THE BISHOP–PHELPS–BOLLOBÁS PROPERTY FOR OPERATORS ON $\mathcal{C}(K)$

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To Rafael Payá on the occasion of his 60th birthday

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ABSTRACT. We provide a version for operators of the Bishop–Phelps–Bollobás theorem when the domain space is the complex space $\mathcal{C}_0(L)$. In fact, we prove that the pair $(\mathcal{C}_0(L), Y)$ will satisfy the Bishop–Phelps–Bollobás property for operators for every Hausdorff locally compact space L and any $\mathbb{C}$-uniformly convex space. As a consequence, this holds for $Y = L_p(\mu)$ $(1 \le p < \infty)$.

1. INTRODUCTION

The Bishop–Phelps theorem states the denseness of the subset of norm-attaining functionals in the (topological) dual of a Banach space. Since the Bishop–Phelps theorem was proved in the 1960s, some interesting papers provided versions of this result for operators. Related to those results are the pioneering work by Lindenstrauss [16], the somehow surprising result obtained by Bourgain [7], and also results for concrete classical Banach spaces (see below). In full generality there is no parallel version of the Bishop–Phelps theorem for operators even if the domain space is c_0 (see [16]). Lindenstrauss also provided in [16] some results of denseness of the subset of norm-attaining operators by assuming some isometric properties either on the domain or on the range space. We mention here two concrete consequences of these results. If the domain space is ℓ_1 or the range space is c_0, every operator can be approximated by norm-attaining operators.

First, Lindenstrauss [16] and later Bourgain [7] proved that certain isomorphic assumptions on the domain space (reflexivity or even Radon–Nikodým property, respectively) imply the denseness of the subset of norm-attaining operators in the corresponding space of linear (bounded) operators. For classical Banach spaces, we only mention some articles containing positive results for specific pairs (see, for instance, [11], [19]) and one containing a counterexample (see [19]). The survey [1] contains the main results on the topic and also longstanding open questions.

Recently, the authors of [2] dealt with "quantitative" versions of the Bishop–Phelps theorem for operators. The motivating result is known nowadays as the *Bishop–Phelps–Bollobás theorem* (see [6]), which has been a very useful tool to study numerical ranges of operators (see, for instance, [6]). This result can be stated as follows.

Let X be a Banach space and let $0 < \varepsilon < 1$. Given $x \in B_X$ and $x^* \in S_{X^*}$ with $|1 - x^*(x)| < \varepsilon^2/4$, there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$.

Here X^* denotes the (topological) dual of the Banach space X and S_X its unit sphere. We write B_X to denote the closed unit ball of X.

Throughout the present article, for two Banach spaces X and Y, $\mathcal{L}(X, Y)$ is the *space of linear bounded operators* from X into Y. We recall that the pair (X, Y) *has the Bishop–Phelps–Bollobás property for operators* (or *BPBp*), if for any $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that, for any $T \in S_{\mathcal{L}(X,Y)}$, if $x_0 \in S_X$ is such that $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist an element $u_0 \in S_X$ and an operator $S \in S_{\mathcal{L}(X,Y)}$ satisfying the following conditions:

$$\|Su_0\| = 1, \qquad \|u_0 - x_0\| < \varepsilon \qquad \text{and} \qquad \|S - T\| < \varepsilon.$$

While [2, Theorem 2.2] proved that, for any space Y satisfying the property β of Lindenstrauss, the pair (X, Y) has the BPBp for operators for every Banach space X. For the domain space, there is no reasonably general property implying a positive result. However, there are some positive results in concrete cases. For instance, there is a characterization of the spaces Y such that the pair (ℓ_1, Y) satisfies the BPBp (see [2]). As a consequence of this result, it is known that this condition is satisfied by finite-dimensional spaces, uniformly convex spaces, $\mathcal{C}(K)$ (K is some compact topological space) and $L_1(\mu)$ (any measure μ).

Now we point out results stating that the pair (X, Y) has the BPBp in the case that the domain space is $\mathcal{C}_0(L)$ (space of continuous functions on a locally compact Hausdorff space L vanishing at infinity). Kim [13] proved that in the real case the pair (c_0, Y) has the BPBp for operators whenever Y is uniformly convex. The paper [3] contains also a positive result for the pair $(\mathcal{C}(K), \mathcal{C}(S))$ in the real case (K and S are compact Hausdorff spaces). Let us point out that in the complex case it is still unknown if the subset of norm-attaining operators from $\mathcal{C}(K)$ to $\mathcal{C}(S)$ is dense in $\mathcal{L}(\mathcal{C}(K), \mathcal{C}(S))$. Recently, Kim, Lee, and Lin in [15] proved that the pair $(L_\infty(\mu), Y)$ has the BPBp whenever Y is a uniformly convex space and μ is any positive measure. The authors also state the analogous result in complex case for the pairs (c_0, Y) and $(L_\infty(\mu), Y)$ (μ is any positive measure)

whenever Y is a $\mathbb{C}$-uniformly convex space. It also holds that the pair $(\mathcal{C}(K), Y)$ has the BPBp in the real case for any uniformly convex space (see [14]).

