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BOUNDEDNESS OF HAUSDORFF OPERATORS ON HARDY SPACES IN THE HEISENBERG GROUP

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Abstract. In the setting of the Heisenberg group, we define weighted Hardy spaces by means of their atomic characterization, and we establish the (sharp) boundedness of Hausdorff operators on power-weighted Hardy spaces. Moreover, we obtain sufficient and necessary conditions for the boundedness of Hausdorff operators on local Hardy spaces in the Heisenberg group.

1. Introduction

The appearance of Hausdorff operators can be traced back to the solution of certain classical problems in analysis. Extensive modern-day research of general Hausdorff operators was started by Galanopoulos and Siskakis [10] in the complex analysis setting, and Liflyand and Móricz [24] in the Fourier transform setting. We refer the reader to a survey paper by Liflyand [22] for more information about the background and development of Hausdorff operators.

Let Φ be an integrable function on $\mathbb{R}^+$. The 1-dimensional Hausdorff operator is defined by

$$h_\Phi(f)(x) = \int_0^\infty \frac{\Phi(t)}{t} f\left(\frac{x}{t}\right) dt, \quad x \in \mathbb{R}.$$

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It is well known that many classical operators in harmonic analysis are special cases of Hausdorff operators if one chooses a suitable Φ. Along with Hardy operators, Hilbert operators, and Hardy–Littlewood–Pólya operators (see [1], [8], [33], [34]), Riemann–Liouville and Weyl fractional integrals can also be derived from 1-dimensional Hausdorff operators (see [36]). In particular, if we consider $x \in \mathbb{R}^n$ in h_Φ directly, it is easy to check that we can derive from h_Φ the weighted Hardy operator and its adjoint operator by choosing appropriate functions Φ, which have been widely studied in recent years (see [9], [35], and the references therein).

A natural n-dimensional version of a Hausdorff operator h_Φ on $\mathbb{R}^n$ is defined by

$$H_\Phi(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f\Big(\frac{x}{|y|}\Big)\,dy.$$

More generally, Lerner and Liflyand [20] studied the operator $H_{\Phi,A}$ for the "good" $n \times n$ matrix functions $A(\cdot)$:

$$H_{\Phi,A}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f\big(A(y)x\big)\,dy.$$

It is easy to verify that if

$$A(y) = \operatorname{diag}\Big[\frac{1}{|y|}, \dots, \frac{1}{|y|}, \frac{1}{|y|}\Big],$$

then

$$H_{\Phi,A}f(x) = H_\Phi f(x).$$

(For the boundedness of Hausdorff operators, we refer the reader to [1], [17], [20], [21], and [23], and the references therein.) Very recently, Ruan and Fan [29] solved an open problem posed by Liflyand in [22]. They established the sharp boundedness of Hausdorff operators on power-weighted Hardy spaces and local Hardy spaces with their atomic decomposition. We will give a nontrivial extension of [29] in the setting of the Heisenberg group since it is a noncommutative nilpotent Lie group, in which the geometric motions are quite different from the Euclidean space due to the loss of interchangeability, and we will modify a typo on the power index in Theorem 1.3(i) in [29].

The theory of Hardy spaces has been a central part of modern harmonic analysis. Here we only focus on such spaces in the setting of the Heisenberg group. It is known that the Heisenberg group plays an important role in several branches of mathematics, such as representation theory, harmonic analysis, several complex variables, partial differential equations, quantum mechanics, and signal theory (see [31] for more details). Coifman and Weiss [3] introduced Hardy spaces H^p for the general class of spaces of homogeneous type using as a definition atomic decompositions. Latter and Uchiyama [19] extended the atomic decomposition to two examples of Hardy spaces including Hardy spaces on the Heisenberg group. Folland and Stein [6] offered a lucid, systematic treatment of the real-variable theory of Hardy spaces in the setting of homogeneous groups, which includes the Heisenberg group. Geller [12] studied the characterization of Hardy operators through Riesz transforms on the Heisenberg group. Christ and Geller [2] later discussed the singular integral characterizations of Hardy spaces on homogeneous groups. In [32], the first-named author and Fan established the boundedness for two kinds of special Hausdorff operators, the Hausdorff–Poisson operator and the Hausdorff–Gauss operator, on Hardy spaces H^p with $0 < p < 1$. This article continues a series of papers about Hausdorff operators on Hardy spaces in the setting of the Heisenberg group.

Now we give a simple introduction to the Heisenberg group (for more details, we refer to [31]). The elements in the Heisenberg group $\mathbb{H}^n$ meet the group law

$$x \cdot y = \Big(x_1 + y_1, x_2 + y_2, \dots, x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + 2\sum_{j=1}^{n}(y_j x_{n+j} - x_j y_{n+j})\Big),$$

where $x = (x_1, x_2, \dots, x_{2n+1})$, $y = (y_1, y_2, \dots, y_{2n+1})$. By definition, we can see that the identity element on $\mathbb{H}^n$ is $0 \in \mathbb{R}^{2n+1}$, while the element inverse to x is $-x$, that is, $x^{-1} = -x$. The corresponding Lie algebra is generated by the left-invariant vector fields

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial x_{2n+1}}, \quad j = 1, \dots, n, \\ X_{n+j} &= \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial x_{2n+1}}, \quad j = 1, \dots, n, \\ X_{2n+1} &= \frac{\partial}{\partial x_{2n+1}}. \end{aligned}$$

The only nontrivial commutator relations are

$$[X_j, X_{n+j}] = -4X_{2n+1}, \quad j = 1, \dots, n.$$

The Heisenberg group $\mathbb{H}^n$ is a homogeneous group with dilations

$$\delta_r(x_1, x_2, \dots, x_{2n}, x_{2n+1}) = (rx_1, rx_2, \dots, rx_{2n}, r^2 x_{2n+1}), \quad r > 0.$$

The Haar measure on $\mathbb{H}^n$ coincides with the usual Lebesgue measure on $\mathbb{R}^{2n} \times \mathbb{R}$. We denote by $|E|$ the measure of any measurable set $E \subset \mathbb{H}^n$. Then

$$\big|\delta_r(E)\big| = r^Q|E|, \qquad d(\delta_r x) = r^Q\,dx,$$

where $Q = 2n + 2$ is called the *homogeneous dimension* of $\mathbb{H}^n$.

The Heisenberg distance derived from the norm

$$|x|_h = \Big[\Big(\sum_{i=1}^{2n} x_i^2\Big)^2 + x_{2n+1}^2\Big]^{\frac{1}{4}},$$

where $x = (x_1, x_2, \dots, x_{2n}, x_{2n+1})$, is given by

$$d(p, q) = d(q^{-1}p, 0) = |q^{-1}p|_h.$$

This distance d is left-invariant in the sense that $d(p, q)$ remains unchanged when p and q are both left-translated by some fixed vector on $\mathbb{H}^n$. Furthermore, d satisfies the triangular inequality (see p. 320 in [18])

$$d(p, q) \le d(p, x) + d(x, q), \quad p, x, q \in \mathbb{H}^n.$$

For $r>0$ and $x\in\mathbb{H}^n$, the ball and sphere with center x and radius r on $\mathbb{H}^n$ are given by

$$B(x,r)=\{y\in\mathbb{H}^n : d(x,y)<r\}$$

and

$$S(x,r)=\{y\in\mathbb{H}^n : d(x,y)=r\},$$

respectively. We have

$$|B(x,r)|=|B(0,r)|=\Omega_Q r^Q,$$

where

$$\Omega_Q=\frac{2\pi^{n+\frac{1}{2}}\Gamma(\frac{n}{2})}{(n+1)\Gamma(n)\Gamma(\frac{n+1}{2})}, \tag{1.1}$$

is the volume of the unit ball $B(0,1)$ on $\mathbb{H}^n$. And according to Lemma 4 in [4],

$$\omega_Q=\int_{S(0,1)} dx=Q\Omega_Q.$$

Now we will provide the definition of Hausdorff operators on the Heisenberg group (see [15], [30]). We denote by $\mathcal{M}_{2n+1}$ the space of $(2n+1)\times(2n+1)$ matrices over $\mathbb{R}$. We identify linear transforms of $\mathbb{R}^{2n+1}$ with their matrices with respect to the canonical basis, and hence think of elements of $\mathcal{M}_{2n+1}$ as either matrices or linear maps, according to context.

Definition 1.1. Let Φ be a locally integrable function on $\mathbb{H}^n$. The *Hausdorff operators* on $\mathbb{H}^n$ are defined by

$$T_\Phi f(x)=\int_{\mathbb{H}^n}\frac{\Phi(y)}{|y|_h^Q} f(\delta_{|y|_h^{-1}}x)\,dy,$$
$$T_{\Phi,A} f(x)=\int_{\mathbb{H}^n}\frac{\Phi(y)}{|y|_h^Q} f\big(A(y)x\big)\,dy,$$

where $A(y)\in\mathcal{M}_{2n+1}$ for all $y\in\mathbb{H}^n$, and we assume that $\det A(y)\neq 0$ almost everywhere in the support of Φ.

Remark 1.2. It is clear that if $A(y)=\operatorname{diag}[1/|y|_h,\dots,1/|y|_h,1/|y|_h^2]$, then $T_{\Phi,A}f(x)=T_\Phi f(x)$.

We should mention two points here. On the one hand, unlike in Euclidean spaces, for any matrix $F\in\mathcal{M}_{2n+1}$ and $x,y\in\mathbb{H}^n$, $(Fy)^{-1}Fx$ may not be equal to $F(y^{-1}x)$ in general due to the noncommutative property. But if $F\in\operatorname{Aut}(\mathbb{H}^n)$, then F and F^{-1} can commute with scalar multiplication; here $\operatorname{Aut}(\mathbb{H}^n)$ denotes the automorphism groups of $\mathbb{H}^n$ (see Proposition 1.21 and Theorem 1.22 in [5]). On the other hand, unlike [29], for $F=(a_{ij})\in\mathcal{M}_{2n+1}$, we cannot use $(\sum_{i,j=1}^{2n+1}|a_{ij}|^2)^{1/2}$ as its norm since it does not satisfy $|Fx|_h\le\|F\||x|_h$ on the Heisenberg group.

