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ON DOMAINS OF UNBOUNDED DERIVATIONS OF GENERALIZED B*-ALGEBRAS

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Abstract. We study properties under which the domain of a closed derivation $\delta : D(\delta) \to A$ of a generalized B*-algebra A remains invariant under analytic functional calculus. For a complete, generalized B*-algebra with jointly continuous multiplication, two sufficient conditions are assumed: that the unit of A belongs to the domain of the derivation, along with a condition related to the coincidence $\sigma_A(x) = \sigma_{D(\delta)}(x)$ of the (Allan) spectra for every element $x \in D(\delta)$. Certain results are derived concerning the spectra for a general element of the domain, in the realm of a domain which is advertibly complete or enjoys the Q-property. For a closed $*$-derivation δ of a complete GB*-algebra with jointly continuous multiplication such that $1 \in D(\delta)$ and x a normal element of the domain, $f(x) \in D(\delta)$ for every analytic function on a neighborhood of the spectrum of x. We also give an example of a closed derivation of a GB*-algebra which does not contain the identity element. A condition for a closed derivation of a GB*-algebra A to be the generator of a one-parameter group of automorphisms of A is provided along with a generalization of the Lumer–Phillips theorem for complete locally convex spaces.

1. Introduction

Generalized B*-algebras (GB*-algebras for short) are (in general, abstract) $*$-algebras consisting of unbounded linear operators on a Hilbert space, and they were first studied in 1967 by Allan in [2]. More precisely, GB*-algebras are

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locally convex $*$-algebras which are generalizations of C^*-algebras. Later, Dixon [9] extended the notion of a GB*-algebra to include nonlocally convex $*$-algebras.

The theory of unbounded $*$-derivations of C^*-algebras is well developed (see [7], [8], [15]). In particular, necessary and sufficient conditions are known which guarantee an unbounded $*$-derivation of a C^*-algebra A to generate a one-parameter automorphism group of A. In connection with this problem, the domains of unbounded $*$-derivations of C^*-algebras have been studied extensively. More importantly, if A is a C^*-algebra and $\delta : D(\delta) \to A$ is a closed, unbounded $*$-derivation of A with $x \in D(\delta)$, then $f(x) \in D(\delta)$ for all analytic functions f on a neighborhood of $\mathrm{Sp}_A(x) = \{\lambda \in \mathbb{C} : x - \lambda 1 \text{ is not invertible in } A\}$. We say that $D(\delta)$ is *closed under analytic functional calculus*.

One-parameter automorphism groups of C^*-algebras represent the time dynamics of quantum mechanical systems, and our physical world consists mainly of unbounded linear operators in a Hilbert space. It is well known that an everywhere defined derivation δ in a C^*-algebra A is bounded. In the case in which δ is an unbounded, densely defined derivation in A, the question of closability of the derivation arises. Motivated by the above, we were led to an investigation of closed, possibly unbounded, $*$-derivations of GB*-algebras. As for everywhere defined derivations in GB*-algebras, we were able to obtain certain results in previous work, for example, with respect to the innerness and continuity of these derivations, in cases where the GB*-algebra satisfied some particular properties (see, e.g., [30], [31]). Nevertheless, the picture is far from complete for the case of a general GB*-algebra.

One of the main results (and motivational forces) of this article is Proposition 3.1, which gives sufficient conditions for the domain of a closed $*$-derivation of a complete GB*-algebra with jointly continuous multiplication to be closed under analytic functional calculus. In Section 4, we prove one of the strongest results of this article, namely, Theorem 4.8: If $A[\tau]$ is a complete GB*-algebra with jointly continuous multiplication and $\delta : D(\delta) \to A$ is a closed $*$-derivation of A such that $1 \in D(\delta)$, then $\sigma_A(x) = \sigma_{D(\delta)}(x)$ for all normal elements $x \in D(\delta)$. (If A is a locally convex $*$-algebra, then $\sigma_A(x)$ represents the Allan spectrum of x; see Section 2.) It follows from this and Proposition 3.1 that the domain of such a closed $*$-derivation is closed under analytic functional calculus for all normal elements $x \in D(\delta)$.

In Section 3, we give some general results involving conditions whereby the domain of a closed $*$-derivation of a GB*-algebra is closed under analytic functional calculus, in which it is not necessarily the case that the corresponding element is normal. If A is a C^*-algebra and $\delta : D(\delta) \to A$ is a closed unbounded $*$-derivation of A, then $1 \in D(\delta)$ (see [15, Theorem 4]). This no longer applies in the more general case of a GB*-algebra, as is demonstrated in Example 3.20.

Also in Section 4, we prove in Theorem 4.1 that if A is a complete locally m-convex algebra and $\delta : D(\delta) \to A$ is a closed $*$-derivation of A such that $1 \in D(\delta)$, then $\mathrm{Sp}_{D(\delta)}(x) = \mathrm{Sp}_A(x)$ for all $x \in A$, which extends the corresponding result of Kissin and Shulman [16, Theorem 5]. From this, it follows that if $A[\tau]$ is a pro-C^*-algebra (i.e., an inverse limit of C^*-algebras) and $\delta : D(\delta) \to A$ is a closed $*$-derivation of A such that $1 \in D(\delta)$, then the domain of δ is closed under analytic functional calculus for all elements $x \in D(\delta)$ (see Corollary 4.3).

In Section 5 of this article, we investigate the problem of finding suitable conditions for a closed $*$-derivation of a GB*-algebra to generate a one-parameter group of $*$-automorphisms of the algebra. In particular, we extend the well-known Lumer–Phillips theorem for Banach spaces (see [19, Theorem 3.1]) to complete locally convex spaces (see Proposition 5.3 and Theorem 5.4). This is applied to prove Proposition 5.5. Let $A[\tau]$ be a pro-C^*-algebra, and let $\delta : D(\delta) \to A$ be a generator of a one-parameter automorphism group $(\alpha_t)_t$ of $*$-automorphisms of A. Let $0 \leq x \in D(\delta)$. If $(p_\alpha)_\alpha$ is a family of C^*-seminorms defining the topology τ on A, then, for all α, there exists a (not necessarily nonzero) continuous positive linear functional f_x^α on A such that $f_x^\alpha(\delta(x)) \leq 0$ and $f_x^\alpha(x) = p_\alpha(x)$. Section 2 below gives all the necessary background material required to establish our results from Section 3 onwards.

2. Preliminaries

Throughout the article, we adopt the convention that all vector spaces are over the field $\mathbb{C}$ of complex numbers, and all topological spaces are assumed to be Hausdorff. Moreover, all algebras are assumed to have an identity element denoted by 1.

The term *topological algebra* is attributed to an algebra which is also a topological vector space such that the multiplication is separately continuous in both variables. If the underlying topological vector space of a topological algebra is metrizable and complete, then the algebra is called a *Fréchet topological algebra*. A topological algebra endowed with a continuous involution $*$ is called a *topological $*$-algebra*. A topological $*$-algebra which is also a locally convex space is called a *locally convex $*$-algebra*. The symbol $A[\tau]$ will stand for a topological ($*$-)algebra A endowed with a given topology τ.

Definition 2.1 ([2, pp. 91–94]). Let $A[\tau]$ be a topological $*$-algebra, and let $\mathcal{B}_A^*$ (or simply $\mathcal{B}^*$ if there is no chance of confusion) denote a collection of subsets B of A with the following properties:

(i) B is absolutely convex, closed, and bounded;
(ii) $1 \in B$, $B^2 \subset B$, and $B^* = B$.

For every $B \in \mathcal{B}^*$, denote by $A[B]$ the linear span of B, which is a normed algebra under the gauge function $\|\cdot\|_B$ of B. If $A[B]$ is complete for every $B \in \mathcal{B}^*$, then $A[\tau]$ is called *pseudocomplete*.

An element $x \in A$ is called *bounded* if for some nonzero complex number λ the set $\{(\lambda x)^n : n = 1, 2, 3, \ldots\}$ is bounded in A. We denote by A_0 the set of all bounded elements in A.

A topological $*$-algebra $A[\tau]$ is called *symmetric* if for every $x \in A$ the element $(1 + x^*x)^{-1}$ exists and belongs to A_0.

The notion of a bounded element is a generalization of the concept of bounded operators on a Banach space, and was used by Allan [1] in order to develop a spectral theory for general locally convex algebras (a relevant notion of boundedness had been given earlier by Warner [29] in the context of locally m-convex algebras). Toward this end, Allan [1] defined an extension of the usual notion of the spectrum of an element of an algebra, which we give in the following definition and to which we will refer as the *Allan spectrum* (a similar notion of spectrum was previously defined by Waelbroeck [28] in the more special setting of commutative b-algebras). Recall that for a general algebra A with an identity element 1, the spectrum of an element $x \in A$ is denoted by $\mathrm{Sp}_A(x)$, where $\mathrm{Sp}_A(x) = \{\lambda \in \mathbb{C} : \lambda 1 - x \text{ has no inverse in } A\}$.