In the present paper, we show that the pair $(\mathcal{C}_0(L), Y)$ satisfies the Bishop–Phelps–Bollobás property for operators in the complex case for every locally compact Hausdorff space L and for any $\mathbb{C}$-uniformly convex (complex) space. Let us note that this is an extension of the result in [15] for the complex case in two ways. First, we consider any space $\mathcal{C}(K)$ instead of $L_\infty(\mu)$ as the domain space and also we consider a strictly more general property on the range space, namely, $\mathbb{C}$-uniform convexity instead of uniform convexity. Our result extends [2, Theorem 5.2] in a satisfactory way and the recent result in [15] for the case that the domain space is $L_\infty(\mu)$. As a consequence, in the complex case the pair $(\mathcal{C}(K), L_1(\mu))$ has the BPBp for every compact Hausdorff space K and any measure μ.

Let us recall again that it is not trivial at all to obtain the result in the complex case from the real case when the domain space is $\mathcal{C}(K)$. As we already pointed out, it is an open problem whether or not the subset of norm-attaining operators between complex spaces $\mathcal{C}(K)$ and $\mathcal{C}(S)$ is dense in $\mathcal{L}(\mathcal{C}(K), \mathcal{C}(S))$. However, a positive result for real $\mathcal{C}(K)$ spaces was proved many years ago (see [11]).

Let us notice that in the case that the range space is a uniform algebra, the paper [8] provides positive results for the BPBp for the class of Asplund operators.

2. The result

Throughout this section, we consider only *complex* normed spaces. For a complex Banach space Y, recall that the $\mathbb{C}$-*modulus of convexity* δ is defined for every $\varepsilon > 0$ by

$$\delta(\varepsilon) = \inf\big\{\sup\big\{\|x + \lambda\varepsilon y\| - 1 : \lambda \in \mathbb{C}, |\lambda| = 1\big\} : x, y \in S_Y\big\}.$$

Recall that the Banach space Y is $\mathbb{C}$-*uniformly convex* if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$ (see [9]). Every uniformly convex complex space is $\mathbb{C}$-uniformly convex and the converse is not true. Globevnik proved that the complex space $L_1(\mu)$ is $\mathbb{C}$-uniformly convex (see [9, Theorem 1]).

We will denote by $\overline{D}(0, 1)$ the closed unit disc in $\mathbb{C}$. Let us notice that for $0 < s < t$ it is satisfied that $\sup\{\|x + \lambda s y\| : \lambda \in \overline{D}(0, 1)\} \leq \sup\{\|x + \lambda t y\| : \lambda \in \overline{D}(0, 1)\}$. Hence, δ is an increasing function and $\delta(t) \leq t$ for every $t > 0$.

In the following, L will be a locally compact Hausdorff topological space and $\mathcal{C}_0(L)$ will be the space of continuous complex valued functions on L vanishing at infinity.

It is convenient to state the next trivial result.

Lemma 2.1. *Assume that* $\lambda, w \in \overline{D}(0, 1), t \in]0, 1[$ *and that* $\operatorname{Re} w\lambda > 1 - t$. *Then* $|w - \overline{\lambda}| < \sqrt{2t}$.

As we already mentioned, the subset of norm-attaining operators between two Banach spaces is not always dense in the corresponding space of operators in the case that the domain space is $\mathcal{C}_0(L)$. Let us notice that there are examples of spaces Y for which the pair (ℓ^2_∞, Y) does not have the Bishop–Phelps–Bollobás property for operators (see [2, Theorem 4.1 and Proposition 3.9], [5,

Corollary 3.3]). For these reasons, some restriction is needed on the range space in order to obtain a BPBp result in case the domain space is $\mathcal{C}_0(L)$.

Schachermayer [19] proved a Bishop–Phelps result in the real case for the subspace of weakly compact operators from any space $\mathcal{C}_0(L)$ into any Banach space. In [4], this result was extended to the complex case. The last result is one of the tools that figures prominently in the proof of the main result. This is our motivation for the next assertion, that might be known, and has interest in itself.

Proposition 2.2. *Let Y be a $\mathbb{C}$-uniformly convex Banach space and let L be any locally compact Hausdorff space. Then every operator from $\mathcal{C}_0(L)$ into Y is weakly compact.*

Proof. By the proof of the James distortion theorem (see, for instance, [17, Proposition 2.e.3]), the space Y cannot contain a copy of c_0 (the space of complex sequences converging to zero, endowed with the usual norm). Otherwise, by considering a convenient multiple of the norm in Y, $\|\|\ \|\|$, that is still $\mathbb{C}$-uniformly convex, one can assume that the usual norm of the copy of c_0 ($\|\ \|$) satisfies

$$\alpha\|\|x\|\| \le \|x\| \le \|\|x\|\|, \quad \forall x \in c_0,$$

for some $\alpha > 0$. By the proof of [17, Proposition 2.e.3], for any $\varepsilon > 0$ there is a block basic sequence (y_n) in Y formed from the usual basis of c_0 satisfying

$$\|\|y_n\|\| = 1 \quad \forall n \in \mathbb{N}, \qquad \left\|\|\sum_{k=1}^{\infty} a_n y_n\|\right\| \le (1+\varepsilon)^2 \big\|(a_n)\big\|_\infty, \quad \forall (a_n) \in c_0$$

and $\|\sum_{k=1}^{\infty} a_n y_n\| = \|(a_n)\|_\infty$ for every $(a_n) \in c_0$. Clearly, the above condition contradicts the $\mathbb{C}$-uniform convexity of Y.