In this article, we establish the sharp boundedness of Hausdorff operators on power-weighted Hardy spaces as well as Hardy spaces.

Theorem 1.3. *Let Φ be a nonnegative function. Suppose that $A(y) \in \mathrm{Aut}(\mathbb{H}^n)$ almost everywhere and that there exists a constant M independent of y such that $\|A^{-1}(y)\| \leq M\|A(y)\|^{-1}$.*

(i) *Let $-Q < \alpha < Q$ and $\alpha \neq 0$. If all entries of the same row of $A(y)$ are nonnegative uniformly or nonpositive uniformly on $y \in \mathrm{supp}(\Phi)$, then $T_{\Phi,A}$ is bounded on $H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)$ if and only if*

$$\int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} \big\|A^{-1}(y)\big\|^\alpha \big|\det A^{-1}(y)\big| \, dy < \infty.$$

(ii) *If there exists at least one row of $A(y)$ such that all entries of such row are nonnegative uniformly or nonpositive uniformly on $y \in \mathrm{supp}(\Phi)$, then $T_{\Phi,A}$ is bounded on $H^1(\mathbb{H}^n)$ if and only if*

$$\int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big| \, dy < \infty.$$

Remark 1.4. The corresponding results in $\mathbb{R}^n$ have been obtained in [29, Theorem 1.3]. However, Theorem 1.3(i) in [29] has a typo on the upper bound for α. In fact, the function $a(x_1, x_2)$ constructed in the proof of Theorem 1.3(i) may not satisfy the cancellation condition when $s = [\alpha] \geq n$.

Corollary 1.5. *Let Φ be a nonnegative-valued function. Then T_Φ is bounded on $H^1(\mathbb{H}^n)$ if and only if $\Phi \in L^1(\mathbb{H}^n)$.*

We also obtain sufficient and necessary conditions for the boundedness of Hausdorff operators on local Hardy spaces.

Theorem 1.6. *Let Φ be a nonnegative function. Suppose that $A(y) \in \mathrm{Aut}(\mathbb{H}^n)$ almost everywhere. If there exists a constant M independent of y such that $\|A^{-1}(y)\| \leq M\|A(y)\|^{-1}$, then $T_{\Phi,A}$ is bounded on $h^1(\mathbb{H}^n)$ if and only if*

$$\int_{\|A^{-1}(y)\|<1} \frac{\Phi(y)}{|y|_h^Q} \big|\det A^{-1}(y)\big| \max\big\{1, \ln\big(\big\|A^{-1}(y)\big\|^{-1}\big)\big\} \, dy + \int_{\|A^{-1}(y)\|\geq 1} \frac{\Phi(y)}{|y|_h^Q} \big|\det A^{-1}(y)\big| \, dy < \infty.$$

Corollary 1.7. *Let Φ be a nonnegative-valued function. Then T_Φ is bounded on $h^1(\mathbb{H}^n)$ if and only if*

$$\int_{|y|_h\geq 1} \Phi(y) \, dy + \int_{|y|_h<1} \Phi(y) \max\big\{1, \ln\big(|y|_h^{-1}\big)\big\} \, dy < \infty.$$

In Section 2, we will introduce the weighted Hardy spaces by means of their atomic characterization and investigate the boundedness of Hausdorff operators on power-weighted Hardy spaces. In Section 3, we will consider the estimates for Hausdorff operators on local Hardy spaces. Further comments will be given in Section 4.

We should mention that, unlike in weighted Hardy spaces, we will use the equivalent characterizations of local Hardy spaces by both atoms and maximal functions for the study of the boundedness of Hausdorff operators. However, on the Heisenberg group, we have not found such equivalence results for weighted local Hardy spaces. That is why we only consider local Hardy spaces with no weights.

Here and throughout this article, we use the notation $\mathcal{A} \preceq \mathcal{B}$ to denote that there exists a constant $C > 0$ independent of all essential values (especially atoms) and variables such that $\mathcal{A} \leq C\mathcal{B}$. And we use the notation $\mathcal{A} \simeq \mathcal{B}$ to denote that there exist positive constants C_1 and C_2 independent of all essential values and variables such that $C_1\mathcal{B} \leq \mathcal{A} \leq C_2\mathcal{B}$. Also, we use $\mathbb{N}$ to denote the set of nonnegative integers.

2. Hausdorff operators on power-weighted Hardy spaces $H^1_{|x|_h^\alpha}(\mathbb{H}^n)$

Recall that if f and g are measurable functions on $\mathbb{H}^n$, their convolution $f * g$ is defined by

$$f * g = \int f(y)g(y^{-1}x)\,dy = \int f(xy^{-1})g(y)\,dy,$$

provided that the integrals converge. The Schwartz class $\mathcal{S}(\mathbb{H}^n)$ is the set of all $\phi \in C^\infty(\mathbb{H}^n)$ that satisfies $P(X^I)\phi$ and that is bounded on $\mathbb{H}^n$ for every polynomial P and every multi-index $I = (x_1, \ldots, x_{2n+1})$, where $X^I = X_1^{i_1} \cdots X_{2n+1}^{2n+1}$. The dual space $\mathcal{S}'(\mathbb{H}^n)$ of $\mathcal{S}(\mathbb{H}^n)$ is the space of tempered distributions on $\mathbb{H}^n$.

For $f \in \mathcal{S}'(\mathbb{H}^n)$ and $\Psi \in \mathcal{S}(\mathbb{H}^n)$, the *nontangential maximal function* $M_\Psi f$ of f with respect to Ψ is defined by

$$M_\Psi f(x) = \sup_{|x^{-1}y|_h < r < \infty} \left| f * \Psi_r(y) \right|, \tag{2.1}$$

where $\Psi_r(y) = r^{-Q}\Psi(\delta_{r^{-1}}y)$. The *nontangential grand maximal function* $M_{(N)}f$ is defined by

$$M_{(N)}f(x) = \sup_{\Psi \in \mathcal{S}, \|\Psi\|_N \leq 1} M_\Psi f(x). \tag{2.2}$$

Folland and Stein [6] introduced Hardy spaces and atomic Hardy spaces on homogeneous groups which include the Heisenberg group.

Definition 2.1. For $0 < p \leq 1$, let $N_p = [Q(1/p-1)]+1$. The *Hardy space* $H^p(\mathbb{H}^n)$ is defined by

$$H^p(\mathbb{H}^n) = \left\{ f \in \mathcal{S}'(\mathbb{H}^n) : M_{(N_p)}f \in L^p(\mathbb{H}^n) \right\}$$

with

$$\|f\|_{H^p(\mathbb{H}^n)} = \|M_{(N_p)}f\|_{L^p(\mathbb{H}^n)}.$$

As is known in [6], the ordered triplet (p, q, α) is called *admissible* if $0 < p \leq 1 \leq q \leq \infty$, $p \neq q$, $\alpha \in \mathbb{N}$, and $\alpha \geq [Q(\frac{1}{p} - 1)]$, where $[\cdot]$ is the integer function. Every polynomial on $\mathbb{H}^n$ can be written uniquely as

$$P = \sum_I a_I x^I \quad (x^I = x_1^{i_1} \cdots x_{2n+1}^{i_{2n+1}}),$$

where $I = (i_1, \dots, i_{2n+1})$ and all but finitely many of coefficients a_I vanish. Clearly, x^I is homogeneous of degree $d(I) := \sum_{k=1}^{2n} i_k + 2i_{2n+1}$. The homogeneous degree of P is defined to be $\max\{d(I) : a_I \neq 0\}$. We denote the space of polynomials of homogeneous degree less than or equal to α by P_α.

Definition 2.2. Let (p, q, α) be an admissible triplet.

(1) A function $a \in L^q(\mathbb{H}^n)$ is called a (p, q, α)-*atom centered at* x_0 if it satisfies the following conditions:
  (i) there exists a ball $B(x_0, r)$ such that $\operatorname{supp}(f) \subset B(x_0, r)$;
  (ii) $\|f\|_{L^q(\mathbb{H}^n)} \leq |B|^{\frac{1}{q}-\frac{1}{p}}$;
  (iii) $\int_{\mathbb{H}^n} f(x)P(x)\,dx = 0$, for $P \in P_\alpha$.

(2) A function $a \in L^q(\mathbb{H}^n)$ is called a *big* (p, q)-*atom centered at* x_0 if there exists a ball $B(x_0, r)$ with $r \geq \frac{1}{2}$ such that it satisfies (i) and (ii).

Definition 2.3. If (p, q, α) is an admissible triplet, the *atomic Hardy space* $H^p_{q,\alpha}(\mathbb{H}^n)$ is defined by the set of all tempered distributions of the form $\sum_j \lambda_j f_j$ (the sum converging in the topology of $\mathcal{S}'$), where each f_j is a (p, q, α)-atom and $\sum_j |\lambda_j|^p < \infty$.

If $f \in H^p_{q,\alpha}(\mathbb{H}^n)$, the quasinorm $\|f\|_{H^p_{q,\alpha}(\mathbb{H}^n)}$ (it is a norm when $p = 1$) is defined by

$$\|f\|_{H^p_{q,\alpha}(\mathbb{H}^n)} = \inf\Big(\sum_j |\lambda_j|^p\Big)^{\frac{1}{p}},$$

where the infimum is taken over all (p, q, α)-atom decompositions of f.

The following equivalent relation is due to Theorem 3.30 in [6].

