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ON THE SUBMULTIPLICATIVITY AND SUBADDITIVITY OF THE SPECTRAL AND ESSENTIAL SPECTRAL RADIUS

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ABSTRACT. Suppose that A and B are positive operators on an ordered Banach space with a normal and generating cone satisfying $0 \leq AB \leq BA$. It is known that, in this case, we have

$$r(AB) \leq r(A)r(B) \quad \text{and} \quad r(A+B) \leq r(A) + r(B).$$

In this article we consider less restrictive assumptions on A and B which give us the same inequalities as above. Moreover, we also prove these inequalities in a more general setting of ordered algebras, and we consider related results in the Banach algebra setting. We apply our results to the essential spectral radius r_{ess} of AM-compact operators and prove the equality $r_{\text{ess}}(AB + BA) = 2r_{\text{ess}}(AB)$ under reasonable assumptions.

1. INTRODUCTION

It is well known that the inequalities

$$r(AB) \leq r(A)r(B) \quad \text{and} \quad r(A+B) \leq r(A) + r(B) \quad (1.1)$$

(for the spectral radius r) hold for arbitrary commuting bounded operators A and B on every Banach space X . It is also known that inequalities in (1.1) hold if A and B are positive operators on an ordered Banach space with a normal and

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generating cone satisfying $0 \leq AB \leq BA$ (see, e.g., [22] or [23]). In this article, we find other less restrictive assumptions under which these inequalities hold. Closely related conditions to our assumptions have already appeared in different contexts (see, e.g., [13], [12], [15], [8], and the references therein).

The paper is organized as follows. In Section 3 we prove (1.1) in a general setting of ordered algebras, and in Section 4 we consider related results in the setting of Banach algebras. In Sections 5 and 6 we apply our results to the spectral radius of positive operators and to the essential spectral radius r_{ess} of AM-compact operators. Moreover, we also prove that the equality $r_{\text{ess}}(AB + BA) = 2r_{\text{ess}}(AB)$ holds under reasonable assumptions (Theorems 6.8 and 6.9).

2. PRELIMINARIES

Let E be an ordered vector space. By E^+ we denote the set of all positive vectors in E . If p is a seminorm on E , then the positive cone E^+ is called p -normal whenever there exists $\gamma > 0$ such that $0 \leq x \leq y$ implies $p(x) \leq \gamma p(y)$. If p is a norm on an ordered vector space E , we will refer to a p -normal cone simply as a *normal cone*.

An ordered vector space \mathcal{A} that is an algebra as well, is said to be an *ordered algebra* whenever $\mathcal{A}^+ \cdot \mathcal{A}^+ \subseteq \mathcal{A}^+$ and $1 \in \mathcal{A}^+$ whenever \mathcal{A} is unital. If \mathcal{A} is a nonunital ordered algebra, we can extend the ordering to the standard unitization \mathcal{A}_1 by $(a, \lambda) \geq (0, 0)$ whenever $a \in \mathcal{A}^+$ and $\lambda \geq 0$. If a is an element in a nonunital algebra \mathcal{A} , then we denote by $a_1 \in \mathcal{A}_1$ the element $(a, 0) \in \mathcal{A}_1$. If p is a seminorm on a nonunital algebra \mathcal{A} , then we denote by p_1 the seminorm on \mathcal{A}_1 defined by $p_1((a, \lambda)) = p(a) + |\lambda|$. If p is submultiplicative on \mathcal{A} , then it should be clear that p_1 is submultiplicative on \mathcal{A}_1 .

Let p be a seminorm on a partially ordered algebra \mathcal{A} . It is not hard to see that whenever \mathcal{A} is a nonunital ordered algebra with a p -normal positive cone, then the positive cone \mathcal{A}_1^+ of the unitization \mathcal{A}_1 is p_1 -normal.

Let \mathcal{A} be an algebra, and let p be a submultiplicative seminorm on \mathcal{A} . The p -spectral radius r_p on \mathcal{A} , defined by

$$r_p(a) = \limsup_{n \rightarrow \infty} p(a^n)^{\frac{1}{n}},$$

satisfies $r_p(a) \leq p(a)$ and is equal to

$$r_p(a) = \lim_{n \rightarrow \infty} p(a^n)^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} p(a^n)^{\frac{1}{n}}.$$

Indeed, if we denote $p_n = p(a^n)$, then submultiplicativity of p implies that we have

$$p_{n+m} \leq p_n p_m \tag{2.1}$$

for all $n, m \in \mathbb{N}$. A well-known calculus lemma asserts that, for any sequence of nonnegative real numbers $\{p_n\}_{n \in \mathbb{N}}$ which satisfy inequalities in (2.1) for all $n, m \in \mathbb{N}$, we have

$$\lim_{n \rightarrow \infty} p_n^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} p_n^{\frac{1}{n}}.$$

It is not hard to see that, for an arbitrary element a in a nonunital algebra \mathcal{A} with a submultiplicative seminorm p , we have $r_{p_1}(a_1) = r_p(a)$. The element $a \in \mathcal{A}$ is called p -quasinilpotent if $r_p(a) = 0$ and it is called p -power bounded if there exists a positive constant $N > 0$ such that $p(a^n) \leq N$ for every $n \in \mathbb{N}$. If a is p -power bounded, then $r_p(a) \leq 1$.

