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DISJOINTNESS-PRESERVING ORTHOGONALLY ADDITIVE OPERATORS IN VECTOR LATTICES

NARIMAN ABASOV[1] and MARAT PLIEV[2*]

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ABSTRACT. In this article, we investigate disjointness-preserving orthogonally additive operators in the setting of vector lattices. First, we present a formula for the band projection onto the band generated by a single positive, disjointness-preserving, order-bounded, orthogonally additive operator. Then we prove a Radon–Nikodým theorem for a positive, disjointness-preserving, order-bounded, orthogonally additive operator defined on a vector lattice E, taking values in a Dedekind-complete vector lattice F. We conclude by obtaining an analytical representation for a nonlinear lattice homomorphism between order ideals of spaces of measurable almost everywhere finite functions.

1. Introduction and preliminaries

Linear disjointness-preserving operators were introduced in some form during the early part of the twentieth century. However, one of the first systematic studies of disjointness-preserving operators dates back to a seminal note by Abramovich, Veksler, and Koldunov [2] published at the end of the 1970s. From that moment until now, interest in disjointness-preserving operators has remained at a rather high level (see the list of references [6], [8], [11], [16], [18], [21], [22], [30]).

Orthogonally additive and in general nonlinear operators in the setting of vector lattices were considered in the latter part of the twentieth century by Mazón and Segura de León [24]. Today, the theory of orthogonally additive operators

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*Corresponding author.
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in ordered spaces is an object of intensive study (see [9], [10], [12], [13], [15], [25], [27]). Orthogonally additive disjointness-preserving operators between vector lattices were introduced and studied in [1] and [29].

Our aim here is to continue this line of investigation. The main objects are abstract Urysohn operators, an important subclass of orthogonally additive operators. In the first part of the article, we obtain a formula for the band projection onto the band generated by a single positive, disjointness-preserving abstract Urysohn operator. In the second part, we prove a Radon–Nikodým theorem for positive, disjointness-preserving abstract Urysohn operators. Finally, in the last part, we obtain an analytical representation of Urysohn lattice homomorphisms between order ideals of spaces of measurable functions. We assume that the reader is familiar with the theory of vector lattices. (For standard information on Banach and vector lattices, we refer to [4].) All vector lattices below are assumed to be Archimedean.

Let E be a vector lattice. A net $(x_\alpha)_{\alpha\in\Lambda}$ in E *order converges* to an element $x\in E$ (notation $x_\alpha \xrightarrow{\text{o}} x$) if there exists a net $(e_\alpha)_{\alpha\in\Lambda}$ in E_+ such that $e_\alpha \downarrow 0$ and $|x_\alpha - x| \le e_\alpha$ for all $\alpha\in\Lambda$ satisfying $\alpha\ge\alpha_0$ for some $\alpha_0\in\Lambda$. Two elements x, y of a vector lattice E are said to be *disjoint* (we use the notation $x\perp y$) if $|x|\wedge|y| = 0$. The sum $x+y$ of two disjoint elements x and y is denoted by $x\sqcup y$. The equality $x = \bigsqcup_{i=1}^n x_i$ means that $x = \sum_{i=1}^n x_i$ and $x_i \perp x_j$ for all $i\ne j$. An element y of E is said to be a *fragment* (in another terminology, a *component*) of an element $x\in E$ if $y\perp(x-y)$. The notation $y\sqsubseteq x$ means that y is a fragment of x. The set of all fragments of an element $x\in E$ is denoted by $\mathcal{F}_x$. A net (x_α) in a vector lattice E *laterally converges* to $x\in E$ if $x_\alpha\sqsubseteq x_\beta\sqsubseteq x$ for all indices $\alpha<\beta$ and $x_\alpha\xrightarrow{\text{o}} x$. In this case, we write $x_\alpha \xrightarrow{\text{lat}} x$. For positive elements x_α, x the condition $x_\alpha\xrightarrow{\text{lat}} x$ means that $x_\alpha\sqsubseteq x$ and $x_\alpha\uparrow x$.

A linear operator $\pi : E\to E$ is said to be an *order projection* if

(1) $\pi\circ\pi = \pi$;
(2) $0\le\pi\le I$, where I is the identity operator on E.

The Boolean algebra of all order projections on a vector lattice E is denoted by $\mathfrak{B}(E)$. The order projection onto the band generated by a subset D of E is denoted by $[D]$. The characteristic function of a set D is denoted by 1_D.

Definition 1.1. Let E be a vector lattice, and let F be a real linear space. An operator $T : E\to F$ is said to be *orthogonally additive* if $T(x+y) = Tx+Ty$ for all $x, y\in E$ with $x\perp y$.

It is clear from the definition that $T(0) = 0$. The set of all orthogonally additive operators is a real vector space with respect to the natural linear operations.

Definition 1.2. Let E and F be vector lattices. An orthogonally additive operator $T : E\to F$ is said to be

- *positive* if $Tx\ge 0$ for all $x\in E$;
- *order-bounded* if T maps order-bounded sets in E to order-bounded sets in F.

An orthogonally additive, order-bounded operator $T : E \to F$ is said to be an *abstract Urysohn* operator.

The importance of this class of operators is determined by applications to the theory of nonlinear integral equations. The classical integral abstract Urysohn operator is presented in the following example.

Example 1.3. Let (A, Σ, μ) and (B, Ξ, ν) be σ-finite complete measure spaces, and let $(A \times B, \mu \times \nu)$ denote the completion of their product measure space. Let $K : A \times B \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the following conditions:[1]

(C_0) $K(s, t, 0) = 0$ for $(\mu \times \nu)$-almost all $(s, t) \in A \times B$;
(C_1) $K(\cdot, \cdot, r)$ is $(\mu \times \nu)$-measurable for all $r \in \mathbb{R}$;
(C_2) $K(s, t, \cdot)$ is continuous on $\mathbb{R}$ for $(\mu \times \nu)$-almost all $(s, t) \in A \times B$.

Given $f \in L_0(B, \Xi, \nu)$, the function $|K(s, \cdot, f(\cdot))|$ is ν-measurable for μ-almost all $s \in A$, and $h_f(s) := \int_B |K(s, t, f(t))| \, d\nu(t)$ is a well-defined and μ-measurable function. Since the function h_f can be infinite on a set of positive measure, we define

$$\mathrm{Dom}_B(K) := \big\{ f \in L_0(\nu) : h_f \in L_0(\mu) \big\}.$$

Then we define an operator $T : \mathrm{Dom}_B(K) \to L_0(\mu)$ by setting

$$(Tf)(s) := \int_B K\big(s, t, f(t)\big) \, d\nu(t) \quad \mu\text{-a.e.} \qquad (\star)$$

Let E and F be order ideals in $L_0(\nu)$ and $L_0(\mu)$, respectively, and let K be a function satisfying (C_0)–(C_2). Then $(\star)$ defines an *orthogonally additive integral operator* from E to F if $E \subseteq \mathrm{Dom}_B(K)$ and $T(E) \subseteq F$.

The set of all abstract Urysohn operators from E to F is denoted by $\mathcal{U}(E, F)$. There is a natural partial order on $\mathcal{U}(E, F)$, namely, $S \le T$, whenever $(T - S) \ge 0$. Then $\mathcal{U}(E, F)$ becomes an ordered vector space. If the vector lattice F is Dedekind-complete, then $\mathcal{U}(E, F)$ is a Dedekind-complete vector lattice.

Theorem 1.4 ([24, Theorem 3.2])**.** *Let E and F be vector lattices, and let F be Dedekind-complete. Then $\mathcal{U}(E, F)$ is a Dedekind-complete vector lattice. Moreover, for each $S, T \in \mathcal{U}(E, F)$ and $x \in E$ the following conditions hold:*

(1) $(T \vee S)(x) = \sup\{T(y) + S(z) : x = y \sqcup z\}$;
(2) $(T \wedge S)(x) = \inf\{T(y) + S(z) : x = y \sqcup z\}$;
(3) $(T)^+(x) = \sup\{Ty : y \sqsubseteq x\}$;
(4) $(T)^-(x) = -\inf\{Ty : y \sqsubseteq x\}$;
(5) $|T|(x) = \sup\{Ty - Tz : x = y \sqcup z\}$.

2. The projection of $\mathcal{U}(E, F)$ onto the band generated by a single disjointness-preserving operator

Order projection is an important tool for investigating operators in vector lattices (see, e.g., [3], [19]). In this section, we associate with an operator $T \in \mathcal{U}(E, F)$ a band in the space $\mathcal{U}(E, F)$, which is called the *shadow* of an operator

[1] (C_1) and (C_2) are called the *Carathéodory conditions.*

T, and we calculate the order projection onto this band. We apply this result for the calculation of the order projection onto the band generated by a single positive, disjointness-preserving, abstract Urysohn operator.

