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THE ISOMORPHIC CLASSIFICATION OF BESOV SPACES OVER $\mathbb{R}^d$ REVISITED

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Abstract. We take advantage of the recent developments in the isomorphic classification of the infinite matrix spaces of mixed norms $\ell_q(\ell_p)$ for the whole range of values $0 < p, q \leq \infty$ to give a unified approach to the classification of Besov spaces over Euclidean spaces. In particular, we show that different Besov spaces with generalized smoothness $\mathring{B}^w_{p,q}(\mathbb{R}^d)$ over the Euclidean space $\mathbb{R}^d$ are isomorphic if and only if the indices p and q match.

1. Introduction and background

In harmonic analysis and partial differential equations, one often wants to place a function $f\colon \Omega \to \mathbb{C}$ defined in some domain $\Omega \subseteq \mathbb{R}^d$ in one or more function spaces in order to quantify its size in some sense. Besov spaces provide a way to measure the size of the differences $f(x+h) - f(x)$ of a function f, for x, $h \in \Omega$. There are various needs and approaches to specify when a function, or more generally a distribution, belongs to a Besov space. The Fourier analytic approach to homogeneous Besov spaces, for instance, yields representations for

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the members of the space $\mathring{\mathrm{B}}^{\sigma}_{p,q}(\mathbb{R}^d)$ of the form

$$f = \sum_{k\in\mathbb{Z}} f_k,$$

in which the summands f_k depend on f linearly. Arranging for a suitable decomposition of the objects in $\mathring{\mathrm{B}}^{\sigma}_{p,q}(\mathbb{R}^d)$ is important to compute their norm and to simplify the analysis of the operators acting on it. It is also the key to be able to identify the members of the space from an abstract point of view. The latter is the angle of functional analysis, which aims at recognizing $\mathring{\mathrm{B}}^{\sigma}_{p,q}(\mathbb{R}^d)$ as an already existing animal in disguise in the Banach space zoo.

The study of the classification of Besov spaces over Euclidean spaces is indeed an important subject in Banach space theory (see [13, Section 6.7.7]). It addresses a very simple question: are the Besov spaces $\mathring{\mathrm{B}}^{\sigma}_{p,q}(\mathbb{R}^d)$ essentially distinct, that is, mutually nonisomorphic?

Historically, a breakthrough in the isomorphic characterization of Besov spaces over the Euclidean d-dimensional space $\mathbb{R}^d$ occurs in [15], where H. Triebel constructed an unconditional basis for the nonhomogeneous Besov space $\mathrm{B}^{\sigma}_{p,q}(\mathbb{R}^d)$. He did not identify the associated sequence space, but he showed that, for $0 < \sigma < \infty$, $1 < p < \infty$, and $1 \le q < \infty$, $\mathrm{B}^{\sigma}_{p,q}(\mathbb{R}^d)$ is isomorphic to the direct ℓ_q-sum of countably many copies of ℓ_p, that is, to the sequence space $(\ell_p \oplus \ell_p \oplus \cdots \oplus \ell_p \oplus \cdots)_q = \ell_q(\ell_p)$. Triebel's techniques remain valid when $-\infty < \sigma < \infty$, $0 < q \le \infty$, and for homogeneous Besov spaces. However, the fact that his methods rely on the boundedness of the Riesz transform rules out the possibility to make them extensive to the extreme cases $p = 1$ or $p = \infty$.

Another significant advance was made by P. G. Lemarié and Y. Meyer in [11]. Here they used an appropriate wavelet basis to obtain a characterization of Besov spaces which led to establishing isomorphisms between homogeneous Besov spaces $\mathring{\mathrm{B}}^{\sigma}_{p,q}(\mathbb{R}^d)$ and the spaces $\ell_q(\ell_p)$ for $1 \le p \le \infty$ and $1 \le q \le \infty$, whence, as J. Peetre already pointed out in [12], the classification of the Banach spaces $\mathring{\mathrm{B}}^{\sigma}_{p,q}(\mathbb{R}^d)$ would be settled once it is proved that different $\ell_q(\ell_p)$ spaces for $1 \le p, q \le \infty$ are not isomorphic. The answer to this question has its roots in [16, p. 242n], where H. Triebel said that he had learned from A. Pełczyński that the above-mentioned conjecture of Peetre holds true for $1 < p < \infty$ and $1 \le q \le \infty$. In 2011, P. Cembranos and J. Mendoza included in [6] a detailed proof of Pełczyński's assertion and extended it to the whole range of values $1 \le p \le \infty$ and $1 \le q \le \infty$. More recently, and initially unaware of the connection of their work with this direction of research, the authors have completed in [1] the picture of the isomorphic classification of $\ell_q(\ell_p)$ spaces by letting the non-locally convex relatives to be part of their natural family and have showed that no two members of the extended class $\{\ell_q(\ell_p) : 0 < q, p \le \infty\}$ are isomorphic.

Besov spaces also have their non-locally convex counterpart. In defining these spaces you just have to allow p or q or both to be less than 1 and you automatically land in quasi-Banach space territory. All of which naturally leads us to wonder if the above neat isomorphic classification of Besov spaces can be extended to the whole range of values $0 < p \le \infty$ and $0 < q \le \infty$. And right away we are faced with an important, yet rather common in the non-locally convex theory, stumbling block: a straightforward extension of the proof of Lemarié and Meyer's Theorem 4.1 in [11] is not possible. This obstruction is mainly due to the fact that their arguments rely on duality techniques and estimates that do not work for $p < 1$ or $q < 1$.