The equality $(ba)^n = b(ab)^{n-1}a$ implies $r_p(ab) = r_p(ba)$ for all $a, b \in \mathcal{A}$. Throughout the article we use the notation $a^0b = b = ba^0$.

For other relevant facts and definitions on ordered vector spaces, ordered algebras, ordered Banach spaces, and Banach lattices and operators acting on them, we refer the reader to [3], [4], and [18].

3. p -SPECTRAL RADIUS INEQUALITIES

The proof of the following result on the monotonicity of the p -spectral radius is straightforward (and similar to the proof of [18, Theorem 4.1]) and is therefore omitted.

Lemma 3.1. *Let \mathcal{A} be an ordered algebra, and let \mathcal{B} be its subalgebra. Let p be a submultiplicative seminorm on \mathcal{B} such that the subcone $\mathcal{B}^+ = \mathcal{B} \cap \mathcal{A}^+$ is p -normal. If $a, b \in \mathcal{A}^+$ and $n_0 \in \mathbb{N}$ are such that $a^n, b^n \in \mathcal{B}^+$ and $a^n \leq b^n$ for all $n \geq n_0$, then $r_p(a) \leq r_p(b)$.*

The following two theorems generalize well-known results on the submultiplicativity and subadditivity of the spectral radius in several directions.

Theorem 3.2. *Let \mathcal{A} be an ordered algebra, let p be a submultiplicative seminorm on \mathcal{A} , and let $k \in \mathbb{N}_0$. Let \mathcal{B} be a subalgebra in \mathcal{A} ordered by a p -normal cone $\mathcal{B}^+ = \mathcal{A}^+ \cap \mathcal{B}$ which satisfies $\mathcal{A}^+\mathcal{B}^+\mathcal{A}^+ \subseteq \mathcal{B}^+$. Let $a, b \in \mathcal{A}^+$, and suppose that there exists a p -power bounded element $c \in \mathcal{A}^+$ such that $ac \leq ca$, $bc \leq cb$, and $ab \leq cba$. If at least one of the elements a and b is contained in \mathcal{B}^+ , then*

$$r_p(c^kba) \leq r_p(a)r_p(b), \quad r_p(ab) \leq r_p(a)r_p(b) \quad \text{and} \quad (3.1)$$

$$r_p(cba - ab) \leq r_p(cba) \leq r_p(a)r_p(b). \quad (3.2)$$

If, in addition, $p(c) < 1$, then elements ab , ba , c^kba , and $cba - ab$ are p -quasinilpotent.

Proof. At first, we will prove by induction on n that the inequality

$$(c^kba)^n \leq c^{\frac{n(n-1+2k)}{2}} b^n a^n \quad (3.3)$$

holds for all $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. For $n = 1$, inequality (3.3) is obvious. Assume now that inequality (3.3) holds for some positive integer n . Then we have

$$(c^kba)^{n+1} \leq c^{\frac{n(n-1+2k)}{2}} b^n a^n c^kba \leq c^{\frac{n(n-1+2k)}{2} + k + n} b^{n+1} a^{n+1} = c^{\frac{(n+1)(n+2k)}{2}} b^{n+1} a^{n+1},$$

and the induction step is completed.

Since the cone \mathcal{B}^+ is p -normal, there exists a positive constant $\gamma > 0$ such that $p(x) \leq \gamma p(y)$ whenever $0 \leq x \leq y$ holds in \mathcal{B} . The p -power boundedness of the element c implies that there exists a positive constant $N > 0$ such that $p(c^n) \leq N$

for every $n \in \mathbb{N}$. Since \mathcal{B}^+ satisfies $\mathcal{A}^+\mathcal{B}^+\mathcal{A}^+ \subseteq \mathcal{B}^+$, it follows that both sides of inequality (3.3) are contained in \mathcal{B}^+ and so

$$p((c^kba)^n) \leq \gamma p(c^{\frac{n(n-1+2k)}{2}} b^n a^n)$$

for all $n \in \mathbb{N}$, $n \geq 2$. Since p is submultiplicative and c is p -power bounded, we have