Definition 2.2 ([1, Definition 3.1]). Let A be a locally convex algebra with an identity element 1, and let $x \in A$. Then the *Allan spectrum* of $x \in A$, denoted by $\sigma_A(x)$ (or simply by $\sigma(x)$ if it is clear which algebra is being considered), is the subset of $\mathbb{C}^*$, that is, the one-point compactification of $\mathbb{C}$, which is described as follows:

$$\sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda 1 - x \text{ has no inverse in } A_0\} \cup \{\infty \text{ if and only if } x \notin A_0\}.$$

The *resolvent set* of x, denoted by $\rho_A(x)$, is the complement of $\sigma_A(x)$ in $\mathbb{C}^*$.

A locally convex $*$-algebra A is said to have *Hermitian* involution if $\sigma_A(x) \subset \mathbb{R}$ for every self-adjoint element $x \in A$ (i.e., $x = x^*$). A pseudocomplete symmetric algebra has Hermitian involution (see [2, p. 93]).

Definition 2.3 ([2, Definition 2.5]). A symmetric pseudocomplete locally convex $*$-algebra $A[\tau]$ such that the collection $\mathcal{B}^*$ has a greatest member denoted by B_0, is called a GB*-*algebra* over B_0.

If A is commutative, then $A_0 = A[B_0]$ (see [2, p. 94]). In general, A_0 is not a $*$-subalgebra of A, and $A[B_0]$ contains all normal elements of A_0, that is, all $x \in A$ such that $xx^* = x^*x$ (see [2, p. 94]). We note that, throughout this paper, by a Fréchet GB*-algebra we mean a GB*-algebra whose underlying locally convex space is metrizable and complete. One of the results which indicates the closeness of a GB*-algebra to a C^*-algebra is the following proposition.

Proposition 2.4 ([2, Theorem 2.6], [6, Theorem 2]). *If $A[\tau]$ is a* GB*-*algebra, then the Banach $*$-algebra $A[B_0]$ is a C^*-algebra which is sequentially dense in A. Moreover, $(1 + x^*x)^{-1} \in A[B_0]$ for every $x \in A$, and B_0 is the unit ball of $A[B_0]$.*

Recall that every C*-algebra is topologically and algebraically $*$-isomorphic to a norm-closed $*$-subalgebra of $B(H)$ for some Hilbert space H. A GB*-algebra is algebraically $*$-isomorphic to a $*$-algebra of closed, possibly unbounded, operators on a Hilbert space (see [9, Theorem 7.11]). Therefore, in light of Proposition 2.4, a GB*-algebra can be thought of as a C^*-algebra with "unbounded elements adjoined to it." For a recent survey on GB*-algebras, the reader is referred to [11].

A particular example of a GB*-algebra is that of a pro-C^*-algebra (see [2, p. 95(3)]. A pro-C^*-algebra $A[\tau]$ is a complete topological $*$-algebra whose topology τ is defined by a family $\Gamma = \{p\}$ of C^*-seminorms, that is, $p(x^*x) = p(x)^2$, for every $p \in \Gamma$ and $x \in A$. A pro-C^*-algebra $A[\tau]$ is, in particular, an m-convex algebra; that is, $p(xy) \leq p(x)p(y)$ for every $x, y \in A$ and for every seminorm $p \in \Gamma$. In general, for a complete locally m-convex ($*$-)algebra $A[\tau_\Gamma]$, the Arens–Michael decomposition gives us that $A = \varprojlim_p A/N_p = \varprojlim_p A_p$ up to topological ($*$-)isomorphisms. In the previous relation, $N_p := \{x \in A : p(x) = 0\}$ and A_p is the completion of the quotient A/N_p with respect to the norm $\|x + N_p\| := p(x), x \in A$. If $A[\tau]$ is a pro-C^*-algebra, then A/N_p is automatically complete (see [10, Theorem 10.24]).

Allan [1] presented a functional calculus for a pseudocomplete locally convex algebra, which we now describe (see Theorem 2.6 below).

Definition 2.5 ([1, p. 414, Definition 5.2]). Let A be a pseudocomplete locally convex algebra A, and let $x \in A$. We denote by F_x the set of all complex-valued functions which are holomorphic on some neighborhood of $\sigma(x)$. We denote by F'_x the quotient set of F_x by the equivalence relation $\sim$: for $f, g \in F_x, f \sim g$ if and only if f equals g on some neighborhood of $\sigma(x)$.

A subset D of $\mathbb{C}^*$ is called a *Cauchy domain* if (i) D is open, (ii) D has a finite number of components the closures of which are pairwise disjoint, and (iii) the boundary ∂D of D is a subset of $\mathbb{C}$ and consists of a finite number of closed rectifiable Jordan curves such that no two intersect.

If $x \in A$ such that $\rho_A(x) \neq \emptyset$, then for any $f \in F_x$, there exists a Cauchy domain D such that (i) $\sigma(x) \subset D$, and (ii) $\mathrm{cl}\, D \subset \Delta(f)$, where $\Delta(f)$ denotes the domain of f and cl denotes the closure of D in $\mathbb{C}^*$. The integral $\int_{+\partial D} f(\lambda) \times (\lambda 1 - x)^{-1}\, d\lambda$ (where $+\partial D$ denotes the positive oriented boundary of D) defines an element of A_0, which is independent of the choice of the Cauchy domain D satisfying (i) and (ii) (see [1, p. 415]).

Theorem 2.6 ([1, Theorem 5.3])**.** *Let $A[\tau]$ be a pseudocomplete locally convex algebra, and let $x \in A$. Then there is a homomorphism $f \mapsto f(x)$ of F'_x into A_0, which is given by the following formulas.*

(i) *If $x \in A_0$, then $f(x) = \int_{+\partial D} f(\lambda)(\lambda 1 - x)^{-1}\, d\lambda$, where D is a Cauchy domain satisfying properties (i) and (ii) as in the immediately preceding paragraph.*
(ii) *If $x \notin A_0$ and $\rho_A(x) \neq \emptyset$, then $f(x) = f(\infty)1 + \int_{+\partial D} f(\lambda)(\lambda 1 - x)^{-1}\, d\lambda$, where D is as before.*
(iii) *If $\rho(x) = \emptyset$, then F_x contains only constant functions. If $f(\lambda) = c$, then $f(x) = c1$.*

We recall that a derivation δ in a topological algebra $A[\tau]$ is a linear map $\delta : D(\delta) \to A$ such that $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in D(\delta)$, where $D(\delta)$ is the domain of the derivation δ which is taken to be a dense subalgebra of A. In the case where A is a topological $*$-algebra, a derivation δ in A is said to be a $*$-*derivation* if $a \in D(\delta)$ implies that $a^* \in D(\delta)$ and $\delta(a^*) = \delta(a)^*$. A derivation δ in A is said to be *closed* if, for any net $\{x_i\}$ in $D(\delta)$ such that $x_i \xrightarrow{\tau} x$ and $\delta(x_i) \xrightarrow{\tau} y$, we have that $x \in D(\delta)$ and $y = \delta(x)$.

3. Domains of closed derivations of GB^*-algebras

Let $\delta : D(\delta) \to A$ be a derivation of a GB^*-algebra $A[\tau]$, and let $x \in D(\delta)$. If f is an analytic function on a neighborhood of $\sigma_A(x)$, the Allan spectrum of x in A, then the question is if the Allan functional calculus $f(x)$ is an element of $D(\delta)$.

Proposition 3.1. *Let $A[\tau]$ be a complete GB^*-algebra with jointly continuous multiplication. Let $\delta : D(\delta) \to A$ be a $(\tau - \tau)$-closed derivation of $A[\tau]$, and let $x \in D(\delta)$. Suppose that δ and x satisfy the following conditions:*

(i) *$1 \in D(\delta)$, and*
(ii) *$(\lambda 1 - x)^{-1} \in D(\delta)$ for all $\lambda \in \partial D$ (assuming that $\rho_A(x) \neq \emptyset$, where D is a Cauchy domain as in Theorem 2.6).*

Then $f(x) \in D(\delta)$ for all $f \in F_x$.

Proof. The proof follows the same argument as that of [8, Corollary 3]. Let

$$I = \frac{1}{2\pi i} \int_{+\partial D} f(\lambda)(\lambda 1 - x)^{-1}\, d\lambda,$$

where D is any Cauchy domain such that $\sigma_A(x) \subseteq D$ and $\operatorname{cl} D \subseteq \Delta(f)$, where $\Delta(f)$ is the domain of f. By [1, Lemma 3.11], we get that $(\lambda 1 - x)^{-1} \in A[B]$ for some $B \in \mathcal{B}$ (where $\mathcal{B}$ denotes the family of all subsets of A which enjoy all the properties of the sets of the family $\mathcal{B}^*$ of Definition 2.1 except that of self-adjointness) and for all $\lambda \in \partial D$. Therefore, I can be approximated by Riemann sums

$$R_n = \frac{1}{2\pi i} \sum_{i=1}^{n} f(\lambda_i)(\lambda_i 1 - x)^{-1},$$

in the sense that R_n converges to I with respect to $\|\cdot\|_B$ on $A[B]$. Hence $R_n \to I$ with respect to the topology τ, since τ is weaker than $\|\cdot\|_B$ on $A[B]$ (see [1, p. 400]). By hypothesis and [1, Lemma 3.11], $R_n \in D(\delta) \cap A[B]$ for all $n \in \mathbb{N}$. By [1, Theorem 3.8], the map $\lambda \mapsto (\lambda 1 - x)^{-1}$ is analytic and hence $(|\cdot| - \tau)$-continuous on $\rho_A(x)$ (since $\rho_A(x) \neq \emptyset$). Using this and the joint continuity of multiplication, we get that the map $\lambda \mapsto f(\lambda)(\lambda 1 - x)^{-1}\delta(x)(\lambda 1 - x)^{-1}$ is continuous on ∂D.