The treatment that M. Frazier and B. Jawerth gave to Besov spaces in [8] was crucial to push forward the non-locally convex case. They obtained an atomic decomposition of Besov spaces valid for $0 < p \le \infty$ and $0 < q \le \infty$, but, unfortunately, their decomposition does not yield directly an isomorphism between $\mathring{\mathrm{B}}^{\sigma}_{p,q}(\mathbb{R}^d)$ and $\ell_q(\ell_p)$. However, as it was pointed out by themselves later in [9], the estimates that they had gotten for the so-called φ-transform also apply for the wavelet transform, and as a consequence it is possible to reconstruct, from Lemarié and Meyer's aforementioned work [11], an isomorphism from $\mathring{\mathrm{B}}^{\sigma}_{p,q}(\mathbb{R}^d)$ onto $\ell_q(\ell_p)$ that is valid for all $0 < p, q \le \infty$. And this completes the picture of the puzzle of the isomorphic classification of classical Besov spaces in both locally convex and non-locally convex cases.

Subsequently, several authors were interested in homogeneous Besov spaces of generalized smoothness (see [4], [5], [7]), which we will denote by $\mathring{\mathrm{B}}^{w}_{p,q}(\mathbb{R}^d)$. Using standard techniques, one can obtain a direct isomorphism from $\mathring{\mathrm{B}}^{w}_{p,q}(\mathbb{R}^d)$ onto $\mathring{\mathrm{B}}^{0}_{p,q}(\mathbb{R}^d)$, so that these Besov spaces are also isomorphic to $\ell_q(\ell_p)$. Thus, it is natural to wonder if Lemarié and Meyer's wavelet technique provides a direct isomorphism from $\mathring{\mathrm{B}}^{w}_{p,q}(\mathbb{R}^d)$ onto the sequence space $\ell_q(\ell_p)$.

In this article we explore the methods of Lemarié and Meyer's and Frazier and Jawerth's in more generality to establish the validity of the above isomorphisms to weighted Besov spaces $\mathring{\mathrm{B}}^{w}_{p,q}(\mathbb{R}^d)$ for the whole range of $0 < p, q \le \infty$, at once. Our results cast light also onto the classical case by providing a unified proof, without having to patch together so many scattered pieces in the literature.

2. Prerequisites: The ingredients of the proof

Throughout this article we employ the notation and terminology commonly used both in Banach space theory (see, e.g., [2]) and in harmonic analysis (see, e.g., [16]). This section is entirely preparatory. With the intention to render our paper as self-contained as possible, for the convenience of the reader we have summarized the relevant concepts that we will need later on.

2.1. **Tempered distributions modulo polynomials.** We will denote by $\mathcal{S}$ the Schwartz class of rapidly decreasing smooth functions in $\mathbb{R}^d$, by $\mathcal{S}'$ its topological dual (i.e., the space of tempered distributions), and by $\langle \cdot, \cdot \rangle \colon \mathcal{S}' \times \mathcal{S} \to \mathbb{C}$ the dual pair between tempered distributions and rapidly decreasing smooth functions. The support of $f \in \mathcal{S}'$, denoted by $\operatorname{supp} f$, is the complement of the largest open set on which f is the null distribution. The Fourier transform can be defined for tempered distributions, and, as is customary, $\hat{f}$ will denote the Fourier transform of $f \in \mathcal{S}'$. Given $f \in \mathcal{S}'$ and $0 < t < \infty$, we define the dilation f_t as $f_t(x) = t^{-d} f(x/t)$. Notice that $\hat{f}_t(\xi) = \hat{f}(t\xi)$.

We will need to consider the closed subspace $\mathcal{S}_0$ of $\mathcal{S}$ consisting of functions with zero moments. There is a natural identification between the topological dual $\mathcal{S}_0'$ of $\mathcal{S}_0$ and the quotient space $\mathcal{S}'/\mathcal{P}$, where $\mathcal{P}$ is the linear space of polynomial functions. Of special interest for us is the subspace $\mathcal{S}_{00}$ of $\mathcal{S}_0$ consisting of functions whose Fourier transform has compact support contained in $\mathbb{R}^d\setminus\{0\}$. For a detailed background on rapidly decreasing functions and tempered distributions, we refer to [14].