$$p((c^kba)^n) \leq \gamma p(c^{\frac{n(n-1+2k)}{2}}) p(b^n) p(a^n) \leq \gamma N p(b^n) p(a^n) \quad (3.4)$$

for all $n \in \mathbb{N}$, $n \geq 2$. By taking the n th roots in inequality (3.4) and letting n go to infinity, we obtain $r_p(c^kba) \leq r_p(b)r_p(a)$. When $k = 0$ we obtain also the second inequality in (3.1) since $r_p(ba) = r_p(ab)$, and when $k = 1$ we obtain the right-hand side of inequality (3.2). Inequality (3.2) follows from the fact that $0 \leq cba - ab \leq cba$ and $0 \leq (cba - ab)^n \leq (cba)^n$ for all $n \in \mathbb{N}$. Since $(cba - ab)^n \in \mathcal{B}^+$ and $(cba)^n \in \mathcal{B}^+$ for all $n \geq 3$, the p -normality of the cone \mathcal{B}^+ implies (3.2) by Lemma 3.1.

Suppose now that $p(c) < 1$ and $k \in \mathbb{N}_0$. From inequality (3.4) we obtain

$$p((c^kba)^n)^{\frac{1}{n}} \leq \gamma^{\frac{1}{n}} p(c)^{\frac{n-1+2k}{2}} p(b^n)^{\frac{1}{n}} p(a^n)^{\frac{1}{n}}. \quad (3.5)$$

Since $p(c) < 1$, the right-hand side of inequality (3.5) tends to zero when $n \rightarrow \infty$, so that we obtain $r_p(c^kba) = 0$. When $k = 0$ we obtain that ab and ba are p -quasinilpotent, and when $k = 1$ we obtain that cba is p -quasinilpotent. Inequality (3.2) implies $r_p(cba - ba) = 0$, which completes the proof. \square

Theorem 3.3. *Let \mathcal{A} be an ordered algebra, and let p be a submultiplicative seminorm on \mathcal{A} such that the cone \mathcal{A}^+ is p -normal. Let a , b , and c be positive elements in \mathcal{A} satisfying $ac \leq ca$, $bc \leq cb$, and $ab \leq cba$. If c is p -power bounded, then*

$$r_p(a + b) \leq r_p(a) + r_p(b). \quad (3.6)$$

Proof. If \mathcal{A} is not unital, then p_1 is a submultiplicative seminorm on \mathcal{A}_1 and the cone \mathcal{A}_1^+ is p_1 -normal. We also have $a_1c_1 \leq c_1a_1$, $b_1c_1 \leq c_1b_1$, and $a_1b_1 \leq c_1b_1a_1$. Since we also have $r_{p_1}(x_1) = r_p(x)$ for all $x \in \mathcal{A}$ and c_1 is p_1 -power bounded, we may assume without loss of generality that \mathcal{A} is unital.

For $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ we define the elements $q_{n,k}(c) \in \mathcal{A}$ in the following way: $q_{n,0}(c) = q_{n,n}(c) = 1$ (the unit element of \mathcal{A}), and for $k \in \mathbb{N}$ let

$$q_{n+1,k}(c) = q_{n,k-1}(c) + q_{n,k}(c)c^k.$$

Note that $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ implies that $q_{n,k}(c)$ is the sum of $\binom{n}{k}$ (not necessarily distinct) powers of the element c .

We will prove by induction on n that

$$(a + b)^n \leq \sum_{k=0}^n q_{n,k}(c) b^{n-k} a^k \quad (3.7)$$

for all $n \in \mathbb{N}$. For $n = 1$ the claim is trivially true. Suppose now that inequality (3.7) holds for some $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
(a+b)^{n+1} &\leq \left(\sum_{k=0}^n q_{n,k}(c) b^{n-k} a^k \right) (a+b) \\
&\leq \sum_{k=0}^n q_{n,k}(c) b^{n-k} a^{k+1} + \sum_{k=0}^n q_{n,k}(c) c^k b^{n-k+1} a^k \\
&= \sum_{k=1}^{n+1} q_{n,k-1}(c) b^{n-k+1} a^k + \sum_{k=0}^n q_{n,k}(c) c^k b^{n-k+1} a^k \\
&= \sum_{k=0}^{n+1} q_{n+1,k}(c) b^{n+1-k} a^k,
\end{aligned}$$

which completes the induction step.

