

ON HARDY-TYPE INEQUALITIES FOR WEIGHTED MEANS

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Communicated by J. Bonet

ABSTRACT. Our aim in this article is to establish weighted Hardy-type inequalities in a broad family of means. In other words, for a fixed vector of weights $(\lambda_n)_{n=1}^{\infty}$ and a weighted mean \mathcal{M} , we search for the smallest extended real number C such that

$$\sum_{n=1}^{\infty} \lambda_n \mathcal{M}((x_1, \dots, x_n), (\lambda_1, \dots, \lambda_n)) \leq C \sum_{n=1}^{\infty} \lambda_n x_n \quad \text{for all } x \in \ell_1(\lambda).$$

The main results provide a complete answer in the case when \mathcal{M} is monotone and satisfies the weighted counterpart of the Kedlaya inequality. In particular, this is the case if \mathcal{M} is symmetric, concave, and the sequence $(\frac{\lambda_n}{\lambda_1 + \dots + \lambda_n})_{n=1}^{\infty}$ is nonincreasing. In addition, we prove that if \mathcal{M} is a symmetric and monotone mean, then the biggest possible weighted Hardy constant is achieved if λ is the constant vector.

1. Introduction

In the first decades of the twentieth century, Hilbert conjectured that the so-called *averaging operator* is a bounded linear map acting on the sequence space ℓ_p for $p > 1$. Motivated by this conjecture, in an equivalent form, several authors proved that

$$\sum_{n=1}^{\infty} \mathcal{P}_p(x_1, \dots, x_n) \leq C(p) \sum_{n=1}^{\infty} x_n, \quad (1.1)$$

Copyright 2019 by the Tusi Mathematical Research Group.

Received Jun. 3, 2018; Accepted Jul. 23, 2018.

First published online Nov. 16, 2018.

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2010 *Mathematics Subject Classification*. Primary 26D15; Secondary 39B62.

Keywords. weighted mean, concave mean, Hardy inequality, Kedlaya inequality.

for every sequence $(x_n)_{n=1}^\infty$ with positive terms, where \mathcal{P}_p denotes the p th *power mean* (extended to the limiting cases $p = \pm\infty$) and

$$C(p) := \begin{cases} 1, & p = -\infty, \\ (1-p)^{-1/p}, & p \in (-\infty, 0) \cup (0, 1), \\ e, & p = 0, \\ \infty, & p \in [1, \infty]. \end{cases}$$

Furthermore, this constant is sharp; that is, it cannot be diminished.

The first result of this type with a nonoptimal constant was established by Hardy [12] in his seminal paper. Later it was improved and extended by Landau [20], Knopp [17], and Carleman [3], whose results are summarized in inequality (1.1). Meanwhile, Copson [4] adopted Elliott's proof in [11] of the Hardy inequality to show (in an equivalent form) that if $\mathcal{P}_p(x, \lambda)$ denotes the p th λ -*weighted power mean of a vector* x , then

$$\sum_{n=1}^{\infty} \lambda_n \mathcal{P}_p((x_1, \dots, x_n), (\lambda_1, \dots, \lambda_n)) \leq C(p) \sum_{n=1}^{\infty} \lambda_n x_n \quad (1.2)$$

for all $p \in (0, 1)$, and sequences $(x_n)_{n=1}^\infty$ and $(\lambda_n)_{n=1}^\infty$ with positive terms. More about the history of the developments related to Hardy-type inequalities is sketched in engaging surveys by Pečarić and Stolarsky [36], Duncan and McGregor [10], and in a relatively recent work by Kufner, Maligranda, and Persson [19].

In a more general setting, for a given mean $\mathcal{M}: \bigcup_{n=1}^\infty I^n \rightarrow I$ (where I is a real interval with $\inf I = 0$), let $\mathcal{H}(\mathcal{M})$ denote the smallest nonnegative extended real number, called the *Hardy constant of* \mathcal{M} , such that

$$\sum_{n=1}^{\infty} \mathcal{M}(x_1, \dots, x_n) \leq \mathcal{H}(\mathcal{M}) \sum_{n=1}^{\infty} x_n$$

for all sequences $(x_n)_{n=1}^\infty$ belonging to I . If $\mathcal{H}(\mathcal{M})$ is finite, then we say that \mathcal{M} is a *Hardy mean*. In this setup, a p th power mean is a Hardy mean if and only if $p \in [-\infty, 1)$ and $\mathcal{H}(\mathcal{P}_p) = C(p)$ for all $p \in [-\infty, \infty]$.

For our investigation of the Hardy property of means, we recall several notions that were partly introduced and used in [34]. Let $I \subseteq \mathbb{R}$ be an interval, and let $\mathcal{M}: \bigcup_{n=1}^\infty I^n \rightarrow I$ be an arbitrary mean. We say that \mathcal{M} is *symmetric*, (*strictly increasing*), and *Jensen convex* (*concave*) if, for all $n \in \mathbb{N}$, the n variable restriction $\mathcal{M}|_{I^n}$ is a symmetric, (*strictly increasing*) in each of its variables, and Jensen convex (*concave*) on I^n , respectively. It is worth mentioning that means are locally bounded functions; therefore, the so-called *Bernstein–Doetsch theorem* (see [1, p. 515]) implies that Jensen convexity (*concavity*) is equivalent to ordinary convexity (*concavity*). If $I = \mathbb{R}_+$, then we can analogously define the notion of homogeneity of \mathcal{M} . Finally, the mean \mathcal{M} is called *repetition-invariant* if, for all $n, m \in \mathbb{N}$ and $(x_1, \dots, x_n) \in I^n$, the following identity is satisfied:

$$\mathcal{M}(\underbrace{x_1, \dots, x_1}_{m\text{-times}}, \dots, \underbrace{x_n, \dots, x_n}_{m\text{-times}}) = \mathcal{M}(x_1, \dots, x_n).$$

With these definitions in hand, let us recall the main theorems of [34, Theorems 3.4, 3.5, Corollary 3.5]. The first result provides a lower estimation of the Hardy constant.