2.2. **Hilbertian wavelet bases.** Next we detail the standard construction of a wavelet basis from a pair (ψ_0, ψ_1) of real valued measurable functions such that

$$\int_{\mathbb{R}} \psi_0(x)\psi_1(x)\,dx = 0, \qquad \text{and} \qquad \int_{\mathbb{R}} \psi_0(x)^2\,dx = 1 = \int_{\mathbb{R}} \psi_1(x)^2\,dx.$$

For each $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in \mathcal{E} := \{0,1\}^d \setminus \{(0,\dots,0)\}$, each $k \in \mathbb{Z}^d$, and each $j \in \mathbb{Z}$, put

$$\psi_\epsilon(x_1,\dots,x_d) = \psi_{\epsilon_1}(x_1)\cdots\psi_{\epsilon_i}(x_i)\cdots\psi_{\epsilon_d}(x_d), \tag{2.1}$$
$$\psi_{\epsilon,k,j}(x) = 2^{jd/2}\psi_\epsilon(2^j x - k), \tag{2.2}$$
$$\Psi(x_1,\dots,x_d) = \psi_0(x_1)\cdots\psi_0(x_i)\cdots\psi_0(x_d), \qquad \text{and} \tag{2.3}$$
$$\Psi_k(x) = \Psi(x-k). \tag{2.4}$$

Let $\Lambda = \mathcal{E}\times\mathbb{Z}^d\times\mathbb{Z}$ and $\Lambda^+ = \{(\epsilon,k,j\} \in \Lambda : k \ge 0\}$. The family $\{\psi_\lambda : \lambda \in \Lambda\}$ is called a *wavelet basis*. In [11], P. G. Lemarié and Y. Meyer constructed a wavelet basis fulfilling the following properties:

- $\psi_\lambda \in \mathcal{S}_{00}$;
- for every $f \in \mathcal{S}_0'$, we have
$$f = \sum_{\lambda\in\Lambda} \langle f, \psi_\lambda\rangle \psi_\lambda, \tag{2.5}$$
  where the (infinite) series is unconditionally convergent; and
- $\langle \psi_\lambda, \psi_\lambda\rangle = 1$ and $\langle \psi_\lambda, \psi_\mu\rangle = 0$ if $\lambda \ne \mu$.

Similarly, the family $\{\Psi_k, k \in \mathbb{Z}^d\} \cup \{\psi_\lambda : \lambda \in \Lambda^+\}$ verifies that

- $\Psi_k \in \mathcal{S}$ and supp $\hat{\Psi}_k$ has compact support;
- for every $f \in \mathcal{S}'$ we have
$$f = \sum_{k\in\mathbb{Z}^d} \langle f, \Psi_k\rangle \Psi_k + \sum_{\lambda\in\Lambda^+} \langle f, \psi_\lambda\rangle\psi_\lambda, \tag{2.6}$$
  where the (infinite) series are unconditionally convergent; and
- $\langle \Psi_k, \Psi_k\rangle = 1$, $\langle \Psi_k, \Psi_l\rangle = 0$ if $k \ne l$ and $\langle \Psi_k, \psi_\lambda\rangle = 0$.

Following [11], such a wavelet basis will be called *Hilbertian*.

Suppose that we have a Hilbertian wavelet basis $\{(\psi_\lambda) : \lambda \in \Lambda\}$. Let X be a quasi-Banach space continuously embedded in $\mathcal{S}_0'$. Define, ignoring the possible dependence on the chosen basis,

$$\mathring{\mathrm{W}}(X) = \left\{ \left(\langle f, \psi_\lambda\rangle\right)_{\lambda\in\Lambda} : f \in X \right\}.$$

Clearly, thanks to the orthogonality of the basis,

$$\mathring{\mathrm{W}}(X) = \Big\{(a_\lambda)_{\lambda\in\Lambda} : \exists f \in X \text{ such that } f = \sum_{\lambda\in\Lambda} a_\lambda \psi_\lambda\Big\},$$

and there is a natural linear bijection between X and $\mathring{\mathrm{W}}(X)$, so that we can transfer the topological structure from X into $\mathring{\mathrm{W}}(X)$. Roughly speaking, $\mathring{\mathrm{W}}(X)$ is a representation of X as a sequence space.

A very general problem is to describe properties or the sequence space $\mathring{\mathrm{W}}(X)$ and, if possible, to be able to characterize it up to isomorphism. For instance, $\{(\psi_\lambda)\colon \lambda\in\Lambda\}$ is an unconditional basic sequence for X if and only if $\mathring{\mathrm{W}}(X)$ is a lattice under the natural ordering for sequence spaces. This problem is successfully discussed in [11] for many classical function spaces in harmonic analysis, such as $\mathrm{H}_1(\mathbb{R}^d)$, $\mathrm{BMO}(\mathbb{R}^d)$, $L_p(\mathbb{R}^d)$ $(1<p<\infty)$, Hölder spaces, Sobolev spaces and locally convex Besov spaces.

2.3. **Band-limited L_p-spaces.** Let $0<p\le\infty$, and suppose that $\mathcal{K}\subseteq\mathbb{R}^d$ is a compact set. Consider

$$L_p[\mathcal{K}] = \big\{f\in L_p(\mathbb{R}^d)\cap\mathcal{S}' : \operatorname{supp}\hat{f}\subseteq\mathcal{K}\big\}.$$

$L_p[\mathcal{K}]$ is a closed subspace of $L_p(\mathbb{R}^d)$ and hence is a quasi-Banach space. We will write $L_p[R] := L_p[\mathbb{D}_R]$, where $\mathbb{D}_R=\{x\in\mathbb{R}^d : |x|\le R\}$ $(0<R<\infty)$. Hereafter we will put

$$\Big|(x_1,\dots,x_i,\dots,x_d)\Big| = \max_{1\le i\le n}|x_i|.$$

In some sense, for a fixed compact set $\mathcal{K}$, the behavior of the spaces $L_p[\mathcal{K}]$ is similar to that of the sequence ℓ_p spaces. By way of example we can state the following result (see [12, Chapter 11, Lemma 1]), whose immediate corollary will be of special interest for our approach to the isomorphic classification of Besov spaces.