Lemma 2.4. *Suppose that* $0 < p \leq 1$. *If* (p, q, α) *is admissible, then*

$$H^p(\mathbb{H}^n) = H^p_{q,\alpha}(\mathbb{H}^n)$$

and

$$\|f\|_{H^p(\mathbb{H}^n)} \simeq \|f\|_{H^p_{q,\alpha}(\mathbb{H}^n)}.$$

In this section, we will also consider weighted Hardy spaces on the Heisenberg group.

Definition 2.5 ([14, p. 369]). A locally integrable function $w : \mathbb{H}^n \to \mathbb{R}^+$ is said to *satisfy* Muckenhaupt's condition $A_p = A_p(\mathbb{H}^n)$, $1 < p < \infty$, if there is a constant $C = C(w, p)$ such that for any ball $B \subset \mathbb{H}^n$,

$$\sup_B \Big(\frac{1}{|B|}\int_B w(x)\,dx\Big)\Big(\frac{1}{|B|}\int_B w(x)^{1-p'}\,dx\Big)^{p-1} \leq C, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

A locally integrable function w is said to *belong* to $A_1(\mathbb{H}^n)$ if there is a constant $C = C(w, p)$ such that

$$|B|^{-1}\int_B w(x)\,dx \leq C \operatorname*{ess\,inf}_{x\in B} w(x)$$

and

$$A_\infty(\mathbb{H}^n) := \bigcup_{1\le p<\infty} A_p.$$

By the standard proofs of Propositions 1.4.1 and 1.4.2 in [27] together with the reverse Hölder inequality on the Heisenberg group in [28], we can get the following results.

Proposition 2.6.

(i) *We have* $A_p(\mathbb{H}^n) \subsetneq A_q(\mathbb{H}^n)$, *for* $1 \le p < q < \infty$.

(ii) *If* $w \in A_p(\mathbb{H}^n)$, $1 < p < \infty$, *then there is an* $\varepsilon > 0$ *such that* $p - \varepsilon > 1$ *and* $w \in A_{p-\varepsilon}(\mathbb{H}^n)$.

An important example of an A_p weight is the power function $|x|_h^\alpha$. By the similar proofs of Propositions 1.4.3 and 1.4.4 in [27], we can get the following properties of power weights.

Proposition 2.7. *Let* $x \in \mathbb{H}^n$. *Then*

(i) $|x|_h^\alpha \in A_1$ *if and only if* $-Q < \alpha \le 0$;

(ii) $|x|_h^\alpha \in A_p$, $1 < p < \infty$, *if and only if* $-Q < \alpha < Q(p-1)$.

Denote by q_w the *critical index* for w, that is, the infimum of all q's such that w satisfies the condition A_q. From Proposition 2.6, we can see that unless $q_w = 1$, w is never an A_{q_w} weight. Also, by Propositions 2.6 and 2.7, we can obtain that if $0 < \alpha < \infty$, then

$$|x|_h^\alpha \in \bigcap_{\frac{Q+\alpha}{Q}<p<\infty} A_p, \tag{2.3}$$

where $(Q+\alpha)/Q$ is the critical index of $|x|_h^\alpha$.

We will define the weighted Hardy spaces $H_w^p(\mathbb{H}^n)$, $0 < p \le 1$, by means of their atomic characterization as follows.

Definition 2.8. Given a weight $w \in A_\infty$ and an admissible triplet $(p, q, [Q(q_w/p-1)]$, a w-$(p, q, [Q(q_w/p-1)]$-atom centered at x_0 with respect to w will be a function a satisfying the following three conditions.

(i) There exists a ball $B(x_0, r)$ such that $\mathrm{supp}(a) \subset B(x_0, r)$.

(ii) We have $\|a\|_{L_w^q(\mathbb{H}^n)} \le w(B(x_0,r))^{\frac{1}{q}-\frac{1}{p}}$, if $q < \infty$ or $\|a\|_{L^\infty(\mathbb{H}^n)} \le w(B(x_0,r))^{-\frac{1}{p}}$, if $q = \infty$.

(iii) We have $\int_{\mathbb{H}^n} a(x)x^I\,dx = 0$ for all multi-indices $I = (i_1, i_2, \dots, i_{2n+1}) \in \mathbb{N}^{2n+1}$ with $|I| = \sum_{k=1}^{2n} i_k + 2i_{2n+1} \le [Q(q_w/p-1)]$.

Definition 2.9. Let $w \in A_\infty$ be a weight, and let $0 < p \le 1 < q \le \infty$. A tempered distribution $f \in \mathcal{S}'$ belongs to $H_w^p(\mathbb{H}^n)$ if and only if f can be written as a series

$$f = \sum_j \lambda_j a_j \tag{2.4}$$

(the sum converging in $\mathcal{S}'$), where each a_j is a w-$(p, q, [Q(q_w/p-1)])$-atom and

$$\sum_j |\lambda_j|^p < \infty. \tag{2.5}$$

Moreover, by setting $\|f\|^p_{H^p_w(\mathbb{H}^n)}$ to be the infimum of the sums (2.5) over all decompositions (2.4), one obtains the norm for such a space.

We obtain the boundedness of Hausdorff operators on power-weighted Hardy spaces.

Proposition 2.10. *Suppose that $A(y) \in \operatorname{Aut}(\mathbb{H}^n)$ almost everywhere.*

(i) *Let $0 < \alpha < \infty$. Then for any $\frac{Q+\alpha}{Q} < q < \infty$,*

$$\big\|T_{\Phi,A}(f)\big\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)} \preceq K_1 \|f\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)},$$

where

$$K_1 = \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \big\|A^{-1}(y)\big\|^{\alpha} \big|\det A^{-1}(y)\big| \Big(\frac{\|A^{-1}(y)\|^Q}{|\det A^{-1}(y)|}\Big)^{q-1} dy$$

and

$$\big\|A^{-1}(y)\big\| = \sup_{x\neq 0} \frac{|A^{-1}(y)x|_h}{|x|_h}.$$

(ii) *Let $-Q < \alpha \le 0$. Then for any $1 < s \le \infty$,*

$$\big\|T_{\Phi,A}(f)\big\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)} \preceq K_2 \|f\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)},$$

where

$$K_2 = \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \big\|A(y)\big\|^{-\alpha} \big|\det A^{-1}(y)\big| \Big(\frac{\|A^{-1}(y)\|^Q}{|\det A^{-1}(y)|}\Big)^{1-\frac{1}{s}} dy.$$

In particular, if $\alpha = 0$, we get the boundedness of $T_{\Phi,A}$ on the Hardy space $H^1(\mathbb{H}^n)$.

Corollary 2.11. *Suppose that $A(y) \in \operatorname{Aut}(\mathbb{H}^n)$ almost everywhere. Then for any $1 < q \le \infty$,*

$$\big\|T_{\Phi,A}(f)\big\|_{H^1(\mathbb{H}^n)} \preceq K_3 \|f\|_{H^1(\mathbb{H}^n)},$$

where

$$K_3 = \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \big\|A^{-1}(y)\big\|^{Q(1-\frac{1}{q})} \big|\det A^{-1}(y)\big|^{\frac{1}{q}} dy.$$

Before proving Proposition 2.10, we need the following results for matrix and A_p weights.

Lemma 2.12. *Suppose that the $(2n+1)\times(2n+1)$ matrix A is invertible. Then*

$$\|A\|^{-Q} \le |\det A^{-1}| \le \|A^{-1}\|^Q, \tag{2.6}$$

where

$$\|A\| = \sup_{x\in\mathbb{H}^n, x\neq 0} \frac{|Ax|_h}{|x|_h}.$$

Proof. It is clear that $|Ax|_h \le \|A\||x|_h$ for any $x \in \mathbb{H}^n$. Then using $A^{-1}x$ instead of x, we can get

$$\|A\|^{-1}|x|_h \le |A^{-1}x|_h \le \|A^{-1}\||x|_h.$$

Therefore,

$$\begin{aligned}\left|\left\{x \in \mathbb{H}^n : \|A\|^{-1}|x|_h \le 1\right\}\right| &\ge \left|\left\{x \in \mathbb{H}^n : |A^{-1}x|_h \le 1\right\}\right| \\ &\ge \left|\left\{x \in \mathbb{H}^n : \|A^{-1}\||x|_h \le 1\right\}\right|,\end{aligned}$$

which implies that

$$\Omega_Q\|A\|^Q \ge \Omega_Q|\det A| \ge \Omega_Q\|A^{-1}\|^{-Q}.$$

Consequently, (2.6) holds. □

By the definition of A_p weight and the Hölder inequality, we can easily get the following result.

Lemma 2.13. *If $w \in A_p(\mathbb{H}^n)$, $1 \le p < \infty$, then for any $f \in L^1_{loc}(\mathbb{H}^n)$ and any ball $B \subset \mathbb{H}^n$,*

$$\frac{1}{|B|}\int_B \big|f(x)\big|\,dx \preceq \Big(\frac{1}{w(B)}\int_B \big|f(x)\big|^p w(x)\,dx\Big)^{\frac{1}{p}}.$$

By taking $f(x) = \chi_E(x)$ in Lemma 2.13, we can deduce the following lemma.

Lemma 2.14. *Let $w \in A_p(\mathbb{H}^n)$, $p \ge 1$. Then for any ball B and its measurable subset $E \subset B$,*

$$\Big(\frac{|E|}{|B|}\Big)^p \preceq \frac{w(E)}{w(B)},$$

where $w(B) = \int_B w(x)\,dx$. Particularly, for any $\lambda > 1$,

$$w\big(B(x_0, \lambda r)\big) \preceq \lambda^{Qp} w\big(B(x_0, r)\big).$$

Proof of Proposition 2.10. Here and in the remainder of the article, for the sake of convenience, sometimes we use $w(x)$ instead of $|x|_h^\alpha$.