Theorem 1.1. *Let $I \subset \mathbb{R}_+$ be an interval with $\inf I = 0$, and let $\mathcal{M}: \bigcup_{n=1}^\infty I^n \rightarrow I$ be a mean. Then for all nonsummable sequences $(x_n)_{n=1}^\infty$ in I ,*

$$\mathcal{H}(\mathcal{M}) \geq \liminf_{n \rightarrow \infty} x_n^{-1} \cdot \mathcal{M}(x_1, x_2, \dots, x_n).$$

In particular,

$$\mathcal{H}(\mathcal{M}) \geq \sup_{y \in I} \liminf_{n \rightarrow \infty} \frac{n}{y} \cdot \mathcal{M}\left(\frac{y}{1}, \frac{y}{2}, \dots, \frac{y}{n}\right) =: \mathcal{C}(\mathcal{M}).$$

Under stronger assumptions for the mean \mathcal{M} , the lower estimate obtained above becomes an equality by the following result.

Theorem 1.2. *For every increasing, symmetric, repetition-invariant, and Jensen concave mean $\mathcal{M}: \bigcup_{n=1}^\infty \mathbb{R}_+^n \rightarrow \mathbb{R}_+$, the equality $\mathcal{H}(\mathcal{M}) = \mathcal{C}(\mathcal{M})$ holds. If, in addition, \mathcal{M} is also homogeneous, then*

$$\mathcal{H}(\mathcal{M}) = \lim_{n \rightarrow \infty} n \cdot \mathcal{M}\left(1, \frac{1}{2}, \dots, \frac{1}{n}\right).$$

In particular, this limit exists.

Upon taking \mathcal{M} to be a power mean in the above theorem, the Hardy–Landau–Knopp–Carleman inequality (1.1) can easily be deduced. (For the details, see [34].) A deeper look into [34] shows that Theorem 1.2 could be split into two parts with an intermediate state of a so-called *Kedlaya mean*. The mean $\mathcal{M}: \bigcup_{n=1}^\infty I^n \rightarrow I$ (I is an interval) is a *Kedlaya mean* if

$$\begin{aligned} &\mathcal{A}(x_1, \mathcal{M}(x_1, x_2), \dots, \mathcal{M}(x_1, x_2, \dots, x_n)) \\ &\leq \mathcal{M}(x_1, \mathcal{A}(x_1, x_2), \dots, \mathcal{A}(x_1, x_2, \dots, x_n)) \end{aligned} \tag{1.3}$$

for every $n \in \mathbb{N}$ and $x \in I^n$. Here and throughout this article, \mathcal{A} will stand for the standard (or weighted) arithmetic mean.

The motivation for the above terminology came from Kedlaya [15], where he proved that the geometric mean satisfies the inequality above; that is, it is a Kedlaya mean. This result provided an affirmative answer to a conjecture by Holland [14]. A more general theorem has recently been established by the authors.

Theorem 1.3 ([34, Theorem 2.1]). *Every symmetric, Jensen concave, and repetition-invariant mean is a Kedlaya mean.*

Moreover, in the proof of Theorem 1.2 the main tool was the following (nowhere explicitly formulated) statement.

Proposition 1.4. *For every monotone Kedlaya mean $\mathcal{M}: \bigcup_{n=1}^\infty I^n \rightarrow I$ (where I is an interval with $\inf I = 0$), the equality $\mathcal{H}(\mathcal{M}) = \mathcal{C}(\mathcal{M})$ holds.*

Obviously, Theorem 1.3, along with Proposition 1.4, implies the first part of Theorem 1.2. To prove the second part, we need to show that the mentioned limit exists.

Recently, the authors presented an approach to weighted Kedlaya inequalities in [35]. In particular, we established a weighted counterpart of Theorem 1.3 (see the notation of $V(\mathbb{Q})$ and $V(\mathbb{R})$ below). It motivated us to look for a weighted analogue of Proposition 1.4. The result obtained in this direction will be presented in Theorem 4.1.

2. Weighted means

We now introduce the notion of weighted means. First, we stress that there is no broad agreement about the definition of weighted means. The one presented below was introduced in [35] in the process of reverse engineering. The main idea was to cover most of the known weighted means (i.e., power, quasiarithmetic, deviation, and quasideviation means) in the abstract setting. This consideration led us to introduce the following new definition.

Definition 2.1 (Weighted means). Let $I \subset \mathbb{R}$ be an arbitrary interval, let $R \subset \mathbb{R}$ be a ring, and, for $n \in \mathbb{N}$, define the set of n -dimensional weight vectors $W_n(R)$ by

$$W_n(R) := \{(\lambda_1, \dots, \lambda_n) \in R^n \mid \lambda_1, \dots, \lambda_n \geq 0, \lambda_1 + \dots + \lambda_n > 0\}.$$

A *weighted mean on I over R* or, in other words, an *R -weighted mean on I* is a function

$$\mathcal{M}: \bigcup_{n=1}^{\infty} I^n \times W_n(R) \rightarrow I$$

satisfying the conditions (i)–(iv) presented below. Elements belonging to I will be called *entries*; elements from R will be called *weights*.

- (i) *Null homogeneity in the weights*: For all $n \in \mathbb{N}$, for all $(x, \lambda) \in I^n \times W_n(R)$, and $t \in R_+$,

$$\mathcal{M}(x, \lambda) = \mathcal{M}(x, t \cdot \lambda).$$

- (ii) *Reduction principle*: For all $n \in \mathbb{N}$ and for all $x \in I^n$, $\lambda, \mu \in W_n(R)$,

$$\mathcal{M}(x, \lambda + \mu) = \mathcal{M}(x \odot x, \lambda \odot \mu),$$

where \odot is a *shuffle operator*¹ defined as

$$(p_1, \dots, p_n) \odot (q_1, \dots, q_n) := (p_1, q_1, \dots, p_n, q_n).$$

- (iii) *Mean value property*: For all $n \in \mathbb{N}$ and for all $(x, \lambda) \in I^n \times W_n(R)$,

$$\min(x_1, \dots, x_n) \leq \mathcal{M}(x, \lambda) \leq \max(x_1, \dots, x_n).$$

- (iv) *Elimination principle*: For all $n \in \mathbb{N}$, for all $(x, \lambda) \in I^n \times W_n(R)$, and for all $j \in \{1, \dots, n\}$ such that $\lambda_j = 0$,

$$\mathcal{M}(x, \lambda) = \mathcal{M}((x_i)_{i \in \{1, \dots, n\} \setminus \{j\}}, (\lambda_i)_{i \in \{1, \dots, n\} \setminus \{j\}}),$$

that is, entries with a zero weight can be omitted.

¹This definition comes from the theory of computation. Perhaps the most famous (folk) result states that the shuffling of two regular languages is again regular (see, e.g., [2]).