Theorem 2.1. *Let $0<p_1<p_2\le\infty$. There exists a constant A such that, for every $0<R<\infty$ and every $f\in L_{p_1}[R]$, we have $f\in L_{p_2}[R]$ and*

$$\|f\|_{p_2}\le AR^{1/p_1-1/p_2}\|f\|_{p_1}.$$

Corollary 2.2. *Suppose $0<p\le 1$. There is a constant B such that if $0<R<\infty$, $f\in L_p[R]$, $g\in L_\infty[R]$, and $x\in\mathbb{R}^d$, then $g(\cdot-x)f(\cdot)\in L_1(\mathbb{R}^d)$ and*

$$\left|\int_{\mathbb{R}^d} g(y-x)f(y)\,dy\right| \le BR^{1/p-1}\Big(\int_{\mathbb{R}^d}\big|g(y-x)f(y)\big|^p\,dy\Big)^{1/p}.$$

Proof. Let $h(y)=g(y-x)f(y)$. Note that

$$\operatorname{supp}\hat{h}\subseteq\operatorname{supp}\hat{g}+\operatorname{supp}\hat{f}\subseteq \mathrm{D}_{2R}.$$

Applying Theorem 2.1 yields the desired result since

$$\left|\int_{\mathbb{R}^d} g(y-x)f(y)\,dy\right| \le \|h\|_1\le A2^{1/p-1}R^{1/p-1}\|h\|_p<\infty. \qquad \square$$

2.4. **Homogeneous Besov spaces.** Under certain conditions, the differences appearing in the classical definition of Besov spaces can be replaced by convolutions $\theta_t * f$, where $\theta \in \mathcal{S}_0$ and $t \in (0, \infty)$ (i.e., it is possible to regard Besov spaces as spaces of tempered distributions and obtain a Littlewood–Paley description of them). For our purposes, among the many different ways to introduce Besov spaces it will be convenient to deal with the more general definition of distributional Besov spaces introduced in [5], which we summarize next.

A doubling weight in $\mathbb{Z}$ is a double sequence $w = (w_j)_{j=-\infty}^{\infty}$ in $(0, \infty)$ such that

$$0 < \inf_{j \in \mathbb{Z}} \frac{w_{j+1}}{w_j} \le \sup_{j \in \mathbb{Z}} \frac{w_{j+1}}{w_j} < \infty.$$

Suppose $\theta \in \mathcal{S}_{00}$ verifies the condition

$$\inf_{\xi \in \mathbb{R}^d \setminus \{0\}} \sum_{j \in \mathbb{Z}} \left|\hat{\theta}(2^{-j}\xi)\right|^2 > 0, \tag{2.7}$$

and let $< 0 < p, q \le \infty$. For $f \in \mathcal{S}_0'$ we define

$$\|f\|_{w,p,q} = \Big(\sum_{j \in \mathbb{Z}} w_j^{-q} \|\theta_{2^{-j}} * f\|_p^q\Big)^{1/q},$$

with the usual modification if $q = \infty$. The *homogeneous Besov space* $\mathring{\mathrm{B}}_{p,q}^w(\mathbb{R}^d)$ is the quasi-Banach space

$$\mathring{\mathrm{B}}_{p,q}^w(\mathbb{R}^d) = \big\{f \in \mathcal{S}_0' : \|f\|_{w,p,q} < \infty\big\}.$$

Besov spaces do not depend on the particular function θ chosen. In fact, for any $\eta \in \mathcal{S}_{00}$ there is a constant C_η such that

$$\Big(\sum_{j \in \mathbb{Z}} w_j^{-q} \|\eta_{2^{-j}} * f\|_p^q\Big)^{1/q} \le C_\eta \|f\|_{w,p,q}. \tag{2.8}$$

If $w_j = 2^{j\sigma}$ for all $j \in \mathbb{Z}$ and a fixed $\sigma \in \mathbb{R}$ we recover the classical homogeneous Besov spaces $\mathring{\mathrm{B}}_{p,q}^\sigma(\mathbb{R}^d)$. We refer to [5] for more details, and to [12], [16], [10], and [4] for connections between this definition and other ways of introducing classical and weighted Besov spaces.

3. The isomorphic classification of weighted Besov spaces: Homogeneous case

Equipped with the machinery introduced above, we are now in a position to tackle our goal and identify isomorphically weighted Besov spaces, recently known as *Besov spaces with generalized smoothness*. The novelty of our approach relies on the effective usage of the properties of band-limited spaces $L_p[\mathcal{K}]$.

Theorem 3.1. *Suppose $0 < p, q \le \infty$, and let w be a doubling weight in $\mathbb{Z}$. We have*

$$\mathring{\mathrm{W}}\big(\mathring{\mathrm{B}}_{p,q}^w(\mathbb{R}^d)\big) = \big\{(a_\lambda)_{\lambda \in \Lambda} : \big\|(a_\lambda)_{\lambda \in \Lambda}\big\|_{w,p,q} < \infty\big\},$$

where

$$\left\|(a_\lambda)_{\lambda\in\Lambda}\right\|_{w,p,q}=\Big(\sum_{j\in\mathbb{Z}}w_j^{-q}2^{jqd(1/2-1/p)}\Big(\sum_{\substack{\epsilon\in\mathcal{E}\\ k\in\mathbb{Z}^d}}|a_{\epsilon,k,j}|^p\Big)^{q/p}\Big)^{1/q},$$

with the usual modifications if $p=\infty$ or $q=\infty$.