Now, in view of the Bessaga–Pelczyński selection principle, if the underlying real space of a complex space contains a real space isomorphic to c_0, then it contains the complex space c_0. So $Y_\mathbb{R}$ does not contain the real space c_0, hence, for any compact space K, every (real) operator from the space $\mathcal{C}(K)$ (real-valued functions) into Y is weakly compact. As a consequence, every operator from the complex space $\mathcal{C}(K)$ into Y is also weakly compact. From here it can be easily deduced that every operator from $\mathcal{C}_0(L)$ into Y is weakly compact, since $\mathcal{C}_0(L)$ is complemented in the space $\mathcal{C}(K)$, with K being the Alexandrov compactification of L. Hence, every operator from $\mathcal{C}_0(L)$ into Y can be extended to an operator from $\mathcal{C}(K)$ into Y. □

For a locally compact Hausdorff topological space L, we denote by $\mathcal{B}(L)$ the space of Borel measurable and bounded complex valued functions defined on L, endowed with the sup norm. If $B \subset L$ is a Borel measurable set, denote by P_B the projection $P_B : \mathcal{B}(L) \longrightarrow \mathcal{B}(L)$ given by $P_B(f) = f\chi_B$ for any $f \in \mathcal{B}(L)$. Of course, in view of the Riesz theorem, the space $\mathcal{B}(L)$ can be identified in a natural way as a subspace of $\mathcal{C}_0(L)^{**}$. As a consequence, for an operator $T \in \mathcal{L}(\mathcal{C}_0(L), Y)$ and a Borel set $B \subset L$, the composition $T^{**}P_B$ makes sense.

Lemma 2.3. *Let Y be a $\mathbb{C}$-uniformly convex space with modulus of $\mathbb{C}$-convexity δ. Let L be a locally compact Hausdorff topological space and let A be a Borel*

subset of L. Assume that for some $0 < \varepsilon < 1$ *and* $T \in S_{\mathcal{L}(\mathcal{C}_0(L),Y)}$, *one has* $\|T^{**}P_A\| > 1 - \frac{\delta(\varepsilon)}{1+\delta(\varepsilon)}$. *Then* $\|T^{**}(I - P_A)\| \leq \varepsilon$.

Proof. Assume that T satisfies the assumptions of the statement. By Proposition 2.2, T is a weakly compact operator, so $T^{**}(\mathcal{C}_0(L))^{**} \subset Y$, and we consider the subspace $\mathcal{B}(K) \subset \mathcal{C}_0(L)^{**}$. We write $\eta = \frac{\delta(\varepsilon)}{1+\delta(\varepsilon)}$. By the assumption there exists $f \in S_{\mathcal{B}(L)}$ such that $f = P_A(f)$ and $\|T^{**}(f)\| > 1 - \eta > 0$. For every $g \in B_{\mathcal{B}(L)}$ we have $\|f + (I - P_A)(g)\| \leq 1$ and so $\|T^{**}(f + \lambda(I - P_A)g)\| \leq 1$ for every $\lambda \in \overline{D}(0,1)$. That is, for any $\lambda \in \overline{D}(0,1)$ we have

$$\left\| \frac{T^{**}(f)}{\|T^{**}(f)\|} + \lambda \frac{T^{**}(I - P_A)(g)}{\|T^{**}(f)\|} \right\| \leq \frac{1}{\|T^{**}(f)\|}$$
$$< \frac{1}{1 - \eta} = 1 + \delta(\varepsilon).$$

As a consequence, $\|T^{**}(I - P_A)(g)\| \leq \varepsilon\|T^{**}(f)\| \leq \varepsilon$. Since g is any function in $B_{\mathcal{B}(L)}$, we deduce that $\|T^{**}(I - P_A)\| \leq \varepsilon$. $\qquad\square$

Theorem 2.4. *The pair* $(\mathcal{C}_0(L), Y)$ *satisfies the Bishop–Phelps–Bollobás property for operators for any locally compact Hausdorff topological space L and any* $\mathbb{C}$*-uniformly convex space Y. Moreover, the function η appearing in the Definition of BPBp depends only on the* $\mathbb{C} =$ *modulus of convexity of Y.*

Proof. Fix $0 < \varepsilon < 1$ and let $\delta(\varepsilon)$ be the modulus of $\mathbb{C}$-convexity of Y. We denote

$$\eta = \frac{\varepsilon^2 \delta(\frac{\varepsilon}{9})^2}{10945(1 + \delta(\frac{\varepsilon}{9}))^2} \qquad \text{and} \qquad s = \frac{\eta(2 - \varepsilon)\varepsilon^2}{2(\varepsilon^2 + 2 \cdot 12^2)}.$$