(i) Since $0 < \alpha < \infty$, by Proposition 2.7, $|x|_h^\alpha \in A_q$ for all $\frac{Q+\alpha}{Q} < q < \infty$. Suppose that $f \in H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)$. It suffices to show that

$$\big\|T_{\Phi,A}(f)\big\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)} \preceq K_1 \sum_j |\lambda_j|$$

uniformly for any atomic decomposition of f:

$$f = \sum_j \lambda_j a_j,$$

where each a_j is a $|\cdot|_h^\alpha$-$(1, q, [\alpha])$-atom and $\sum_j |\lambda_j| < \infty$.

By the Minkowski inequality,

$$\big\|T_{\Phi,A}(f)\big\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)} \preceq \sum_j |\lambda_j| \big\|T_{\Phi,A}(a_j)\big\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)}.$$

Thus it suffices to show that

$$\big\|T_{\Phi,A}(a)\big\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)} \preceq K_1$$

uniformly for all $|\cdot|_h^\alpha$-$(1,q,[\alpha])$-atoms a. Using the Minkowski inequality again, we obtain

$$\big\|T_{\Phi,A}(a)\big\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)} \leq \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q}\big\|a\big(A(y)\cdot\big)\big\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)}\,dy.$$

Therefore, it remains to verify that

$$\big\|a\big(A(y)\cdot\big)\big\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)} \preceq \big\|A^{-1}(y)\big\|^\alpha\big|\det A^{-1}(y)\big|\Big(\frac{\|A^{-1}(y)\|^Q}{|\det A^{-1}(y)|}\Big)^{q-1}$$

uniformly for a. For this we only need to show that

$$a_y(x) := \big\|A^{-1}(y)\big\|^{-\alpha}\big|\det A(y)\big|\Big(\frac{\|A^{-1}(y)\|^Q}{|\det A^{-1}(y)|}\Big)^{1-q} a\big(A(y)x\big)$$

is a $|\cdot|_h^\alpha$-$(1,q,[\alpha])$-atom up to a constant factor which is independent of a.

Suppose that the smallest ball that satisfies Definition 2.8 for a is $B(x_0,r)$. Since $A(y)\in \mathrm{Aut}(\mathbb{H}^n)$ almost everywhere, we have

$$\begin{aligned}\operatorname{diam}\big(A^{-1}(y)B(x_0,r)\big) &= \sup_{x,z\in B(x_0,r)}\big|\big(A^{-1}(y)z\big)^{-1}\big(A^{-1}(y)x\big)\big|_h\\ &\leq 2\big\|A^{-1}(y)\big\|r. \end{aligned} \tag{2.7}$$

Therefore,

$$\operatorname{supp} a_y(x) = \operatorname{supp} a\big(A(y)x\big) \subset B\big(\widetilde{x}_0, \big\|A^{-1}(y)\big\|r\big)$$

for some $\widetilde{x}_0\in\mathbb{H}^n$.

By a change of variables, we have

$$\begin{aligned}&\|a_y\|_{L^q_{|\cdot|_h^\alpha}(\mathbb{H}^n)}\\ &\quad= \big\|A^{-1}(y)\big\|^{-\alpha}\big|\det A(y)\big|\Big(\frac{\|A^{-1}(y)\|^Q}{|\det A^{-1}(y)|}\Big)^{1-q}\Big(\int_{\mathbb{H}^n}\big|a\big(A(y)x\big)\big|^q|x|_h^\alpha\,dx\Big)^{\frac{1}{q}}\\ &\quad\preceq \big\|A^{-1}(y)\big\|^{\alpha(\frac{1}{q}-1)}\big|\det A(y)\big|^{1-\frac{1}{q}}\Big(\frac{\|A^{-1}(y)\|^Q}{|\det A^{-1}(y)|}\Big)^{1-q}\Big(\int_{\mathbb{H}^n}\big|a(z)\big|^q|z|_h^\alpha\,dz\Big)^{\frac{1}{q}}\\ &\quad\preceq \big\|A^{-1}(y)\big\|^{\alpha(\frac{1}{q}-1)}\big|\det A(y)\big|^{1-\frac{1}{q}}\Big(\frac{\|A^{-1}(y)\|^Q}{|\det A^{-1}(y)|}\Big)^{1-q}\Big(\int_{B(x_0,r)}|z|_h^\alpha\,dz\Big)^{\frac{1}{q}-1}. \end{aligned} \tag{2.8}$$

Since $w(x)=|x|_h^\alpha\in A_q$ and $A^{-1}(y)B(x_0,r)\subset B(\widetilde{x}_0,\|A^{-1}(y)\|r)$, by Lemma 2.14, we have

$$\Big(\frac{|A^{-1}(y)B(x_0,r)|}{|B(\widetilde{x}_0,\|A^{-1}(y)\|r)|}\Big)^q \preceq \frac{w(A^{-1}(y)B(x_0,r))}{w(B(\widetilde{x}_0,\|A^{-1}(y)\|r))}.$$

Thus,

$$\begin{aligned} & w\big(B\big(\widetilde{x}_0, \|A^{-1}(y)\|r\big)\big) \\ & \quad \preceq \Big(\frac{|B(\widetilde{x}_0, \|A^{-1}(y)\|r)|}{|A^{-1}(y)B(x_0,r)|}\Big)^q w\big(A^{-1}(y)B(x_0,r)\big) \\ & \quad = \Big(\frac{\|A^{-1}(y)\|^Q}{|\det A^{-1}(y)|}\Big)^q \int_{A^{-1}(y)B(x_0,r)} |x|_h^\alpha\, dx \\ & \quad \leq \Big(\frac{\|A^{-1}(y)\|^Q}{|\det A^{-1}(y)|}\Big)^q \big\|A^{-1}(y)\big\|^\alpha \big|\det A^{-1}(y)\big| \int_{B(x_0,r)} |z|_h^\alpha\, dz. \end{aligned} \tag{2.9}$$

Therefore,

$$\int_{B(x_0,r)} |z|_h^\alpha\, dz \succeq \Big(\frac{\|A^{-1}(y)\|^Q}{|\det A^{-1}(y)|}\Big)^{-q} \big\|A^{-1}(y)\big\|^{-\alpha} \big|\det A(y)\big| w\big(B\big(\widetilde{x}_0, \big\|A^{-1}(y)\big\|r\big)\big).$$

Substituting it into (2.8), we obtain

$$\|a_y\|_{L^q_{|\cdot|_h^\alpha}(\mathbb{H}^n)} \preceq w\big(B\big(\widetilde{x}_0, \big\|A^{-1}(y)\big\|r\big)\big)^{\frac{1}{q}-1}.$$

Since $A(y) \in \mathrm{Aut}(\mathbb{H}^n)$, for any homogeneous polynomial P in $P_{[\alpha]}$, $P \circ A^{-1}(y)$ is also a homogeneous polynomial in $P_{[\alpha]}$, which ensures that a_y still satisfies the cancellation condition. Therefore, a_y is indeed a $|\cdot|_h^\alpha$-$(1, q, [\alpha])$-atom up to a constant factor which is independent of a. Proposition 2.10(i) is proved.

(ii) Since $-Q < \alpha \leq 0$, we have $|x|_h^\alpha \in A_1(\mathbb{H}^n)$. Replacing the $|\cdot|_h^\alpha$-$(1, q, [\alpha])$-atom a and a_y in the proof of (1) by the $|\cdot|_h^\alpha$-$(1, s, 0)$-atom b, $1 < s \leq \infty$, and

$$b_y(x) := \big\|A(y)\big\|^\alpha \big|\det A(y)\big| \Big(\frac{\|A^{-1}(y)\|^Q}{|\det A^{-1}(y)|}\Big)^{\frac{1}{s}-1} b\big(A(y)x\big),$$

respectively, we just need to show that

$$\|b_y\|_{L^s_{|\cdot|_h^\alpha}(\mathbb{H}^n)} \preceq w\big(B\big(\widetilde{x}_0, \big\|A^{-1}(y)\big\|r\big)\big)^{\frac{1}{s}-1}.$$

In fact, as in (2.8), we have

$$\|b_y\|_{L^s_{|\cdot|_h^\alpha}(\mathbb{H}^n)} \preceq \big\|A(y)\big\|^{\alpha(1-\frac{1}{s})} \big|\det A(y)\big|^{1-\frac{1}{s}} \Big(\frac{\|A^{-1}(y)\|^Q}{|\det A^{-1}(y)|}\Big)^{\frac{1}{s}-1} \Big(\int_{B(x_0,r)} |z|_h^\alpha\, dz\Big)^{\frac{1}{s}-1}.$$

By the similar estimate to (2.9) with $q = 1$, we obtain

$$\int_{B(x_0,r)} |z|_h^\alpha\, dz \succeq \Big(\frac{\|A^{-1}(y)\|^Q}{|\det A^{-1}(y)|}\Big)^{-1} \big\|A(y)\big\|^\alpha \big|\det A(y)\big| w\big(B\big(\widetilde{x}_0, \big\|A^{-1}(y)\big\|r\big)\big).$$

Therefore,

$$\|b_y\|_{L^s_{|\cdot|_h^\alpha}(\mathbb{H}^n)} \preceq w\big(B\big(\widetilde{x}_0, \big\|A^{-1}(y)\big\|r\big)\big)^{\frac{1}{s}-1}.$$

The proof is complete. $\square$

Proof of Theorem 1.3. (i) Since $\|A^{-1}(y)\| \le M\|A(y)\|^{-1}$, by Lemma 2.12, we have

$$\big\|A(y)\big\|^{-Q} \simeq \big|\det A^{-1}(y)\big| \simeq \big\|A^{-1}(y)\big\|^{Q}.$$

Then the sufficient part follows immediately from Proposition 2.10.

Next, we will show the necessary part. For simplicity, we will only consider the case $n = 1$. The proof of the case $n > 1$ is only a notational difference and we leave it to the reader.