Proof. Define the norm in the Besov space from $\theta\in\mathcal{S}_{00}$ verifying (2.7). Fix a Hilbertian wavelet basis, for which we will use the notation introduced in (2.1) and (2.2). Let $R>0$ such that $\operatorname{supp}\hat{\psi}_\epsilon\subseteq\mathbb{D}_{R_0}$ for every $\epsilon\in\mathcal{E}$ and some $R_0<R$. Choose $\eta\in\mathcal{S}_{00}$ such that $\operatorname{supp}\hat{\eta}\subseteq\mathbb{D}_R$ and $\hat{\eta}(\xi)=1$ for $|\xi|\le R_0$. Notice that

$$\langle f,\psi_{\epsilon,k,j}\rangle=\int_{\mathbb{R}^d}2^{jd/2}\eta_{2^{-j}}*f(x)\psi_\epsilon(2^jx-k)\,dx.$$

Consider

$$D=\max_{y\in\mathbb{R}^d}\Big(\sum_{\substack{\epsilon\in\mathcal{E}\\ k\in\mathbb{Z}^d}}\left|\psi_\epsilon(y-k)\right|^{\overline{p}}\Big)^{1/p},$$

where $\overline{p}=\min\{1,p\}$, and

$$E=\max_{\epsilon\in\mathcal{E}}\|\psi_\epsilon\|_1.$$

Suppose first that $0<p\le 1$. By Corollary 2.2,

$$\Big(\sum_{\substack{\epsilon\in\mathcal{E}\\ k\in\mathbb{Z}^d}}\left|\langle f,\psi_{\epsilon,k,j}\rangle\right|^p\Big)^{1/p}\le BDR^{1/p-1}2^{jd/2}2^{jd(1/p-1)}\|\eta_{2^{-k}}*f\|_p.$$

The corresponding estimate for $1\le p\le\infty$ is obtained as in [11]. However, being faithful to our motivation to supply a unified approach, we include a proof.

By Hölder's inequality,

$$\Big(\sum_{\substack{\epsilon\in\mathcal{E}\\ k\in\mathbb{Z}^d}}\left|\langle f,\psi_{\epsilon,k,j}\rangle\right|^p\Big)^{1/p}\le DE^{1-1/p}2^{jd/2}2^{jd(1/p-1)}\|\eta_{2^{-k}}*f\|_p.$$

In any case, with $C_0=BDR^{1/p-1}$ for $p\le1$, and $C_0=DE^{1-1/p}$ for $p>1$ we have

$$\Big(\sum_{\substack{\epsilon\in\mathcal{E}\\ k\in\mathbb{Z}^d}}\left|\langle f,\psi_{\epsilon,k,j}\rangle\right|^p\Big)^{1/p}\le C_02^{jd(1/p-1/2)}\|\eta_{2^{-k}}*f\|_p.$$

Therefore,

$$\left\|\big(\langle f,\psi_\lambda\rangle\big)_{\lambda\in\Lambda}\right\|_{w,p,q}\le C_0C_\eta\|f\|_{w,p,q}.$$

Conversely, take $(a_\lambda)_{\lambda\in\Lambda}$ such that $\|(a_\lambda)_{\lambda\in\Lambda}\|_{w,p,q}<\infty$. It is straightforward to check that $\sum_{\lambda\in\Lambda}a_\lambda\psi_\lambda$ converges in $\mathcal{S}_0'$ to a tempered distribution f. Put

$$\eta_{\epsilon,i}=\theta_{2^{-i}}*\psi_\epsilon.$$

Notice that there is $N \in \mathbb{N}$ such that $\eta_{\epsilon,i} = 0$ for $|i| > N$. Let us denote

$$F = \max_{y\in\mathbb{R}^d} \sum_{\substack{|i|\le N\\ \epsilon\in\mathcal{E}\\ k\in\mathbb{Z}^d}} \Big|\eta_{\epsilon,i}(y-k)\Big|,$$

$$G = \max_{\substack{|i|\le N\\ \epsilon\in\mathcal{E}}} \|\eta_{\epsilon,i}\|_p,$$

$$H = \max_{\substack{|i|\le N\\ \epsilon\in\mathcal{E}}} \|\eta_{\epsilon,i}\|_1,$$

and

$$L = \sup_{\substack{j\in\mathbb{Z}\\ |i|\le N}} \frac{w_j}{w_{j+i}}.$$

Now, notice that

$$\theta_{2^{-j}} * f(x) = 2^{jd/2} \sum_{\substack{-N\le i\le N\\ \epsilon\in\mathcal{E}\\ k\in\mathbb{Z}^d}} a_{\epsilon,k,i+j}\eta_{\epsilon,i}(2^j x - k).$$