Definition 2.1. Let E, F be vector lattices. An abstract Urysohn operator $T : E \to F$ is considered disjointness-preserving if $Tx \perp Ty$ for all disjoint $x, y \in E$.

The following is a classical example of an orthogonally additive disjointness-preserving operator.

Example 2.2. Let (Ω, Σ, μ) be a σ-finite and complete measure space. Let E be a vector sublattice of the space $L_0(\mu)$ of all μ-measurable and μ-almost everywhere finite functions on A, where μ-almost everywhere equal functions are identified. Consider a function $N : A \times \mathbb{R} \to \mathbb{R}$ satisfying

(1) $N(t, 0) = 0$ for μ-almost all $t \in A$;
(2) $N(\cdot, f(\cdot))$ is μ-measurable for every $f \in E$.

Then an operator $T : E \to L_0(\mu)$ defined by $(Tx)(t) := N(t, x(t))$ is the projection commuting. Indeed, if an operator $\sigma : L_0(\mu) \to L_0(\mu)$ is an order projection, then there is a μ-measurable subset $V \subset A$ such that $\sigma f = f 1_V$ for every $f \in L_0(\mu)$. Now we may write

$$\begin{aligned}(T\sigma f)(t) &= T(f 1_V)(t) = N\big(t, f(t)1_V(t)\big)\\ &= N\big(t, f(t)\big)1_V(t) = (Tf)(t)1_V(t) = (\sigma T f)(t).\end{aligned}$$

Recall that for a measurable function $\psi \in L_0(\mu)$, we denote by $\operatorname{supp}(\psi)$ the measurable set $\{t \in \Omega : f(t) \neq 0\}$. Take any two disjoint elements $f, g \in E$, and assume that $V := \operatorname{supp}(f)$ and $H := \operatorname{supp}(g)$. Clearly, V and H are disjoint μ-measurable sets. Then we have

$$\begin{aligned}T(f+g) &= T\big((f+g)(1_V + 1_H)\big)\\ &= N\big(t, (f+g)(t)\big)(1_V + 1_H)(t)\\ &= N\big(t, (f+g)(t)\big)1_V(t) + N\big(t, (f+g)(t)\big)1_H(t)\\ &= N\big(t, f(t)\big) + N\big(t, g(t)\big) = Tf + Tg;\\ (Tf)(t) &= T(f1_V)(t) = N\big(t, f(t)1_V(t)\big) = N\big(t, f(t)\big)1_V(t);\\ (Tg)(t) &= T(g1_H)(t) = N\big(t, g(t)1_H(t)\big) = N\big(t, g(t)\big)1_H(t)\\ &\Rightarrow Tf \perp Tg.\end{aligned}$$

Thus T is a disjointness-preserving, orthogonally additive operator.

These operators are known in the literature as *nonlinear superposition operators* or *Nemytskii operators* (see [5]).

Definition 2.3. Let E be a vector lattice. The subset D of E is said to be a *lateral ideal* if the following conditions hold:

(1) if $x \in D$, then $y \in D$ for every $y \in \mathcal{F}_x$;
(2) if $x, y \in D$, $x \perp y$, then $x + y \in D$.

A subset D of the vector lattice E is said to be *laterally closed* in E if for every laterally convergence net $(x_\alpha)_{\alpha\in\Lambda} \subset D$ such that $x_\alpha \xrightarrow{\text{lat}} x$, we have $x \in D$. A lateral ideal D is said to be a *lateral band* in E if it is laterally closed in E.

Example 2.4. Let E be a vector lattice. Then every order ideal in E is a lateral ideal.

Example 2.5. Let E, F be vector lattices, and let $T \in \mathcal{U}_+(E,F)$. Then $\mathcal{N}_T := \{e \in E : T(e) = 0\}$ is a lateral ideal.

Example 2.6 ([7, Lemma 3.5]). Let E be a vector lattice, and let $x \in E$. Then $\mathcal{F}_x$ is a lateral ideal.

Example 2.7. Let E be a vector lattice. Then every band D in E is a lateral band.

Example 2.8. Let E be a vector lattice, and let $x \in E$. Then the set $\mathcal{F}_x$ is a lateral band.

The set of all lateral bands of a vector lattice E is denoted by $\mathfrak{Lb}(E)$.

Lemma 2.9 ([26, Propositions 3.8, 3.9])**.** *Let E be a vector lattice. Then the binary relation $\sqsubseteq$ is a partial order on E, and $x \sqsubseteq y$ if and only if $x^+ \sqsubseteq y^+$ and $x^- \sqsubseteq y^-$ for any $x, y \in E$.*

Lemma 2.10 ([26, Propositions 3.10, 3.11])**.** *Let E be a Dedekind-complete vector lattice, let $y \in E$, and let D be a lateral band in E. Then the set $\mathcal{F}_y$ with the partial order $u \sqsubseteq w$ is a Dedekind-complete Boolean algebra, and the set $D(y) := \mathcal{F}_y \cap D$ contains a maximal element with respect to the partial order $\sqsubseteq$.*

We denote by y^D the maximal element of the set $D(y)$ in the Boolean algebra $\mathcal{F}_y$.

Lemma 2.11. *Let E be a vector lattice, and let $x, y, z \in E$ and $z \sqsubseteq x \sqcup y$. Then there exist elements $z_1, z_2 \in E$ such that*

(i) $z = z_1 \sqcup z_2$;
(ii) $z_1 \sqsubseteq x$; $z_2 \sqsubseteq y$.

Proof. By Lemma 2.9, we have

$$z^+ \sqsubseteq (x+y)^+ = x^+ + y^+;$$
$$z^- \sqsubseteq (x+y)^- = x^- + y^-.$$

Now, by the Riesz decomposition property (see [4, Theorem 1.13]) there exist

$$z_i^+, z_i^- \in E_+; \quad i \in \{1, 2\}$$

such that

$$\begin{aligned} z_1^+ &\sqsubseteq x^+; & z_1^- &\sqsubseteq x^-; \\ z_2^+ &\sqsubseteq y^+; & z_2^- &\sqsubseteq y^-; \\ z^+ &= z_1^+ \sqcup z_2^+; & z^- &= z_1^- \sqcup z_2^-. \end{aligned}$$

Set $z_1 = z_1^+ - z_1^-$; $z_2 = z_2^+ - z_2^-$. It is clear that $z = z_1 \sqcup z_2$. Hence

$$z_1 = z_1^+ - z_1^- \sqsubseteq x,$$
$$z_2 = z_2^+ - z_2^- \sqsubseteq y,$$

and the proof is finished. □

There is the following natural projection of a Dedekind-complete vector lattice onto a lateral band having nice properties.

Lemma 2.12. *Let D be a lateral band of a Dedekind complete vector lattice E. Then the map $\mathfrak{p}_D : E \to E$ defined by setting, for every $x \in E$,*

$$\mathfrak{p}_D x = x^D \tag{2.1}$$

is:

(1) *a projection of E onto D such that $\mathfrak{p}_D x \sqsubseteq x$ for all $x \in E$;*
(2) *a disjointness-preserving operator;*
(3) *an orthogonally additive operator.*

Proof.

(1) The fact that $\mathfrak{p}_D$ is a projection of E onto D is proved in ([14, Theorem 3]), and the property $\mathfrak{p}_D(x) \sqsubseteq x$ for all $x \in E$ is obvious.
(2) The properties are proved in [14, Theorem 3].
(3) Fix any $x, y \in E$ with $x \perp y$ and $z \in \mathcal{F}_{x+y} \cap D$. Then by Lemma 2.11, there exists a pair of elements $z_1, z_2 \in E$ such that $z_1 \in \mathcal{F}_x \cap D$, $z \in \mathcal{F}_y \cap D$, and $z = z_1 \sqcup z_2$. Thus

$$z = z_1 + z_2 \sqsubseteq \mathfrak{p}_D x + \mathfrak{p}_D y,$$

and passing to the supremum with respect to the partial order $\sqsubseteq$ over all $z \in \mathfrak{F}_{x+y} \cap D$ in the left-hand side of the above formula, we obtain $\mathfrak{p}_D(x + y) \sqsubseteq \mathfrak{p}_D x + \mathfrak{p}_D y$. On the other hand, for every $z_1 \in \mathcal{F}_x \cap D$ and $z_2 \in \mathcal{F}_y \cap D$, the sum $z_1 + z_2$ belongs to $\mathcal{F}_{x+y} \cap D$ and therefore

$$z_1 + z_2 = z \sqsubseteq \mathfrak{p}_D(x + y).$$

Passing to the supremum with respect to the partial order $\sqsubseteq$ in the left-hand side of the above formula, first over all $z_1 \in \mathcal{F}_x \cap D$ and then over all $z_2 \in \mathcal{F}_y \cap D$, we obtain $\mathfrak{p}_D x + \mathfrak{p}_D y \sqsubseteq \mathfrak{p}_D(x + y)$. Finally, we have the equality

$$\mathfrak{p}_D x + \mathfrak{p}_D y = \mathfrak{p}_D(x + y)$$

and the proof is finished. □

Remark 2.13. If in Lemma 2.12 D is a band, then $\mathfrak{p}_D$ is a band projection onto the band D.