The τ-convergence of the Riemann sums $\delta(R_n) = \frac{1}{2\pi i}\sum_{i=1}^{n} f(\lambda_i)(\lambda_i 1 - x)^{-1} \times \delta(x)(\lambda_i 1 - x)^{-1}$ is concluded along the lines of the normed case (see, e.g., [13, Theorem 3.3.2, p. 63]). Indeed, for the sake of brevity, let $g(\lambda) = f(\lambda)(\lambda 1 - x)^{-1} \times \delta(x)(\lambda 1 - x)^{-1}$. Hence, as already observed, g is a $(|\cdot| - \tau)$-continuous map from ∂D into A. Let $\partial D = \gamma_1 \cup \cdots \cup \gamma_m$, where $\gamma_k, k = 1, \ldots, m$, are non-intersecting, closed, rectifiable Jordan curves. We want to show that for each $k = 1, \ldots, m$, the Riemann sums $S_{\pi^k} = \sum_{i=1}^{n} g(z_k(t_{ik}))(z_k(t_{ik}) - z_k(t_{(i-1)k}))$ form a Cauchy net in A, where $\pi^k = \{t_{ik}\}_{i=0}^{n}$ is a partition of an interval, say, $[a_k, b_k]$, $|\pi^k| = \max_{1\le i\le n} |t_{ik} - t_{(i-1)k}|$, and $z_k(t), t \in [a_k, b_k]$ defines the curve γ_k. Let $\epsilon > 0$ be given, and let p be an arbitrary but fixed seminorm in Γ_τ. Since g is continuous on ∂D, for every $z_k(t), t \in [a_k, b_k]$, there is $\delta_{z_k(t)} > 0$ such that

$g(U(z_k(t), \delta_{z_k(t)})) \subset U_p(g(z_k(t)), \frac{\epsilon}{2})$, where

$$U\big(z_k(t), \delta_{z_k(t)}\big) = \big\{\mu \in \mathbb{C} : \big|\mu - z_k(t)\big| < \delta_{z_k(t)}\big\},$$
$$U_p\Big(g\big(z_k(t)\big), \frac{\epsilon}{2}\Big) = \Big\{x \in A : p\big(x - g\big(z_k(t)\big)\big) < \frac{\epsilon}{2}\Big\}.$$

Since ∂D is compact, if λ is the Lebesgue number of the open covering $\{U(z_k(t), \delta_{z_k(t)}) : t \in [a_k, b_k]\}$ of γ_k, then for $z_k(t_1), z_k(t_2) \in \gamma_k$ such that $|z_k(t_1) - z_k(t_2)| < \lambda$, we have that $z_k(t_1), z_k(t_2) \in U(z_k(t), \delta_{z_k(t)})$ for some $t \in [a_k, b_k]$. Therefore, $p(g(z_k(t_1)) - g(z_k(t_2))) < \epsilon$. Thus, for partitions π_1^k, π_2^k such that $|\pi_1^k|, |\pi_2^k| < \frac{\lambda}{2}$, we have that

$$p(S_{\pi_1^k} - S_{\pi_2^k}) < \epsilon \operatorname{Var}\big(z_k(t)\big),$$

where $\operatorname{Var}(z_k(t))$ is the total variation of the curve γ_k which is finite since the curve is assumed rectifiable. Since the seminorm p was arbitrary, we get that S_{π^k} is τ-Cauchy in A, hence convergent in A. Since the above considerations hold for all $k = 1, \ldots, m$, we conclude that

$$\delta(R_n) \to \frac{1}{2\pi i} \int_{+\partial D} f(\lambda)(\lambda 1 - x)^{-1} \delta(x)(\lambda 1 - x)^{-1}\, d\lambda$$

with respect to the topology τ.

Since δ is $(\tau - \tau)$-closed, it follows that $I \in D(\delta)$. By hypothesis, $1 \in D(\delta)$, and so $f(x) \in D(\delta)$ in all three cases (i)–(iii) of Theorem 2.6. □

We now show that condition (ii) in Proposition 3.1 is also a necessary condition. Again, let A be a pseudocomplete locally convex algebra. Let $x \in A$ with $\rho_A(x) \neq \emptyset$. Let $\delta : D(\delta) \to A$ be a $*$-derivation such that $f(x) \in D(\delta)$ for all analytic functions f on a neighborhood surrounding $\sigma_A(x)$.

Let $\lambda \notin \sigma_A(x)$, and let $f(\mu) = \frac{1}{\lambda - \mu}$. Then f is analytic on $\mathbb{C}^* \setminus \{\lambda\}$, and hence on a neighborhood U of $\sigma_A(x)$ with $\lambda \notin U$. We could take U to be a Cauchy domain D. Then $f(x) = (\lambda 1 - x)^{-1}$, whether or not $x \in A_0$. The functional calculus is a homomorphism, and $\int_{+\partial D} f(\mu)(\mu 1 - x)^{-1}\, d\mu$ is (closed) curve-independent (see also [26, Theorem 7.2] in this regard). By assumption $(\lambda 1 - x)^{-1} = f(x) \in D(\delta)$, whether or not $x \in A_0$.

If A above is a C^*-algebra, then it is well known (see [8, Theorem 2]) that $(\lambda 1 - x)^{-1} \in D(\delta)$ for all $\lambda \in \rho_A(x)$. The proof of this result, as given in [8], relies on expressing $(\lambda 1 - x)^{-1}$ as a power series, especially making use of the Neumann series when $\lambda > r_A(x)$. The proof is concluded with an analytic continuation argument. All of this works since a C^*-algebra consists of bounded linear operators. However, a GB*-algebra generally has unbounded linear operators. The following proposition shows that Bratteli and Robinson's proof of [8, Theorem 2] is generally not conceivable in the world of unbounded operators due to there being no Neumann series (since it is not a Q-algebra unless it is a C^*-algebra). We recall that a unital topological algebra is a Q-algebra if the group of invertible elements is open. Motivated by the above discussion, we give the following proposition, for which the proof follows immediately.

Proposition 3.2. *Let $A[\tau]$ be a unital topological algebra for which there is a neighborhood U of $0 \in A$ such that $\sum_{n=0}^{\infty} x^n$ converges to $(1-x)^{-1}$ for all $x \in U$. Then $A[\tau]$ is a Q-algebra.*

Corollary 3.3. *Let $A[\tau]$ be a Fréchet* GB**-algebra for which there is a neighborhood U of $0 \in A$ such that $\sum_{n=0}^{\infty} x^n$ converges to $(1-x)^{-1}$ for all $x \in U$. Then $A[\tau]$ is a C^*-algebra.*

Proof to Proposition 3.2. Every Fréchet Q-algebra has continuous inversion (see [33, Corollary 7.8]) and is barreled. Therefore, by Proposition 3.2, [1, Corollary 4.2], and [2, Corollary 2.8], it follows that $A[\tau]$ is a C^*-algebra. □

Let $(\alpha_t)_{t\in\mathbb{R}}$ be a τ-continuous one-parameter group of $*$-automorphisms of a locally convex $*$-algebra $A[\tau]$. Then

$$D(\delta_\alpha) = \left\{x \in A : \tau - \lim_{t\to 0} \frac{\alpha_t(x) - x}{t} \text{ exists}\right\}$$

is dense in A. Let

$$\delta_\alpha(x) = \tau - \lim_{t\to 0} \frac{\alpha_t(x) - x}{t}$$

for all $x \in D(\delta_\alpha)$. It is well known that δ_α is an unbounded $*$-derivation of A, and we say that δ_α is the *generator* of the group of automorphisms $(\alpha_t)_{t\in\mathbb{R}}$.

Remark 3.4. Let $A[\|\cdot\|]$ be a C*-algebra, and let $(\alpha_t)_t$ be a strongly continuous one-parameter subgroup of $*$-automorphisms of A. In [22], an element a of A is defined to be *analytic* if $t \to \alpha_t(a)$ is analytic. This is equivalent to $a \in \bigcap_{n=1}^{\infty} D(\delta_\alpha^n)$ and $\sum_{n=0}^{\infty}(\frac{\|\delta_\alpha^n(a)\|}{n!})s^n < +\infty$ for some $s > 0$ (see [22]). From the definition of analytic element, we see that if an element a is analytic, then $a \in D(\delta_\alpha)$.

If $a \in A$ is analytic, then we can define an A-valued complex analytic function f on neighborhood of zero in $\mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty}(\frac{\delta^n(a)}{n!})z^n$, where $|z| < s$ (see [22]). Furthermore, $f(t) = \alpha_t(a)$ for $|t| < s$. Lastly, Sakai [22, pp. 428–429] proves that the set of analytic elements is a $*$-subalgebra of A.