If $0 < p \le 1$, since $L_p(\mathbb{R}^d)$ is a p-Banach space,

$$\|\theta_{2^{-j}} * f\|_p \le G 2^{jd/2} 2^{-jd/p} \Big(\sum_{\substack{-N\le i\le N\\ \epsilon\in\mathcal{E}\\ k\in\mathbb{Z}^d}} |a_{\epsilon,k,i+j}|^p \Big)^{1/p}.$$

If $1 \le p \le \infty$, an appeal to Hölder's inequality yields

$$\Big|\theta_{2^{-j}} * f(x)\Big| \le F^{1-1/p} 2^{jd/2} \Big(\sum_{\substack{-N\le i\le N\\ \epsilon\in\mathcal{E}\\ k\in\mathbb{Z}^d}} |a_{\epsilon,k,i+j}|^p \Big|\eta_{\epsilon,i}(2^j x - k)\Big| \Big)^{1/p};$$

hence

$$\|\theta_{2^{-j}} * f\|_p \le F^{1-1/p} H^{1/p} 2^{jd/2} 2^{-jd/p} \Big(\sum_{\substack{-N\le i\le N\\ \epsilon\in\mathcal{E}\\ k\in\mathbb{Z}^d}} |a_{\epsilon,k,i+j}|^p \Big)^{1/p}.$$

Let $C_1 = G$ for $p \le 1$ and $C_1 = F^{1-1/p} H^{1/p}$ for $p > 1$, so that we have a similar inequality for any value of p. Summing in j and putting $C_2 = C_1 L 2^{Nd(1/2-1/p)}$ gives

$$\|f\|_{w,p,q} \le C_2 \Big(\sum_{j\in\mathbb{Z}} \Big(\sum_{\substack{-N\le i\le N\\ \epsilon\in\mathcal{E}\\ k\in\mathbb{Z}^d}} \big(w_{j+i} 2^{(j+i)(1/2-1/p)} |a_{\epsilon,k,i+j}| \big)^p \Big)^{q/p} \Big)^{1/q}.$$

Let $M = (2N+1)^{\max\{0,1/p-1/q\}}$, so that

$$\Big(\sum_{|i|\le N} |y_i| \Big)^{q/p} \le M^q \sum_{|i|\le N} |y_i|^{q/p}.$$

Finally, putting $C_3 = C_2 M$ yields

$$\begin{aligned}\|f\|_{w,p,q} &\le C_3\Big(\sum_{\substack{-N\le i\le N\\ j\in\mathbb{Z}}}\Big(\sum_{\substack{\epsilon\in\mathcal{E}\\ k\in\mathbb{Z}^d}}\big(w_{j+i}2^{(j+i)(1/2-1/p)}|a_{\epsilon,k,i+j}|\big)^p\Big)^{q/p}\Big)^{1/q}\\ &= C_3(2N+1)^{1/q}\Big\|(a_\lambda)_{\lambda\in\Lambda}\Big\|_{w,p,q}.\end{aligned}$$ □

The previous theorems yields a direct proof of the isomorphic characterization of homogeneous Besov spaces, as wished.

Corollary 3.2. *Let $d \in \mathbb{N}$, $0 < p, q \le \infty$, and let w be a doubling weight in $\mathbb{Z}$. Then $\mathring{\mathrm{B}}^w_{p,q}(\mathbb{R}^d)$ is isomorphic to $\ell_q(\ell_p)$.*

Proof. From Theorem 3.1 it follows readily that $\mathring{\mathrm{W}}(\mathring{\mathrm{B}}^w_{p,q}(\mathbb{R}^d))$ is isomorphic to $\ell_q(\ell_p)$. □

Corollary 3.3. *Let $d_i \in \mathbb{N}$, $0 < p_i, q_i \le \infty$, and let w_i be doubling weights in $\mathbb{Z}$ $(i = 1, 2)$. Then $\mathring{\mathrm{B}}^{w_1}_{p_1,q_1}(\mathbb{R}^{d_1})$ is isomorphic to $\mathring{\mathrm{B}}^{w_2}_{p_2,q_2}(\mathbb{R}^{d_2})$ if and only if $p_1 = p_2$ and $q_1 = q_2$.*

Proof. It is straightforward from Corollary 3.2 and [1, Theorem 2.4]. □

Remark 3.4. We want to emphasize the difference between the approach to Besov spaces presented in this article and that of M. Frazier and B. Jawerth for classical Besov spaces in [8]. Consider a function $\phi \in \mathcal{S}_{00}$ such that $\sum_{j\in\mathbb{Z}} |\hat{\phi}(2^j\xi)| = 1$ for every $\xi \in \mathbb{R}^d \setminus \{0\}$. Construct, for $k \in \mathbb{Z}^d$ and $j \in \mathbb{Z}$, the functions

$$\varphi_{k,j} \colon \mathbb{R}^d \to \mathbb{C}, \qquad \varphi_{k,j}(x) = 2^{jd/2}\varphi(2^j x - k).$$

Define the so-called φ-transform of f:

$$S_\varphi(f) = \big(\langle f, \varphi_{k,j}\rangle\big)_{j\in\mathbb{Z},k\in\mathbb{Z}^d}, \quad f \in \mathcal{S}_0'.$$

Conversely, given a sequence $\alpha = (a_{k,j})_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ consider, when possible,

$$T_\varphi(\alpha) = \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^d} a_{k,j}\varphi_{k,j}.$$

It follows from [8, Lemma 2.1] that

$$T_\varphi\big(S_\varphi(f)\big) = f, \quad f \in \mathcal{S}_0'.$$

However, there is not a unique sequence α such that $T_\varphi(\alpha) = f$.