Set $A(y) = (a_{ij})_{3\times 3}$. By hypothesis, without loss of generality, we may assume that $a_{1j} \ge 0$, $a_{2j} \le 0$, $a_{3j} \ge 0$, $j = 1, 2, 3$. Set

$$D = \big\{x = (x_1, x_2, x_3) \in \mathbb{H}^1 \;\big|\; x_j > 0, j = 1, 2, 3\big\}.$$

Then for all $x \in D$, we have

$$A(y)x \in \big\{z = (z_1, z_2, z_3) \in \mathbb{H}^1 \;\big|\; z_1 > 0, z_2 < 0, z_3 > 0\big\},$$

and $|A(y)D| \ne 0$, which is due to the invertible property of $A(y)$.

We now construct an atom $a(x)$ on $\mathbb{H}^1$ such that it has the cancellation property for all polynomials of homogeneous degree less than Q. Let

$$\widetilde{a}(x) = \frac{\alpha + Q}{w_Q}\chi_{D\cap B(0,1)}(x).$$

Then extend it to a function $a(x)$ on $\mathbb{H}^1$ such that $a(x)$ is odd in each variable (for odd atoms on $\mathbb{R}$, we refer to [7]). It is clear that $a(x)$ satisfies the support and size conditions. Also, we have

$$\int_{\mathbb{H}^1} a(x_1, x_2, x_3)x_1^{i_1}x_2^{i_2}x_3^{i_3}\,dx_1\,dx_2\,dx_3 = 0$$

for $(i_1, i_2, i_3) \in \mathbb{N}^3$ with $i_1 + i_2 + 2i_3 \le [\alpha] < 4$. In fact, if $[\alpha] \ge 4$, the function $a(x_1, x_2, x_3)x_1^{i_1}x_2^{i_2}x_3^{i_3}$ may not be odd in any variable and then the integral cannot be zero. Therefore, $a(x)$ is a $|\cdot|_h^\alpha$-$(1, \infty, \beta)$-atom, where $\beta = 0$ or $\beta = [\alpha]$, and $\|a\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^1)} < \infty$.

Suppose that $T_{\Phi,A}$ is bounded on $H^1_{|\cdot|_h^\alpha}(\mathbb{H}^1)$. We have

$$\begin{aligned}
\|a\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^1)} &\succeq \big\|T_{\Phi,A}(a)\big\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^1)} \succeq \big\|T_{\Phi,A}(a)\big\|_{L^1_{|\cdot|_h^\alpha}(\mathbb{H}^1)}\\
&\ge \int_D \Big|\int_{\mathbb{H}^1} \frac{\Phi(y)}{|y|_h^Q} a\big(A(y)x\big)\,dy\Big| |x|_h^\alpha\,dx\\
&= \int_{\mathbb{H}^1} \frac{\Phi(y)}{|y|_h^Q}\big\|A(y)\big\|^{-\alpha}\big|\det A^{-1}(y)\big| \int_{A(y)D} \big|a(z)\big||z|_h^\alpha\,dz\,dy\\
&\succeq \int_{\mathbb{H}^1} \frac{\Phi(y)}{|y|_h^Q}\big\|A^{-1}(y)\big\|^{\alpha}\big|\det A^{-1}(y)\big|\,dy,
\end{aligned}$$

which leads to Theorem 1.3(i).

(ii) Similar to (i), the sufficient part follows immediately from Proposition 2.10.

On the other hand, without loss of generality, we may assume that all entries of the first row of A(y) are nonnegative. Set

$$D = \{x = (x_1, x_2, \dots, x_{2n+1}) \in \mathbb{H}^n \mid x_j > 0, j = 1, 2, \dots, 2n+1\}.$$

Then for any $x \in D$, we have

$$A(y)x \in \{z = (z_1, z_2, \dots, z_{2n+1}) \in \mathbb{H}^n \mid z_1 > 0\},$$

and $|A(y)D| \neq 0$.

Let f_0 be a function with support on $B(0,1)$ satisfying

$$f_0(x) = \begin{cases} \frac{1}{\Omega_Q}, & x \in B(0,1) \cap \{x = (x_1, x_2, \dots, x_{2n+1}) \mid x_1 > 0\}, \\ 0, & x \in B(0,1) \cap \{x = (x_1, x_2, \dots, x_{2n+1}) \mid x_1 = 0\}, \\ -\frac{1}{\Omega_Q}, & x \in B(0,1) \cap \{x = (x_1, x_2, \dots, x_{2n+1}) \mid x_1 < 0\}. \end{cases}$$

It is clear that f_0 is a $(1,\infty,0)$-atom and $\|f_0\|_{H^1(\mathbb{H}^n)} < \infty$. Suppose that $T_{\Phi,A}$ is bounded on $H^1(\mathbb{H}^n)$. Then

$$\begin{aligned} \|f_0\|_{H^1(\mathbb{H}^n)} &\succeq \big\|T_{\Phi,A}(f_0)\big\|_{H^1(\mathbb{H}^n)} \succeq \big\|T_{\Phi,A}(f_0)\big\|_{L^1(\mathbb{H}^n)} \\ &\geq \int_D \bigg| \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} f_0\big(A(y)x\big)\, dy \bigg|\, dx \\ &= \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big| \int_{A(y)D} \big|f_0(z)\big|\, dz\, dy \\ &\succeq \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|\, dy. \end{aligned}$$

Therefore,

$$\int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|\, dy < \infty.$$

The proof of Theorem 1.3 is finished. □

3. Boundedness of Hausdorff operators on local Hardy spaces $h^1(\mathbb{H}^n)$

The theory of local Hardy spaces on $\mathbb{R}^n$ was introduced by Goldberg [13]. The corresponding local Hardy spaces on the Heisenberg group and on the stratified Lie group have been introduced in [25] and [26], respectively. Now let us recall the definition of local Hardy space and some basic properties. The local maximal functions are defined by taking the supremum over $0 < r \leq 1$ instead of $0 < r < \infty$ in (2.1) and (2.2):

$$\begin{aligned} \widetilde{M}_\Psi f(x) &= \sup_{|x^{-1}y|_h < r \leq 1} \big|f * \Psi_r(y)\big|, \\ \widetilde{M}_{(N)} f(x) &= \sup_{\Psi \in \mathcal{S}, \|\Psi\|_N \leq 1} \widetilde{M}_\Psi f(x). \end{aligned}$$

Definition 3.1. Let $0 < p \le 1$. The *local Hardy space* $h^p(\mathbb{H}^n)$ is defined by

$$h^p(\mathbb{H}^n) = \big\{ f \in \mathcal{S}'(\mathbb{H}^n) : \widetilde{M}_{(N_p)} f \in L^p(\mathbb{H}^n) \big\}$$

with

$$\|f\|_{h^p(\mathbb{H}^n)} = \|\widetilde{M}_{(N_p)} f\|_{L^p(\mathbb{H}^n)}.$$

According to [25], the local Hardy space also has atomic characterization.

Lemma 3.2. *Let $0 < p \le 1 < q \le \infty$. A distribution $f \in h^p(\mathbb{H}^n)$ if and only if f can be written as*

$$f = \sum_k \lambda_k a_k + \sum_k \mu_k b_k$$

(the sum converging in the sense of distributions and in $h^p(\mathbb{H}^n)$-norm), where $\sum_k |\lambda_k|^p + |\mu_k|^p < \infty$, and where each a_k is a (p, q, α)-atom and each b_k is a big (p, q)-atom. Moreover,

$$\|f\|_{h^p(\mathbb{H}^n)} \simeq \inf \Big(\sum_k |\lambda_k|^p + |\mu_k|^p \Big)^{\frac{1}{p}},$$

where the infimum is taken over all decompositions of f as above.

Let $\widetilde{M}_\Psi^+ f$ and $\widetilde{M}_{(N)}^+ f$ be the *local radial maximal function* and *local radial grand maximal function* of f, respectively, which are defined by

$$\widetilde{M}_\Psi^+ f(x) = \sup_{0<r\le 1} \big| f * \Psi_r(y) \big|,$$
$$\widetilde{M}_{(N)}^+ f(x) = \sup_{\Psi \in \mathcal{S}, \|\Psi\|_N \le 1} \widetilde{M}_\Psi^+ f(x).$$

A function Ψ is called a *commutative approximate identity* if $\Psi \in \mathcal{S}(\mathbb{H}^n)$ and satisfies the following properties:

$$\int_{\mathbb{H}^n} \Psi(x)\, dx = 1, \qquad \Psi_t * \Psi_s = \Psi_s * \Psi_t.$$

Similar to Hardy spaces, by the same argument as in [6], the local Hardy spaces also have the equivalent definitions (see also [25]).

Lemma 3.3. *Suppose that $f \in \mathcal{S}'(\mathbb{H}^n)$ and $0 < p \le 1$. Let Ψ be a commutative approximate identity, and let $N \ge N_p$ be fixed. Then*

$$\|\widetilde{M}_{(N)} f\|_{L^p(\mathbb{H}^n)} \simeq \|\widetilde{M}_{(N)}^+ f\|_{L^p(\mathbb{H}^n)} \simeq \|\widetilde{M}_\Psi f\|_{L^p(\mathbb{H}^n)} \simeq \|\widetilde{M}_\Psi^+ f\|_{L^p(\mathbb{H}^n)}.$$

We can construct a commutative approximate identity from the heat kernel on the Heisenberg group. Let us denote by $h(x, t)$, $x \in \mathbb{H}^n$, $t \in (0, \infty)$, the heat kernel for $\mathbb{H}^n$, which is the fundamental solution of the heat operator

$$\partial_t - \frac{1}{2} \Delta_H,$$

where

$$\Delta_H = \sum_{j=1}^{2n} X_j^2.$$

We know from [11] and [16] that $h(x,t)$ is given explicitly (for $t>0$) by

$$h(x,t)=\frac{1}{(2\pi t)^{\frac{Q}{2}}}\int_{-\infty}^{+\infty}\exp\Big(i\frac{uz}{2t}-\frac{\sum_{j=1}^{2n}x_j^2}{2t}z\coth z\Big)\Big(\frac{z}{\sinh z}\Big)^n\,dz,$$

where $x=(x_1,x_2,\dots,x_{2n+1})$. Let

$$\Psi(x)=h(x,1). \tag{3.1}$$

According to p. 128 in [6], $\Psi\in\mathcal{S}(\mathbb{H}^n)$, $\int_{\mathbb{H}^n}\Psi(x)\,dx=1$, and $\Psi_t(x)=h(x,t^2)$, and hence

$$\Psi_t*\Psi_s=h(x,t^2+s^2)=\Psi_s*\Psi_t.$$

Thus, Ψ is a commutative approximate identity.