Corollary 3.5. *Let δ be a closed derivation of a complete* GB**-algebra $A[\tau]$ having jointly continuous multiplication, which is also the generator of a τ-continuous one-parameter group of $*$-automorphisms of A. Then, for all $x \in D(\delta)$ and $f \in F_x$, we have that $f(x) \in D(\delta)$.*

Proof. We show that conditions (i) and (ii) of Proposition 3.1 are satisfied. The fact that $1 \in D(\delta_\alpha)$ is trivial, and this verifies condition (i). As for condition (ii), let $x \in D(\delta_\alpha)$. Observe that if $\lambda \in \rho_A(x)$, then

$$\begin{aligned}
&\frac{\alpha_t((\lambda 1 - x)^{-1}) - (\lambda 1 - x)^{-1}}{t} \\
&\quad = \frac{(\lambda 1 - x)^{-1}[(\lambda 1 - x) - (\lambda 1 - \alpha_t(x))]\alpha_t((\lambda 1 - x)^{-1})}{t} \\
&\quad = (\lambda 1 - x)^{-1}\frac{\alpha_t(x) - x}{t}\alpha_t\big((\lambda 1 - x)^{-1}\big) \to (\lambda 1 - x)^{-1}\delta_\alpha(x)(\lambda 1 - x)^{-1}
\end{aligned}$$

as $t \to 0$, and therefore $(\lambda 1 - x)^{-1} \in D(\delta_\alpha)$. □

The last corollary therefore motivates the question as to when a closed derivation of a complete GB*-algebra $A[\tau]$, with jointly continuous multiplication, is the generator of a continuous one-parameter group of $*$-automorphisms of A. One answer to this question, for a complete GB*-algebra, is given with Theorems 5.2 and 5.4.

The remainder of this section is devoted to finding some necessary and sufficient conditions under which conditions (i) and (ii) of Proposition 3.1 are satisfied. We first investigate condition (i) with the culmination of Corollary 3.8 below.

The proof of the following theorem is an adaptation of the proof of [16, Theorem 4].

Theorem 3.6. *Let $A[\tau]$ be a complete* GB*-*algebra with jointly continuous multiplication. If $\delta : D(\delta) \to A$ is a $(\tau - \tau)$-closed derivation with domain $D(\delta)$ such that $\overline{D(\delta) \cap A[B_0]}^{\|\cdot\|_{B_0}} = A[B_0]$, then $1 \in D(\delta)$.*

Proof. Since $A[\tau]$ has jointly continuous multiplication, it follows that the multiplications $A[B_0] \times A \to A$ and $A \times A[B_0] \to A$ are $(\|\cdot\|_{B_0} \times \tau - \tau)$-continuous and $(\tau \times \|\cdot\|_{B_0} - \tau)$-continuous, respectively. Therefore, since $A[\tau]$ is also complete, by [21, Proposition 3.4] the topology τ can be defined by a family of seminorms $(p_\alpha)_{\alpha\in\Lambda}$ such that $p_\alpha(ax) \le \|a\|_{B_0} p_\alpha(x)$ and $p_\alpha(xa) \le \|a\|_{B_0} p_\alpha(x)$ for all $a \in A[B_0]$, $x \in A$, and $\alpha \in \Lambda$.

Since $\overline{D(\delta) \cap A[B_0]}^{\|\cdot\|_{B_0}} = A[B_0]$, there exists $y \in D(\delta) \cap A[B_0]$ such that $\|1 - y\|_{B_0} = \epsilon < 1$. For every $n \in \mathbb{N}$, let

$$x_n = 1 - (1-y)^n = \sum_{k=1}^{n} C_n^k (-1)^{k+1} y^k,$$

where the C_n^k are the ordinary binomial coefficients. Then $x_n \in A[B_0] \cap D(\delta)$ for all $n \in \mathbb{N}$, and $x_n \to 1$ with respect to $\|\cdot\|_{B_0}$. Therefore, $x_n \to 1$ with respect to the topology τ.

We show that $\delta(x_n) \to 0$ with respect to the topology τ. Observe that

$$\begin{aligned} x_{n+1} &= 1 - (1-y)(1-y)^n \\ &= 1 - (1-y)(1-x_n) \\ &= y + x_n - yx_n. \end{aligned}$$

Therefore,

$$\begin{aligned} \delta(x_{n+1}) &= \delta(y) + \delta(x_n) - y\delta(x_n) - \delta(y)x_n \\ &= \delta(y)(1-x_n) + (1-y)\delta(x_n). \end{aligned}$$

Choose an arbitrary $\alpha \in \Lambda$ and keep it fixed. It follows that

$$\begin{aligned} p_\alpha\big(\delta(x_{n+1})\big) &\le p_\alpha\big(\delta(y)(1-x_n)\big) + p_\alpha\big((1-y)\delta(x_n)\big) \\ &\le \|1-x_n\|_{B_0} p_\alpha\big(\delta(y)\big) + \|1-y\|_{B_0} p_\alpha\big(\delta(x_n)\big) \\ &\le \epsilon^n p_\alpha\big(\delta(y)\big) + \epsilon p_\alpha\big(\delta(x_n)\big). \end{aligned}$$

Let $t_{\alpha,n} = p_\alpha(\delta(x_n))$ and $C_\alpha = p_\alpha(\delta(y))$ for all $n \in \mathbb{N}$ and $\alpha \in \Lambda$. Then $t_{\alpha,n+1} \leq \epsilon^n C_\alpha + \epsilon t_{\alpha,n}$. By induction,

$$t_{\alpha,n+1} \leq n\epsilon^n C_\alpha + \epsilon^n t_{\alpha,1} \to 0$$

as $n \to \infty$ (write $\epsilon = \frac{1}{\epsilon_0}$; then $\epsilon_0 > 1$, and hence $n\epsilon^n = \frac{n}{\epsilon_0^n} \to 0$ as $n \to \infty$). Therefore, $p_\alpha(\delta(x_n)) \to 0$ as $n \to \infty$. Since $\alpha \in \Lambda$ was chosen arbitrarily, the last statement holds for all $\alpha \in \Lambda$. So $\delta(x_n) \to 0$ with respect to the topology τ. Since δ is $(\tau - \tau)$-closed, it follows that $1 \in D(\delta)$. □

Remark 3.7.

(1) It would be interesting to know when the condition $\overline{D(\delta) \cap A[B_0]}^{\|\cdot\|_{B_0}} = A[B_0]$ in Theorem 3.6 holds, and we give an answer to this in Corollary 4.14 below.

(2) Theorem 3.6 is especially true for all Fréchet GB*-algebras, C^*-like algebras, and particularly pro-C^*-algebras (since these algebras are complete and have jointly continuous multiplication; for the definition of a C^*-like algebra, see [14]).

(3) In Theorem 3.6, one only requires the condition that $\overline{D(\delta) \cap A[B_0]}^{\|\cdot\|_{B_0}} = A[B_0]$ in order to know that there exists $y \in A[B_0] \cap D(\delta)$ such that $\|1 - y\|_{B_0} < 1$, that is, $1 \in \overline{D(\delta) \cap A[B_0]}^{\|\cdot\|_{B_0}}$. We therefore have the following result.

Corollary 3.8. *Let $A[\tau]$ be a complete* GB**-algebra with jointly continuous multiplication. Let $\delta : D(\delta) \to A$ be a $(\tau - \tau)$-closed ($*$-)derivation with domain $D(\delta)$. The following conditions are equivalent:*

(i) $1 \in D(\delta)$*, and*

(ii) $1 \in \overline{D(\delta) \cap A[B_0]}^{\|\cdot\|_{B_0}}$.

Proof. The proof of (ii) ⇒ (i) is exactly the same as the proof of Theorem 3.6 (see Remark 3.7(3) above). The implication (i) ⇒ (ii) is trivial. □

Let $x \in A[\tau]$, and let $\delta : D(\delta) \to A$ be a closed $*$-derivation of A with $x \in D(\delta)$. Then the question of whether $(\lambda 1 - x)^{-1}$ is in $D(\delta)$ for all $\lambda \in \rho_A(x) \cap \mathbb{C}$ is implied by $\mathrm{Sp}_{D(\delta)}(x) = \mathrm{Sp}_A(x)$. Since we want to find conditions under which Proposition 3.1(ii) is satisfied, we now turn our attention to the question of whether $\mathrm{Sp}_{D(\delta)}(x) = \mathrm{Sp}_A(x)$ is true for all $x \in D(\delta)$.

If $A[\tau]$ is a topological algebra which is not necessarily unital, then we use the symbol G_A^q to denote the set of quasi-invertible elements in A. If A is unital, then the symbol G_A denotes the set of all invertible elements in A. Furthermore, if A is unital, then $x \in G_A^q$ if and only if $1 - x \in G_A$.

Recall that a topological algebra $A[\tau]$ is said to be *advertibly complete* if every Cauchy net (x_λ) in A, having the property that $x_\lambda \circ x \to 0$ and $x \circ x_\lambda \to 0$ for some $x \in A$, converges in A (see [10, Definition 6.1]).