Assume that $T \in S_{\mathcal{L}(\mathcal{C}_0(L),Y)}$ and $f_0 \in S_{\mathcal{C}_0(L)}$ satisfy

$$\|Tf_0\| > 1 - s.$$

Our goal is to find an operator $S \in S_{\mathcal{L}(\mathcal{C}_0(L),Y)}$ and $g \in S_{\mathcal{C}_0(L)}$ such that

$$\big\|S(g)\big\| = 1, \qquad \|S - T\| < \varepsilon, \qquad \text{and} \qquad \|g - f_0\| < \varepsilon.$$

We can choose $y_1^* \in S_{Y^*}$ such that

$$\operatorname{Re} y_1^*(Tf_0) = \|Tf_0\| > 1 - s. \tag{2.1}$$

We identify $\mathcal{C}_0(L)^*$ with the space $M(L)$ of Borel regular complex measures on L in view of the Riesz theorem. We write $\mu_1 = T^*(y_1^*) \in M(L)$.

Now we give a very brief sketch of the proof. First we will show that on a large measurable subset A of L, the function f_0 is close to the conjugate of $g_1 = \frac{d\mu_1}{d|\mu_1|}$. Then we approximate A by a compact subset $B \subset A$ where g_1 is continuous. Next we use the denseness of the subset of norm-attaining operators from $C(B)$ into $L_1(\mu)$ to obtain an operator S_2 close of the restriction of T to $C(B)$ and attaining its norm at some continuous function on B. Finally, we extend the operator S_2 to a new operator S defined on $\mathcal{C}_0(L)$ and we obtain a function g satisfying the desired conditions.

Since μ_1 is absolutely continuous with respect to its variation $|\mu_1|$, by the Radon–Nikodým theorem there is a Borel measurable function $g_1 \in \mathcal{B}(L)$ such that $|g_1| = 1$ and such that

$$\mu_1(f) = \int_L f g_1 \, d|\mu_1|, \quad \forall f \in \mathcal{C}_0(L).$$

We write $\beta = \frac{\varepsilon^2}{2 \cdot 12^2}$ and we denote by A the set given by

$$A = \big\{ t \in L : \operatorname{Re} f_0(t) g_1(t) > 1 - \beta \big\}.$$

By Lemma 2.1, we have

$$\big\| (f_0 - \overline{g_1}) \chi_A \big\|_\infty \le \sqrt{2\beta} = \frac{\varepsilon}{12}. \tag{2.2}$$

Clearly, A is also Borel measurable, and we know that

$$
\begin{aligned}
1 - s &< \operatorname{Re} y_1^*(T f_0) \\
&= \operatorname{Re} \mu_1(f_0) \\
&= \operatorname{Re} \int_L f_0 g_1 \, d|\mu_1| \\
&= \operatorname{Re} \int_A f_0 g_1 \, d|\mu_1| + \operatorname{Re} \int_{L \setminus A} f_0 g_1 \, d|\mu_1| \\
&\le |\mu_1|(A) + (1 - \beta)|\mu_1|(L \setminus A) \\
&= |\mu_1|(L) - \beta|\mu_1|(L \setminus A) \\
&\le 1 - \beta|\mu_1|(L \setminus A).
\end{aligned}
$$

Hence,

$$|\mu_1|(L \setminus A) \le \frac{s}{\beta} = \frac{\eta(2 - \varepsilon)12^2}{\varepsilon^2 + 2 \cdot 12^2}.$$

By Lusin's theorem (see, for instance, [18, Theorem 2.23]) and by the inner regularity of μ_1 there is a compact set $B \subset A$ such that the restriction of g_1 to B is continuous, and $|\mu_1|(A \setminus B) \le \frac{\varepsilon\eta}{2}$, and so

$$|\mu_1|(L \setminus B) \le |\mu_1|(L \setminus A) + |\mu_1|(A \setminus B) \le \frac{s}{\beta} + \frac{\varepsilon\eta}{2}.$$

From (2.1) and the previous estimate, we obtain

$$|\mu_1|(B) = |\mu_1|(L) - |\mu_1|(L \setminus B) > 1 - s - \frac{s}{\beta} - \frac{\varepsilon\eta}{2} = 1 - \eta. \tag{2.3}$$

Hence,

$$
\begin{aligned}
\|T^{**} P_B\| &\ge |\mu_1|(B) \\
&> 1 - \eta \\
&> 1 - \frac{\delta\left(\frac{\varepsilon}{9}\right)}{1 + \delta\left(\frac{\varepsilon}{9}\right)}.
\end{aligned}
$$

By applying Lemma 2.3, we deduce

$$\left\|T^{**}(I - P_B)\right\| \leq \frac{\varepsilon}{9}. \tag{2.4}$$

By Proposition 2.2, T is a weakly compact operator and so $T^{**}(\mathcal{C}_0(L)^{**}) \subset Y$. So we can define the operator $\tilde{S} \in \mathcal{L}(\mathcal{C}_0(L), Y)$ by

$$\tilde{S}(f) = T^{**}(f\chi_B) + \varepsilon_1 y_1^*\big(T^{**}(f\chi_B)\big)T^{**}(\overline{g_1}\chi_B) \quad \big(f \in \mathcal{C}_0(L)\big),$$

where $\varepsilon_1 = \frac{1}{6}\frac{\delta(\frac{\varepsilon}{9})}{1+\delta(\frac{\varepsilon}{9})}$.