Let $-\infty < \sigma < \infty$ and $0 < p, q \le \infty$. Consider the sequence space

$$\mathring{\mathrm{b}}^\sigma_{p,q} = \Big\{(a_{k,j})_{j\in\mathbb{Z},k\in\mathbb{Z}^d} : \Big(\sum_{j\in\mathbb{Z}} 2^{jq(d/2-d/p+\sigma)}\Big(\sum_{k\in\mathbb{Z}^d}|a_{k,j}|^p\Big)^{q/p}\Big)^{1/q} < \infty\Big\}.$$

It follows from [8, Theorem 2.6] that S_φ is bounded from $\mathring{\mathrm{B}}^\sigma_{p,q}(\mathbb{R}^d)$ into $\mathring{\mathrm{b}}^\sigma_{p,q}$, while [8, Theorem 3.1] leads to the boundedness of T_φ from $\mathring{\mathrm{b}}^\sigma_{p,q}$ onto $\mathring{\mathrm{B}}^\sigma_{p,q}(\mathbb{R}^d)$. From a functional analytic point of view, these results imply that $\mathring{\mathrm{B}}^\sigma_{p,q}(\mathbb{R}^d)$ embeds complementably in $\mathring{\mathrm{b}}^\sigma_{p,q}$, which is isomorphic to $\ell_q(\ell_p)$. But, unless $p = q$, this does not necessarily mean that $\mathring{\mathrm{B}}^\sigma_{p,q}(\mathbb{R}^d)$ is isomorphic to $\ell_q(\ell_p)$.

4. What about nonhomogeneous Besov spaces?

The wavelet decomposition (with respect to a compactly supported wavelet basis of Daubechies type) of nonhomogeneous Besov spaces of generalized smoothness was achieved by A. Almeida (see [3]) The intention in the definition of nonhomogeneous Besov spaces is the same as in the homogeneous case but deals only with differences $f(x+h)-f(x)$ for *small values* of h (i.e., $|h| \le 1$). If we introduce Besov spaces in terms of convolutions $\theta_t * f$ we must consider only values of the dilation parameter t in the interval $(0,1]$. Finally, if we use a *discrete* definition in terms of convolutions $\theta_{2^{-k}} * f$ we must consider nonnegative values of the parameter $k \in \mathbb{Z}$. To give a precise definition, consider a doubling weight in $\mathbb{Z}^+$, that is, a weight $w = (w_j)_{j=0}^{\infty}$ in $(0,\infty)$ such that

$$0 < \inf_{j \in \mathbb{Z}+} \frac{w_{j+1}}{w_j} \le \sup_{j \in \mathbb{Z}^+} \frac{w_{j+1}}{w_j} < \infty,$$

and a pair of functions (Θ, θ) such that $\theta \in \mathcal{S}_{00}$, $\Theta \in \mathcal{S}$, $\hat{\Theta}$ has compact support and the condition

$$\inf_{\xi \in \mathbb{R}^d \setminus \{0\}} \left| \hat{\Theta}(\xi) \right|^2 + \sum_{j=0}^{\infty} \left| \hat{\theta}(2^{-j}\xi) \right|^2 > 0$$

holds. Define, for $< 0 < p, q \le \infty$,

$$\mathrm{B}_{p,q}^{w}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}' : \|f\|_{w,p,q} < \infty \right\},$$

where, for $f \in \mathcal{S}'$,

$$\|f\|_{w,p,q} = \|\Theta * f\|_p + \Big(\sum_{j=0}^{\infty} w_j^{-q} \|\theta_{2^{-j}} * f\|_p^q \Big)^{1/q},$$

with the usual modification if $q = \infty$.

As for homogeneous Besov spaces, $\mathrm{B}_{p,q}^{w}(\mathbb{R}^d)$ is a quasi-Banach space that does not depend on the particular pair of functions (Θ, θ) chosen.

The study of the wavelet decomposition of nonhomogeneous Besov spaces, using Lamarié and Meyer's wavelet system instead of a compactly supported one, follows the same steps as the homogeneous case, with the exception that we must use (2.6) instead of (2.5). Therefore, in what follows, we will only present the definitions and state the results, leaving out the proofs.

Consider a Hilbertian wavelet basis constructed from a pair (ψ_0, ψ_1). Let $\{\Psi_k, : k \in \mathbb{Z}^d\}$ and $\{\psi_\lambda : \lambda \in \Lambda^+\}$ as in (2.2) and (2.4). Given a quasi-Banach space X continuously embedded in $\mathcal{S}'$, define

$$\mathrm{W}(X) = \Big\{ \Big(\big(\langle f, \Psi_k \rangle \big)_{k \in \mathbb{Z}^d}, \big(\langle f, \psi_\lambda \rangle \big)_{\lambda \in \Lambda^+} \Big) : f \in X \Big\}.$$

$\mathrm{W}(X)$ is a quasi-Banach space with the topology that inherits from X.

The following result is analogous to Theorem 3.1 for the nonhomogeneous case.