Proposition 3.9 ([10, Proposition 6.2])**.** *Let $A[\tau]$ be an advertibly complete topological algebra (not necessarily unital) such that the completion $\widetilde{A}[\widetilde{\tau}]$ of $A[\tau]$ is a topological algebra. If $x \in A$, then $x \in G_A^q$ if and only if $x \in G_{\widetilde{A}}^q$.*

Corollary 3.10. *Let $A[\tau]$ be a complete topological algebra, and let $\delta : D(\delta) \to A$ be a derivation of A such that $D(\delta)$ is advertibly complete and $1 \in D(\delta)$. If $x \in D(\delta)$, then $(\lambda 1 - x)^{-1} \in D(\delta)$ for all $\lambda \notin \mathrm{Sp}_A(x)$.*

Proof. Observe that $D(\delta)$ is an advertibly complete topological algebra in the relative topology inherited from the topology τ on A, whose completion is $A[\tau]$. The completion of the advertibly complete topological algebra $D(\delta)$ is therefore a topological algebra. Let $x \in D(\delta)$. By Proposition 3.9, we have that $x \in G_{D(\delta)}$ if and only if $x \in G_A$. It follows that $(\lambda 1 - x)^{-1} \in D(\delta)$ for all $\lambda \notin \mathrm{Sp}_A(x)$. □

Every Q-algebra is advertibly complete, and the concepts of advertibly complete and Q-algebra are equivalent for normed algebras (see [10, Theorem 6.5]). An advertibly complete algebra need not be a Q-algebra.

If $\delta : D(\delta) \to A$ is a closed derivation of a Banach algebra A, then $D(\delta)$ is a Q-algebra (see [16, Theorem 5]). This result does not extend to GB*-algebras. Let $A[\tau]$ be a Fréchet GB*-algebra which is not a Q-algebra (i.e., not a C^*-algebra, by Corollary 3.3). Take any continuous derivation $\delta : A \to A$ of A (e.g., an inner derivation). By the closed graph theorem, δ is closed. However, $D(\delta) = A$ is not a Q-algebra.

For the next result, which is an extension of [16, Theorem 1], we have to recall that a Fréchet GB*-algebra $A[\tau]$ is a Q-algebra if and only if it is a C^*-algebra (see Corollary 3.3). So if we want to look at a GB*-algebra $A[\tau]$ which is not a C^*-algebra and behaves like a Q-algebra, then one has to put some other topology τ' on A such that $A[\tau']$ is a Q-algebra (although not necessarily a GB*-algebra).

Proposition 3.11. *Let $A[\tau']$ be a complete Q-algebra. Let B be a subalgebra of A which is also τ'-dense in A. The following statements are equivalent.*

(i) *The algebra $B[\tau'_B]$ is a Q-algebra, where τ'_B is the relative topology on B induced by the topology τ' on A.*
(ii) *The algebra $B[\tau'_B]$ is advertibly complete.*
(iii) *We have that $\mathrm{Sp}_B(x) = \mathrm{Sp}_A(x)$ for all $x \in B$.*

Proof. (i) $\Rightarrow$ (iii) This is [10, Proposition 6.16].

(iii) $\Rightarrow$ (i) Condition (iii) is equivalent to $G_B = B \cap G_A$, where G_B and G_A denote the set of invertible elements in B and A, respectively, with inverses in B and A, respectively. By hypothesis, G_A is τ'-open in A, and hence G_B is τ'_B-open in B.

(i) $\Rightarrow$ (ii) This follows from [10, Theorem 6.5]. Now (ii) $\Rightarrow$ (iii) follows from [10, Proposition 6.2]. □

Observe that the implication (iii) $\Rightarrow$ (ii) in the above proposition holds under less stringent conditions. If $A[\tau]$ is a topological algebra such that there is a topology τ' on A such that $A[\tau']$ is a Q-algebra (not necessarily complete), and B is a subalgebra of A (not necessarily τ'-dense in A), then (iii) implies (ii). The proof is the same as that above.

Remark 3.12. Proposition 3.11 is especially valid if B is the domain of a closed derivation of A, which is the motivation for the proof of the proposition.

We now give two examples demonstrating that there are GB*-algebras $A[\tau]$ having Q-algebra topologies τ' and $A[\tau]$ is not a C^*-algebra.

Example 3.13 ([34, Example 1]). Let A be the commutative algebra of all complex-valued continuous functions on $[0, 1]$. Let A be equipped with the compact-open topology τ, defined by countable compact subsets of $[0, 1]$. Then $A[\tau]$ is a pro-C^*-algebra, and it is a C^*-algebra with respect to the topology τ' defined by the norm on A. Now $A[\tau']$ is a Q-algebra, and $A[\tau]$ is not a Q-algebra. So $A[\tau]$ is not a C^*-algebra.

Example 3.14. The noncommutative algebra $B(H)$ of all bounded linear operators on a Hilbert space H is a Q-algebra with respect to the norm topology, and $B(H)[\tau_w]$ is a GB*-algebra with respect to the weak operator topology τ_w on $B(H)$, but it is not a C^*-algebra.

We require the following result, which is a special case of [34, Corollary 3], in the proof of Theorem 3.16 below.

Theorem 3.15. *Let $A[\tau]$ be a commutative barreled pro-C^*-algebra. Then $A[\tau]$ is a Q-algebra if and only if $\mathrm{Sp}_A(x)$ is bounded for every $x \in A$.*

Theorem 3.16. *Let $A[\tau]$ be a metrizable commutative pro-C^*-algebra. The following statements are equivalent.*

(i) *We have that $\mathrm{Sp}_A(x)$ is bounded for all $x \in A$.*
(ii) *Every $(\tau - \tau)$-closed derivation $\delta : D(\delta) \to A$ has the properties that $\mathrm{Sp}_{D(\delta)}(x) = \mathrm{Sp}_A(x)$ for all $x \in D(\delta)$, and $D(\delta)[\tau_{D(\delta)}]$ is a Q-algebra.*
(iii) *Every $(\tau - \tau)$-closed derivation $\delta : D(\delta) \to A$ has the property that $D(\delta)[\tau_{D(\delta)}]$ is a Q-algebra.*

Proof. (i) $\Rightarrow$ (ii) By Theorem 3.15, we get that $A[\tau]$ is a Fréchet Q-algebra. Therefore, by Corollary 3.3, it follows that $A[\tau]$ is a C^*-algebra.

Since $\delta : D(\delta) \to A$ is $\|\cdot\|$-closed, we have that $\mathrm{Sp}_{D(\delta)}(x) = \mathrm{Sp}_A(x)$ for all $x \in D(\delta)$. It follows from Proposition 3.11 that $D(\delta)[\tau_{D(\delta)}]$ is a Q-algebra.

(iii) $\Rightarrow$ (i) Assume that (iii) holds. Observe that the zero derivation δ on A is $(\tau-\tau)$-continuous, and hence $(\tau-\tau)$-closed (since it is defined on the whole of A). Therefore, by (iii), $A[\tau] = D(\delta)$ is a Q-algebra. Therefore, $\mathrm{Sp}_A(x)$ is bounded for all $x \in A$ (this is true for any Q-algebra). The implication (ii) $\Rightarrow$ (iii) is obvious. $\square$

In the current literature, there exist various characterizations of C^*-algebras within the class of locally convex $*$-algebras. The following proposition is another such result with the surprising outcome that it is the nature of the unbounded $*$-derivations which completely determines whether or not a given Fréchet GB*-algebra is a C^*-algebra.

Proposition 3.17. *Let $A[\tau]$ be a Fréchet* GB**-algebra. The following statements are equivalent.*

(i) *We have that $A[\tau]$ is a C^*-algebra.*
(ii) *Every $(\tau - \tau)$-closed $*$-derivation $\delta : D(\delta) \to A$ has the properties that $\sigma_{D(\delta)}(x) = \sigma_A(x)$ for all $x \in D(\delta)$, and $D(\delta)[\tau_{D(\delta)}]$ is a Q-algebra.*

(iii) *Every $(\tau-\tau)$-closed derivation $\delta : D(\delta) \to A$ has the property that $D(\delta)[\tau_{D(\delta)}]$ is a Q-algebra.*

In the above, the symbol $\tau_{D(\delta)}$ denotes the relative topology on the domain $D(\delta)$ of δ induced by τ.

Proof. (i) $\Rightarrow$ (ii) Suppose that (i) holds, and let $\delta : D(\delta) \to A$ be a $(\tau-\tau)$-closed unbounded $*$-derivation of A. Then τ is defined by the C^*-norm $\|\cdot\|$, say. Then $\delta : D(\delta) \to A$ is $(\|\cdot\|-\|\cdot\|)$-closed, and hence $\sigma_{D(\delta)}(x) = \sigma_A(x)$ for all $x \in D(\delta)$ (see [8, Theorem 2]), and $D(\delta)$ is a Q-algebra with respect to the relative topology coming from the norm topology on A (proved in [16, Theorem 5]).

(iii) $\Rightarrow$ (i) Assume that (iii) holds. Now the zero derivation δ on A is $(\tau-\tau)$-continuous, and hence $(\tau-\tau)$-closed (since it is defined on the whole of A). Therefore, by (iii), $A[\tau] = D(\delta)$ is a Q-algebra. Therefore, by Corollary 3.3, $A[\tau]$ is a C^*-algebra. The implication (ii) $\Rightarrow$ (iii) is obvious. □

For the proof of the following proposition, we recall that if $A[\tau]$ is a complete commutative m-convex algebra (i.e., a commutative Arens–Michael algebra), then $x \in G_A$ if and only if $f(x) \neq 0$ for all continuous characters f on A, since $A[\tau]$, being complete, is advertibly complete (see [10, Proposition 6.10(6)]).