Inequality (3.7) and the fact that the cone \mathcal{A}_+ is p -normal imply that there exists $\gamma > 0$ such that

$$p((a+b)^n) \leq \gamma p\left(\sum_{k=0}^n q_{n,k}(c) b^{n-k} a^k\right)$$

for all $n \in \mathbb{N}$. Since c is p -power bounded, there exists $M \geq 1$ such that

$$p((a+b)^n) \leq \gamma M \left(p(b^n) + \sum_{k=1}^{n-1} \binom{n}{k} p(a^k) p(b^{n-k}) + p(a^n) \right) \quad (3.8)$$

for all $n \in \mathbb{N}$. Let us choose now an arbitrary $\epsilon > 0$. Then there exists $L > 0$ such that for any $n \in \mathbb{N}$ we have $p(a^n) \leq L(r_p(a) + \epsilon)^n$ and $p(b^n) \leq L(r_p(b) + \epsilon)^n$. Hence, applying (3.8) it follows that

$$\begin{aligned}
p((a+b)^n) &\leq \gamma M L \sum_{k=0}^n \binom{n}{k} (r_p(a) + \epsilon)^k (r_p(b) + \epsilon)^{n-k} \\
&= \gamma M L (r_p(a) + r_p(b) + 2\epsilon)^n.
\end{aligned}$$

By taking the n th roots and letting n go to infinity, we obtain $r_p(a+b) \leq r_p(a) + r_p(b) + 2\epsilon$. Since $\epsilon > 0$ was arbitrary, we finally have $r_p(a+b) \leq r_p(a) + r_p(b)$. \square

The following result follows from Theorems 3.2 and 3.3.

Corollary 3.4. *Let \mathcal{A} be an ordered algebra, and let p be a submultiplicative seminorm on \mathcal{A} . Let \mathcal{B} be a subalgebra in \mathcal{A} ordered by a p -normal cone $\mathcal{B}^+ = \mathcal{A}^+ \cap \mathcal{B}$ which satisfies $\mathcal{A}^+ \mathcal{B}^+ \mathcal{A}^+ \subseteq \mathcal{B}^+$. Let $a, b \in \mathcal{A}^+$ be such that $ab \geq ba$ or $ba \geq ab$. If at least one of the elements a and b is contained in \mathcal{B}^+ , then*

$$r_p(ab - ba) \leq r_p(ab) \leq r_p(a)r_p(b). \quad (3.9)$$

If a and b are both in \mathcal{B}^+ , then (3.6) holds.

Proof. As in the proof of Theorem 3.3, we may assume without loss of generality that \mathcal{A} is unital. In this case the result follows from Theorems 3.2 and 3.3 by taking c to be the unit element of \mathcal{A} . \square

Remark 3.5. In particular, Theorem 3.2 and Corollary 3.4 hold whenever \mathcal{A} is an ordered algebra and p is a submultiplicative seminorm on \mathcal{A} such that the positive cone \mathcal{A}^+ is p -normal.

4. RELATED RESULTS IN BANACH ALGEBRAS

Recall that the spectrum of an element a of a nonunital Banach algebra \mathcal{A} is defined as the spectrum of the element a_1 in the standard unitization \mathcal{A}_1 of \mathcal{A} . Let $r(a)$ denote the spectral radius of $a \in \mathcal{A}$.

The following theorem is a version of our results in the setting of Banach algebras. Since its proof is very similar to the proofs of Theorem 3.2 and Theorem 3.3, we omit it.

Theorem 4.1. *Let \mathcal{A} be a Banach algebra, and let $k \in \mathbb{N}_0$. Suppose that the elements a , b , and c satisfy $ac = ca$, $bc = cb$, and $ab = cba$. If c is power-bounded, then*

$$r(ab) \leq r(a)r(b), \quad r(c^kba) \leq r(a)r(b), \quad \text{and} \quad r(a+b) \leq r(a) + r(b).$$

If $\|c\| < 1$, then ab , ba , and c^kba are quasinilpotent.

Note that the conditions $ac = ca$, $bc = cb$, and $ab = cba$, where c is power-bounded, are natural. Operators satisfying closely related conditions have already been studied in the context of spectral algebras and related invariant subspace problems (see, e.g., [13, Proposition 2.3], [12], [15], [8], and the references therein).

An important example of a Banach algebra is the Calkin algebra. Let X be a Banach space, and let π be the canonical projection of $\mathcal{L}(X)$ to the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$, where $\mathcal{L}(X)$ and $\mathcal{K}(X)$ denote the set of all bounded and compact linear operators on X , respectively. The *essential spectral radius* $r_{\text{ess}}(T)$ of the operator T is defined as the spectral radius of the element $\pi(T)$ in the Calkin algebra. An operator T is said to be a *Riesz operator* whenever $r_{\text{ess}}(T) = 0$.

Applying Theorem 4.1 to the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$, we obtain the following result for the essential spectral radius r_{ess} .

