due to Eldred, Kirk, and Veeramani.

4. A common best proximity pair theorem

In this section we establish a common best proximity pair theorem for a commuting family of noncyclic relatively nonexpansive mappings in strictly convex metric spaces. We begin with the following notion.

Definition 4.1. Let A be a nonempty subset in a convex metric space $(X, d, \mathcal{W})$, and let $T : A \to A$ be a mapping. We say that T is *semiaffine*, provided that

$$T\Big(\mathcal{W}\Big(x, y, \frac{1}{2}\Big)\Big) = \mathcal{W}\Big(Tx, Ty, \frac{1}{2}\Big)$$

for any $x, y \in A$.

Let $T : A \cup B \to A \cup B$ be a noncyclic mapping, where (A, B) is a nonempty pair of subsets of a convex metric space $(X, d, \mathcal{W})$. In what follows, we shall adopt the following notation:

$$\begin{aligned}\mathcal{F}_A(T) &:= \{x \in A : Tx = x\} \quad \& \quad \mathcal{F}_B(T) := \{y \in B : Ty = y\},\\ \operatorname{Prox}_{A\times B}(T) &:= \big\{(x, y) \in A \times B : Tx = x, Ty = y \text{ and } d(x, y) = \operatorname{dist}(A, B)\big\}.\end{aligned}$$

We now state our main result of this section.

Theorem 4.2. *Let (A, B) be a nonempty, bounded, closed, and proximal compactness pair in a strictly convex metric space $(X, d, \mathcal{W})$. Suppose that $\mathfrak{I}$ is a commuting family of noncyclic relatively nonexpansive mappings defined on $A \cup B$ such that for any $T \in \mathfrak{I}$ both $T|_A$ and $T|_B$ are continuous and semiaffine self-mappings. If (A, B) is a T-regular reflexive pair for each $T \in \mathfrak{I}$ and (A, B) has NPNS, then the family has a common best proximity pair.*

Proof. It follows from Lemma 3.6 that (A_0, B_0) is a nonempty, closed, and T-regular pair for any $T \in \mathfrak{I}$. We divide the proof of the theorem into the following steps:

- $\mathrm{Prox}_{A\times B}(T)$ *is nonempty for each* $T \in \mathfrak{I}$ *and*

$$\mathrm{Prox}_{A\times B}(T) \subseteq \big(\mathcal{F}_A(T) \cap A_0\big) \times \big(\mathcal{F}_B(T) \cap B_0\big), \quad \forall T \in \mathfrak{I}.$$

Proof. By Theorem 3.12 the result follows.

- *Let* T, S *be two elements of* $\mathfrak{I}$*. Then* $(\mathcal{F}_A(T) \cap A_0, \mathcal{F}_B(T) \cap B_0)$ *is a nonempty, bounded, closed, and* S*-regular pair.*

Proof. Since $\mathrm{Prox}_{A\times B}(T)$ is nonempty, $(\mathcal{F}_A(T)\cap A_0, \mathcal{F}_B(T)\cap B_0)$ is also a nonempty pair. Let $\{x_n\}$ be a sequence in $\mathcal{F}_A(T) \cap A_0$ such that $x_n \to p$. Thus, $Tx_n = x_n$ for all $n \in \mathbb{N}$ and there exists $y_n \in B_0$ so that $d(x_n, y_n) = \mathrm{dist}(A, B)$. We have

$$d(x_n, Ty_n) = d(x_n, y_n) = \mathrm{dist}(A, B), \quad \forall n \in \mathbb{N},$$

which implies that $Ty_n = y_n$ for any $n \in \mathbb{N}$ by the strict convexity of the convex metric space $(X, d, \mathcal{W})$. Since (A, B) is a proximal compactness pair, there exists a subsequence $\{(x_{n_i}, y_{n_i})\}$ of the sequence $\{(x_n, y_n)\}$ such that $(x_{n_i}, y_{n_i}) \to (p, q)$. It now follows from the continuity of $T|_A$ that

$$p = \lim_i x_{n_i} = \lim_i T(x_{n_i}) = Tp.$$

Similarly, by the continuity of $T|_B$, we have $Tq = q$. Also,

$$d(p, q) = \lim_i d(x_{n_i}, y_{n_i}) = \mathrm{dist}(A, B).$$

Therefore, $(p, q) \in \mathrm{Prox}_{A\times B}(T) \subseteq (\mathcal{F}_A(T) \cap A_0) \times (\mathcal{F}_B(T) \cap B_0)$; that is, $p \in \mathcal{F}_A(T) \cap A_0$, and so $\mathcal{F}_A(T) \cap A_0$ is closed. By a similar manner, we can see that $\mathcal{F}_B(T)\cap B_0$ is also closed. We now prove that the pair $(\mathcal{F}_A(T)\cap A_0)\times(\mathcal{F}_B(T)\cap B_0)$ is an S-regular pair. Let $x \in \mathcal{F}_A(T) \cap A_0$. Since $S(A_0) \subseteq A_0$, $Sx \in A_0$. Moreover, $TSx = STx = Sx$, and so $Sx \in \mathcal{F}_A(T) \cap A_0$. Thus, $S(\mathcal{F}_A(T) \cap A_0) \subseteq \mathcal{F}_A(T) \cap A_0$. Similarly, $S(\mathcal{F}_B(T) \cap B_0) \subseteq \mathcal{F}_B(T) \cap B_0$, and hence S is noncyclic on $(\mathcal{F}_A(T) \cap A_0) \cup (\mathcal{F}_B(T) \cap B_0)$. Besides, for $(x, y) \in \mathcal{F}_A(T) \cap A_0 \times \mathcal{F}_B(T) \cap B_0$ by the fact that both $T|_A$ and $T|_B$ are semiaffine, we have