Proposition 3.18 below is an analogue of [16, Theorem 5]. In what follows, we will use the term "character" to mean "multiplicative linear functional."

Proposition 3.18. *Let $A[\tau]$ be a commutative pro-C^*-algebra, and let $\delta : D(\delta) \to A$ be a $(\tau-\tau)$-closed derivation such that $\mathrm{Sp}_{D(\delta)}(x) = \mathrm{Sp}_A(x)$ for all $x \in D(\delta)$. Then $D(\delta)[\tau_{D(\delta)}]$ is advertibly complete.*

Proof. By hypothesis, $G_{D(\delta)} = D(\delta) \cap G_A$. Let $x \in D(\delta)$ be invertible with inverse in $D(\delta)$, that is, $x \in G_{D(\delta)}$. Then $x \in G_A$. So, for all continuous characters f on A, we have $f(x) \neq 0$. Since $D(\delta)$ is dense in A, every continuous character g on $D(\delta)$ extends to a continuous character f on A. So $g(x) \neq 0$ for all continuous characters g on $D(\delta)$.

Now suppose that $x \in D(\delta)$, and for all continuous characters g on $D(\delta)$, we have $g(x) \neq 0$. Every continuous character g on $D(\delta)$ extends to a continuous character f on A. All continuous characters on A are continuous extensions of continuous characters of $D(\delta)$. Therefore, $f(x) \neq 0$ for all continuous characters f on A, and hence $x \in G_A$. Since $G_{D(\delta)} = D(\delta) \cap G_A$, it follows that $x \in G_{D(\delta)}$. Therefore, $x \in D(\delta)$ is in $G_{D(\delta)}$ if and only if $g(x) \neq 0$ for all continuous characters g on $D(\delta)$.

By [10, Proposition 6.10(6)], $D(\delta)[\tau_{D(\delta)}]$ is advertibly complete. □

Motivated by Theorem 3.16(ii), it would be interesting to know if all $(\tau-\tau)$-closed derivations $\delta : D(\delta) \to A$ of a commutative barreled pro-C^*-algebra with $1 \in D(\delta)$, and $\mathrm{Sp}_{D(\delta)}(x) = \mathrm{Sp}_A(x)$ for all $x \in D(\delta)$, have the property that $D(\delta)$ is a Q-algebra. The following example answers this question in the negative.

Example 3.19. Let A_1 be an incomplete, unital, commutative, normed Q-algebra which is also a pre-C^*-algebra (i.e., a C^*-algebra without the completeness property). For every $1 < n \in \mathbb{N}$, let A_n be a commutative C^*-algebra. Then $A = \prod_{n=1}^{\infty} A_n$ is an advertibly complete commutative C^*-convex algebra (i.e., a pro-C^*-algebra without the completeness property) in the product topology τ which is not a Q-algebra (this is a special case of [10, Example 6.12(1)]). So $\mathrm{Sp}_A(x) = \mathrm{Sp}_{\widetilde{A}}(x)$ for all $x \in A$ (by Proposition 3.9), where $\widetilde{A}$ is the completion of A. In fact, $\widetilde{A} = \prod_{n=1}^{\infty} B_n$, where B_1 is the completion of A_1, and $B_n = A_n$ for all $1 < n \in \mathbb{N}$. Therefore, $\widetilde{A}$ is a commutative metrizable pro-C^*-algebra.

For every $n > 1$, let $\delta_n : A_n \to A_n$ be the zero derivation. For $n = 1$, let δ_1 be any closed derivation of the C^*-algebra B_1 with $D(\delta_1) = A_1$. Since δ_n is continuous and $D(\delta_n) = A_n$ for all $n > 1$, one has that δ_n is a closed derivation of A_n for all $n > 1$ (for $n > 1$, recall that A_n is complete).

Let $\delta : A \to \widetilde{A}$ be the derivation defined by $\delta((x_n)_n) = (\delta_n(x_n))_n$ for all $(x_n)_n \in A$. Then δ is closed in the product topology. Let $(y_m) = (a_n^m)_n \in A$ with $y_m \to y = (a_n)_n \in \widetilde{A}$, and let $\delta(y_m) \to b = (b_n)_n \in \widetilde{A}$ as $m \to \infty$.

Then $a_n^m \to a_n$ for all $n \in \mathbb{N}$ as $m \to \infty$, and

$$\begin{aligned} \delta(y_m) \to b \quad &\Rightarrow \quad \delta\big((a_n^m)_n\big) \to (b_n)_n \\ &\Rightarrow \quad \delta_n(a_n^m) \to b_n \\ &\Rightarrow \quad a_n \in A_n \text{ and } b_n = \delta_n(a_n) \\ &\Rightarrow \quad y = (a_n)_n \in A \text{ and } b = (b_n)_n = \big(\delta_n(a_n)\big)_n = \delta(y). \end{aligned}$$

Therefore, δ is closed; note also that $D(\delta) = A$. So, $D(\delta)$ is advertibly complete, not a Q-algebra, and $\mathrm{Sp}_{D(\delta)}(x) = \mathrm{Sp}_A(x)$ for all $x \in D(\delta)$ (for this last fact, see Proposition 3.9). This is the same as saying that, for all $x \in D(\delta)$ and $\lambda \notin \mathrm{Sp}_A(x)$, one has that $(\lambda 1 - x)^{-1} \in D(\delta)$.

Example 3.20. Consider the pro-C^*-algebra $\widetilde{A}[\tau]$ and the closed derivation $\delta : D(\delta) \to \widetilde{A}$ as in Example 3.19, where $D(\delta)$ is the $*$-subalgebra A as in Example 3.19. Then $1 \in D(\delta)$. Furthermore, if (p_n) is the family of C^*-seminorms on $\widetilde{A}$ defining the topology τ (as in Example 3.19), then $\sup_n p_n(x_n) \leq 1$ if and only if $p_n(x_n) \leq 1$ for all $n \in \mathbb{N}$ and all $(x_n)_n \in \widetilde{A}$. Therefore

$$\begin{aligned} \widetilde{A}[B_0] \cap D(\delta) = \widetilde{A}[B_0] \cap A \\ &= \Big\{(x_n)_n \in A : \sup_n p_n(x_n) < \infty\Big\} \\ &= \Big\{(x_n)_n \in A : \sup_{n\geq 2} p_n(x_n) < \infty\Big\}. \end{aligned}$$

Note that $\widetilde{A}[B_0]$ is a C^*-algebra with respect to the norm $\|(x_n)\| = \sup_n p_n(x_n)$. Let $x = (x_n) \in \widetilde{A}[B_0] \subseteq \widetilde{A}$. Then $\sup_n p_n(x_n) < \infty$, so $\sup_{n\geq 2} p_n(x_n) < \infty$. There is a sequence (a_m) in A_1 such that $a_m \to x_1$. Let $b_m = (b_n^{(m)})_n$ be defined as follows: $b_1^{(m)} = a_m$ for all $m \in \mathbb{N}$, and $b_n^{(m)} = x_n$ for all $n \geq 2$ and all $m \in \mathbb{N}$. Then $b_m \in A$ for all $m \in \mathbb{N}$. By the three displayed equalities above, and the fact that $\sup_{n\geq 2} p_n(x_n) < \infty$, it follows that $b_m \in \widetilde{A}[B_0] \cap A$. Now

$$\|b_m - x\| = \sup_n p_n(b_n^{(m)} - x_n) = p_1(a_m - x_1) \to 0.$$

Now p_1 is the C^*-norm on A_1. Therefore, $\overline{\widetilde{A}[B_0] \cap D(\delta)}^{\|\cdot\|} = \widetilde{A}[B_0]$. Observe that the pro-$C^*$-algebra $\widetilde{A}[\tau]$ is not a C^*-algebra.

We conclude this section by giving an example of a GB*-algebra admitting a closed $*$-derivation $\delta : A[\tau] \to A$ for which $1 \notin D(\delta)$ (see Example 3.24). For this, we require the following two propositions, of which the following proposition is an immediate consequence of [5, Proposition 3.6]. In regard to the term "GNS-representation" used in what follows, we briefly recall that if B is a general unital $*$-algebra and f is a state on B, the GNS-representation of B is constructed as follows. Consider $N_f = \{x \in B : f(x^*x) = 0\}$. Then the quotient B/N_f is a pre-Hilbert space under the inner product $\langle x + N_f, y + N_f\rangle = f(y^*x), x, y \in B$. Let H_f be the Hilbert space completion of B/N_f with respect to the inner product. Then the GNS-representation of B is the $*$-representation of B on H_f which is given by $\pi_f : B \to \mathcal{L}(B/N_f) : \pi_f(x)(y+N_f) = xy+N_f, x, y \in B$, where $\mathcal{L}(B/N_f)$ denotes all linear operators X from B/N_f into B/N_f such that the domain of X^* contains B/N_f and such that $X^*(B/N_f) \subset B/N_f$ (see, e.g., [24, p. 227]).