Theorem 4.1. *Let $< 0 < p, q \le \infty$, and let w be a doubling weight in $\mathbb{Z}^+$. We have*

$$\mathrm{W}\big(\mathrm{B}_{p,q}^{w}(\mathbb{R}^d)\big) = \ell_p(\mathbb{Z}^d) \oplus \left\{ (a_\lambda)_{\lambda \in \Lambda^+} : \left\| (a_\lambda)_{\lambda \in \Lambda} \right\|_{w,p,q} < \infty \right\},$$

where

$$\left\|(a_\lambda)_{\lambda\in\Lambda}\right\|_{w,p,q} = \Big(\sum_{j=0}^{\infty} w_j^{-q} 2^{jqd(1/2-1/p)} \Big(\sum_{\substack{\epsilon\in\mathcal{E}\\ k\in\mathbb{Z}^d}} |a_{\epsilon,k,j}|^p\Big)^{q/p}\Big)^{1/q},$$

with the usual modifications if $p=\infty$ *or* $q=\infty$.

Corollary 4.2. *Let* $d\in\mathbb{N}$, $< 0 < p,q \le \infty$, *and let* w *be a doubling weight in* $\mathbb{Z}^+$. *Then* $\mathrm{B}^w_{p,q}(\mathbb{R}^d)$ *is isomorphic to* $\ell_q(\ell_p)$.

Corollary 4.3. *For* $i=1,2$, *let* $d_i\in\mathbb{N}$, $0<p_i,q_i\le\infty$, *and* w_i *be doubling weights in* $\mathbb{Z}$. *Then* $\mathrm{B}^{w_1}_{p_1,q_1}(\mathbb{R}^{d_1})$ *is isomorphic to* $\mathrm{B}^{w_2}_{p_2,q_2}(\mathbb{R}^{d_2})$ *if and only if* $p_1=p_2$ *and* $q_1=q_2$.

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References

1. F. Albaic and J. L. Ansorena, *On the mutually non isomorphic $\ell_p(\ell_q)$ spaces, II*, Math. Nachr. **288** (2015), no. 1, 5–9. MR3310494. DOI 10.1002/mana.201300161. 109, 116
2. F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Grad. Texts in Math. **233**, Springer, New York, 2006. MR2192298. 110
3. A. Almeida, *Wavelet bases in generalized Besov spaces*, J. Math. Anal. Appl. **304** (2005), no. 1, 198–211. MR2124658. DOI 10.1016/j.jmaa.2004.09.017. 117
4. J. L. Ansorena and O. Blasco, *Characterization of weighted Besov spaces*, Math. Nachr. **171** (1995), 5–17. Zbl 0813.46023. MR1316348. DOI 10.1002/mana.19951710102. 110, 113
5. J. L. Ansorena and O. Blasco, *Convolution multipliers on weighted Besov spaces*, Bol. Soc. Mat. Mexicana (3) **4** (1998), no. 1, 47–68. Zbl 0946.42007. MR1625577. 110, 113
6. P. Cembranos and J. Mendoza, *On the mutually non isomorphic $\ell_p(\ell_q)$ spaces*, Math. Nachr. **284** (2011), no. 16, 2013–2023. MR2844675. DOI 10.1002/mana.201010056. 109
7. W. Farkas and H. Leopold, *Characterisations of function spaces of generalised smoothness*, Ann. Mat. Pura Appl. (4) **185** (2006), no. 1, 1–62. Zbl 1116.46024. MR2179581. DOI 10.1007/s10231-004-0110-z. 110
8. M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. **34** (1985), no. 4, 777–799. Zbl 0551.46018. MR0808825. DOI 10.1512/iumj.1985.34.34041. 110, 116
9. M. Frazier and B. Jawerth, "Applications of the ϕ and wavelet transforms to the theory of function spaces" in *Wavelets and Their Applications*, Jones and Bartlett, Boston, 1992, 377–417. MR1187350. 110
10. M. Frazier, B. Jawerth, and G. Weiss, *Littlewood-Paley Theory and the Study of Function Spaces*, CBMS Regional Conference Series in Mathematics **79**, Amer. Math. Soc., Providence, 1991. MR1107300. 113
11. P. G. Lemarié and Y. Meyer, *Ondelettes et bases hilbertiennes*, Rev. Mat. Iberoamericana **2** (1986), no. 1–2, 1–18. Zbl 0657.42028. MR0864650. DOI 10.4171/RMI/22. 109, 110, 111, 112, 114
12. J. Peetre, *New Thoughts on Besov Spaces*, Duke Univ. Math. Ser. **1**, Department of Mathematics, Duke University, Durham, NC, 1976. MR0461123. 109, 112, 113
13. A. Pietsch, *History of Banach Spaces and Linear Operators*, Birkhäuser, Boston, 2007. Zbl 1121.46002. MR2300779. 109

14. W. Rudin, *Functional Analysis*, 2nd ed., Internat. Ser. Pure Appl. Math., McGraw-Hill, New York, 1991. MR1157815. 111
15. H. Triebel, *Über die Existenz von Schauderbasen in Sobolev-Besov-Räumen: Isomorphiebeziehungen*, Studia Math. **46** (1973), 83–100 (German). Zbl 1108.42007. MR0338771. 109
16. H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978. MR0503903. 109, 110, 113

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