Corollary 4.2. *Let X be a Banach space, let $k \in \mathbb{N}_0$, and let A , B , C be bounded operators on X that satisfy $AC = CA$, $BC = CB$, and $AB = CBA$. If C is power-bounded, then the following inequalities hold:*

$$r_{\text{ess}}(AB) \leq r_{\text{ess}}(A)r_{\text{ess}}(B), \quad r_{\text{ess}}(C^kBA) \leq r_{\text{ess}}(A)r_{\text{ess}}(B), \quad \text{and} \\ r_{\text{ess}}(A+B) \leq r_{\text{ess}}(A) + r_{\text{ess}}(B).$$

If $\|C\| < 1$, then AB , BA , and C^kBA are Riesz operators.

5. SPECTRAL RADIUS INEQUALITIES FOR POSITIVE OPERATORS

A linear operator T on an ordered Banach space E is said to be *positive* whenever the positive cone E^+ is invariant under T . An operator T on E is said to be a *regular operator* (notation $T \in \mathcal{L}_r(E)$) whenever it is a linear combination of positive operators. A positive operator on an ordered Banach space with a closed and generating cone is bounded (see [4]). In particular, a positive operator on a Banach lattice is bounded. If the cone E^+ of an ordered Banach space is generating, then we can introduce a partial ordering on $\mathcal{L}(E)$ by $A \geq B$ whenever $A - B$ is a positive operator on E , so that, in this case, $\mathcal{L}(E)$ with a positive cone $\mathcal{L}^+(E)$ becomes an ordered algebra.

Let E be a Banach lattice. The set $\mathcal{L}(E)$ of all bounded operators on E equipped with the operator norm is an example of an ordered normed algebra. It is well known that the operator norm on the positive cone of positive operators on a Banach lattice is monotone.

An operator T on a Banach lattice is said to be an *AM-compact operator* whenever T maps order intervals to relatively compact sets. It is obvious that every compact operator is an AM-compact operator; however, there exist AM-compact operators that are not compact. For example, the identity operator on l^p -space ($1 \leq p < \infty$) is an AM-compact operator that is not a compact operator (see [2, Problem 10.3.7]).

The following theorem states that the cone of positive operators in $\mathcal{L}_r(E)$ is normal whenever the cone of the underlying ordered Banach space E is normal, closed, and generating. This result follows immediately from a result of Wickstead [20, Theorem 3.1]. Since it enables us to construct new examples of ordered Banach spaces with normal cones from given ones, we include its short proof.

Theorem 5.1. *Let E be an ordered Banach space with a closed, generating, and normal cone. Then the cone of all positive operators on E is a normal cone of $\mathcal{L}_r(E)$.*

Proof. By the Krein-Šmulian theorem (see [4, Theorem 2.37]), there exists a constant $\rho > 0$ such that, for each $x \in E$, there exist positive vectors $y, z \in E$ satisfying

$$x = y - z, \quad \|y\| \leq \rho\|x\|, \quad \text{and} \quad \|z\| \leq \rho\|x\|. \quad (5.1)$$

Since the cone of E is normal as well, [4, Theorem 2.38] implies that there exists a constant $c > 0$ such that $\|x\| \leq c \max\{\|y\|, \|z\|\}$ whenever $y \leq x \leq z$.

Let us choose a positive vector x with $\|x\| = 1$, and let $A \leq B$ be positive operators on E . There exist positive vectors $y, z \in E$ satisfying $x = y - z$, $\|y\| \leq \rho$, and $\|z\| \leq \rho$. Since the operator A is positive, we have $-Az \leq Ay - Az \leq Ay$, so that

$$\begin{aligned} \|Ax\| &= \|Ay - Az\| \leq c \max\{\|Ay\|, \|Az\|\} \leq c^2 \max\{\|By\|, \|Bz\|\} \\ &\leq c^2 \|B\| \max\{\|y\|, \|z\|\} \leq c^2 \rho \|B\|. \end{aligned}$$

Since x with $\|x\| = 1$ was arbitrary, we conclude that $\|A\| \leq c^2 \rho \|B\|$ and that the cone of positive operators is normal in $\mathcal{L}_r(E)$. \square

Therefore, if positive operators A , B , and C on an ordered Banach space E with a closed, normal, and generating cone E^+ satisfy the assumptions of Theorem 3.2, then inequalities $r(AB) \leq r(A)r(B)$ and $r(A+B) \leq r(A) + r(B)$ for the spectral radius hold. In particular, this holds when E is a Banach lattice.

If we omit the assumption that the cone of E^+ is closed, then the cone of positive operators of the space $\mathcal{L}_r(E)$ is not necessarily normal. However, inequalities from Theorem 3.2 and Theorem 3.3 still hold (see Theorem 5.2 below). Before we prove this result, we recall some general results on local spectral radii of bounded operators.