The following result is from Proposition 8.11 in [6]. For multiple index $I=(i_1,i_2,\dots,i_{2n+1})\in\mathbb{N}^{2n+1}$, we denote $X^I=X_1^{i_1}X_2^{i_2}\cdots X_{2n+1}^{i_{2n+1}}$.

Lemma 3.4. *If $\Phi\in\mathcal{S}(\mathbb{H}^n)$, then*

(i) $|X^I\partial_t^j\Phi_t(x)|\le C_{Ij}(t+|x|_h)^{-(Q+|I|+j)}$, *where* $C_{Ij}=\sup_{t+|x|_h=1}|X^I\times\partial_t^j\Phi(x,t)|$;

(ii) $X^I\Phi(x)\le C_I(1+|x|)^{-(Q+|I|+1)}$, *where* $C_I=\sup_{|x|_h=1,0\le t\le 1}|X^I\Phi(x,t)|/t$.

With the above equivalence of the maximal definition and the atomic decomposition of functions in local Hardy spaces, we obtain the boundedness of Hausdorff operators on such spaces.

Proposition 3.5. *Suppose that $A(y)\in\mathrm{Aut}(\mathbb{H}^n)$ almost everywhere. Then for any $1<q\le\infty$,*

$$\big\|T_{\Phi,A}(f)\big\|_{h^1(\mathbb{H}^n)}\preceq K_4\|f\|_{h^1(\mathbb{H}^n)},$$

where

$$\begin{aligned}K_4=&\int_{\|A^{-1}(y)\|\ge 1}\frac{|\Phi(y)|}{|y|_h^Q}\big\|A^{-1}(y)\big\|^{Q(1-\frac{1}{q})}\big|\det A^{-1}(y)\big|^{\frac{1}{q}}\,dy\\&+\int_{\|A^{-1}(y)\|<1}\frac{|\Phi(y)|}{|y|_h^Q}\big|\det A^{-1}(y)\big|\\&\times\max\Big\{\Big(\frac{\|A^{-1}(y)\|^Q}{|\det A^{-1}(y)|}\Big)^{1-\frac{1}{q}},\ln\big(\big\|A^{-1}(y)\big\|^{-1}\big)\Big\}\,dy.\end{aligned}$$

Proof. Like in Proposition 2.10, by Lemma 3.2, it suffices to show that the result holds for any atomic decomposition of $f\in h^1(\mathbb{H}^n)$:

$$f=\sum_k\lambda_k a_k+\sum_k\mu_k b_k,$$

where each a_k is a $(1,q,0)$-atom, each b_k is a big $(1,q)$-atom, and $\sum_k|\lambda_k|+|\mu_k|<\infty$.

By the Minkowski inequality,

$$\big\|T_{\Phi,A}(f)\big\|_{h^1(\mathbb{H}^n)}\le\sum_k|\lambda_k|\big\|T_{\Phi,A}(a_k)\big\|_{h^1(\mathbb{H}^n)}+\sum_k|\mu_k|\big\|T_{\Phi,A}(b_k)\big\|_{h^1(\mathbb{H}^n)}.$$

Therefore, it suffices to show that

$$\left\|T_{\Phi,A}(a)\right\|_{h^1(\mathbb{H}^n)} \preceq K_4, \qquad \left\|T_{\Phi,A}(b)\right\|_{h^1(\mathbb{H}^n)} \preceq K_4$$

uniformly for all $(1,q,0)$-atoms a and all big $(1,q)$-atoms b, respectively.

By the same discussion as that in the proof of Proposition 2.10 (with $\alpha = 0$), we can obtain that

$$\left\|T_{\Phi,A}(a)\right\|_{h^1(\mathbb{H}^n)} \preceq K_2 \leq K_4.$$

Thus, it remains to estimate $\|T_{\Phi,A}(b)\|$.

By the Minkowski inequality again,

$$\left\|T_{\Phi,A}(b)\right\|_{h^1(\mathbb{H}^n)} \leq \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \left\|b\big(A(y)\cdot\big)\right\|_{h^1(\mathbb{H}^n)}\, dy.$$

Suppose that the smallest ball that satisfies Definition 2.2 for b is $B(x_0, r)$ with $r \geq \frac{1}{2}$. By the same discussion as in (2.7), we have

$$\operatorname{supp} b\big(A(y)x\big) \subset A^{-1}(y)B(x_0,r) \subset B\big(\widetilde{x}_0, \left\|A^{-1}(y)\right\| r\big), \tag{3.2}$$

for some $\widetilde{x}_0 \in \mathbb{H}^n$. We also have

$$\begin{aligned}
\left\|b\big(A(y)\cdot\big)\right\|_{L^q(\mathbb{H}^n)} &= \Big(\int_{A^{-1}(y)B(x_0,r)} \left|b\big(A(y)x\big)\right|^q dx\Big)^{\frac{1}{q}} \\
&= \left|\det A^{-1}(y)\right|^{\frac{1}{q}} \Big(\int_{B(x_0,r)} \left|b(z)\right|^q dz\Big)^{\frac{1}{q}} \\
&\leq \left|\det A^{-1}(y)\right|^{\frac{1}{q}} \left|B(x_0,r)\right|^{\frac{1}{q}-1} \\
&= \left\|A^{-1}(y)\right\|^{Q(1-\frac{1}{q})} \left|\det A^{-1}(y)\right|^{\frac{1}{q}} \left|B\big(\widetilde{x}_0, \left\|A^{-1}(y)\right\| r\big)\right|^{\frac{1}{q}-1}.
\end{aligned} \tag{3.3}$$

Therefore, when $\|A^{-1}(y)\| \geq 1$, then $\|A^{-1}(y)\|^{Q(\frac{1}{q}-1)} |\det A(y)|^{\frac{1}{q}} b(A(y)x)$ is still a big $(1,q)$-atom. Thus

$$\begin{aligned}
&\int_{\|A^{-1}(y)\|\geq 1} \frac{|\Phi(y)|}{|y|_h^Q} \left\|b\big(A(y)\cdot\big)\right\|_{h^1(\mathbb{H}^n)}\, dy \\
&\quad \preceq \int_{\|A^{-1}(y)\|\geq 1} \frac{|\Phi(y)|}{|y|_h^Q} \left\|A^{-1}(y)\right\|^{Q(1-\frac{1}{q})} \left|\det A^{-1}(y)\right|^{\frac{1}{q}}\, dy.
\end{aligned}$$

When $\|A^{-1}(y)\| \le 1$, take Ψ as in (3.1); then Ψ is a commutative approximate identity. Denote $\widetilde{r} = \|A^{-1}(y)\|r$. By Lemma 3.3, we have

$$\begin{aligned}
&\big\|b\big(A(y)\cdot\big)\big\|_{h^1(\mathbb{H}^n)}\\
&\simeq \Big\|\sup_{0<s\le 1}\big|b\big(A(y)\cdot\big)*\Psi_s\big|\Big\|_{L^1(\mathbb{H}^n)}\\
&= \int_{|x^{-1}\widetilde{x}_0|_h<2\widetilde{r}} \sup_{0<s\le 1}\Big|\int_{\mathbb{H}^n} b\big(A(y)z\big)\Psi_s(z^{-1}x)\,dz\Big|\,dx\\
&\quad + \int_{2\widetilde{r}\le|x^{-1}\widetilde{x}_0|_h<2r} \sup_{0<s\le 1}\Big|\int_{A^{-1}(y)B(x_0,r)} b\big(A(y)z\big)\Psi_s(z^{-1}x)\,dz\Big|\,dx\\
&\quad + \int_{|x^{-1}\widetilde{x}_0|_h\ge 2r} \sup_{0<s\le 1}\Big|\int_{A^{-1}(y)B(x_0,r)} b\big(A(y)z\big)\Psi_s(z^{-1}x)\,dz\Big|\,dx\\
&= I_1+I_2+I_3.
\end{aligned}$$

Using the Hölder inequality, we have

$$\begin{aligned}
I_1 &\le \big|B(\widetilde{x}_0,2\widetilde{r})\big|^{1-\frac{1}{q}}\Big(\int_{|x^{-1}\widetilde{x}_0|_h<2\widetilde{r}} \sup_{0<s\le 1}\Big|\int_{\mathbb{H}^n} b\big(A(y)z\big)\Psi_s(z^{-1}x)\,dz\Big|^q\,dx\Big)^{\frac{1}{q}}\\
&\le \big|B(\widetilde{x}_0,2\widetilde{r})\big|^{1-\frac{1}{q}}\big\|M\big(b\big(A(y)\cdot\big)\big)\big\|_{L^q(\mathbb{H}^n)},
\end{aligned}$$

where $M(b(A(y)\cdot))$ is the Hardy–Littlewood maximal function of $b(A(y)\cdot)$, which is defined by

$$M(f)(x) = \sup_{B\ni x}\frac{1}{|B|}\Big|\int_B f(y)\,dy\Big|.$$

Therefore, by (3.3),

$$I_1 \le \big|B(\widetilde{x}_0,2\widetilde{r})\big|^{1-\frac{1}{q}}\big\|b\big(A(y)\cdot\big)\big\|_{L^q(\mathbb{H}^n)} \preceq \big\|A^{-1}(y)\big\|^{Q(1-\frac{1}{q})}\big|\det A^{-1}(y)\big|^{\frac{1}{q}}. \tag{3.4}$$