Definition 2.14. Lateral bands D_1 and D_2 are said to be *disjoint* if $D_1 \cap D_2 = \{0\}$. Let E be a vector lattice, and let $D_1, \dots, D_n \in \mathfrak{Lb}(E)$. The set $\{\bigsqcup_{i=1}^n x_i : x_i \in D_i; 1 \le i \le n\}$ is said to be *a disjoint sum* of $D_1, \dots, D_n$ and is denoted by $\bigsqcup_{i=1}^n D_i$. If $n = 2$, we use the notation $D_1 \sqcup D_2$. We denote by $\mathfrak{D}_0(E)$ (or $\mathfrak{D}_0$ for short) the set of all finite disjoint sums (D_i) of mutually disjoint lateral bands in E such that $E = \bigsqcup_{i=1}^n D_i$, $n \in \mathbb{N}$.

Example 2.15. Let E be a vector lattice with the projection property, and let D be a subset of E. Then $\{D\}^{\perp\perp}$ and $\{D\}^{\perp}$ are disjoint bands and consequently disjoint lateral bands, and there is the decomposition

$$E = \{D\}^{\perp\perp} \oplus \{D\}^{\perp}.$$

The next lemma provides another example of the decomposition of a vector lattice E into the disjoint sum of lateral bands.

Lemma 2.16. *Let E be a Dedekind-complete vector lattice, and let D be a lateral band of E. Put*

$$D^{\perp} = \big\{y \in E : \mathcal{F}_y \cap \mathcal{F}_x = \{0\}, x \in D\big\}.$$

Then D and $D^{\perp}$ are disjoint lateral bands, and there is the decomposition

$$E = D \oplus D^{\perp}.$$

Proof. First, we show that $D^{\perp}$ is a lateral band. We only need to check that the sum of two disjoint elements of $D^{\perp}$ belongs to $D^{\perp}$. Take a pair of disjoint elements $u, v \in D^{\perp}$, and let $z \sqsubseteq u + v$. Assume that there exists $x \in D$ with a nonzero fragment w of z such that $w \in \mathcal{F}_x \cap \mathcal{F}_z$. Then $w \sqsubseteq u + v$, and by Lemma 2.11 there exist elements w_1, w_2 such that $w_1 \sqsubseteq u$, $w_2 \sqsubseteq v$, and $w = w_1 \sqcup w_2$. Either w_1 or w_2 is a nonzero element. Assume that it is w_1. Then $w_1 \in \mathcal{F}_x \cap \mathcal{F}_u$, and we have a contradiction. It is clear that lateral band $D^{\perp}$ is disjoint to D. Take an arbitrary element $y \in E$. Since the vector lattice E is Dedekind-complete, then by Lemma 2.10 there exist y^D and $y^{D^{\perp}}$ and there is the decomposition of the element y into the sum of two disjoint fragments $y = y^D + y^{D^{\perp}}$. □

Remark 2.17. If in Lemma 2.16 a lateral band D coincides with $\mathcal{F}_x$ for some $x \in E$, then there is the decomposition of the vector lattice E into the disjoint sum of nonlinear sets $\mathcal{F}_x$ and $\mathcal{F}_x^{\perp}$.

Now we need some auxiliary lemmas.

Lemma 2.18. *Let E be a vector lattice. Then the set $\mathfrak{D}_0(E)$ is directed by an inclusion.*

Proof. Let $(D_i), (D'_j) \in \mathfrak{D}_0(E)$, $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$. Then $(D_i \cap D'_j) \in \mathfrak{D}_0$. Indeed, $D_i \cap D'_j$ is a lateral band for every pair (i, j), $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$. Lateral bands $D_i \cap D'_j$ and $D_l \cap D'_s$ are disjoint for different pairs (i, j) and (l, s). Let us show that $E = \bigsqcup_{i,j} D_i \cap D'_j$. Fix an arbitrary element $x \in E$. By the definition of (D_i), there exist $x_i \in D_i$, $i \in \{1, \dots, n\}$ such that $x = \bigsqcup_{i=1}^{n} x_i$ and $x_i = \bigsqcup_{j=1}^{m} y_i^j$ for some $y_i^j \in D'_j$, $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$. It is clear that $y_i^j \in D_i \cap D'_j$ for every (i, j), $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$. Consequently, any $x \in E$ has a representation $x = \bigsqcup_{j,i=1}^{m,n} y_i^j$, and therefore $(D_i \cap D'_j) \in \mathfrak{D}_0$. □

Lemma 2.19. *Let E, F be Dedekind-complete vector lattices, and let $D_1, D_2 \subset E$ be disjoint lateral bands. Then $T\mathfrak{p}_{D_1} \perp T\mathfrak{p}_{D_2}$ for any $T \in \mathcal{U}_+(E, F)$.*

Proof. Take an element $x \in E$. Then by Lemma 2.10, there exists $x^{D_1} \in \mathcal{F}_x \cap D_1$. Put $x' = x - x^{D_1}$. It is clear that $\mathcal{F}_{x'} \cap D_1 = \{0\}$, $\mathcal{F}_{x^{D_1}} \cap D_2 = \{0\}$, and $x = x^{D_1} \sqcup x'$. Now we may write

$$\begin{aligned}(T\mathfrak{p}_{D_1} \wedge T\mathfrak{p}_{D_2})x &= \inf\{T\mathfrak{p}_{D_1}y + T\mathfrak{p}_{D_2}z : x = y \sqcup z\} \\ &\leq T\mathfrak{p}_{D_1}x' + T\mathfrak{p}_{D_2}x^{D_1} = 0.\end{aligned}$$

□

Lemma 2.20. *Let E be a vector lattice, and let $x \in E_+$ and $x_1, x_2 \in \mathcal{F}_x$. Then $x_1 \wedge x_2 \in \mathcal{F}_{x_i}$, $i \in \{1,2\}$.*

Proof. First, we prove that $x_1 \wedge x_2 \in \mathcal{F}_{x_1}$:

$$\begin{aligned}|x_1 - x_1 \wedge x_2| \wedge |x_1 \wedge x_2| &= \big|(x - x_1) \vee (x - x_2)\big| \wedge |x_1 \wedge x_2| \\ &\leq \big(|x - x_1| \vee |x - x_2|\big) \wedge \big(|x_1| \wedge |x_2|\big) \\ &= |x - x_1| \wedge \big(|x_1| \wedge |x_2|\big) \vee |x - x_2| \wedge \big(|x_1| \wedge |x_2|\big) \\ &\leq \big(|x - x_1| \wedge |x_1|\big) \vee \big(|x - x_2| \wedge |x_2|\big) = 0.\end{aligned}$$

The same arguments show that $x_1 \wedge x_2 \in \mathcal{F}_{x_2}$. □

Lemma 2.21. *Let E, F be Dedekind-complete vector lattices, and let $D_1, D_2 \subset E$ be disjoint lateral bands. Then $D_1 \sqcup D_2$ is also a lateral band and $\mathfrak{p}_{D_1 \sqcup D_2} = \mathfrak{p}_{D_1} + \rho_{D_2}$.*

Proof. First, we check condition (1) from Definition 2.3. Let $z \sqsubseteq x_1 \sqcup x_2$, where $x_i \in D_i$, $i \in \{1,2\}$. Then by Lemma 2.11, there exist z_1, z_2 such that $z_1 \sqsubseteq x_1$, $z_2 \sqsubseteq x_2$, and $z = z_1 \sqcup z_2$. Hence $z_i \in D_i$, $i \in \{1,2\}$ and $z \in D_1 \sqcup D_2$. Let us prove condition (2) from Definition 2.3. Take elements $x, y \in D_1 \sqcup D_2$, $x \perp y$. Then there exist $z_1, h_1 \in D_1$ and $z_2, h_2 \in D_2$, so that $x = z_1 \sqcup z_2$, $y = h_1 \sqcup h_2$. So we may write

$$x + y = z_1 + z_2 + h_1 + h_2 = w_1 \sqcup w_2 \in D_1 \sqcup D_2,$$

where $w_i = z_i \sqcup h_i$, $i \in \{1,2\}$, and $w_1 \perp w_2$. Clearly, the set $D_1 \sqcup D_2$ is laterally closed. Now we show that $\mathfrak{p}_{D_1 \sqcup D_2} = \mathfrak{p}_{D_1} + \rho_{D_2}$. Denote the set $D_1 \sqcup D_2$ by H, and fix $x \in E$. Then for every $y \sqsubseteq x$, $y \in H$, we have $y = y_1 \sqcup y_2$, $y_i \in D_i$, $i \in \{1,2\}$, and the following relations

$$\begin{aligned}y = y_1 + y_2 &\sqsubseteq \mathfrak{p}_{D_1}x + \mathfrak{p}_{D_2}x, \\ \mathfrak{p}_H x &\sqsubseteq \mathfrak{p}_{D_1}x + \mathfrak{p}_{D_2}x\end{aligned}$$

hold for every $x \in E$. Observe that if $u, v \in \mathcal{F}_x$, $v \in D_1$, $v \in D_2$, then $u \perp v$. Indeed, assume that $|u| \wedge |v| > 0$. Then we may write

$$\begin{aligned}0 < |v| \wedge |v| &= (u_+ + u_-) \wedge (v_+ + v_-) \\ &\leq u_+ \wedge v_+ + u_+ \wedge v_- + u_- \wedge v_+ + u_- \wedge v_-.\end{aligned}$$