Given a real or complex Banach space X , the *local spectral radius* $r_T(x)$ of a bounded linear operator T on X at vector x is defined as

$$r_T(x) = \limsup_{n \rightarrow \infty} \|T^n x\|^{\frac{1}{n}}.$$

It is easy to see that $r_T(x) \leq r(T)$ for every $x \in X$. Daneš [6] proved that there exists $x \in X$ with $r_T(x)$ equal to $r(T)$. Moreover, he actually proved that the set

$$\{x \in X : r_T(x) = r(T)\}$$

is a second category set (by Baire). Förster and Nagy [10] proved that if E is an ordered Banach space with a generating cone E^+ and T is a positive operator on E , then

$$r(T) = \max_{x \in E^+} r_T(x). \quad (5.2)$$

We present only an outline of the proof of the following result, since we apply similar arguments as in the proofs of Theorems 3.2 and 3.3.

Theorem 5.2. *Let E be an ordered Banach space with a normal and generating cone, and let $k \in \mathbb{N}_0$. Let A and B be positive operators on E , and suppose that there exists a positive power-bounded operator C on E satisfying $AC \leq CA$, $BC \leq CB$, and $AB \leq CBA$. Then*

$$\begin{aligned} r(C^k BA) &\leq r(A)r(B), & r(AB) &\leq r(A)r(B), \\ r(A+B) &\leq r(A) + r(B) & \text{and} & \quad r(CBA - AB) \leq r(CBA) \leq r(A)r(B). \end{aligned}$$

If, in addition, $\|C\| < 1$, then the operators AB , BA , $C^k BA$, and $CBA - AB$ are quasinilpotent.

Proof. By applying (3.3), the power-boundedness of C , and the normality of the cone E^+ , it follows that

$$r_{C^k BA}(x) \leq r(A)r(B) \quad (5.3)$$

holds for all $x \in E^+$. The inequality $r(C^k BA) \leq r(A)r(B)$ follows from (5.3) and (5.2).

By applying (3.7), the inequality $r(A+B) \leq r(A) + r(B)$ is proved in a similar manner. The rest of the proof is omitted. \square

Corollary 5.3. *Let E be an ordered Banach space with a normal and generating cone. Let A and B be positive operators on E with a positive or negative commutator $AB - BA$. Then we have*

$$r(AB - BA) \leq r(AB) \leq r(A)r(B) \quad \text{and} \quad r(A+B) \leq r(A) + r(B). \quad (5.4)$$

Inequalities $r(AB - BA) \leq r(AB) \leq r(A)r(B)$ appeared in [7] in the case of positive operators A and B with a positive commutator $AB - BA$ on Banach lattices. Zima ([23, Theorem 2.1] and [22]) proved that the same inequalities hold in the case of ordered Banach spaces with normal and generating cones. Moreover, Zima [23] proved that

$$r(AB + BA) = 2r(AB). \quad (5.5)$$

The following example shows that the assumption on the power-boundedness of the operator C in Theorem 5.2 cannot be omitted.

Example 5.4. Let $E = \mathbb{C}^2$,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}, \quad \text{and} \quad C = 4I.$$

Then $r(A) = 1$, $r(B) = 2$,

$$CBA = \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix}, \quad AB = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \quad AB - BA = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

and

$$A + B = \begin{bmatrix} 0 & 2 \\ 5 & 0 \end{bmatrix}.$$

Thus $CBA \geq AB$, $CA = AC$, and $CB = BC$, but $r(CBA) = 16 > r(CBA - AB) = 15 > r(AB) = 4 > 2 = r(A)r(B)$ and $r(A+B) = \sqrt{10} > 3 = r(A) + r(B)$.

Example 5.4 shows also that in Corollary 5.3 the positivity of the commutator cannot be omitted. The following example illustrates that the assumptions of Theorem 5.2 are sensible if we compare them with the assumptions of Corollary 5.3.

Example 5.5. Let $E = \mathbb{C}^2$,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad C = 2I.$$

Then $r(B) = 2$, $r(A) = 0.5(5 + \sqrt{33})$,

$$CBA = \begin{bmatrix} 8 & 12 \\ 8 & 12 \end{bmatrix}, \quad AB = \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix}, \quad AB - BA = \begin{bmatrix} -1 & -3 \\ 3 & 1 \end{bmatrix}$$

and

$$A + B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

Therefore $AB - BA$ and $BA - AB$ are not positive; $CBA \geq AB$, $CA = AC$, $CB = BC$, and $r(AB) = 10 < r(A)r(B)$; and $r(A+B) = 0.5(7 + \sqrt{57}) < r(A) + r(B)$.