Let us note that $\tilde{S}^{**} = \tilde{S}^{**}P_B$, and we have

$$\begin{aligned}
\|\tilde{S}\| &\geq \left|y_1^*\big(\tilde{S}^{**}(\overline{g_1}\chi_B)\big)\right| \\
&= \left|y_1^*\big(T^{**}(\overline{g_1}\chi_B)\big) + \varepsilon_1 y_1^*\big(T^{**}(\overline{g_1}\chi_B)\big)y_1^*\big(T^{**}(\overline{g_1}\chi_B)\big)\right| \\
&\geq \left|y_1^*\big(T^{**}(\overline{g_1}\chi_B)\big)\right|\left|1 + \varepsilon_1 y_1^*\big(T^{**}(\overline{g_1}\chi_B)\big)\right| \\
&\geq |\mu_1|(B)\big(1 + \varepsilon_1|\mu_1|(B)\big) \\
&> (1 - \eta)\big(1 + \varepsilon_1(1 - \eta)\big) \quad \big[\text{by } (2.3)\big].
\end{aligned}$$

As a consequence,

$$1 \leq 1 - \eta + \varepsilon_1(1 - \eta)^2 \leq \|\tilde{S}\| \leq 1 + \varepsilon_1, \tag{2.5}$$

and so

$$\big|1 - \|\tilde{S}\|\big| \leq \varepsilon_1. \tag{2.6}$$

For every $h \in \mathcal{C}(B)$, we will denote by $h\chi_B$ the natural extension of h to L, which is a Borel function on L. Let S_1 be the operator given by

$$S_1(h) = \tilde{S}^{**}(h\chi_B) \quad \big(h \in \mathcal{C}(B)\big),$$

which is clearly an operator from $\mathcal{C}(B)$ into Y. Since $\tilde{S}^{**} = \tilde{S}^{**}P_B$, it is clear that $\|S_1\| = \|\tilde{S}\|$. We know that B is a compact set and $\tilde{S}$ is weakly compact, and by [4, Theorem 2] there is an operator $S_2 \in \mathcal{L}(\mathcal{C}(B), Y)$ and $h_1 \in S_{\mathcal{C}(B)}$ satisfying that

$$\|\tilde{S}\| = \|S_2\| = \left\|S_2(h_1)\right\| \quad \text{and} \quad \|S_2 - S_1\| < \frac{\varepsilon\eta}{2}. \tag{2.7}$$

We can choose $y_2^* \in S_{Y^*}$ such that

$$y_2^*\big(S_2(h_1)\big) = \|S_2\|. \tag{2.8}$$

By rotating the elements h_1 and y_2^*, if needed we can also assume that $y_1^*(T^{**}(h_1\chi_B)) \in \mathbb{R}_0^+$. In view of (2.7), the choice of y_2^*, and by using that $y_1^*(T^{**}(h_1\chi_B)) \in \mathbb{R}_0^+$, we have

$$\begin{aligned}
\|\tilde{S}\| - \frac{\varepsilon\eta}{2} &\leq \operatorname{Re} y_2^*\big(S_1(h_1)\big) \\
&= \operatorname{Re} y_2^*\big(\tilde{S}^{**}(h_1\chi_B)\big) \\
&= \operatorname{Re} y_2^*\big(T^{**}(h_1\chi_B)\big) + \varepsilon_1 \operatorname{Re} y_1^*\big(T^{**}(h_1\chi_B)\big)y_2^*\big(T^{**}(\overline{g_1}\chi_B)\big) \\
&\leq 1 + \varepsilon_1 \operatorname{Re} y_2^*\big(T^{**}(\overline{g_1}\chi_B)\big).
\end{aligned}$$

Combining this inequality with the estimate (2.5), we deduce that

$$\operatorname{Re} y_2^*\big(T^{**}(\overline{g_1}\chi_B)\big) \geq (1-\eta)^2 - \frac{\eta(2+\varepsilon)}{2\varepsilon_1}.$$

As a consequence, we obtain

$$\operatorname{Re} y_2^*\big(\tilde{S}^{**}(\overline{g_1}\chi_B)\big) = \operatorname{Re} y_2^*\big(T^{**}(\overline{g_1}\chi_B)\big) + \varepsilon_1 \operatorname{Re} y_1^*\big(T^{**}(\overline{g_1}\chi_B)\big)y_2^*\big(T^{**}(\overline{g_1}\chi_B)\big)$$

$$\geq (1-\eta)^2 - \frac{\eta(2+\varepsilon)}{2\varepsilon_1} + \varepsilon_1|\mu_1|(B)\Big((1-\eta)^2 - \frac{\eta(2+\varepsilon)}{2\varepsilon_1}\Big)$$

$$\geq \Big((1-\eta)^2 - \frac{\eta(2+\varepsilon)}{2\varepsilon_1}\Big)\big(1 + \varepsilon_1(1-\eta)\big) \quad \big[\text{by } (2.3)\big].$$