Then by Lemmas 2.9 and 2.20, we deduce that there exists a nonzero fragment $h \in \mathcal{F}_u \cap \mathcal{F}_v$ and therefore that $h \in D_1 \cap D_2$, which is a contradiction. Hence, if $x \in E$, $u, v \in \mathcal{F}_x$, $u \in D_1$, $v \in D_2$, then $u \perp v$ and

$$u + v \sqsubseteq \mathfrak{p}_H x.$$

Passing to the supremum with respect to the partial order $\sqsubseteq$ over all $u \in \mathcal{F}_x \cap D_1$ and over all $v \in \mathcal{F}_x \cap D_2$ in the left-hand side of the above formula, we obtain $\mathfrak{p}_{D_1}x + \mathfrak{p}_{D_2}x \sqsubseteq \mathfrak{p}_H x$. Thus $\mathfrak{p}_{D_1}x + \mathfrak{p}_{D_2}x = \mathfrak{p}_H x$, and the proof is finished. $\square$

Now we are ready to introduce the main technical tool of this section.

Definition 2.22. Let E, F be Dedekind-complete vector lattices. The *shadow* of an abstract Urysohn operator $S \in \mathcal{U}(E,F)$ is defined to be the set

$$\mathfrak{Sh}(S) = \big\{T \in \mathcal{U}(E,F) : \big[T\mathfrak{p}_D(E)\big] \leq \big[S\mathfrak{p}_D(E)\big]; D \in \mathfrak{Lb}(E)\big\}.$$

The next lemma is important for further consideration.

Lemma 2.23. *Let E, F be Dedekind-complete vector lattices. Then for any operator $S \in \mathcal{U}(E,F)$, the following assertions hold:*

(a) *$\mathfrak{Sh}(S)$ is a band in $\mathcal{U}(E,F)$;*
(b) *$\mathfrak{Sh}(S) = \mathfrak{Sh}(|S|) \supset \{S\}^{\perp\perp}$;*
(c) *the band projection T^S of an operator $T \in \mathcal{U}(E,F)$ to $\mathfrak{Sh}(S)$ is calculated by the rule*

$$T^S = \inf\Big\{\sum_i \big[S\mathfrak{p}_{D_i}(E)\big]T\mathfrak{p}_{D_i}(E) : (D_i) \in \mathfrak{D}_0\Big\}. \tag{2.2}$$

Moreover, the set of fragments in (2.2) is downward directed.

Proof. First, we prove assertion (a). If $G_i \subset G$, $i \in \{1,2\}$, are bands in a vector lattice F, then we have that $(G_1 \cup G_2)^{\perp\perp} \subset G$ and therefore that $\mathfrak{Sh}(S)$ is a vector subspace of $\mathcal{U}(E,F)$. If $T \in \mathfrak{Sh}(S)$ and D is a lateral band in E, then by Theorem 1.4 $T^+\mathfrak{p}_D x = \sup\{Ty : y \sqsubseteq x^D\}$ for every $x \in E$. Hence

$$\big[T^+\mathfrak{p}_D(E)\big] = \sup\big\{[Ty] : y \sqsubseteq x^D, x \in E\big\} \leq \big[\pi^D S(E)\big].$$

Thus $\mathfrak{Sh}(S)$ is a vector sublattice. It is clear that $\mathfrak{Sh}(S)$ is also an order ideal in $\mathcal{U}(E,F)$. Now, if $0 \leq T_\alpha \uparrow T$ and $T_\alpha \in \mathfrak{Sh}(S)$, then $[T\mathfrak{p}_D x] = \sup[T_\alpha \mathfrak{p}_D x] \leq [S\mathfrak{p}_D(E)]$. We now prove assertion (b). It is clear that $\mathfrak{Sh}(T) \subset \mathfrak{Sh}(S)$ for every operator $T \in \mathfrak{Sh}(S)$. The inclusions $S^+, S^- \in \mathfrak{Sh}(S) \cap \mathfrak{Sh}(|S|)$ imply that $\mathfrak{Sh}(S) = \mathfrak{Sh}(|S|)$. If $0 \leq T \in \{S\}^{\perp\perp} = \{|S|\}^{\perp\perp}$, then $T = \sup_n n|S| \wedge T$. Thus, we obtain

$$\begin{aligned}[T\mathfrak{p}_D x] &= \sup_n \big[\big(n|S| \wedge T\big)\mathfrak{p}_D x\big] \\ &\leq \sup_n \big[\big(n|S|\big)\mathfrak{p}_D x\big] = \big[|S|\mathfrak{p}_D(E)\big]\end{aligned}$$

for all elements $x \in E$. Consequently, $T \in \mathfrak{Sh}(|S|)$. Finally, we prove the last assertion (c). Let

$$h_S(T) = \inf\Big\{\sum_i \big[S\mathfrak{p}_{D_i}(E)\big]T\mathfrak{p}_{D_i} : (D_i) \in \mathfrak{D}_0\Big\}.$$

We must check the following relations for any $T \in \mathcal{U}_+(E,F)$:

(1) $0 \leq h_S(T) \leq T$;
(2) $h_S(h_S(T)) = h_S(T)$;

(3) $h_S(T) = T \Leftrightarrow T \in \mathfrak{Sh}(S)$;
(4) h_S is a linear operator.

By Lemma 2.21, for every $(D_i) \in \mathfrak{D}_0$ we have $\mathfrak{p}_{\bigsqcup D_i} = I$, where I is an identity operator. Therefore we have the operator $0 \leq h_S(T) \leq T$ and formula (1) is proved. Moreover, by Lemma 2.19 the operator $\sum_i [S\mathfrak{p}_{D_i}(E)]T\mathfrak{p}_{D_i}$ is a fragment of T for every $(D_i) \in \mathfrak{D}_0$. Observe that by Lemma 2.18, the finer the partition (D_i) is, the smaller this fragment becomes. Hence, $h_S(T)$ is the (o)-limit of the decreasing net of fragments $T_{(D_i)}$. The formula (3) is a consequence of the following chain of equivalent assertions:

$$\begin{aligned} h_S(T) = T &\Leftrightarrow \forall (D_i) \in \mathfrak{D}_0 \sum_i \big[S\mathfrak{p}_{D_i}(E)\big]T\mathfrak{p}_{D_i} = T \\ &= \sum_i T\mathfrak{p}_{D_i} \Leftrightarrow \forall D \in \mathfrak{Lb}(E)\big[S\mathfrak{p}_D(E)\big]T\mathfrak{p}_D = T\mathfrak{p}_D \\ &\Leftrightarrow \forall D \in \mathfrak{Lb}(E)\big[T\mathfrak{p}_D(E)\big] \leq \big[S\mathfrak{p}_D(E)\big]. \end{aligned}$$

If $T_1, T_2 \geq 0$, then for any $(D_i), (D_k) \in \mathfrak{D}_0$ we have

$$\begin{aligned} &\sum_i \big[S\mathfrak{p}_{D_i}(E)\big]T_1\mathfrak{p}_{D_i} + \sum_j \big[S\mathfrak{p}_{D_j}(E)\big]T_2\mathfrak{p}_{D_j} \\ &\quad \geq \sum_k \big[S\mathfrak{p}_{D_k}(E)\big](T_1 + T_2)\mathfrak{p}_{D_k} \\ &\quad = \sum_k \big[S\mathfrak{p}_{D_k}(E)\big]T_1\mathfrak{p}_{D_k} + \sum_k \big[S\mathfrak{p}_{D_k}(E)\big]T_2\mathfrak{p}_{D_k}, \end{aligned}$$

where $(D_k) \in \mathfrak{D}_0$ is finer than (D_i) and (D_j). Taking the infimum, we obtain

$$h_S(T_1) + h_S(T_2) = h_S(T_1 + T_2).$$

It remains to verify the equality (2). Suppose that $W = h_S(T)$, with $T \in \mathcal{U}_+(E, F)$. For any $D \in \mathfrak{Lb}(E)$, we have

$$\begin{aligned} W\mathfrak{p}_D &= \inf\Big\{\sum_i \big[S\mathfrak{p}_{D_i}(E)\big]T\mathfrak{p}_{D_i}\mathfrak{p}_D : (D_i) \in \mathfrak{D}_0\Big\} \\ &= \inf\Big\{\sum_i \big[S\mathfrak{p}_{D'_i}(E)\big]T\mathfrak{p}_{D'_i}\mathfrak{p}_D : \bigsqcup(D'_i) = D\Big\}. \end{aligned}$$

Thus, $[W\mathfrak{p}_D(E)] \leq [S\mathfrak{p}_D(E)]$ for every $D \in \mathfrak{Lb}(E)$. By the equivalence (3) established above, we obtain $W = h_S(W)$. □

Let E be a vector lattice with a principal projection property, and let F be a Dedekind-complete vector lattice. Then the infimum in Theorem 1.4 can be calculated by the formula

$$(T \wedge S)x := \inf\big\{T\sigma x + S\sigma^{\perp}x : \sigma \in \mathcal{B}(E)\big\}.$$

Observe that every Dedekind-complete vector lattice E has a principal projection property. Now, we are ready to present the main result of this section.