Proposition 3.21. *Let $\delta : D(\delta) \to A$ be a derivation of a* GB*-*algebra $A[\tau]$. Assume that there is a continuous linear functional f on A such that $f|_{D(\delta)}$ is a state on $D(\delta)$ and*

(i) *$f \circ \delta$ is continuous on $D(\delta)$,*
(ii) *the GNS-representation π_f of $D(\delta)$ is faithful.*

Then δ is closable.

The strategy of the proof of this proposition given in [5] is to prove that

$$D(\delta') = \big\{f \in A' : \delta'(f) \text{ has a continuous extension to } A\big\}$$

is $\sigma(A', A)$-dense in A'. By [17, p. 34], this is equivalent to δ being $(\tau - \tau)$-closed, not only $(\tau - \tau)$-closable. One therefore has the following result.

Proposition 3.22. *Let $\delta : D(\delta) \to A$ be a derivation of a* GB*-*algebra $A[\tau]$. Assume that there is a continuous linear functional f on A such that $f|_{D(\delta)}$ is a state on $D(\delta)$ and*

(i) *$f \circ \delta$ is continuous on $D(\delta)$,*
(ii) *the GNS-representation π_f of $D(\delta)$ is faithful.*

Then δ is closed.

An example of an unbounded derivation of a C^*-algebra satisfying the conditions of Proposition 3.22 is every inner limit derivation of a uniformly hyperfinite algebra (see [8, Corollaries 6 and 9 and their proofs]). For Example 3.24, we require the following lemma. We recall that a C^*-algebra M is called a *W^*-algebra* if there is a Banach space M_* such that the dual $(M_*)^*$ of M_* is M.

Lemma 3.23. *Let $A[\tau]$ be a* GB*-*algebra whose $A[B_0]$ is a W^*-algebra. If I is a two-sided ideal of A, then I is a $*$-ideal of A.*

Proof. Let $x \in I$. Then $x(1 + x^*x)^{-1} \in I \cap A[B_0]$. Observe that $I \cap A[B_0]$ is a two-sided ideal of the W^*-algebra $A[B_0]$. Since any two-sided ideal of a W^*-algebra is a $*$-ideal, it follows that $[x(1+x^*x)^{-1}]^* \in I \cap A[B_0]$, that is, $(1+x^*x)^{-1}x^* \in I \cap A[B_0]$. Therefore

$$x^* = (1+x^*x)(1+x^*x)^{-1}x^* \in I,$$

completing the proof. □

It should be noted that it is known that every closed two-sided ideal of a GB*-algebra is a $*$-ideal (see [18]). In the preceding lemma, the assumption that $A[B_0]$ is a W^*-algebra makes the assumption "closed" for the ideal I to be redundant in order to conclude that I is a $*$-ideal.

Example 3.24. Let $\delta : D(\delta) \to A$ be a derivation of a GB*-algebra $A[\tau]$ with jointly continuous multiplication, and suppose that there is a $*$-subalgebra B of $D(\delta)$ which is dense in A and such that $1 \notin B$ (if $1 \notin D(\delta)$, then we can take $B = D(\delta)$). Assume that there is a continuous linear functional f on A such that $f|_{D(\delta)}$ is a state on $D(\delta)$ and

(i) $f \circ \delta$ is continuous on $D(\delta)$,

(ii) the GNS-representation π_f *of* $D(\delta)$ is faithful.

If δ_0 is the restriction of δ to B, then conditions (i) and (ii) still hold for δ_0. That condition (i) is still valid is trivial. Concerning (ii), the restriction of π_f to B is the map $(\pi_f)_B : B \to B(\mathcal{H})$ defined by $(\pi_f)_B(a)(x + (N_f \cap B)) = ax + (N_f \cap B)$ for all $a, x \in B$. Since π_f is faithful, it follows easily that $(\pi_f)_B$ is faithful. By Proposition 3.22, δ_0 is closed. By Theorem 3.6, $\overline{D(\delta_0) \cap A[B_0]}^{\|\cdot\|_{B_0}} \neq A[B_0]$ (since $1 \notin B$).

If A is a C^*-algebra, then a $*$-subalgebra B as described above does not exist, since every closed unbounded derivation of a C^*-algebra has 1 in its domain. Therefore, the example can only exist in the realm of nonnormed GB*-algebras.

Now, in this example, let $A[\tau]$ be a commutative metrizable pro-C^*-algebra which is not a C^*-algebra (with respect to the topology τ), and which has a (continuous) positive linear functional f such that π_f is faithful. Assume also that $A[B_0]$ is a W^*-algebra.

Observe that $A[\tau]$ is not a Q-algebra since it is not a C^*-algebra (by Corollary 3.3), and therefore, A has a dense maximal ideal M, say (see [34, Corollary 3]). Observe that $1 \notin M$ (since M is a maximal ideal, which is always properly contained in the algebra). Now let δ (as above) denote the everywhere defined zero derivation on A. Observe that condition (i) above is then trivially satisfied. Let δ_0 (as above) be the restriction of δ to M; that is, we are taking B above to be M. Since $A[B_0]$ is a W^*-algebra, it follows from Lemma 3.23 that M is a $*$-ideal, hence a $*$-subalgebra, of A. Then δ_0 is a closed $*$-derivation of A with $1 \notin M = B = D(\delta_0)$.

Furthermore, M is not closed under analytic functional calculus either. Assume that there exists $x \in M$ such that $\rho_A(x) \neq \emptyset$. Let $f(\lambda) = 1 \in \mathbb{C}$ for all λ in $\mathbb{C}^*$. Then f is an analytic function on $\mathbb{C}^*$, and $f(x) = 1 \in A$. Therefore, $f(x) \notin M = D(\delta_0)$.

In the example above, we might also have an example of a GB*-algebra $A[\tau]$ having a derivation $\delta : A \to A$ which is not closed under analytic functional calculus for self-adjoint elements of $D(\delta)$, with $1 \notin D(\delta)$. Is there a closed derivation for which the domain contains $1 \in A$ and which is not closed under analytic functional calculus? While such an example seems, at present, to be out of reach, were it to exist, then, by Corollary 3.5, it would have no significance to quantum physics and quantum statistical mechanics (and perhaps also not to the general theory of Lie groups and Lie algebras).

4. Domains of closed derivations of pro-C^*-algebras

In this section, we detect certain features (such as continuity of inversion or commutativity) which, when applied to a complete GB*-algebra with jointly continuous multiplication, are sufficient to fulfill Proposition 3.1(ii). The empowerment of the latter property resulted either from the proven coincidence of the algebraic spectra or the Allan spectra (see Theorem 4.4 and Theorem 4.8, respectively) depending on the initial assumptions that we set for the algebra.

We start off by showing that if $\delta : D(\delta) \to A$ is a closed $*$-derivation of a complete locally m-convex $*$-algebra $A[\tau]$ with $1 \in D(\delta)$, then $\mathrm{Sp}_{D(\delta)}(x) = \mathrm{Sp}_A(x)$ for all $x \in D(\delta)$. We first recall the following fact. Let $A[\tau_\Gamma]$ be a complete locally m-convex $*$-algebra with Γ a family of submultiplicative seminorms defining the topology τ_Γ of A. Consider $M_2(A)$, the set of all 2×2 matrices with elements from A, endowed with the usual matrix operations, involution $(a_{ij})_{i,j} \mapsto (a^*_{ji})_{i,j}$ and equipped with the topology induced by the seminorms $\{S_{\tilde{p}}\}_{p\in\Gamma}$, where $S_{\tilde{p}}(\left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{smallmatrix}\right))$, for each matrix $\left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{smallmatrix}\right)$ in $M_2(A)$, is defined by the following equalities:

$$\begin{aligned}&\sup\left\{\tilde{p}\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{pmatrix}\begin{pmatrix} a_1 \\ a_2\end{pmatrix}\right) : a_1, a_2 \in A; p(a_1) + p(a_2) \le 1\right\}\\ &\quad = \sup\bigl\{p(x_{11}a_1 + x_{12}a_2) + p(x_{21}a_1 + x_{22}a_2) : a_1, a_2 \in A; p(a_1) + p(a_2) \le 1\bigr\},\end{aligned}$$

where $\tilde{p}(a, b) := p(a) + p(b)$ for every $(a, b) \in A \times A, p \in \Gamma$. Then $M_2(A)$ is a complete locally m-convex $*$-algebra such that $M_2(A) = \varprojlim_p M_2(A_p)$ within a topological $*$-isomorphism (see [20, p. 467]). Note that, for every $p \in \Gamma$, the topology on $M_2(A_p)$ is the one induced by the norm

$$\begin{aligned}S_{\|\cdot\|_p}\left(\begin{pmatrix} a_p & b_p \\ c_p & d_p\end{pmatrix}\right) := \sup\bigl\{&\|a_px_p + b_py_p\|_p + \|c_px_p + d_py_p\|_p :\\ &(x_p, y_p) \in A_p \times A_p; \|x_p\|_p + \|y_p\|_p \le 1\bigr\},\end{aligned}$$

$\left(\begin{smallmatrix} a_p & b_p \\ c_p & d_p\end{smallmatrix}\right) \in M_2(A_p)$ (see [20, p. 467] and the comments therein).