From now on, I is an arbitrary interval and R stands for an arbitrary subring of \mathbb{R} . Let us introduce some natural properties of weighted means. A weighted mean \mathcal{M} is said to be *symmetric* if, for all $n \in \mathbb{N}$, $x \in I^n$, $\lambda \in W_n(R)$, and $\sigma \in S_n$,

$$\mathcal{M}(x, \lambda) = \mathcal{M}(x \circ \sigma, \lambda \circ \sigma).$$

We will call a weighted mean \mathcal{M} *Jensen concave* if, for all $n \in \mathbb{N}$, $x, y \in I^n$, and $\lambda \in W_n(R)$,

$$\mathcal{M}\left(\frac{x+y}{2}, \lambda\right) \geq \frac{1}{2}(\mathcal{M}(x, \lambda) + \mathcal{M}(y, \lambda)). \tag{2.1}$$

A weighted mean \mathcal{M} is said to be *continuous in the weights* if, for all $n \in \mathbb{N}$ and $x \in I^n$, the mapping $\lambda \mapsto \mathcal{M}(x, \lambda)$ is continuous on $W_n(R)$. Finally, a weighted mean \mathcal{M} is *monotone* if, for all $n \in \mathbb{N}$, $x \in I^n$, and $\lambda \in W_n(R)$, the mapping $x_i \mapsto \mathcal{M}(x, \lambda)$ is increasing for all $i \in \{1, \dots, n\}$.

For the sake of convenience, we will use the sum-type abbreviation. If \mathcal{M} is an R -weighted mean on I , $n \in \mathbb{N}$, and $(x, \lambda) \in I^n \times W_n(R)$, then we denote

$$\mathcal{M}_{i=1}^n(x_i, \lambda_i) := \mathcal{M}((x_1, \dots, x_n), (\lambda_1, \dots, \lambda_n)).$$

Let us recall some basic properties of weighted means defined in this way. The first result binds nonweighted, repetition-invariant means and \mathbb{Z} -weighted means.

Theorem 2.2 ([35, Theorem 2.3]). *If \mathcal{M} is a repetition-invariant mean on I , then the formula*

$$\tilde{\mathcal{M}}((x_1, \dots, x_n), (\lambda_1, \dots, \lambda_n)) := \mathcal{M}(\underbrace{x_1, \dots, x_1}_{\lambda_1 \text{ entries}}, \dots, \underbrace{x_n, \dots, x_n}_{\lambda_n \text{ entries}}) \tag{2.2}$$

defines a weighted mean on I over \mathbb{Z} . Conversely, if $\tilde{\mathcal{M}}$ is a \mathbb{Z} -weighted mean on I , then

$$\mathcal{M}(x_1, \dots, x_n) := \tilde{\mathcal{M}}((x_1, \dots, x_n), \underbrace{(1, \dots, 1)}_{n \text{ entries}}) \tag{2.3}$$

is a repetition-invariant mean on I . Furthermore, these transformations are inverses of each other.

Moreover, the following two easy statements were explicitly worded.

Theorem 2.3. *If \mathcal{M} is a symmetric repetition-invariant mean on I , then the function $\tilde{\mathcal{M}}$ defined by the formula (2.2) is a symmetric weighted mean on I over \mathbb{Z} . Conversely, if $\tilde{\mathcal{M}}$ is a symmetric \mathbb{Z} -weighted mean on I , then the function \mathcal{M} defined by (2.3) is a symmetric repetition-invariant mean on I .*

Theorem 2.4. *If \mathcal{M} is a Jensen concave repetition-invariant mean on I , then the function $\tilde{\mathcal{M}}$ defined by the formula (2.2) is a Jensen concave weighted mean on I over \mathbb{Z} . Conversely, if $\tilde{\mathcal{M}}$ is a Jensen concave \mathbb{Z} -weighted mean on I , then the function \mathcal{M} defined by (2.3) is a Jensen concave repetition-invariant mean on I .*

We will also need an extension theorem from [35, Theorem 2.2].

Theorem 2.5. *Let I be an interval, let $R \subset \mathbb{R}$ be a ring, and let \mathcal{M} be a weighted mean defined on I over R . Then there exists a unique mean $\tilde{\mathcal{M}}$ defined on I over R^* (which denotes the quotient field of R) such that*

$$\tilde{\mathcal{M}}|_{\bigcup_{n=1}^{\infty} I^n \times W_n(R)} = \mathcal{M}.$$

Moreover, if \mathcal{M} is symmetric/monotone, then so is $\tilde{\mathcal{M}}$.

With this, we can extend means defined in Theorem 2.2 to the field \mathbb{Q} . Let us recall that the weighted power mean $\mathcal{P}_p: \bigcup_{n=1}^{\infty} \mathbb{R}_+^n \times W_n(\mathbb{R}) \rightarrow \mathbb{R}_+$ is defined for all $p \in \mathbb{R}$ by

$$\mathcal{P}_p(x, \lambda) := \begin{cases} \left(\frac{\lambda_1 x_1^p + \lambda_2 x_2^p + \dots + \lambda_n x_n^p}{\lambda_1 + \lambda_2 + \dots + \lambda_n}\right)^{1/p} & \text{if } p \neq 0, \\ (x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n})^{1/(\lambda_1 + \lambda_2 + \dots + \lambda_n)} & \text{if } p = 0, \end{cases}$$

and admits all properties (i)–(iv).

In a more general setting, in the spirit of [13], we can define weighted quasiarithmetic means as follows. If I is an arbitrary interval and $f: I \rightarrow \mathbb{R}$ is continuous and monotone, then the weighted quasiarithmetic mean $\mathcal{A}_f: \bigcup_{n=1}^{\infty} I^n \times W_n(\mathbb{R}) \rightarrow I$ is a function such that for all $n \in \mathbb{N}$ and a pair $x \in I^n$ with weights $\lambda \in W_n(\mathbb{R})$,

$$\mathcal{A}_f(x, \lambda) := f^{-1}\left(\frac{\lambda_1 f(x_1) + \lambda_2 f(x_2) + \dots + \lambda_n f(x_n)}{\lambda_1 + \lambda_2 + \dots + \lambda_n}\right).$$

This sequence of generalization could be continued to Bajraktarević means, to deviation (Daróczy) means, and to quasideviation means. Investigating these families, however, lies outside the scope of this paper and we refer the reader to a series of papers by Losonczi [21]–[26] (for Bajraktarević means); by Daróczy [5], [6], Daróczy and Losonczi [7], and Daróczy and Páles [8], [9] (for deviation means); and by Páles [27]–[33] (for deviation and quasideviation means).