6. ESSENTIAL SPECTRAL RADIUS INEQUALITIES FOR POSITIVE OPERATORS

In this section we apply our general results of Section 3 to the case of the essential spectral radius of positive AM-compact operators on Banach lattices.

The essential spectral radius is closely related to the *measure of noncompactness* $\beta : \mathcal{L}(X) \rightarrow [0, \infty)$ defined by

$$\beta(T) = \inf \left\{ \delta > 0 : \exists x_1, \dots, x_n \in X \text{ such that } T(B_X) \subseteq \bigcup_{i=1}^n B(x_i, \delta) \right\}.$$

Here B_X denotes the closed unit ball of X , and $B(x_i, \delta)$ denotes the closed ball with the center x_i and radius δ .

The measure of noncompactness is a submultiplicative seminorm on $\mathcal{L}(X)$ that satisfies $\beta(T) \leq \|T\|$ (see [1, Lemma 7.59] and its preceding remark; see also [19] or [14]). Nussbaum [16] proved that

$$r_{\text{ess}}(T) = \lim_{n \rightarrow \infty} \beta(T^n)^{\frac{1}{n}}, \quad (6.1)$$

which is precisely the β -spectral radius $r_\beta(T)$ of the operator T . The proof of the following lemma is straightforward, and so we omit it.

Lemma 6.1. *Let S, T_1, T_2 , and R be bounded operators on a Banach lattice E . If T_1 and T_2 are AM-compact operators and R is order-bounded, then ST_1R and $T_1 + T_2$ are AM-compact operators.*

Theorem 6.2. *Let $k \in \mathbb{N}_0$, let E be a Banach lattice such that E and E^* have order-continuous norms, and let A, B , and C be positive operators on E satisfying $AC \leq CA$, $BC \leq CB$, and $AB \leq CBA$. Suppose that the operator C is β -power-bounded. If at least one of the operators A and B is AM-compact, then*

$$\begin{aligned} r_{\text{ess}}(C^k BA) &\leq r_{\text{ess}}(A)r_{\text{ess}}(B), \\ r_{\text{ess}}(CBA - AB) &\leq r_{\text{ess}}(CBA) \leq r_{\text{ess}}(A)r_{\text{ess}}(B). \end{aligned} \quad (6.2)$$

If A and B are AM-compact, then

$$r_{\text{ess}}(A + B) \leq r_{\text{ess}}(A) + r_{\text{ess}}(B). \quad (6.3)$$

If $\beta(C) < 1$, then $AB, C^k BA$, and $CBA - AB$ are Riesz operators. In particular, if $\|C\| < 1$, then $AB, C^k BA$, and $CBA - AB$ are Riesz operators.

Proof. Let \mathcal{B}^+ be the set of all positive AM-compact operators on E , and let us define $\mathcal{B} := \mathcal{B}^+ - \mathcal{B}^+$. By Lemma 6.1 the set \mathcal{B} is an ordered algebra and its positive cone \mathcal{B}^+ satisfies $\mathcal{L}_r(E)^+ \mathcal{B}^+ \mathcal{L}_r(E)^+ \subseteq \mathcal{B}^+$. Since the measure of noncompactness is a submultiplicative seminorm on \mathcal{B} which is also monotone on \mathcal{B}^+ by [17, Corollary 2.6]; that is, for AM-compact operators $0 \leq S \leq T$ we have $\beta(S) \leq \beta(T)$, and the result follows by applying (6.1) and Theorems 3.2 and 3.3. \square

Corollary 6.3. *Let A and B be positive operators with a positive commutator $AB - BA$ on a Banach lattice E with the property that E and E^* have order-continuous norms. If at least one of the operators A and B is AM-compact, then*

$$r_{\text{ess}}(AB - BA) \leq r_{\text{ess}}(AB) \leq r_{\text{ess}}(A)r_{\text{ess}}(B). \quad (6.4)$$

If A and B are AM-compact, then

$$r_{\text{ess}}(A + B) \leq r_{\text{ess}}(A) + r_{\text{ess}}(B). \quad (6.5)$$

The following result follows directly from Corollary 6.3.

Corollary 6.4. *Let E be a Banach lattice such that E and E^* have order-continuous norms, and let A and B be positive operators on E with a positive commutator $AB - BA$. Suppose that at least one of the operators A and B is AM-compact. Then the following statements hold:*

- (a) *If at least one of the operators A and B is a Riesz operator, then AB and $AB - BA$ are also Riesz operators.*
- (b) *If A and B are AM-compact, then the operator $A + B$ is a Riesz operator if and only if A and B are Riesz operators.*

Remark 6.5. In Corollary 4.2, we can, by (6.1) and as in Theorem 6.2, replace the assumptions that C is power-bounded and $\|C\| < 1$ with weaker assumptions that C is β -power-bounded and $\beta(C) < 1$. We omit the proof.