$$T\Big(\mathcal{W}\Big(x, Sx, \frac{1}{2}\Big)\Big) = \mathcal{W}\Big(Tx, TSx, \frac{1}{2}\Big) = \mathcal{W}\Big(x, STx, \frac{1}{2}\Big) = \mathcal{W}\Big(x, Sx, \frac{1}{2}\Big),$$

and, equivalently, $T(\mathcal{W}(y, Sy, \frac{1}{2})) = \mathcal{W}(y, Sy, \frac{1}{2})$. Thereby,

$$\Big(\mathcal{W}\Big(x, Sx, \frac{1}{2}\Big), \mathcal{W}\Big(y, Sy, \frac{1}{2}\Big)\Big) \in \big(\mathcal{F}_A(T) \cap A_0\big) \times \big(\mathcal{F}_B(T) \cap B_0\big),$$

and so $(\mathcal{F}_A(T) \cap A_0, \mathcal{F}_B(T) \cap B_0)$ is S-regular.

- S *and* T *have a common best proximity pair.*

Proof. By the above discussion, $S : (\mathcal{F}_A(T) \cap A_0) \cup (\mathcal{F}_B(T) \cap B_0) \to (\mathcal{F}_A(T) \cap A_0)\cup(\mathcal{F}_B(T)\cap B_0)$ is a noncyclic relatively nonexpansive mapping, and $(\mathcal{F}_A(T)\cap A_0, \mathcal{F}_B(T) \cap B_0)$ is an S-regular reflexive pair and has NPNS. This implies that S has a best proximity pair in $(\mathcal{F}_A(T) \cap A_0) \times (\mathcal{F}_B(T) \cap B_0)$ from Theorem 3.12.

Therefore, there exists a point $(p,q) \in (\mathcal{F}_A(T) \cap A_0) \times (\mathcal{F}_B(T) \cap B_0)$ such that $p = Sp$, $q = Sq$, and $d(p,q) = \operatorname{dist}(A,B)$. Also, we have $p = Tp$, $q = Tq$; hence,

$$(p,q) \in \operatorname{Prox}_{A\times B}(T) \cap \operatorname{Prox}_{A\times B}(S) \neq \emptyset.$$

- *Every finite family $\{T_j\}_{j=1}^n$ of $\mathfrak{I}$ has a common best proximity pair.*

Proof. The previous argument concludes that $\operatorname{Prox}_{A\times B}(T_1) \cap \operatorname{Prox}_{A\times B}(T_2) \neq \emptyset$. We now have that

$$\begin{aligned} T_3 : &\Big(\Big(\bigcap_{i=1}^{2} \mathcal{F}_A(T_i)\Big) \cap A_0\Big) \cup \Big(\Big(\bigcap_{i=1}^{2} \mathcal{F}_B(T_i)\Big) \cap B_0\Big) \\ &\to \Big(\Big(\bigcap_{i=1}^{2} \mathcal{F}_A(T_i)\Big) \cap A_0\Big) \cup \Big(\Big(\bigcap_{i=1}^{2} \mathcal{F}_B(T_i)\Big) \cap B_0\Big) \end{aligned}$$

is noncyclic relatively nonexpansive, and $((\bigcap_{i=1}^2 \mathcal{F}_A(T_i)) \cap A_0, (\bigcap_{i=1}^2 \mathcal{F}_B(T_i)) \cap B_0)$ is a T_3-regular reflexive pair and has NPNS. Thus, $\bigcap_{j=1}^3 \operatorname{Prox}_{A\times B}(T_j)$ is nonempty. Continuing this process, we obtain

$$\bigcap_{j=1}^{n} \operatorname{Prox}_{A\times B}(T_j) \neq \emptyset.$$

- *The family $\mathfrak{I}$ has a common best proximity pair.*

Proof. Since every finite family of $\mathfrak{I}$ has a common best proximity pair, we conclude that the family $\{(\mathcal{F}_A(T) \cap A_0, \mathcal{F}_B(T) \cap B_0)\}_{T\in\mathfrak{I}}$ consists of nonempty, bounded, closed, and S-regular pairs, for any $S \in \mathfrak{I}$ has the finite intersection property. By the fact that (A,B) is a T-regular reflexive pair for any $T \in \mathfrak{I}$, we must have $(\bigcap_{T\in\mathfrak{I}}(\mathcal{F}_A(T) \cap A_0), \bigcap_{T\in\mathfrak{I}}(\mathcal{F}_B(T) \cap B_0))$ as a nonempty and closed pair. Thus,

$$\bigcap_{T\in\mathfrak{I}} \operatorname{Prox}_{A\times B}(T) \neq \emptyset. \qquad \square$$

The next corollaries are deduced immediately from Theorem 4.2.

Corollary 4.3. *Let (A,B) be a nonempty, bounded, and closed pair in a uniformly convex Banach space X. Suppose that $\mathfrak{I}$ is a commuting family of noncyclic relatively nonexpansive mappings defined on $A \cup B$ such that for any $T \in \mathfrak{I}$ both $T|_A$ and $T|_B$ are continuous and semiaffine self-mappings, and let (A,B) be a T-regular pair for any $T \in \mathfrak{I}$. Then the family has a common best proximity pair.*

Corollary 4.4. *Let A be a nonempty, bounded, and closed subset of a uniformly convex Banach space X. Suppose that $\mathfrak{I}$ is a commuting family of nonexpansive self-mappings defined on A such that any $T \in \mathfrak{I}$ is semiaffine. If A is T-regular for any $T \in \mathfrak{I}$, then the family has a common fixed point.*

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