2.1. Weighted Kedlaya property. As in [35], we introduce the notion of the weighted Kedlaya inequality. To have a weighted counterpart of the Kedlaya inequality, we have to take weight sequences λ from R with a positive first member. Therefore, for a given ring R , we define

$$W^0(R) := \{\lambda \in R^{\mathbb{N}} \mid \lambda_1 > 0, \lambda_2, \lambda_3, \dots \geq 0\}.$$

For a weight sequence $\lambda \in W^0(R)$, we say that a weighted mean $\mathcal{M}: \bigcup_{n=1}^{\infty} I^n \times W_n(R) \rightarrow I$ satisfies the λ -weighted Kedlaya inequality (or λ -Kedlaya inequality for short) if

$$\mathcal{A}_{\mathcal{M}}\left(\mathcal{M}(x_i, \lambda_i), \lambda_k\right) \leq \mathcal{M}\left(\mathcal{A}(x_i, \lambda_i), \lambda_k\right) \quad (n \in \mathbb{N}, x \in I^n).$$

In fact, the nonincreasingness of the ratio sequence $\left(\frac{\lambda_i}{\lambda_1 + \dots + \lambda_i}\right)$ will be a key assumption for Kedlaya-type inequalities; therefore, we also set

$$V(R) := \left\{ \lambda \in W^0(R) \mid \left(\frac{\lambda_i}{\lambda_1 + \dots + \lambda_i}\right)_{i=1}^{\infty} \text{ is nonincreasing} \right\}.$$

Actually, in 1999 Kedlaya [16] proved that the geometric mean satisfies the λ -weighted Kedlaya inequality for all $\lambda \in V(\mathbb{R})$. This result has been generalized recently by the authors [35, Corollary 3.2, Proposition 3.7] to the family of symmetric, Jensen concave means. More precisely, the following theorem has been established.

Theorem 2.6. *Every symmetric and Jensen concave \mathbb{Q} -weighted mean (resp., \mathbb{R} -weighted mean which is continuous in the weights) satisfies the λ -weighted Kedlaya inequality for all $\lambda \in V(\mathbb{Q})$ (resp., $\lambda \in V(\mathbb{R})$).*

In fact, we will sometimes assume that a mean is a λ -Kedlaya mean, and the above theorem delivers us a sufficient condition (cf. Theorem 4.1 and related Corollaries 4.2, 4.3).

2.2. Weighted Hardy property. As in [34], the Kedlaya inequality leads us to the Hardy property (with an optimal constant). Nevertheless, to take advantage of the weighted Kedlaya inequality in dealing with the Hardy property, we need to define its weighted counterpart. Such a definition is a natural extension of the nonweighted setup.

Definition 2.7 (Weighted Hardy property). Let I be an interval with $\inf I = 0$, and let $R \subset \mathbb{R}$ be a ring. For an R -weighted mean \mathcal{M} on I and weights $\lambda \in W^0(R)$, let C be the smallest extended real number such that, for all sequences (x_n) in I ,

$$\sum_{n=1}^{\infty} \lambda_n \cdot \mathcal{M}(x_i, \lambda_i) \leq C \cdot \sum_{n=1}^{\infty} \lambda_n x_n.$$

We call C the λ -weighted Hardy constant of \mathcal{M} (or λ -Hardy constant of \mathcal{M} for short) and denote it by $\mathcal{H}_\lambda(\mathcal{M})$. Whenever this constant is finite, then \mathcal{M} is called a λ -weighted Hardy mean or simply a λ -Hardy mean.

This definition is an extension of the Hardy constant (and consequently, the Hardy property). Indeed, by null homogeneity in the weights, we may assume without loss of generality that $1 \in R$, whenever it is useful. In this setup for $\mathbf{1} := (1, 1, 1, \dots)$, by Theorem 2.2, the R -weighted mean $\tilde{\mathcal{M}}$ with weights $\mathbf{1}$ could be associated with the nonweighted mean \mathcal{M} , and (in the setting of this theorem) the equality

$$\mathcal{H}_1(\tilde{\mathcal{M}}) = \mathcal{H}(\mathcal{M}).$$

is valid. A question naturally arises: What is a relation between being λ -Hardy and $\mathbf{1}$ -Hardy? Luckily, we have a simple (in its wording) property which generalizes the result (1.2) of Elliott [11] and Copson [4].

Theorem 2.8. *For every symmetric and monotone mean \mathcal{M} on I over R , we have*

$$\mathcal{H}_1(\mathcal{M}) = \sup_{\lambda \in W^0(R)} \mathcal{H}_\lambda(\mathcal{M}).$$

The technical and quite lengthy proof of this theorem is deferred to the last section. As an immediate consequence, we obtain the following.

Corollary 2.9. *Let $\tilde{\mathcal{M}}$ be a symmetric and monotone mean on I over R . Then the following conditions are equivalent:*

- (i) $\tilde{\mathcal{M}}$ is a λ -Hardy mean for all $\lambda \in W^0(R)$;
- (ii) $\tilde{\mathcal{M}}$ is a $\mathbf{1}$ -Hardy mean;
- (iii) \mathcal{M} defined by (2.3) is a Hardy mean.

3. Auxiliary results

In this section, we prove a number of technical lemmas which will be useful in the forthcoming sections. We first establish a purely analytic fact. This is then followed by results that are directly related to the weighted Hardy property. Throughout this section, let $\lambda \in W^0(\mathbb{R})$, and set $\Lambda_n := \lambda_1 + \cdots + \lambda_n$ for $n \in \mathbb{N}$.

Lemma 3.1. *The series $\sum \lambda_n$ and $\sum \lambda_n/\Lambda_n$ are equiconvergent (either both convergent or both divergent).*

Indeed, if $\sum_{n=1}^{\infty} \lambda_n < \infty$, then $\sum_{n=1}^{\infty} \lambda_n/\Lambda_n \leq \sum_{n=1}^{\infty} \lambda_n/\Lambda_1 < \infty$. The reverse implication is due to Abel [18, p. 125].

Now we turn to results directly related to means. The first two statements are about properties of the Hardy constant, while the last one is a sort of rearranging property of a weighted mean in the case of a nonincreasing function. The following lemma is somehow related to the so-called *Hardy sequence* (cf. [34, Proposition 3.1]).