We continue this section with a result concerning quasinilpotence of positive commutators of positive operators on Banach lattices. In [5], Bračič et al. considered spectral properties of positive commutators of positive operators on Banach lattices. In [5, Theorem 2.2], the authors proved that a positive commutator of positive compact operators is necessarily quasinilpotent, and they asked whether a positive commutator of positive operators is quasinilpotent when only one of the operators A and B is compact. This problem was independently solved in [9] and [11]. Partial results were previously obtained in [7]. We now recall the main result from [7].

Theorem 6.6. *Let A and B be positive operators on a Banach lattice E such that $A + B$ is a Riesz operator. If the commutator $AB - BA$ is a power compact positive operator, then it is a quasinilpotent operator.*

Since the preceding result does not follow from results obtained in [9] and [11], it is of independent interest, as a positive commutator of positive operators can also be quasinilpotent under different assumptions than the ones from [9] and [11].

The following result extends the class of positive operators with positive quasinilpotent commutators.

Corollary 6.7. *Let E be a Banach lattice such that E and E^* have order continuous norms, and let A and B be positive AM-compact operators on E with a positive commutator $AB - BA$. If A and B are Riesz operators and $AB - BA$ is power-compact, then $AB - BA$ is quasinilpotent.*

Proof. Since A and B are Riesz operators, Corollary 6.4 implies that the operator $A + B$ is also a Riesz operator. An application of Theorem 6.6 shows that $AB - BA$ is quasinilpotent. \square

We conclude this paper with a result on the essential spectral radius version of the equality (5.5). If one of the operators A and B is a compact operator on a Banach space, then we obviously have $r_{\text{ess}}(AB + BA) = r_{\text{ess}}(AB) = 0$. The

equality $r_{\text{ess}}(AB + BA) = 2r_{\text{ess}}(AB)$ is also obvious in the case of commuting operators A and B on a Banach space.

If E is a Banach lattice and A and B are positive power-compact operators on E with a positive commutator $AB - BA$, then the equality $r_{\text{ess}}(AB + BA) = r_{\text{ess}}(AB) = 0$ holds as well. Indeed, [7, Lemma 2.2] implies that the operator $A + B$ is power-compact, so that the operator $(A + B)^2$ is power-compact as well. Therefore, the operator $(A + B)^{2n}$ is a compact operator on E for some positive integer n . Inequalities

$$0 \leq AB \leq AB + BA \leq A^2 + AB + BA + B^2 = (A + B)^2 \quad (6.6)$$

and [3, Theorem 5.13] imply that $(AB)^{3n}$ and $(AB + BA)^{3n}$ are compact operators, so that AB and $AB + BA$ are power-compact operators. It follows that $r_{\text{ess}}(AB + BA) = r_{\text{ess}}(AB) = 0$.

We finish this article with the following two results.

Theorem 6.8. *Let E be a Banach lattice such that E and E^* have order-continuous norms, and let A and B be positive operators on E with a positive commutator $AB - BA$. If at least one of the operators A and B is AM-compact, then*

$$r_{\text{ess}}(AB + BA) = 2r_{\text{ess}}(AB).$$

Proof. The fact that the commutator $AB - BA$ is a positive operator on E implies that

$$0 \leq 2BA \leq AB + BA \leq 2AB.$$

By the assumption, operators BA , AB , and $AB + BA$ are AM-compact, so that the monotonicity of the essential spectral radius on the positive cone of the ordered algebra generated by positive AM-compact operators gives us

$$0 \leq 2r_{\text{ess}}(BA) \leq r_{\text{ess}}(AB + BA) \leq 2r_{\text{ess}}(AB).$$

To finish the proof we apply the equality $r_{\text{ess}}(AB) = r_{\text{ess}}(BA)$, which holds for all bounded operators on Banach spaces. \square

Similarly, one can prove the following generalization of Theorem 6.8

Theorem 6.9. *Let \mathcal{A} be an ordered algebra, and let p be a submultiplicative seminorm on \mathcal{A} . Let \mathcal{B} be a subalgebra in \mathcal{A} ordered by a p -normal cone $\mathcal{B}^+ = \mathcal{A}^+ \cap \mathcal{B}$ which satisfies $\mathcal{A}^+ \mathcal{B}^+ \mathcal{A}^+ \subseteq \mathcal{B}^+$. Let $a, b \in \mathcal{A}^+$ be such that $ab \leq ba$ or $ba \leq ab$. If at least one of the elements a and b is contained in \mathcal{B}^+ , then*

$$r_p(ab + ba) = 2r_p(ab).$$

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