Theorem 2.24. *Let E, F be Dedekind-complete vector lattices. Then the equality $\mathfrak{Sh}(S) = \{S\}^{\perp\perp}$ holds for any disjointness-preserving operator $S \in \mathcal{U}_+(E, F)$. In particular, the following band projection formula is valid for every $T \in \mathcal{U}_+(E, F)$:*

$$[S]T = \inf\Big\{\sum_i \big[S\mathfrak{p}_{D_i}(E)\big]T\mathfrak{p}_{D_i} : (D_i) \in \mathfrak{D}_0\Big\}. \tag{2.3}$$

Proof. Let D be a band in the vector lattice E. We remark that $T\mathfrak{p}_D = T\sigma_D$, where σ_D is an order projection onto the band D. Note that the equality $[S\sigma_D(E)]T = T\sigma_D$ is valid for every operator $T \in \mathfrak{Sh}(S)$ and every band $D \subset E$. Indeed, since S is a disjointness-preserving operator, the order projections $[S\sigma(E)]$ and $[S\sigma^\perp(E)]$ are disjoint. Taking into account that

$$\begin{gathered}\big[S\sigma(E)\big]S\sigma^\perp(E) = 0; \qquad \big[S\sigma^\perp(E)\big]S\sigma(E) = \big[S\sigma(E)\big]^\perp S\sigma(E) = 0;\\ \big[T\sigma(E)\big] \le \big[S\sigma(E)\big]; \qquad \big[T\sigma^\perp(E)\big] \le \big[S\sigma^\perp(E)\big],\end{gathered}$$

we have

$$\begin{aligned}\big[S\sigma(E)\big]Tx &= \big[S\sigma(E)\big]T(\sigma x + \sigma^\perp x) = \big[S\sigma(E)\big]T\sigma x + \big[S\sigma(E)\big]T\sigma^\perp x\\ &= \big[S\sigma(E)\big]T\sigma x = \big(\big[S\sigma(E)\big] + \big[S\sigma(E)\big]^\perp\big)T\sigma x = T\sigma x\end{aligned}$$

for every $x \in E$. Now, take a positive operator $T \in \mathfrak{Sh}(S)$ such that $T \wedge S = 0$. Assume that there exists $e \in E$ such that $Te > 0$. Let $D = \mathcal{F}_e$. We remark that $[S\mathfrak{p}_D(E)] = [Se]$ and $[T\mathfrak{p}_D(E)] = [Te]$. By our assumption, $[T\mathfrak{p}_D(E)] \le [S\mathfrak{p}_D(E)]$. Hence $Se > 0$ and $Se \wedge Te > 0$. Observe that for a vector lattice F with the projection property, the following inequality holds:

$$\begin{gathered}u \wedge v \le \varrho u + \varrho^\perp v;\\ u, v \in F_+; \qquad \varrho \in \mathfrak{B}(F).\end{gathered}$$

Indeed,

$$\begin{aligned}(u \wedge v) &= (\varrho + \varrho^\perp)(u \wedge v)\\ &= \varrho(u \wedge v) + \varrho^\perp(u \wedge v) \le \varrho u + \varrho^\perp v.\end{aligned}$$

Now, for any $\sigma \in \mathcal{B}(E)$, we may write

$$\begin{aligned}&Te \wedge Se \le T\sigma e + S\sigma^\perp e = \big[S\sigma(E)\big]Te + \big[S\sigma^\perp(E)\big]Se;\\ &Te \wedge Se \le \inf_{\sigma \in \mathcal{P}(E)}\{T\sigma e + S\sigma^\perp e\} = (T \wedge S)e = 0.\end{aligned}$$

Hence for all elements $e \in E$, we have $Te = 0$. Thus the inclusion $T \in \{S\}^{\perp\perp}$ is proved. The equality 2.3 is the consequence of Lemma 2.23. □

Note that for linear regular operators, a similar theorem was proved by Kolesnikov in [20, p. 515].

3. The Radon–Nikodým-type theorem for a positive disjointness-preserving operator

The Radon–Nikodým theorem is a well-known classical result of functional analysis. The aim of this section is to prove the Radon–Nikodým-type theorem for a positive, disjointness-preserving abstract Urysohn operator.

First we show (as in the linear case) that the module and the positive and negative parts of a disjointness-preserving abstract Urysohn operator can be evaluated pointwise.

Lemma 3.1. *Let E and F be vector lattices with F Dedekind-complete, and let $T \in \mathcal{U}(E,F)$ be a disjointness-preserving operator. Then for every $x \in E$, the following conditions hold:*

(1) $|T|x = |Tx|$;
(2) $T^+x = (Tx)^+$;
(3) $T^-x = (Tx)^-$;
(4) $T^+x \wedge T^-x = 0$.

Proof. First we prove the equality $|T|x = |Tx|$, $x \in E$. By Theorem 1.4, we have that

$$|T|x = \sup\{Ty - Tz : x = y \sqcup z\} \geq Tx \vee (-Tx) = |Tx|.$$

We need to prove the reverse inequality. Take $y, z \in E$ such that $x = y \sqcup z$. Then $Ty \perp Tz$ and we may write

$$Ty - Tz \leq |Ty - Tz| = |Ty + Tz| = \big|T(y+z)\big| = |Tx|.$$

Passing to the supremum on the left-hand side of the above inequality over all fragments y, z of x such that $x = y \sqcup z$, we deduce that $|T|x \leq |Tx|$. Hence we prove that $|T|x = |Tx|$ for any $x \in E$. Since

$$T^+ = \frac{1}{2}\big(|T| + T\big); \qquad T^- = \frac{1}{2}\big(|T| - T\big),$$

we have

$$T^+x = \frac{1}{2}\big(|T| + T\big)x = \frac{1}{2}\big(|T|x + Tx\big) = \frac{1}{2}\big(|Tx| + Tx\big) = (Tx)^+;$$
$$T^-x = \frac{1}{2}\big(|T| - T\big)x = \frac{1}{2}\big(|T|x - Tx\big) = \frac{1}{2}\big(|Tx| - Tx\big)(Tx)^+ = (Tx)^-.$$

Finally,

$$T^+x \wedge T^-x = (Tx)^+ \wedge (Tx)^- = 0$$

and the proof is finished. □

Now we need some auxiliary lemmas.

Lemma 3.2. *Let E, F be vector lattices with F Dedekind-complete, let $T \in \mathcal{U}_+(E,F)$ be a disjointness-preserving operator, and let $S \in \mathcal{U}_+(E,F)$ and $Sx \in \{Tx\}^{\perp\perp}$ for all $x \in E$. Then $S \in \{T\}^{\perp\perp}$.*

Proof. Since $0 \leq S \in \{T\}^{\perp\perp}$, we have that $S \wedge nT \uparrow S$. Thus $(S \wedge nT)x \uparrow Sx$ for every $x \in E$. Then

$$0 \leq (S \wedge nT)x \leq nTx \Rightarrow (S \wedge nT)x \in \{Tx\}^{\perp\perp}$$

and we deduce that $Sx \in \{Tx\}^{\perp\perp}$. □

Lemma 3.3. *Let E, F be vector lattices with F Dedekind-complete, and let $S, T \in \mathcal{U}_+(E, F)$ be disjointness-preserving operators. Then $T + S$ is a disjointness-preserving operator if and only if $Sx \perp Ty$ for every pair of disjoint elements $x, y \in E$.*