So in view of (2.7) and (2.5), we have

$$\operatorname{Re} y_2^*\big(S_2(\overline{g_1}_{|B})\big) \geq \operatorname{Re} y_2^*\big(S_1(\overline{g_1}_{|B})\big) - \|S_2 - S_1\|$$

$$\geq \operatorname{Re} y_2^*\big(\tilde{S}^{**}(\overline{g_1}\chi_B)\big) - \|S_2 - S_1\|$$

$$\geq \Big((1-\eta)^2 - \frac{\eta(2+\varepsilon)}{2\varepsilon_1}\Big)\big(1 + \varepsilon_1(1-\eta)\big) - \frac{\eta\varepsilon}{2}$$

$$\geq \Big((1-\eta)^2 - \frac{\eta(2+\varepsilon)}{2\varepsilon_1}\Big)\big(1 + \varepsilon_1(1-\eta)\big) - \frac{\eta\varepsilon\|S_2\|}{2}.$$

Let us write $R_2 = \frac{S_2}{\|S_2\|}$ and $\mu_2 = R_2^*(y_2^*) \in M(B)$. Let $g_2 = \frac{d\mu_2}{d|\mu_2|}$, and we can assume that $|g_2| = 1$. From the previous inequality, by using again (2.7) and (2.5), we obtain

$$\operatorname{Re} y_2^*\big(R_2(\overline{g_1}_{|B})\big)$$

$$\geq \frac{\big((1-\eta)^2 - \frac{\eta(2+\varepsilon)}{2\varepsilon_1}\big)(1 + \varepsilon_1(1-\eta))}{\|S_2\|} - \frac{\eta\varepsilon}{2}$$

$$\geq \frac{\big((1-\eta)^2 - \frac{\eta(2+\varepsilon)}{2\varepsilon_1}\big)(1 + \varepsilon_1(1-\eta))}{1 + \varepsilon_1} - \frac{\eta\varepsilon}{2}$$

$$= 1 - \frac{2\eta - 2\eta^2 + \varepsilon_1(1 - (1-\eta)^3) + \frac{2\eta+\eta\varepsilon}{2\varepsilon_1} + \frac{\eta\varepsilon}{2}(2 + \varepsilon_1 - \eta)}{1 + \varepsilon_1}$$

$$> 1 - 6\eta - 2\frac{\eta}{\varepsilon_1} - \varepsilon\eta. \tag{2.9}$$

We consider the measurable subset C of L given by

$$C = \big\{t \in B : \operatorname{Re}\big(\overline{g_1}(t) + h_1(t)\big)g_2(t) > 2 - \beta\big\}.$$

In view of (2.8) and (2.9), we have

$$2 - 6\eta - 2\frac{\eta}{\varepsilon_1} - \varepsilon\eta < \operatorname{Re}\mu_2(h_1 + \overline{g_1}_{|B})$$

$$= \int_C \operatorname{Re}(h_1 + \overline{g_1})g_2\, d|\mu_2| + \int_{B\setminus C} \operatorname{Re}(h_1 + \overline{g_1})g_2\, d|\mu_2|$$

$$\leq 2|\mu_2|(C) + (2 - \beta)|\mu_2|(B\setminus C)$$

$$= 2|\mu_2|(B) - \beta|\mu_2|(B\backslash C)$$
$$\leq 2 - \beta|\mu_2|(B\backslash C).$$

Hence,

$$|\mu_2|(B\backslash C) \leq \frac{6\eta + 2\frac{\eta}{\varepsilon_1} + \varepsilon\eta}{\beta}. \tag{2.10}$$

On the other hand, in view of Lemma 2.1, we have

$$\big\|(g_1 - g_2)\chi_C\big\|_\infty \leq \sqrt{2\beta} = \frac{\varepsilon}{12} \qquad \text{and} \qquad \big\|(h_1 - \overline{g_2})\chi_C\big\|_\infty \leq \sqrt{2\beta} = \frac{\varepsilon}{12}.$$

From the previous inequality and (2.2), it follows that

$$\big\|(h_1 - f_0)\chi_C\big\|_\infty$$
$$\leq \big\|(h_1 - \overline{g_2})\chi_C\big\|_\infty + \big\|(\overline{g_2} - \overline{g_1})\chi_C\big\|_\infty + \big\|(\overline{g_1} - f_0)\chi_C\big\|_\infty$$
$$\leq \frac{\varepsilon}{4}. \tag{2.11}$$