Lemma 3.2. *Let \mathcal{M} be an R -weighted mean on I . Then for all $n \in \mathbb{N}$ and $x \in I^n$,*

$$\sum_{i=1}^n \lambda_i \cdot \mathcal{M}(x_j, \lambda_j) \leq \mathcal{H}_\lambda(\mathcal{M}) \sum_{i=1}^n \lambda_i x_i. \quad (3.1)$$

Proof. Take $\varepsilon \in I$ and $x_m := \min(\varepsilon/(\lambda_m 2^m), \varepsilon)$ for $m > n$. Then we have

$$\begin{aligned} \sum_{i=1}^n \lambda_i \cdot \mathcal{M}(x_j, \lambda_j) &\leq \sum_{i=1}^{\infty} \lambda_i \cdot \mathcal{M}(x_j, \lambda_j) \leq \mathcal{H}_\lambda(\mathcal{M}) \sum_{i=1}^{\infty} \lambda_i x_i \\ &\leq \mathcal{H}_\lambda(\mathcal{M}) \left(\sum_{i=1}^n \lambda_i x_i + \sum_{i=n+1}^{\infty} \frac{\varepsilon}{2^i} \right) \leq \mathcal{H}_\lambda(\mathcal{M}) \left(\varepsilon + \sum_{i=1}^n \lambda_i x_i \right). \end{aligned}$$

Now we can pass the limit $\varepsilon \rightarrow 0$ to obtain (3.1). \square

With this already proved, we can present a weighted analogue of [34, Theorem 3.3]. Stolz's theorem (see [37, pp. 173–175]) allows for a significantly shortened proof.

Lemma 3.3. *Let \mathcal{M} be an R -weighted mean on I . If $\sum_{n=1}^{\infty} \lambda_n x_n = \infty$, then*

$$\mathcal{H}_\lambda(\mathcal{M}) \geq \liminf_{n \rightarrow \infty} \frac{1}{x_n} \mathcal{M}(x_i, \lambda_i).$$

Proof. By Stolz's theorem and Lemma 3.2, we have

$$\begin{aligned} \mathcal{H}_\lambda(\mathcal{M}) &\geq \liminf_{N \rightarrow \infty, \lambda_N > 0} \frac{\sum_{n=1}^N \lambda_n \cdot \mathcal{M}_{i=1}^n(x_i, \lambda_i)}{\sum_{n=1}^N \lambda_n x_n} \\ &\geq \liminf_{n \rightarrow \infty, \lambda_n > 0} \frac{\lambda_n \mathcal{M}_{i=1}^n(x_i, \lambda_i)}{\lambda_n x_n} \geq \liminf_{n \rightarrow \infty} \frac{\mathcal{M}_{i=1}^n(x_i, \lambda_i)}{x_n}, \end{aligned}$$

which was to be shown. □

4. Main results

In this section, we will prove an important relation between the λ -Kedlaya and λ -Hardy properties. With this in hand, we will use the notation of $V(\mathbb{R})$ and $V(\mathbb{Q})$ to present a handy characterization of the λ -Hardy property. In fact, a lot of statements will depend on the summability of the weight sequence (λ_n) .

Theorem 4.1. *Let \mathcal{M} be an R -weighted mean on I , and let $\lambda \in W^0(R)$. Define*

$$\mathcal{C}_\lambda(\mathcal{M}) := \sup_{y > 0} \liminf_{n \rightarrow \infty} \frac{\lambda_1 + \lambda_2 + \dots + \lambda_n}{y} \cdot \mathcal{M}_{k=1}^n\left(\frac{y}{\lambda_1 + \lambda_2 + \dots + \lambda_k}, \lambda_k\right).$$

- (i) *If $\sum_{n=1}^\infty \lambda_n = \infty$, then $\mathcal{H}_\lambda(\mathcal{M}) \geq \mathcal{C}_\lambda(\mathcal{M})$.*
- (ii) *If \mathcal{M} is monotone and satisfies the λ -Kedlaya inequality, then $\mathcal{H}_\lambda(\mathcal{M}) \leq \mathcal{C}_\lambda(\mathcal{M})$.*

Proof. Denote the partial sum of $\lambda_1 + \dots + \lambda_k$ by Λ_k . In the first part, Lemma 3.1 implies that

$$\sum_{n=1}^\infty \lambda_n \cdot \frac{y}{\Lambda_n} = \infty \quad \text{for all } y > 0.$$

Consequently, by Lemma 3.3,

$$\mathcal{H}_\lambda(\mathcal{M}) \geq \liminf_{n \rightarrow \infty} \frac{\Lambda_n}{y} \mathcal{M}_{k=1}^n\left(\frac{y}{\Lambda_k}, \lambda_k\right) \quad \text{for all } y > 0.$$

Finally, we can take the supremum over all positive y 's and obtain $\mathcal{H}_\lambda(\mathcal{M}) \geq \mathcal{C}_\lambda(\mathcal{M})$.

To prove part (ii), let $x \in \ell^1(\lambda)$ be a sequence of positive numbers, and let $y_0 := \sum_{n=1}^\infty \lambda_n x_n$. Then

$$m_k = \frac{1}{\Lambda_k} \sum_{i=1}^k \lambda_i \cdot x_i \leq \frac{y_0}{\Lambda_k}, \quad k \in \mathbb{N}.$$

The (n, λ) -Kedlaya inequality applied to the vector (x_1, x_2, \dots, x_n) and the monotonicity of \mathcal{M} imply that

$$\sum_{k=1}^n \lambda_k \cdot \mathcal{M}_{i=1}^k(x_i, \lambda_i) \leq \Lambda_n \cdot \mathcal{M}_{k=1}^n(m_k, \lambda_k) \leq \Lambda_n \cdot \mathcal{M}_{k=1}^n\left(\frac{y_0}{\Lambda_k}, \lambda_k\right).$$

Upon taking the \liminf as n tends to ∞ , we obtain

$$\sum_{k=1}^{\infty} \lambda_k \cdot \mathcal{M}_{i=1}^k(x_i, \lambda_i) \leq \left(\liminf_{n \rightarrow \infty} \frac{\Lambda_n}{y_0} \cdot \mathcal{M}_{k=1}^n\left(\frac{y_0}{\Lambda_k}, \lambda_k\right) \right) \cdot y_0 \leq \mathcal{C}_\lambda(\mathcal{M}) \sum_{n=1}^{\infty} \lambda_n x_n.$$

Therefore, the desired inequality $\mathcal{H}_\lambda(\mathcal{M}) \leq \mathcal{C}_\lambda(\mathcal{M})$ follows. □

At this moment, using Theorem 2.6, we obtain two direct consequences of Theorem 4.1.