Proof. Take a pair of disjoint elements $x, y \in E$, and assume that $T + S$ is a disjointness-preserving operator. Then we may write

$$0 \leq |Sx| \wedge |Ty| = Sx \wedge Ty \leq (S+T)x \wedge (S+T)y = 0.$$

Hence $Sx \perp Ty$. On the other hand,

$$Sx \wedge Sy = Tx \wedge Ty = Sx \wedge Ty = Sy \wedge Tx = 0.$$

Thus

$$\begin{aligned} 0 &\leq (S+T)x \wedge (S+T)y \\ &\leq (Sx \wedge Sy) \wedge (Tx \wedge Ty) \wedge (Sx \wedge Ty) \wedge (Sy \wedge Tx) = 0. \end{aligned}$$ □

Lemma 3.4. *Let E, F be vector lattices with F Dedekind-complete, let $T \in \mathcal{U}_+(E, F)$ be a disjointness-preserving operator, and let $S \in \mathcal{U}_+(E, F)$ satisfy $Sx \in \{Tx\}^{\perp\perp}$ for all $x \in E$. Then S is a disjointness-preserving operator.*

Proof. If $x, y \in E$, $x \perp y$, then we have

$$0 \leq (Sx \wedge nTx) \wedge (Sy \wedge mTy) \leq (n+m)(Tx \wedge Ty) = 0.$$

Thus

$$0 \leq (Sx \wedge nTx) \wedge (Sy \wedge mTy) = 0$$

for any $n, m \in \mathbb{N}$. Since $Sx \in \{Tx\}^{\perp\perp}$ and $Sy \in \{Ty\}^{\perp\perp}$, we have that $Sx \wedge nTx \uparrow Sx$, $Sy \wedge mTy \uparrow Sy$, and therefore that $Sx \wedge Sy = 0$. □

Remark 3.5. Observe by Lemma 3.2 that if $0 \leq S \in \{T\}^{\perp\perp}$, then S is a disjointness-preserving operator.

Lemma 3.6. *Let E, F be vector lattices with F Dedekind-complete, let $T \in \mathcal{U}_+(E, F)$ be a disjointness-preserving operator, and let $0 \leq S_1, S_2 \in \{T\}^{\perp\perp}$. Then*

$$(S_1 \wedge S_2)x = S_1x \wedge S_2x.$$

Proof. Let $S_1' = S_1 - S_1 \wedge S_2$ and $S_2' = S_2 - S_1 \wedge S_2$. Put $S' = S_1' - S_2'$. Since $S_1', S_2' \in \{T\}^{\perp\perp}$, then $|S'| \in \{T\}^{\perp\perp}$ and therefore $|S'|$ is a disjointness-preserving operator. From $S_1' \wedge S_2' = 0$, we deduce that $(S')^+ = S_1'$ and $(S')^- = S_2'$. By Lemma 3.1, we get $S_1'x \wedge S_2'x = 0$ for every $x \in E$. Now we may write

$$\big(S_1x - (S_1 \wedge S_2)x\big) \wedge \big(S_2x - (S_1 \wedge S_2)x\big) = 0,$$

and we deduce that

$$(S_1 \wedge S_2)x = S_1x \wedge S_2x$$

for all $x \in E$. ☐

Lemma 3.7. *Let E, F be vector lattices with F Dedekind-complete, let $S_1, S_2 \in \mathcal{U}_+(E, F)$, and let S_1+S_2 be a disjointness-preserving operator. Then the following statements are equivalent:*

1. $S_1 \wedge S_2 = 0$*;*
2. $S_1x \wedge S_2x = 0$ *for all* $x \in E$*.*

Proof. (2) ⇒ (1): Take an arbitrary element $x \in E$. By Theorem 1.4,

$$(S_1 \wedge S_2)x = \inf\{S_1y + S_2z : x = y \sqcup z\} \leq S_1x \wedge S_2x = 0$$

and the implication is proved. (1) ⇒ (2): Applying Lemma 3.7 to $T = S_1 + S_2$, we have

$$S_1x \wedge S_2x = (S_1 \wedge S_2)x = 0$$

and the proof if finished. ☐

The following theorem is the main result of this section.

Theorem 3.8. *Let E, F be vector lattices with F Dedekind-complete, let $T \in \mathcal{U}_+(E, F)$ be a disjointness-preserving operator, and let $S \in \mathcal{U}_+(E, F)$. Then the following statements are equivalent:*

1. $S \in \{T\}^{\perp\perp}$*;*
2. $Sx \in \{Tx\}^{\perp\perp}$ *for all* $x \in E$*.*

Proof. The implication (1) ⇒ (2) is proved in Lemma 3.2. Now we prove the implication (2) ⇒ (1). By Lemma 3.3, S is a disjointness-preserving operator. We claim that $Sx \perp Ty$ for any disjoint elements $x, y \in E$. Actually, $Tx \wedge Ty$, and hence $\{Tx\}^{\perp\perp} \cap \{Ty\}^{\perp\perp} = \{0\}$. Consequently, $Sx \in \{Tx\}^{\perp\perp}$ and $Ty \in \{Ty\}^{\perp\perp}$. Now, by Lemma 3.3 we have that $S + T$ is a disjointness-preserving operator. Taking into account the decomposition

$$\mathcal{U}(E, F) = \{T\}^{\perp\perp} \oplus \{T\}^{\perp},$$

we have that there is the representation $S = S_1 + S_2$, where $0 \leq S_1 \in \{T\}^{\perp\perp}$ and $0 \leq S_2 \in \{T\}^{\perp}$. Since $0 \leq S_2 + T \leq S + T$, we get that $S_2 + T$ is a disjointness-preserving operator. By Lemma 3.3, we have that $S_2x \in \{Tx\}^{\perp}$. On the other hand, from $0 \leq S_2x \leq Sx$ and $Sx \in \{Tx\}^{\perp\perp}$ for all $x \in E$, we deduce that $S_2x = 0$ for all elements $x \in E$ and therefore that $S_2 = 0$. Thus $S = S_1 \in \{T\}^{\perp\perp}$ and the proof is completed. ☐

We remark that the Radon–Nikodým-type theorem for linear lattice homomorphisms and linear operators having the Maharam property was proved in [23] (see also [17]).

4. Urysohn lattice homomorphisms in order ideals of spaces of measurable functions

In this section, we investigate Urysohn lattice homomorphisms on order ideals of spaces of all measurable functions, and we obtain the analytical representation for this class of operators.

Definition 4.1. Let E be a vector lattice, and let X be a vector space. An orthogonally additive map $T : E \to X$ is said to be *even* if $T(x) = T(-x)$ for any $x \in E$. If E, F are vector lattices, then the set of all even abstract Urysohn operators from E to F is denoted by $\mathcal{U}^{\mathrm{ev}}(E, F)$.

If E, F are vector lattices with F Dedekind-complete, then the space $\mathcal{U}^{\mathrm{ev}}(E, F)$ is a Dedekind-complete sublattice of $\mathcal{U}(E, F)$ (see [28, Lemma 3.2]).

Now we are ready to give the following definition.

Definition 4.2. Let E and F be vector lattices. The operator $T \in \mathcal{U}^{\mathrm{ev}}_+(E, F)$ is called a *Urysohn lattice homomorphism* if the following conditions hold:

(1) $T(x \vee y) = Tx \vee Ty$ for every $x, y \in E_+$;
(2) $T(x \wedge y) = Tx \wedge Ty$ for every $x, y \in E_+$.

It is clear that a Urysohn lattice homomorphism is an increasing operator on E_+.

Example 4.3. If $E = F = \mathbb{R}$, then the set of all Urysohn lattice homomorphisms on $\mathbb{R}$ coincides with the set of all even, nondecreasing on $\mathbb{R}_+$ functions $f : \mathbb{R} \to \mathbb{R}_+$ such that $f(0) = 0$ and $f([a, b])$ is a bounded set for every $a, b \in \mathbb{R}$.

Now we need some auxiliary lemmas.