By the inner regularity of μ_2 there is a compact set $K_1 \subset C$ such that

$$|\mu_2|(C\backslash K_1) < \frac{\eta\varepsilon}{2}. \tag{2.12}$$

Let us note that

$$\|R_2^{**}P_{K_1}\| \geq \|y_2^* R_2^{**} P_{K_1}\| = |\mu_2|(K_1)$$
$$= |\mu_2|(B) - |\mu_2|(B\backslash C) - |\mu_2|(C\backslash K_1)$$
$$\geq \operatorname{Re} y_2^*\big(R_2(\overline{g_1}_{|B})\big) - |\mu_2|(B\backslash C) - |\mu_2|(C\backslash K_1)$$
$$\geq \operatorname{Re} y_2^*\big(R_2(\overline{g_1}_{|B})\big) - |\mu_2|(B\backslash C) - \frac{\eta\varepsilon}{2} \quad \big[\text{by } (2.12)\big]$$
$$\geq 1 - 6\eta - 2\frac{\eta}{\varepsilon_1} - \varepsilon\eta - \frac{6\eta + 2\frac{\eta}{\varepsilon_1} + \varepsilon\eta}{\beta} - \frac{\eta\varepsilon}{2} \quad \big[\text{by } (2.9) \text{ and } (2.10)\big]$$
$$> 1 - 2\frac{6\eta + 2\frac{\eta}{\varepsilon_1} + \varepsilon\eta}{\beta} - \frac{\eta\varepsilon}{2}$$
$$> 1 - \frac{\delta(\frac{\varepsilon}{9})}{1 + \delta(\frac{\varepsilon}{9})} > 0.$$

Hence, $K_1 \neq \varnothing$.

In view of Lemma 2.3, we obtain

$$\big\|R_2^{**}(P_B - P_{K_1})\big\| \leq \frac{\varepsilon}{9}. \tag{2.13}$$

We denote by T_2 the element in $\mathcal{L}(\mathcal{C}_0(L), Y)$ defined by

$$T_2(f) = R_2(f_{|B}) \quad \big(f \in \mathcal{C}_0(L)\big).$$

Clearly, $\|T_2^{**}(I - P_{K_1})\| = \|R_2^{**}(P_B - P_{K_1})\|$, and since it satisfied $T_2^{**}(P_B - P_{K_1}) = T_2^{**}(I - P_{K_1})P_B$ in view of (2.13), we obtain

$$\big\|T_2^{**}(P_B - P_{K_1})\big\| \leq \frac{\varepsilon}{9}. \tag{2.14}$$

We also write $R(f) = T^{**}(f\chi_B)$ for every $f \in \mathcal{C}(B)$ and so we have

$$\left\|(T_2^{**} - T^{**})P_B\right\| = \|R_2 - R\|. \tag{2.15}$$

By the definition of S_1 we know that

$$\|S_1 - R\| \le \varepsilon_1. \tag{2.16}$$

Since $K_1 \ne \varnothing$, let us fix $t_0 \in K_1$. Since $K_1 \subset C$, we have that $|h_1(t_0)| > 1 - \beta > 1 - \varepsilon/2$. So we can choose an open set V in B such that $t_0 \in V \subset \{t \in B : |h_1(t)| > 1 - \varepsilon/2\}$ and a function $v \in \mathcal{C}(B)$ satisfying $v(B) \subset [0,1], v(t_0) = 1$ and $\operatorname{supp} v \subset V$. So there are functions $h_i \in \mathcal{C}(B)$ $(i = 2,3)$ such that

$$h_2(t) = h_1(t) + v(t)\big(1 - |h_1(t)|\big)\frac{h_1(t)}{|h_1(t)|} \quad (t \in B)$$

and

$$h_3(t) = h_1(t) - v(t)\big(1 - |h_1(t)|\big)\frac{h_1(t)}{|h_1(t)|} \quad (t \in B).$$

It is clear that $h_i \in B_{\mathcal{C}(B)}$ for $i = 2,3$ and $h_1 = \frac{1}{2}(h_2 + h_3)$. By using the fact that the operator R_2 attains its norm at h_1, we clearly have that

$$\left\|R_2(h_2)\right\| = 1 \qquad \text{and} \qquad \left|h_2(t_0)\right| = 1. \tag{2.17}$$

Since $\operatorname{supp} v \subset V \subset \{t \in B : |h_1(t)| > 1 - \varepsilon/2\}$, we obtain for $t \in V$ that

$$\left|h_2(t) - h_1(t)\right| \le 1 - \left|h_1(t)\right| < \frac{\varepsilon}{2}.$$

For $t \in B \backslash V$, $h_2(t) = h_1(t)$ so $\|h_2 - h_1\| < \varepsilon/2$. In view of (2.11), we obtain

$$\begin{aligned}
\|h_2 - f_{0|C}\| &\le \|h_2 - h_1\| + \|h_1 - f_{0|C}\| \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \\
&= \frac{3\varepsilon}{4}.
\end{aligned} \tag{2.18}$$

Since $B \subset L$ is a compact subset, there is a function $f_2 \in \mathcal{C}_0(L)$ that extends the function h_2 to L (see, for instance, [10, Corollary 9.15 and Theorem 12.4], [12, Theorems 17 and 18]). Since the function $\Phi : \mathbb{C} \longrightarrow \mathbb{C}$ given by $\Phi(z) = z$ if $|z| \le 1$ and $\Phi(z) = \frac{z}{|z|}$ if $|z| > 1$ is continuous, by using $\Phi \circ f_2$ instead of f_2 if needed, and the fact that $h_2 \in S_{\mathcal{C}(B)}$, we can also assume that $f_2 \in S_{\mathcal{C}_0(L)}$. Since f_2 is an extension of h_2, by using (2.18) there is an open set $G \subset L$ such that $K_1 \subset G$ and satisfying also that

$$\left\|(f_2 - f_0)\chi_G\right\|_\infty < \frac{7\varepsilon}{8}. \tag{2.19}$$

By Urysohn's lemma, there is a function $u \in \mathcal{C}_0(L)$ such that $u(L) \subset [0,1]$, $u_{|K_1} = 1$ and $\operatorname{supp} u \subset G$. We define the function f_3 by

$$f_3 = uf_2 + (1 - u)f_0,$$

that clearly belongs to $B_{\mathcal{C}_0(L)}$.