In I_2, for any x that satisfies $2\widetilde{r} \le |x^{-1}\widetilde{x}_0|_h < 2r$ and $z \in A^{-1}(y)B(x_0,r) \subset B(\widetilde{x}_0,\widetilde{r})$, we have

$$|z^{-1}x|_h \ge d(x,\widetilde{x}_0) - d(z,\widetilde{x}_0) \ge \frac{1}{2}|x^{-1}\widetilde{x}_0|_h.$$

Using Lemma 3.4, we obtain

$$\big|\Psi_s(z^{-1}x)\big| \preceq \big(s+|z^{-1}x|_h\big)^{-Q} \preceq |x^{-1}\widetilde{x}_0|_h^{-Q}.$$

Therefore, by Hölder's inequality and the penultimate line of (3.3),

$$\begin{aligned}
I_2 &\le \Big(\int_{2\widetilde{r}\le|x^{-1}\widetilde{x}_0|_h<2r} |x^{-1}\widetilde{x}_0|_h^{-Q}\,dx\Big)\Big(\int_{A^{-1}(y)B(x_0,r)} \big|b\big(A(y)z\big)\big|^q\,dz\Big)^{\frac{1}{q}}\\
&\qquad \times \big|A^{-1}(y)B(x_0,r)\big|^{1-\frac{1}{q}}\\
&= \omega_Q \ln\big(\big\|A^{-1}(y)\big\|^{-1}\big)\big|\det A^{-1}(y)\big|.
\end{aligned} \tag{3.5}$$

In I_3, for any x that satisfies $|x^{-1}\widetilde{x}_0|_h \geq 2r$ and $z \in A^{-1}(y)B(x_0,r) \subset B(\widetilde{x}_0,\widetilde{r})$, we also have

$$|z^{-1}x|_h \geq \frac{1}{2}|x^{-1}\widetilde{x}_0|_h.$$

Since $\Psi \in \mathcal{S}(\mathbb{H}^n)$ and $0 < s \leq 1$, using Lemma 3.4 again, we have

$$\begin{aligned}\big|\Psi_s(z^{-1}x)\big| &= \frac{1}{s^Q}\big|\Psi\big(\delta_{s^{-1}}(z^{-1}x)\big)\big| \preceq \frac{1}{s^Q}\Big(1+\frac{|z^{-1}x|_h}{s}\Big)^{-(Q+1)} \\ &\preceq |x^{-1}\widetilde{x}_0|_h^{-(Q+1)}.\end{aligned}$$

Thus, as in (3.5),

$$\begin{aligned}I_3 &\preceq \Big(\int_{|x^{-1}\widetilde{x}_0|_h\geq 2r} |x^{-1}\widetilde{x}_0|_h^{-Q-1}\,dx\Big)\Big(\int_{A^{-1}(y)B(x_0,r)}\big|b\big(A(y)z\big)\big|\,dz\Big) \\ &\preceq r^{-1}\big|\det A^{-1}(y)\big| \preceq \big|\det A^{-1}(y)\big|, \end{aligned}\tag{3.6}$$

where the last inequality is due to the fact that $r \geq \frac{1}{2}$.

Consequently, by (2.6) and (3.4)–(3.6), we have

$$\begin{aligned}&\int_{\|A^{-1}(y)\|<1}\frac{|\Phi(y)|}{|y|_h^Q}\big\|b\big(A(y)\cdot\big)\big\|_{h^1(\mathbb{H}^n)}\,dy \\ &\quad\preceq \int_{\|A^{-1}(y)\|<1}\frac{|\Phi(y)|}{|y|_h^Q}\big|\det A^{-1}(y)\big| \\ &\qquad\times \max\Big\{\Big(\frac{\|A^{-1}(y)\|^Q}{|\det A^{-1}(y)|}\Big)^{1-\frac{1}{q}}, \ln\big(\big\|A^{-1}(y)\big\|^{-1}\big)\Big\}\,dy.\end{aligned}$$

The proposition is proved. □

Proof of Theorem 1.6. Since $\|A^{-1}(y)\| \leq M\|A(y)\|^{-1}$, by Lemma 2.12, we have

$$\big\|A(y)\big\|^{-Q} \simeq \big|\det A^{-1}(y)\big| \simeq \big\|A^{-1}(y)\big\|^Q. \tag{3.7}$$

Then the sufficient part follows immediately from Proposition 3.5.

On the other hand, suppose that $T_{\Phi,A}$ is bounded on $h^1(\mathbb{H}^n)$. Set $f_0(x) = \chi_{B(0,1)}(x)$. By an easy computation, we can obtain that

$$\|f_0\|_{h^1(\mathbb{H}^n)} \preceq 1 < \infty.$$

By Fubini's theorem,

$$\begin{aligned}\|f_0\|_{h^1(\mathbb{H}^n)} &\succeq \big\|T_{\Phi,A}(f_0)\big\|_{h^1(\mathbb{H}^n)} \geq \big\|T_{\Phi,A}(f_0)\big\|_{L^1(\mathbb{H}^n)} \\ &= \int_{\mathbb{H}^n}\Big|\int_{\mathbb{H}^n}\frac{\Phi(y)}{|y|_h^Q}f_0\big(A(y)x\big)\,dy\Big|\,dx \\ &= \Big(\int_{\mathbb{H}^n}\frac{\Phi(y)}{|y|_h^Q}\big|\det A^{-1}(y)\big|\,dy\Big)\Big(\int_{B(0,1)}f_0(z)\,dz\Big) \\ &= \Omega_Q\int_{\mathbb{H}^n}\frac{\Phi(y)}{|y|_h^Q}\big|\det A^{-1}(y)\big|\,dy.\end{aligned}$$

Therefore,

$$\int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} \left|\det A^{-1}(y)\right| dy < \infty. \tag{3.8}$$

Next, take $\Psi(x) = \frac{2}{\omega_Q \Gamma(\frac{Q}{2})} e^{-|x|_h^2}$. Then $\Psi \in \mathcal{S}(\mathbb{H}^n)$ and $\int_{\mathbb{H}^n} \Psi = 1$. Obviously, Ψ is a radial function. By Proposition 4.28 in [6], Ψ is a commutative approximate identity. Therefore,

$$\begin{aligned}
\left\|T_{\Phi,A}(f_0)\right\|_{h^1(\mathbb{H}^n)} &\simeq \int_{\mathbb{H}^n} \sup_{0<s\leq 1} \left|T_{\Phi,A}(f_0) * \Psi_s(x)\right| dx \\
&\succeq \int_{B(0,1)} \left|T_{\Phi,A}(f_0) * \Psi_{|x|_h}(x)\right| dx.
\end{aligned}$$

By Fubini's theorem and a change of variables, we have

$$\begin{aligned}
\left|T_{\Phi,A}(f_0) * \Psi_{|x|_h}(x)\right| &= \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} \int_{\mathbb{H}^n} f_0\big(A(y)z\big) \Psi_{|x|_h}(z^{-1}x)\, dz\, dy \\
&= \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} \left|\det A^{-1}(y)\right| \int_{\mathbb{H}^n} f_0(u) \Psi_{|x|_h}\Big(\big(A^{-1}(y)u\big)^{-1}x\Big)\, du\, dy \\
&= \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} \left|\det A^{-1}(y)\right| \int_{|u|_h<1} \Psi_{|x|_h}\Big(\big(A^{-1}(y)u\big)^{-1}x\Big)\, du\, dy.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\|T_{\Phi,A}(f_0)\right\|_{h^1(\mathbb{H}^n)} \\
&\quad\succeq \int_{B(0,1)} \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} \left|\det A^{-1}(y)\right| \int_{|u|_h<1} \Psi_{|x|_h}\Big(\big(A^{-1}(y)u\big)^{-1}x\Big)\, du\, dy\, dx \\
&\quad\succeq \int_{\|A^{-1}(y)\|<1} \frac{\Phi(y)}{|y|_h^Q} \left|\det A^{-1}(y)\right| \\
&\quad\quad\times \int_{|u|_h<1} \int_{|x|_h<1} \frac{1}{|x|_h^Q} \Psi\Big(\delta_{|x|_h^{-1}}\Big(\big(A^{-1}(y)u\big)^{-1}x\Big)\Big)\, dx\, du\, dy \\
&\quad\succeq \int_{\|A^{-1}(y)\|<1} \frac{\Phi(y)}{|y|_h^Q} \left|\det A^{-1}(y)\right| \\
&\quad\quad\times \int_{|u|_h<1} \int_{|u|_h<|z|_h\leq \frac{1}{\|A^{-1}(y)\|}} \frac{1}{|z|_h^Q} \\
&\quad\quad\times \Psi\Big(\delta_{|A^{-1}(y)z|_h^{-1}}\Big(\big(A^{-1}(y)u\big)^{-1}\big(A^{-1}(y)z\big)\Big)\Big)\, dz\, du\, dy,
\end{aligned}$$

where the last inequality is deduced by (3.7).