Corollary 4.2. *Let \mathcal{M} be a symmetric, monotone, and Jensen-concave \mathbb{Q} -weighted mean, and let $\lambda \in V(\mathbb{Q})$. Then $\mathcal{H}_\lambda(\mathcal{M}) \leq \mathcal{C}_\lambda(\mathcal{M})$. Furthermore, if $\sum_{n=1}^{\infty} \lambda_n = \infty$, then $\mathcal{H}_\lambda(\mathcal{M}) = \mathcal{C}_\lambda(\mathcal{M})$.*

Corollary 4.3. *Let \mathcal{M} be a symmetric, monotone, and Jensen-concave \mathbb{R} -weighted mean which is continuous in the weights, and let $\lambda \in V(\mathbb{R})$. Then $\mathcal{H}_\lambda(\mathcal{M}) \leq \mathcal{C}_\lambda(\mathcal{M})$. Furthermore, if $\sum_{n=1}^{\infty} \lambda_n = \infty$, then $\mathcal{H}_\lambda(\mathcal{M}) = \mathcal{C}_\lambda(\mathcal{M})$.*

We can also apply this theorem to justify the λ -Hardy property.

Corollary 4.4. *Let \mathcal{M} be an R -weighted mean on I , and let $\lambda \in W^0(R)$.*

- (i) *If $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\mathcal{C}_\lambda(\mathcal{M}) = \infty$, then \mathcal{M} is not a λ -Hardy mean.*
- (ii) *If \mathcal{M} is a monotone mean which satisfies the λ -Kedlaya inequality and $\mathcal{C}_\lambda(\mathcal{M})$ is finite, then \mathcal{M} is a λ -Hardy mean.*

5. Proof of Theorem 2.8

Let us mention some further definitions and notation from [35]. Instead of explicitly writing down weights, we can consider a function with finite range as the argument of the given mean. Let R be a subring of \mathbb{R} . We will denote its quotient field (the smallest field generated by R) by R^* . We say that $D \subseteq \mathbb{R}$ is an R -interval if D is of the form $[a, b)$, where $a, b \in R$.

Given an R -interval $D = [a, b)$, a function $f: D \rightarrow I$ is called R -simple if there exist $n \in \mathbb{N}$ and a partition of D into R -intervals $\{D_i\}_{i=1}^n$ such that $\sup D_i = \inf D_{i+1}$ for $i \in \{1, \dots, n-1\}$ and f is constant on each subinterval D_i . Then for an R -weighted mean \mathcal{M} on I , we define

$$\mathcal{M} \int_a^b f(x) dx := \mathcal{M}_{i=1}^n(f|_{D_i}, |D_i|) = \mathcal{M}((f|_{D_1}, \dots, f|_{D_n}), (|D_1|, \dots, |D_n|)).$$

In this setting, \mathcal{M} is symmetric if and only if for every pair of R -simple functions $f, g: D \rightarrow I$ which have the same distribution, the equality $\mathcal{M} \int f(x) dx = \mathcal{M} \int g(x) dx$ holds. Similarly, \mathcal{M} is monotone if and only if for every pair of R -simple functions $f, g: D \rightarrow I$ with $f \leq g$, the inequality $\mathcal{M} \int f(x) dx \leq \mathcal{M} \int g(x) dx$ is valid. Furthermore, for an R -interval $[p, q) \subset D$ and function f like above, we will keep all integral-type convolutions. For instance,

$$\mathcal{M} \int_{[p,q)} f(x) dx = \mathcal{M} \int_p^q f(x) dx.$$

Let us now present some simple results related to decreasing functions.

Lemma 5.1. *Let \mathcal{M} be a R^* -weighted, monotone mean on I , and let $a \in R^* \cap (0, \infty)$. Then for any nonincreasing R^* -simple function $f: [0, a] \rightarrow I$, the mapping $F: R^* \cap (0, a] \rightarrow I$ given by $F(u) := \mathcal{M}_0^u f(t) dt$ is nonincreasing.*

Proof. Fix $p, q \in R^* \cap (0, a]$ with $q < p$. As f is decreasing, we know that $f(\frac{p}{q}t) \leq f(t)$ for all $t \in [0, q]$. Therefore, by null homogeneity in the weights and monotonicity,

$$F(p) = \mathcal{M}_0^p f(t) dt = \mathcal{M}_0^q f\left(\frac{p}{q} \cdot t\right) dt \leq \mathcal{M}_0^q f(t) dt = F(q),$$

which was to be proved. □

By Theorem 2.5, we know that \mathcal{M} has a unique extension to an R^* -weighted mean on I . As the weights are fixed (and belong to R), one can assume without loss of generality that we are dealing with a weight sequence from R^* . Consequently, as it is handy, \mathcal{M} is an R^* -weighted mean.

To verify Theorem 2.8, it suffices to prove that, for all $N \in \mathbb{N}$, $\lambda \in W^0(R^*)$ and $x \in I^N$, there holds

$$\sum_{n=1}^N \lambda_n \mathcal{M}_n(x_i, \lambda_i) \leq \mathcal{H}_1(\mathcal{M}) \sum_{n=1}^N \lambda_n x_n. \tag{5.1}$$

Indeed, if we pass the limit $N \rightarrow \infty$, this inequality would imply that $\mathcal{H}_\lambda(\mathcal{M}) \leq \mathcal{H}_1(\mathcal{M})$. This proof is split into two parts. In fact, each part can be formulated as a separate lemma.

Lemma 5.2. *Let \mathcal{M} be a monotone R^* -weighted mean on I . Then for all $N \in \mathbb{N}$ and all nonincreasing sequences $x \in I^N$ and weights $\lambda \in W_N^0(R^*)$, the inequality (5.1) is valid.*

Lemma 5.3. *Let \mathcal{M} be a symmetric and monotone R -weighted mean on I . Then for all $N \in \mathbb{N}$ and all vectors $x \in I^N$ and weights $\lambda \in W_N^0(R)$, there exist $M \in \mathbb{N}$, a nonincreasing sequence $y \in I^M$, and a weight sequence $\psi \in W_M^0(R)$ such that*

$$\sum_{n=1}^N \lambda_n x_n = \sum_{m=1}^M \psi_m y_m$$

and

$$\sum_{n=1}^N \lambda_n \mathcal{M}_n(x_i, \lambda_i) \leq \sum_{m=1}^M \psi_m \mathcal{M}_m(y_i, \psi_i). \tag{5.2}$$

Moreover,

$$\sum_{\{n: x_n=t\}} \lambda_n = \sum_{\{m: y_m=t\}} \psi_m \quad \text{for all } t \in \mathbb{R}. \tag{5.3}$$

We emphasize that the fact that the sum of the λ 's and ψ 's is equal is not used in the proof of main theorem; however, it could be potentially useful in another setting.