Lemma 4.4 ([1, Lemma 2.1])**.** *Let E, F be vector lattices, and let $T \in \mathcal{U}^{\mathrm{ev}}_+(E, F)$. Then the following statements are equivalent:*

(1) *$T(x \vee y) = Tx \vee Ty$ for every $x, y \in E_+$;*
(2) *$T(x \wedge y) = Tx \wedge Ty$ for every $x, y \in E_+$;*
(3) *T is a Urysohn lattice homomorphism from E to F.*

Lemma 4.5. *Let E, F be vector lattices, and let $T \in \mathcal{U}^{\mathrm{ev}}_+(E, F)$. Then the following statements are equivalent:*

(1) *$T \in \mathcal{U}^{\mathrm{ev}}_+(E, F)$ is a disjointness-preserving and an increasing operator on E_+;*
(2) *T is a Urysohn lattice homomorphism from E to F.*

Proof. The implication (2) $\Rightarrow$ (1) is obvious. Let us prove (1) $\Rightarrow$ (2). By Lemma 4.4, it is enough to prove that $T(x \wedge y) = Tx \wedge Ty$ for all $x, y \in E_+$. Identify E with a vector sublattice of the Dedekind-complete vector lattice $C_\infty(Q)$ of all extended real-valued continuous functions on some extremally disconnected compact space Q. Put $A = \{t \in Q : x(t) \leq y(t)\}$ and $B = \{t \in Q : y(t) < x(t)\}$. Observe that A and B are clopen, disjoint subsets of Q. Then

$$x \wedge y = x1_A + y1_B; \qquad x \vee y = x1_B + y1_A;$$
$$x = x1_A + x1_B; \qquad y = y1_A + y1_B.$$

Now we have

$$T(x \wedge y) = T(x1_A + y1_B) = Tx1_A + Ty1_B.$$

On the other hand, taking into account that $x1_B \perp y1_A$, $x1_A \perp y1_B$, we may write

$$\begin{aligned} Tx \wedge Ty &= T(x1_A + x1_B) \wedge T(y1_A + y1_B) \\ &= (Tx1_A + Tx1_B) \wedge (Ty1_A + Ty1_B) \\ &\leq Tx1_A \wedge Ty1_A + Tx1_A \wedge Ty1_B \\ &\quad + Tx1_B \wedge Ty1_A + Tx1_B \wedge Ty1_B \\ &= Tx1_A + Ty1_B. \end{aligned}$$

Thus $T(x \wedge y) \geq Tx \wedge Ty$ and the equality $T(x \wedge y) = Tx \wedge Ty$ is proved. □

The following lemma provides a typical example of a Urysohn lattice homomorphism on the space of all measurable functions.

Lemma 4.6. *Let (A, Σ, μ) be σ-finite complete measure spaces, let $L_0(A, \Sigma, \mu)$ (or $L_0(\mu)$ for brevity) be a vector space of all measurable μ-almost everywhere finite real-valued functions on A, and let $N : A \times \mathbb{R} \to \mathbb{R}_+$ be a function satisfying the following conditions:*

- (C_0) *$N(t, 0) = 0$ for μ-almost all $t \in A$;*
- (C_1) *$N(\cdot, r)$ is μ-measurable for all $r \in \mathbb{R}$;*
- (C_2) *$N(t, \cdot)$ is continuous on $\mathbb{R}$ for μ-almost all $t \in A$;*
- (C_3) *$N(\cdot, r) = N(\cdot, -r)$ for μ-almost all $t \in A$ and every $r \in \mathbb{R}$;*
- (C_4) *$N(\cdot, x) < N(\cdot, r)$ for μ-almost all $t \in A$ and every $x, r \in \mathbb{R}_+$, $x < r$.*

Then the operator T defined by

$$(Tf)(t) = N\big(t, f(t)\big); \quad f \in L_0(\mu)$$

is a Urysohn lattice homomorphism from $L_0(\mu)$ to $L_0(\mu)$.

Proof. Recall that a function $N : A \times \mathbb{R} \to \mathbb{R}$ is considered a Carathéodory function if it satisfies conditions (C_1)–(C_2). It is well known that for every Carathéodory function N and every $f \in L_0(\mu)$, the function $N(\cdot, f(\cdot))$ also belongs to $L_0(\mu)$ (see, e.g., [5, Chapter 1.4]). Thus the operator $T : L_0(\mu) \to L_0(\mu)$ is well defined. In Example 2.2 it was proved that T is an orthogonally additive operator on $L_0(\mu)$. It is clear that T is a positive abstract Urysohn operator. Let us show that $T(f) = T(-f)$ for every $f \in L_0(\mu)$. Put $D_r := \{t \in A : N(t, r) \neq N(t, -r)\}$, where $r \in \mathbb{Q}$, and let $D = \bigcup_{r \in \mathbb{Q}} D_r$. By the condition (C_3) we have $\mu(D_r) = 0$ and therefore $\mu(D) = 0$. Take $x \in \mathbb{R}$, and let $t \notin D$. Then there exists a sequence $(r_n) \subset \mathbb{Q}$ with $x = \lim_{n\to\infty} r_n$ $(-x = \lim_{n\to\infty}(-r_n))$ and by the condition (C_2), we deduce that $N(t, x) = N(t, -x)$. Hence for all $t \notin D$, we have $N(t, f(t)) = N(t, -f(t))$, $f \in L_0(\mu)$. Thus $T \in \mathcal{U}_+^{\mathrm{ev}}(L_0(\mu))$. Take $f, g \in L_0(\mu)$ with $f \perp g$. It means that $\mu\{t \in A : t \in \operatorname{supp}(f) \cap \operatorname{supp}(g)\} = 0$. Let $D_1 := \operatorname{supp}(f)$ and $D_2 := \operatorname{supp}(g)$. Now we may write

$$(Tf)(t) = N\big(t, f(t)\big) = N\big(t, f(t)1_{D_1}\big)$$

$$= N\big(t, f(t)\big)1_{D_1}(t) = (Tf)(t)1_{D_1}(t);$$
$$(Tg)(t) = (Tg)(t)1_{D_2}(t).$$

Hence T is a disjointness-preserving operator. Let us show that T is an increasing operator on $L_0(\mu)_+$. Set $\mathbb{Q}_+^2 = \{(p,q) : p,q \in \mathbb{Q}_+, p < q\}$. With every $(p,q) \in \mathbb{Q}_+^2$ we associate a measurable set $G_{p,q} = \{t \in A : N(t,p) > N(t,q)\}$. It is clear that $\mu(G_{p,q}) = 0$. Put $G = \bigcup_{(p,q)\in\mathbb{Q}_+^2} G_{p,q}$. Then $\mu(G) = 0$ and taking into account the condition (C_2), we have that $N(t,x) \leq N(t,y)$ for every $t \notin G$ and $x, y \in \mathbb{R}_+$, $x \leq y$. Take $f, g \in L_0(\mu)_+$ with $f \leq g$, and let $H = \{t \in A : f(t) > g(t)\}$. Clearly, $\mu(H) = 0$. Thus $N(t, f(t)) \leq N(t, g(t))$ for all $t \notin H \cup G$ and the assertion is proved. Then by Lemma 4.5, T is a Urysohn lattice homomorphism and the proof is finished. □

A map $N : A \times \mathbb{R} \to \mathbb{R}_+$ is said to be an $\mathfrak{H}$-*function* if it satisfies conditions (C_0)–(C_4) of Lemma 4.6. We need some information about Boolean homomorphisms.

Definition 4.7. Let $\mathfrak{A}, \mathfrak{B}$ be Boolean algebras. A map $\varphi : \mathfrak{A} \to \mathfrak{B}$ is said to be a *Boolean homomorphism* if the following conditions hold:

(1) $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ for all $x, y \in \mathfrak{A}$;
(2) $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$ for all $x, y \in \mathfrak{A}$;
(3) $\varphi(\mathbf{0}_{\mathfrak{A}}) = \mathbf{0}_{\mathfrak{B}}$ and $\varphi(\mathbf{1}_{\mathfrak{A}}) = \mathbf{1}_{\mathfrak{B}}$.

If, moreover, $\varphi(\bigvee_{i=1}^{\infty} x_i) = \bigvee_{i=1}^{\infty} \varphi(x_i)$ for every countable family (x_i) of mutually disjoint elements of $\mathfrak{A}$, then φ is called a σ-*Boolean homomorphism*, or σ-*homomorphism* for short.

Let (A, Σ, μ) and (B, Ξ, ν) be σ-finite and complete measure spaces. Let $\varphi : \Sigma \to \Xi$ be a σ-homomorphism of Boolean algebras Σ and Ξ. With $\varphi : \Sigma \to \Xi$ is associated a linear operator $\mathcal{S}_\varphi : L_0(\mu) \to L_0(\nu)$. Indeed, take a Σ-simple function $f = \sum_{i=1}^{n} r_i 1_{A_i} \geq 0$, where $A_i, i \in \{1, \dots, n\}$ are mutually disjoint μ-measurable subsets of A. Let

$$\mathcal{S}_\varphi(f) := \sum_{i=1}^{n} r_i 1_{\varphi(A_i)}.$$

Since φ is a σ-homomorphism, the function $\mathcal{S}_\varphi(f)$ is well defined and ν-measurable. For every μ-measurable function $f \geq 0$ there exists a nondecreasing sequence (f_n) of simple functions such that $f = \sup_n f_n$. Thus we put

$$\mathcal{S}_\varphi(f) = \sup_n \mathcal{S}_\varphi(f_n).$$

Taking into account that φ is a σ-homomorphism, we deduce that $\mathcal{S}_\varphi(f)$ is a well defined ν-measurable function. Finally, we put

$$\mathcal{S}_\varphi(f) = \mathcal{S}_\varphi(f_+) - \mathcal{S}_\varphi(f_-)$$

for every μ-measurable function f. It is clear that $\mathcal{S}_\varphi$ is a linear, order-continuous operator from $L_0(\mu)$ to $L_0(\nu)$.