For any $|u|_h < 1$ and $|u|_h < |z|_h \leq \frac{1}{\|A^{-1}(y)\|}$, since $A(y) \in \mathrm{Aut}(\mathbb{H}^n)$, we have

$$\begin{aligned}
&\left|\delta_{|A^{-1}(y)z|_h^{-1}}\left(\left(A^{-1}(y)u\right)^{-1}\left(A^{-1}(y)z\right)\right)\right|_h \\
&\quad = \frac{|A^{-1}(y)(u^{-1}z)|_h}{|A^{-1}(y)z|_h} \preceq \frac{\|A^{-1}(y)\||u^{-1}z|_h}{\|A(y)\|^{-1}|z|_h} \\
&\quad \preceq M\frac{|z|_h + |u|_h}{|z|_h} \preceq 2M.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|f_0\|_{h^1(\mathbb{H}^n)} &\succeq \left\|T_{\Phi,A}(f_0)\right\|_{h^1(\mathbb{H}^n)} \\
&\succeq \int_{\|A^{-1}(y)\|<1} \frac{\Phi(y)}{|y|_h^Q}\left|\det A^{-1}(y)\right| \int_{|u|_h<1} \int_{1<|z|_h\leq\frac{1}{\|A^{-1}(y)\|}} \frac{1}{|z|_h^Q}\, dz\, du\, dy \\
&= \omega_Q\Omega_Q \int_{\|A^{-1}(y)\|<1} \frac{\Phi(y)}{|y|_h^Q}\left|\det A^{-1}(y)\right| \ln\left\|A^{-1}(y)\right\|^{-1} dy.
\end{aligned}$$

Thus,

$$0 < \int_{\|A^{-1}(y)\|<1} \frac{\Phi(y)}{|y|_h^Q}\left|\det A^{-1}(y)\right| \ln\left\|A^{-1}(y)\right\|^{-1} dy < \infty. \tag{3.9}$$

The necessary part of this theorem can be deduced immediately by (3.8) and (3.9). □

4. Adjoint Hausdorff operators and further comments

In this section, we focus on the corresponding results for the adjoint operators of Hausdorff operators.

We can easily find the adjoint operator T^* as the one satisfying, for appropriate functions f and g,

$$\int_{\mathbb{R}^n} (Tf)(x)g(x)\, dx = \int_{\mathbb{R}^n} (T^*g)(x)f(x)\, dx.$$

It can be checked that the adjoint operator $T^*_{\Phi,A}$ is defined as

$$T^*_{\Phi,A} = \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q}\left|\det A^{-1}(y)\right| f\left(A^{-1}(y)x\right) dy, \quad x \in \mathbb{H}^n.$$

In particular,

$$T^*_{\Phi} = \int_{\mathbb{H}^n} \Phi(y) f(\delta_{|y|_h} x)\, dy, \quad x \in \mathbb{H}^n.$$

Obviously, $T^*_{\Phi,A}$ is also a Hausdorff operator, and it can be written as $T_{\widetilde{\Phi},\widetilde{A}}$, where $\widetilde{\Phi}(y) = \Phi(y)|\det A^{-1}(y)|$ and $\widetilde{A}(y) = A^{-1}(y)$. When $A(y) = \mathrm{diag}[1/|y|_h, \ldots, 1/|y|_h, 1/|y|_h^2]$, $T_{\widetilde{\Phi},\widetilde{A}} = T^*_{\Phi}$. Therefore, $T^*_{\Phi,A}$ and T^*_{Φ} also have the corresponding results as in the above sections. We will list them (without proof) in the following.

Theorem 4.1. *Suppose that $A(y) \in \mathrm{Aut}(\mathbb{H}^n)$ almost everywhere.*

(i) *Let* $0<\alpha<\infty$. *Then for any* $\frac{Q+\alpha}{Q}<q<\infty$,

$$\big\|T^*_{\Phi,A}(f)\big\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)} \preceq \widetilde{K}_1\|f\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)},$$

where

$$\widetilde{K}_1=\int_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}\big\|A(y)\big\|^\alpha\Big(\frac{\|A(y)\|^Q}{|\det A(y)|}\Big)^{q-1}dy$$

and

$$\big\|A(y)\big\|=\sup_{x\neq 0}\frac{|A(y)x|_h}{|x|_h}.$$

(ii) *Let* $-Q<\alpha\le 0$. *Then for any* $1<s\le\infty$,

$$\big\|T^*_{\Phi,A}(f)\big\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)} \preceq \widetilde{K}_2\|f\|_{H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)},$$

where

$$\widetilde{K}_2=\int_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}\big\|A^{-1}(y)\big\|^{-\alpha}\Big(\frac{\|A(y)\|^Q}{|\det A(y)|}\Big)^{1-\frac{1}{s}}dy.$$

Since the dual space of $H^1(\mathbb{H}^n)$ is $\mathrm{BMO}(\mathbb{H}^n)$, together with Corollary 2.11, we can get the following result.

Corollary 4.2. *Suppose that* $A(y)\in\mathrm{Aut}(\mathbb{H}^n)$ *almost everywhere. Then for any* $1<q\le\infty$,

$$\big\|T^*_{\Phi,A}(f)\big\|_{H^1(\mathbb{H}^n)} \preceq \widetilde{K}_3\|f\|_{H^1(\mathbb{H}^n)},$$
$$\big\|T^*_{\Phi,A}(f)\big\|_{\mathrm{BMO}(\mathbb{H}^n)} \preceq K_3\|f\|_{\mathrm{BMO}(\mathbb{H}^n)},$$
$$\big\|T_{\Phi,A}(f)\big\|_{\mathrm{BMO}(\mathbb{H}^n)} \preceq \widetilde{K}_3\|f\|_{\mathrm{BMO}(\mathbb{H}^n)},$$

where

$$\widetilde{K}_3=\int_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}\big\|A(y)\big\|^{Q(1-\frac{1}{q})}\big|\det A(y)\big|^{\frac{1}{q}-1}dy,$$

and K_3 *is defined as in Corollary 2.11.*

Theorem 4.3. *Let* Φ *be a nonnegative function. Suppose that* $A(y)\in\mathrm{Aut}(\mathbb{H}^n)$ *almost everywhere and that there exists a constant* M *independent of* y *such that* $\|A(y)\|\le M\|A^{-1}(y)\|^{-1}$.

(i) *Let* $-Q<\alpha<Q$ *and* $\alpha\neq 0$. *If all entries of the same row of* $A^{-1}(y)$ *are nonnegative uniformly or nonpositive uniformly on* $y\in\mathrm{supp}(\Phi)$, *then* $T^*_{\Phi,A}$ *is bounded on* $H^1_{|\cdot|_h^\alpha}(\mathbb{H}^n)$ *if and only if*

$$\int_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}\big\|A(y)\big\|^\alpha dy<\infty.$$

(ii) *If there exists at least one row of* $A^{-1}(y)$ *such that all entries of such row are nonnegative uniformly or nonpositive uniformly on* $y \in \operatorname{supp}(\Phi)$*, then* $T^*_{\Phi,A}$ *is bounded on* $H^1(\mathbb{H}^n)$ *if and only if*

$$\int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q}\, dy < \infty.$$

From Theorem 4.3 and Corollary 1.5, we have the following characterization for boundedness of Hausdorff operators.

Corollary 4.4. *Let* Φ *be a nonnegative-valued function. Then*

(i) T^*_Φ *is bounded on* $H^1(\mathbb{H}^n)$ *if and only if* $\int_{\mathbb{H}^n} \Phi(y)|y|_h^{-Q}\, dy < \infty$;
(ii) T^*_Φ *is bounded on* $\mathrm{BMO}(\mathbb{H}^n)$ *if and only if* $\int_{\mathbb{H}^n} \Phi(y)\, dy < \infty$;
(iii) T_Φ *is bounded on* $\mathrm{BMO}(\mathbb{H}^n)$ *if and only if* $\int_{\mathbb{H}^n} \Phi(y)|y|_h^{-Q}\, dy < \infty$.

Theorem 4.5. *Suppose that* $A(y) \in \mathrm{Aut}(\mathbb{H}^n)$ *almost everywhere. Then for any* $1 < q \le \infty$,

$$\big\|T_{\Phi,A}(f)\big\|_{h^1(\mathbb{H}^n)} \preceq \widetilde{K}_4 \|f\|_{h^1(\mathbb{H}^n)},$$

where

$$\widetilde{K}_4 = \int_{\|A(y)\|\ge 1} \frac{|\Phi(y)|}{|y|_h^Q} \big\|A(y)\big\|^{Q(1-\frac{1}{q})} \big|\det A(y)\big|^{\frac{1}{q}-1}\, dy + \int_{\|A(y)\|<1} \frac{|\Phi(y)|}{|y|_h^Q} \max\Big\{\Big(\frac{\|A(y)\|^Q}{|\det A(y)|}\Big)^{1-\frac{1}{q}}, \ln\big(\big\|A(y)\big\|^{-1}\big)\Big\}\, dy.$$

Theorem 4.6. *Let* Φ *be a nonnegative function. Suppose that* $A(y) \in \mathrm{Aut}(\mathbb{H}^n)$ *almost everywhere. If there exists a constant* M *independent of* y *such that* $\|A(y)\| \le M\|A^{-1}(y)\|^{-1}$*, then* $T^*_{\Phi,A}$ *is bounded on* $h^1(\mathbb{H}^n)$ *if and only if*

$$\int_{\|A(y)\|\ge 1} \frac{\Phi(y)}{|y|_h^Q}\, dy + \int_{\|A(y)\|<1} \frac{\Phi(y)}{|y|_h^Q} \max\big\{1, \ln\big(\big\|A(y)\big\|^{-1}\big)\big\}\, dy < \infty.$$

Corollary 4.7. *Let* Φ *be a nonnegative-valued function. Then* T^*_Φ *is bounded on* $h^1(\mathbb{H}^n)$ *if and only if*

$$\int_{|y|_h\le 1} \frac{\Phi(y)}{|y|_h^Q}\, dy + \int_{|y|_h>1} \frac{\Phi(y)}{|y|_h^Q} \max\big\{1, \ln|y|_h\big\}\, dy < \infty.$$

Finally, we offer some closing comments on weighted local Hardy spaces on the Heisenberg group. In the proof of boundedness of Hausdorff operators on local Hardy spaces, we used the equivalence of their two kinds of definitions, that is, Lemma 3.2. However, on the Heisenberg group, we have not found such results for weighted local Hardy spaces; thus we have not been able to obtain the sharp boundedness for Hausdorff operators on these spaces. Therefore, to establish the sharpness of the conditions on Φ and A to ensure the boundedness for Hausdorff operators $T_{\Phi,A}$ on weighted local Hardy spaces on the Heisenberg group would be an interesting question.

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