Note also that

$$f_3(t) = f_2(t) = h_2(t) \quad \forall t \in K_1, \qquad f_3(t) = f_0(t) \quad \forall t \in L\backslash G \qquad (2.20)$$

and that

$$\big|f_3(t) - f_0(t)\big| = u(t)\big|f_2(t) - f_0(t)\big|, \quad \forall t \in G\backslash K_1.$$

In view of (2.19), we obtain

$$\|f_3 - f_0\| < \varepsilon. \qquad (2.21)$$

We write $\lambda_0 = \overline{h_2(t_0)}$ and we know that $|\lambda_0| = 1$. Here we define the operator $S \in \mathcal{L}(\mathcal{C}_0(L), Y)$ given by

$$S(f) = R_2^{**}\big((f\chi_{K_1})_{|B}\big) + \lambda_0 f(t_0) R_2^{**}(h_2 \chi_{B\backslash K_1}) \quad \big(f \in \mathcal{C}_0(L)\big).$$

Since R_2 is weakly compact, S is well defined. For every $f \in B_{\mathcal{C}_0(L)}$, we have that $|\lambda_0 f(t_0)| \le 1$ and so

$$\big\|(f\chi_{K_1})_{|B} + \lambda_0 f(t_0) h_2 \chi_{B\backslash K_1}\big\|_\infty \le 1.$$

Since $\|R_2\| \le 1$, then

$$\big\|S(f)\big\| = \big\|R_2^{**}\big((f\chi_{K_1})_{|B} + \lambda_0 f(t_0) h_2 \chi_{B\backslash K_1}\big)\big\| \le 1.$$

It is also satisfied that

$$\begin{aligned}
S(f_3) &= R_2^{**}\big((f_3 \chi_{K_1})_{|B}\big) + \lambda_0 f_3(t_0) R_2^{**}(h_2 \chi_{B\backslash K_1}) \\
&= R_2^{**}(h_2) \quad \big[\text{by } (2.20)\big] \\
&= R_2(h_2)
\end{aligned}$$

and, in view of (2.17), we obtain $\|S(f_3)\| = \|R_2(h_2)\| = 1$. Hence, $S \in S_{\mathcal{L}(\mathcal{C}_0(L),Y)}$ and it attains its norm at f_3. We also know that $\|f_3 - f_0\| < \varepsilon$ by inequality (2.21). It suffices to check that S is close to T. Indeed, we obtain the following estimate:

$$\begin{aligned}
\|S - T\| &\le \|S^{**} - T^{**}P_B\| + \big\|T^{**}(I - P_B)\big\| \\
&\le \|T_2^{**}P_{K_1} - T^{**}P_B\| + \big\|R_2^{**}(P_B - P_{K_1})\big\| + \frac{\varepsilon}{9} \quad \big[\text{by } (2.4)\big] \\
&= \big\|(T_2^{**} - T^{**})P_B\big\| + \big\|T_2^{**}(P_B - P_{K_1})\big\| + \frac{2\varepsilon}{9} \quad \big[\text{by } (2.13)\big] \\
&\le \big\|(T_2^{**} - T^{**})P_B\big\| + \frac{\varepsilon}{3} \quad \big[\text{by } (2.14)\big] \\
&= \|R_2 - R\| + \frac{\varepsilon}{3} \quad \big[\text{by } (2.15)\big] \\
&\le \|R_2 - S_2\| + \|S_2 - S_1\| + \|S_1 - R\| + \frac{\varepsilon}{3} \\
&\le \big|1 - \|S_2\|\big| + \frac{\eta\varepsilon}{2} + \varepsilon_1 + \frac{\varepsilon}{3} \quad \big[\text{by } (2.7) \text{ and } (2.16)\big] \\
&\le 2\varepsilon_1 + \frac{\eta\varepsilon}{2} + \frac{\varepsilon}{3} < \varepsilon \quad \big[\text{by } (2.6) \text{ and } (2.7)\big]. \qquad \square
\end{aligned}$$

Since the complex spaces $L_p(\mu)$ $(1 \le p < \infty)$ are $\mathbb{C}$-uniformly convex, we obtain the following result.

Corollary 2.5. *In the complex case, the pair* $(\mathcal{C}_0(L), L_p(\mu))$ *has the Bishop–Phelps–Bollobás property for operators for every Hausdorff locally compact space* L, *every positive measure* μ, *and* $1 \leq p < \infty$.

As we already mentioned, we extended in a nontrivial way a result by Kim, Lee, and Lin [15, Theorem 6] where they consider any (complex) space $L_\infty(\nu)$ as the domain space.

Open problem. In the real case it is not known whether or not the pair (c_0, ℓ_1) has the BPBp for operators.

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