Definition 4.8. Let E and F be order ideals in vector lattices $L_0(A, \Sigma, \mu)$ and $L_0(B, \Xi, \nu)$, respectively, and let $\varphi : \Sigma \to \Xi$ be a σ-homomorphism of Boolean algebras Σ and Ξ. We say that $\mathfrak{H}$-function $N : B \times \mathbb{R} \to \mathbb{R}_+$ is a (φ, E, F)-type if $N(s, \mathcal{S}_\varphi(f)(s)) \in F$ for every $f \in E$.

Definition 4.9. An orthogonally additive operator T from vector lattice E to vector lattice F is called *laterally full* if for every $x \in E$, $z \sqsubseteq Tx$ there exists $y \sqsubseteq x$ such that $Ty = z$.

Definition 4.10. An operator $T \in \mathcal{U}_+^{\mathrm{ev}}(E, F)$ is said to be *strictly positive* if the relation $Tx = 0$ implies that $x = 0$ for every $x \in E_+$.

The next theorem is the main result of this section.

Theorem 4.11. *Let (A, Σ, μ) and (B, Ξ, ν) be finite measure spaces, let E, F be order ideals in $L_0(\mu)$ and $L_0(\nu)$, respectively, and let $T \in \mathcal{U}_+^{\mathrm{ev}}(E, F)$. Then the following statements are equivalent.*

(1) *T is a strictly positive, laterally full, order-continuous Urysohn lattice homomorphism.*
(2) *There exist an $\mathfrak{H}$-function of (φ, E, F)-type $N : B \times \mathbb{R} \to \mathbb{R}_+$ and a surjective σ-homomorphism φ from the Boolean algebra Σ onto the Boolean algebra $\Xi_0 = \{A \cap \operatorname{supp} N : A \in \Xi\}$ such that for every $f \in E$ the following equality holds:*

$$(Tf)(s) = N\big(s, \mathcal{S}_\varphi(f)(s)\big). \tag{4.1}$$

Proof. (1) $\Rightarrow$ (2). First we prove that $\operatorname{supp} T(x1_V) = \operatorname{supp} T(y1_V)$ for any measurable subset V of A and any $x, y \in \mathbb{R}_+$. The sum $V \cup H$ of two measurable disjoint sets is denoted by $V \sqcup H$. Assume that for some $x < y$ the measurable set $\operatorname{supp} T(x1_V)$ is a proper subset of $\operatorname{supp} T(y1_V)$. Then there exists a decomposition $\operatorname{supp} T(y1_V) = \operatorname{supp} T(x1_V) \sqcup G$, where $\operatorname{supp} T(x1_V)$ and G are disjoint ν-measurable sets. Since the operator T is laterally full, there exists a μ-measurable subset H of V such that $\mu(H) > 0$, $T(y1_H) = T(y1_V)1_D$, and $T(y1_{V\setminus H}) = T(y1_V)1_{\operatorname{supp} T(x1_V)}$. Taking into account that the operator T is strictly positive, we deduce that $T(x1_H) > 0$. Elements $x1_H$ and $y1_{V\setminus H}$ are disjoint, but $\operatorname{supp}(Tx1_H) \subset \operatorname{supp}(Tx1_H)$, which is a contradiction; consequently, measurable sets $\operatorname{supp} T(x1_V)$ and $\operatorname{supp} T(y1_V)$ are coincident. Put $G = \operatorname{supp} T(1_A)$. We show that there is a surjective σ-homomorphism $\varphi : \Sigma \to \Xi_0$, where $\Xi_0 = \{H \cap G : H \in \Xi\}$. Indeed, for every $V \in \Sigma$ a map $\varphi : \Sigma \to \Xi_0$ is defined by

$$\varphi(V) = \operatorname{supp} T(1_V), \quad V \in \Sigma.$$

It is clear that φ maps μ-null subsets of A to ν-null subsets of B and $\varphi(A) = G$. For any μ-measurable subset V of A, we have

$$\begin{aligned}\varphi(A) &= \varphi\big((A\setminus V) \sqcup V)\big) = \operatorname{supp} T(1_{(A\setminus V)\sqcup V}) \\ &= \operatorname{supp} T(1_{A\setminus V}) \sqcup \operatorname{supp} T(1_V) = \varphi(A\setminus V) \sqcup \varphi(V) \\ &\Rightarrow \varphi(A\setminus V) = \varphi(A)\setminus\varphi(V).\end{aligned}$$

Take two disjoint μ-measurable sets V_1 and V_2. Then

$$\varphi(V_1 \sqcup V_2) = \operatorname{supp} T(1_{V_1 \sqcup V_2}) = \operatorname{supp} T(1_{V_1} + 1_{V_2}).$$

Taking into account that T is a disjointness-preserving operator, we get that measurable sets $\operatorname{supp} T(1_{V_1})$ and $\operatorname{supp} T(1_{V_2})$ are disjoint and $\operatorname{supp} T(1_{V_1} + 1_{V_2}) = \operatorname{supp} T(1_{V_1}) \sqcup \operatorname{supp} T(1_{V_2})$. Hence

$$\varphi(V_1 \vee V_2) = \varphi(V_1 \sqcup V_2) = \varphi(V_1) \sqcup \varphi(V_2) = \varphi(V_1) \vee \varphi(V_2),$$

and consequently φ is a Boolean homomorphism. Since the operator T is laterally full, then φ is a surjective homomorphism. Additionally, taking into account that the operator T is order-continuous, we deduce that φ is a σ-homomorphism. Now, put

$$N(s, x) = T(x 1_A)(s), \quad x \in \mathbb{R}.$$

It is clear that $N(\cdot, 0) = 0$ for ν-almost all $s \in B$ and that the function $N(\cdot, x)$ is ν-measurable for all $x \in \mathbb{R}$. Moreover, since the operator T is a Urysohn lattice homomorphism, we obtain that N satisfies conditions (C_0)–(C_4) of Lemma 4.6. Hence N is a $\mathfrak{H}$-function.

Now, take a Σ-simple function $f = \sum_{i=1}^n x_i 1_{A_i}$, where the A_i's are mutually disjoint μ-measurable subsets of A and $x_i \in \mathbb{R}$, $1 \leq i \leq n$. Then

$$\begin{aligned} T(f) &= T\Big(\sum_{i=1}^n x_i 1_{A_i}\Big) = \sum_{i=1}^n T(x_i 1_{A_i}) \\ &= \sum_{i=1}^n N(s, x_i) 1_{\varphi(1_{A_i})} = \sum_{i=1}^n N(s, x_i 1_{\varphi(1_{A_i})}) \\ &= N\Big(s, \sum_{i=1}^n x_i 1_{\varphi(1_{A_i})}\Big) = N\Big(s, S_\varphi\Big(\sum_{i=1}^n x_i 1_{A_i}\Big)\Big). \end{aligned}$$

Assume that f is an arbitrary element of E. There there exists a sequence of Σ-simple functions (f_n) which order-converges to f. Thus

$$Tf = \lim_{n\to\infty} T(f_n) = \lim_{n\to\infty} N(s, S_\varphi(f_n) = N\big(s, S_\varphi(f)\big).$$

Clearly, N is an $\mathfrak{H}$-function of (φ, E, F)-type and we get an analytical representation for the Urysohn lattice homomorphism T.

(2) $\Rightarrow$ (1) It is not difficult to check that the operator T defined by formula (4.1) is a strictly positive, order-continuous Urysohn lattice homomorphism. Let us prove that T is a laterally full operator. Indeed, take an element $f \in E$, and assume that g is a fragment of Tf. Then there exists a measurable set $D \in \Xi_0$ such that $g = (Tf)1_D$. On the other hand, since φ is a surjective σ-homomorphism, there exists a measurable set $H \in \Sigma$ such that $D = \varphi(H)$. Then we may write

$$\begin{aligned} (Tf)1_D(s) &= N\big(s, \mathcal{S}_\varphi(f)(s)\big)1_D(s) = N\big(s, \mathcal{S}_\varphi(f)1_D(s)\big) \\ &= N\big(s, \mathcal{S}_\varphi(f)1_{\varphi(H)}(s)\big) = N\big(s, \mathcal{S}_\varphi(f1_H)(s)\big) = (Tf1_H)(s). \end{aligned}$$

Hence T is a laterally full operator. □

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[1]MAI—Moscow Aviation Institute (National Research University), str. Orshanskaya 3, Moscow, 121552 Russia.
E-mail address: abasovn@mail.ru

[2]Southern Mathematical Institute of the Russian Academy of Sciences, str. Markusa 22, Vladikavkaz, 362027 Russia and RUDN University, 6 Miklukho-Maklaya st., Moscow, 117198 Russia.
E-mail address: plimarat@yandex.ru