Having these two lemmas, for a given sequence $x = (x_1, x_2, \dots)$ with weights $\lambda \in W^0(R^*)$ and $N \in \mathbb{N}$, we can apply Lemma 5.3 and then Lemma 5.2 to a vector $y \in I^M$ with corresponding weights ψ to obtain

$$\sum_{n=1}^N \lambda_n \mathcal{M}_{i=1}^n(x_i, \lambda_i) \leq \sum_{m=1}^M \psi_m \mathcal{M}_{i=1}^m(y_i, \psi_i) \leq \mathcal{H}_1(\mathcal{M}) \sum_{n=1}^M \psi_n y_n = \mathcal{H}_1(\mathcal{M}) \sum_{n=1}^N \lambda_n x_n.$$

Then, if we pass the limit $N \rightarrow \infty$, we get

$$\sum_{n=1}^{\infty} \lambda_n \mathcal{M}_{i=1}^n(x_i, \lambda_i) \leq \mathcal{H}_1(\mathcal{M}) \sum_{n=1}^{\infty} \lambda_n x_n \quad (\lambda \in W^0(R^*)),$$

which obviously implies that $\mathcal{H}_\lambda(\mathcal{M}) \leq \mathcal{H}_1(\mathcal{M})$. As $\mathbf{1} \in W^0(R)$, the equality in Theorem 2.8 easily follows.

In order to make the proofs more compact, define $\Lambda_n := \lambda_1 + \dots + \lambda_n$ for $n \in \{1, \dots, N\}$. In view of the null homogeneity of the mean \mathcal{M} , we may also assume that $\Lambda_N = 1$. Now, define the function $f: [0, 1) \rightarrow \mathbb{R}$ as

$$f|_{[\Lambda_{n-1}, \Lambda_n)} := x_n, \quad n \in \{1, \dots, N\}.$$

Then we have that

$$\mathcal{M}_0^{\Lambda_n} f(x) dx = \mathcal{M}_{i=1}^n(x_i, \lambda_i), \quad n \in \{1, \dots, N\}.$$

Proof of Lemma 5.2. First observe that if $\mathcal{H}_1(\mathcal{M}) = \infty$, then this lemma is trivial. From now on, suppose that $\mathcal{H}_1(\mathcal{M})$ is finite. Define, for $j \in \mathbb{N}$, the function $f_j: [0, 1) \rightarrow I$ by

$$f_j|_{[n/j, (n+1)/j)} := f\left(\frac{n}{j}\right) \quad \text{for all } n \in \{0, \dots, j-1\}.$$

As the sequence x is nonincreasing, thus f is nonincreasing too. Therefore, $f \leq f_j$ and f_j is nonincreasing for every $j \in \mathbb{N}$. Thus, by Lemma 5.1, also nonincreasing is the function $C_j: [0, 1) \rightarrow I$ given by

$$C_j(t) := \begin{cases} \inf_{\substack{s \leq t \\ s \in R^*}} \mathcal{M}_0^s f_j(x) dx & \text{if } t \in (0, 1), \\ x_1 & \text{if } t = 0 \end{cases} \quad (j \in \mathbb{N}).$$

As C_j is monotonic, it is also Riemann integrable. Using these properties, we get

$$\begin{aligned} \lambda_n \cdot \mathcal{M}_{i=1}^n(x_i, \lambda_i) &= \lambda_n \cdot \mathcal{M}_0^{\Lambda_n} f(x) dx \leq \lambda_n \cdot \mathcal{M}_0^{\Lambda_n} f_j(x) dx \\ &= \lambda_n \cdot C_j(\Lambda_n) = \int_{\Lambda_{n-1}}^{\Lambda_n} C_j(\Lambda_n) dx \leq \int_{\Lambda_{n-1}}^{\Lambda_n} C_j(x) dx. \end{aligned}$$

Therefore, for all $j \in \mathbb{N}$,

$$\sum_{n=1}^N \lambda_n \cdot \mathcal{M}_{i=1}^n(x_i, \lambda_i) \leq \int_0^1 C_j(x) dx. \quad (5.4)$$

We now majorize the right-hand side of this inequality. Observe first that for all $j \in \mathbb{N}$,

$$\int_0^{\frac{1}{j}} C_j(x) dx \leq \frac{1}{j} \cdot C_j(0) = \frac{x_1}{j}. \tag{5.5}$$

Furthermore, for all $j, n \in \mathbb{N}$ such that $n < j$,

$$\int_{\frac{n}{j}}^{\frac{n+1}{j}} C_j(x) dx \leq \frac{1}{j} \cdot C_j\left(\frac{n}{j}\right) = \frac{1}{j} \cdot \mathcal{M}_{\frac{n}{j}} f_j(x) dx = \frac{1}{j} \cdot \mathcal{M}\left(f_j\left(\frac{i}{j}\right), 1\right). \tag{5.6}$$

If we now sum up (5.5) and (5.6) for all $n \in \{1, \dots, j - 1\}$, then we obtain, for all $j \geq 2$,

$$\int_0^1 C_j(x) dx \leq \frac{1}{j} \left(x_1 + \sum_{n=1}^{j-1} \mathcal{M}\left(f_j\left(\frac{i}{j}\right), 1\right) \right). \tag{5.7}$$

However, \mathcal{M} is a **1**-weighted Hardy mean. In this setting, by [34, Proposition 3.1], we have that the finite estimation announced in the definition of Hardy constant remains valid for finite sequences too. That is,

$$\sum_{n=1}^{j-1} \mathcal{M}\left(f_j\left(\frac{i}{j}\right), 1\right) \leq \mathcal{H}_1(\mathcal{M}) \cdot \sum_{n=0}^{j-1} f_j\left(\frac{n}{j}\right) \quad (j \geq 2). \tag{5.8}$$

Moreover, as f is nonincreasing, we have

$$\frac{1}{j} \sum_{n=0}^{j-1} f_j\left(\frac{n}{j}\right) = \frac{1}{j} \sum_{n=0}^{j-1} f\left(\frac{n}{j}\right) \leq \frac{x_1}{j} + \int_0^1 f(x) dx = \frac{x_1}{j} + \sum_{n=1}^N \lambda_n x_n \quad (j \geq 2). \tag{5.9}$$

Now combining (5.4), (5.7), (5.8), and (5.9), for $j \geq 2$, we obtain

$$\begin{aligned} \sum_{n=1}^N \lambda_n \cdot \mathcal{M}_{\frac{n}{j}}(x_i, \lambda_i) &\leq \int_0^1 C_j(x) dx \leq \frac{1}{j} \left(x_1 + \sum_{n=1}^{j-1} \mathcal{M}\left(f_j\left(\frac{i}{j}\right), 1\right) \right) \\ &\leq \frac{1}{j} \left(x_1 + \mathcal{H}_1(\mathcal{M}) \sum_{n=0}^{j-1} f_j\left(\frac{n}{j}\right) \right) \\ &\leq \frac{(1 + \mathcal{H}_1(\mathcal{M}))x_1}{j} + \mathcal{H}_1(\mathcal{M}) \sum_{n=1}^N \lambda_n x_n. \end{aligned}$$

Finally, as $j \rightarrow \infty$, we get (5.1). □

Now we turn to the proof of Lemma 5.3. Let us stress that in this lemma, the assumptions for the mean \mathcal{M} are more restrictive. More precisely, we assume \mathcal{M} to be not only monotone but also symmetric. On the other hand, we need \mathcal{M} to be R -weighted instead of R^* -weighted only. However, in view of Theorem 2.5, this difference is rather a technical one.

Proof of Lemma 5.3. Throughout this proof, let us denote by g^* the right continuous nonincreasing rearrangement of an R -simple function $g: D \rightarrow \mathbb{R}$. It is easy to observe that g^* is again R -simple.

Without loss of generality, we may assume that the members of the sequence λ are positive. Consider a strictly increasing sequence $(\Psi_m)_{m=0}^M \in R^{M+1}$ such that $\Psi_0 = 0$, $\Psi_M = \Lambda_N$, $(\Lambda_n)_{n=0}^N$ is a subsequence of $(\Psi_m)_{m=0}^M$, and f^* is constant on all intervals $[\Psi_{m-1}, \Psi_m)$, where $m \in \{1, 2, \dots, M\}$.

Set $\psi_m := \Psi_m - \Psi_{m-1}$ and y_m to be the value of f^* on $[\Psi_{m-1}, \Psi_m)$; $m \in \{1, \dots, M\}$. Furthermore, for every $n \in \{0, \dots, N\}$ there exists a unique $i_n \in \{0, \dots, M\}$ such that $\Psi_{i_n} = \Lambda_n$. As $\Psi_M = \Lambda_N$, we obtain $i_N = M$; furthermore, by $\Lambda_0 = 0 = \Psi_0$, we get $i_0 = 0$.

Obviously (y_m) is nonincreasing, $\sum_{n=1}^N \lambda_n = \Lambda_N = \Psi_M = \sum_{m=1}^M \psi_m$, and

$$\sum_{n=1}^N \lambda_n x_n = \int_0^{\Lambda_N} f(x) dx = \int_0^{\Psi_M} f(x) dx = \int_0^{\Psi_M} f^*(x) dx = \sum_{m=1}^M \psi_m y_m.$$

Therefore, the only property which remains to be proved is (5.2). One can easily see that

$$(f|_{[0,u]})^*(x) \leq f^*(x), \quad x \in [0, u), u \in R \cap [0, \Lambda_N).$$

Thus, by the monotonicity of \mathcal{M} ,

$$\mathcal{M}_0^u(f|_{[0,u]})^*(x) dx \leq \mathcal{M}_0^u f^*(x) dx, \quad u \in R \cap [0, \Lambda_N).$$

But, by the definition, $(f|_{[0,u]})^*$ and $f|_{[0,u]}$ have the same distribution. Whence, applying the symmetry of \mathcal{M} , we arrive at

$$\mathcal{M}_0^u f(x) dx \leq \mathcal{M}_0^u f^*(x) dx, \quad u \in R \cap [0, \Lambda_N).$$

Therefore, applying this inequality for $u = \Lambda_n$, we obtain

$$\sum_{n=1}^N \lambda_n \mathcal{M}_{i=1}^n(x_i, \lambda_i) = \sum_{n=1}^N \lambda_n \mathcal{M}_0^{\Lambda_n} f(x) dx \leq \sum_{n=1}^N \lambda_n \mathcal{M}_0^{\Lambda_n} f^*(x) dx. \quad (5.10)$$

We now note that

$$\lambda_n = \Lambda_n - \Lambda_{n-1} = \Psi_{i_n} - \Psi_{i_{n-1}} = \sum_{m=i_{n-1}+1}^{i_n} (\Psi_m - \Psi_{m-1}).$$

Therefore, by Lemma 5.1, the definition of (i_n) , and the identity above, we obtain

$$\begin{aligned} \sum_{n=1}^N \lambda_n \cdot \mathcal{M}_0^{\Lambda_n} f^*(x) dx &= \sum_{n=1}^N \left(\sum_{m=i_{n-1}+1}^{i_n} (\Psi_m - \Psi_{m-1}) \right) \mathcal{M}_0^{\Psi_{i_n}} f^*(x) dx \\ &\leq \sum_{n=1}^N \sum_{m=i_{n-1}+1}^{i_n} (\Psi_m - \Psi_{m-1}) \mathcal{M}_0^{\Psi_m} f^*(x) dx \\ &= \sum_{m=1}^M (\Psi_m - \Psi_{m-1}) \mathcal{M}_0^{\Psi_m} f^*(x) dx = \sum_{m=1}^M \psi_m \cdot \mathcal{M}_{i=1}^m(y_i, \psi_i). \end{aligned}$$

But this inequality combined with (5.10) is exactly what (5.2) states. As this was the only remaining property to be verified, the proof is complete. \square

Acknowledgments. The first author's work was partially supported by Hungarian Scientific Research Fund (OTKA) grant K-111651 and by the EFOP-3.6.2-16-2017-00015 project. This project is co-financed by the European Union and the European Social Fund.

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