For the proof of the following result, we use an adaptation of the proof of [15, Theorem 1.3].

Theorem 4.1. *Let $A[\tau]$ be a complete locally m-convex $*$-algebra, and let $\delta : D(\delta) \to A$ be a closed $*$-derivation of A such that $1 \in D(\delta)$. Then $\mathrm{Sp}_{D(\delta)}(x) = \mathrm{Sp}_A(x)$ for all $x \in D(\delta)$.*

Proof. On the one hand, it is immediate that $\mathrm{Sp}_A(x) \subset \mathrm{Sp}_{D(\delta)}(x)$. We show the other inclusion. We consider the following subalgebras of $M_2(A)$:

$$\mathcal{A}_\delta = \left\{ \begin{pmatrix} a & \delta(a) \\ 0 & a \end{pmatrix} : a \in D(\delta) \right\} \qquad \text{and} \qquad \mathfrak{B} = \left\{ \begin{pmatrix} a & c \\ 0 & a \end{pmatrix} : a, c \in A \right\},$$

where $\mathfrak{B}$ is a closed subalgebra of $M_2(A)$. Indeed, if $\{ \left(\begin{smallmatrix} b_i & c_i \\ 0 & b_i \end{smallmatrix}\right) \}_{i \in I}$ is a net in $\mathfrak{B}$ with $\left(\begin{smallmatrix} b_i & c_i \\ 0 & b_i \end{smallmatrix}\right) \to \left(\begin{smallmatrix} b & c \\ d & f \end{smallmatrix}\right) \in M_2(A)$ with respect to $\{S_{\tilde{p}}\}_{p \in \Gamma}$, then for every $p \in \Gamma$, we have that

$$\begin{aligned} p(b_i - b) &= p\big((b_i - b)1 + (c_i - c)0\big) \\ &\leq \sup\big\{ p\big((b_i - b)a_1 + (c_i - c)a_2\big) + p\big(-da_1 + (b_i - f)a_2\big) : \\ &\qquad (a_1, a_2) \in A \times A; \tilde{p}(a_1, a_2) = p(a_1) + p(a_2) \leq 1 \big\} \\ &= S_{\tilde{p}}\left(\begin{pmatrix} b_i & c_i \\ 0 & b_i \end{pmatrix} - \begin{pmatrix} b & c \\ d & f \end{pmatrix} \right) \underset{i}{\to} 0, \end{aligned}$$

where the inequality is based on the fact that we can assume without loss of generality that $p(1) = 1$ for every $p \in \Gamma$ (see [10, Theorem 2.3]). Therefore, $b_i \to b$ with respect to τ. Similarly, we show that $b_i \to f$ and that $d = 0$.

So, $\mathfrak{B}$, being a closed subalgebra of $M_2(A)$, is a complete locally m-convex algebra. Therefore, by the Arens–Michael decomposition of $\mathfrak{B}$, we have that $\mathfrak{B} = \varprojlim_p \mathfrak{B}[S_{\tilde{p}}]/N_{S_{\tilde{p}}} = \varprojlim_p \mathfrak{B}_{S_{\tilde{p}}}$, up to topological isomorphisms, where $\mathfrak{B}_{S_{\tilde{p}}}$ stands for the completion of the normed algebra $\mathfrak{B}[S_{\tilde{p}}]/N_{S_{\tilde{p}}}$ with respect to the norm $\|\cdot\|_{S_{\tilde{p}}}$: $\|B + N_{S_{\tilde{p}}}\|_{S_{\tilde{p}}} = S_{\tilde{p}}(B), B \in \mathfrak{B}$.

We next show that $\mathfrak{B}_{S_{\tilde{p}}} = \{ \left(\begin{smallmatrix} a_p & b_p \\ N_p & a_p \end{smallmatrix}\right) : a_p, b_p \in A_p \}$. Toward this end, we consider the map

$$\Phi : \mathfrak{B}[S_{\tilde{p}}]/N_{S_{\tilde{p}}} \to M_2(A_p) : \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + N_{S_{\tilde{p}}} \mapsto \begin{pmatrix} a + N_p & b + N_p \\ N_p & a + N_p \end{pmatrix}.$$

The map Φ is well defined since if, for $a, b \in A$, we have that $S_{\tilde{p}}(\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)) = 0$, then $p(a) = \tilde{p}(\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)) \leq S_{\tilde{p}}(\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)) = 0$, and thus that $a \in N_p$. Moreover, $p(b) = p(a \cdot 0 + b \cdot 1) + p(a) = \tilde{p}(\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)) \leq S_{\tilde{p}}(\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)) = 0$, and hence $b \in N_p$ also. Furthermore, Φ is clearly injective and an algebra homomorphism. In addition, Φ is isometric as can be seen by the following relations:

$$\begin{aligned} &S_{\|\cdot\|_p}\left(\Phi\left[\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + N_{S_{\tilde{p}}} \right] \right) \\ &\quad = S_{\|\cdot\|_p}\left(\begin{pmatrix} a + N_p & b + N_p \\ N_p & a + N_p \end{pmatrix} \right) \\ &\quad = \sup\big\{ \big\|(a + N_p)x_p + (b + N_p)y_p\big\|_p + \big\|(a + N_p)y_p\big\|_p : \\ &\qquad (x_p, y_p) \in A_p \times A_p; \|x_p\|_p + \|y_p\|_p \leq 1 \big\} \\ &\quad = \sup\big\{ \big\|(a + N_p)(x + N_p) + (b + N_p)(y + N_p)\big\|_p + \big\|(a + N_p)(y + N_p)\big\|_p : \\ &\qquad (x + N_p, y + N_p) \in A/N_p \times A/N_p; \|x + N_p\|_p + \|y + N_p\|_p \leq 1 \big\} \end{aligned}$$

$$= \sup\big\{p(ax+by)+p(ay):(x,y)\in A^2; p(x)+p(y)\leq 1\big\}$$
$$= S_{\tilde{p}}\left(\begin{pmatrix} a & b \\ 0 & a\end{pmatrix}\right) = \left\|\begin{pmatrix} a & b \\ 0 & a\end{pmatrix} + N_{S_{\tilde{p}}}\right\|_{S_{\tilde{p}}}.$$

Hence every element $\left(\begin{smallmatrix} a & b \\ 0 & a\end{smallmatrix}\right) + N_{S_{\tilde{p}}} \in \mathfrak{B}/N_{S_{\tilde{p}}}$ can be identified with the element $\left(\begin{smallmatrix} a+N_p & b+N_p \\ N_p & a+N_p\end{smallmatrix}\right) \in M_2(A_p)$. Moreover, Φ extends to an isometric algebra homomorphism from $\mathfrak{B}_{S_{\tilde{p}}}$ into $M_2(A_p)$.

Then, for $\{\left(\begin{smallmatrix} a_i & b_i \\ 0 & a_i\end{smallmatrix}\right) + N_{S_{\tilde{p}}}\}_i$ a $\|\cdot\|_{S_{\tilde{p}}}$-Cauchy net in $\mathfrak{B}/N_{S_{\tilde{p}}}$, we deduce that $(a_i+N_p)_i, (b_i+N_p)_i$ are $\|\cdot\|_p$-Cauchy nets. So, there are $a_p, b_p \in A_p$ such that $a_i+N_p \to a_p, b_i+N_p\to b_p$ with respect to $\|\cdot\|_p$. Thus we have that

$$\begin{aligned}
&S_{\|\cdot\|_p}\left(\begin{pmatrix} a_i+N_p & b_i+N_p \\ N_p & a_i+N_p\end{pmatrix} - \begin{pmatrix} a_p & b_p \\ N_p & a_p\end{pmatrix}\right)\\
&\quad= \sup\big\{\big\|\big[(a_i+N_p)-a_p\big]x_p + \big[(b_i+N_p)-b_p\big]y_p\big\|_p + \big\|\big[(a_i+N_p)-a_p\big]y_p\big\|_p :\\
&\qquad (x_p,y_p)\in A_p\times A_p; \|x_p\|_p+\|y_p\|_p\leq 1\big\}\\
&\quad\leq \big\|(a_i+N_p)-a_p\big\|_p + \big\|(b_i+N_p)-b_p\big\|_p \to 0.
\end{aligned}$$

So, we conclude that

$$\lim_{\|\cdot\|_{S_{\tilde{p}}}}\left[\begin{pmatrix} a_i & b_i \\ 0 & a_i\end{pmatrix} + N_{S_{\tilde{p}}}\right] \equiv \lim_{S_{\|\cdot\|_p}}\begin{pmatrix} a_i+N_p & b_i+N_p \\ N_p & a_i+N_p\end{pmatrix} = \begin{pmatrix} a_p & b_p \\ N_p & a_p\end{pmatrix},$$

where the first identification is derived from the fact that the extension of Φ is isometric as noted above.

Since the derivation δ is closed, we deduce that $\mathcal{A}_\delta$ is a closed subalgebra of $\mathfrak{B}$, and hence a complete locally m-convex algebra. Then $\mathcal{A}_\delta = \varprojlim_p(\mathcal{A}_\delta)_p$ up to topological isomorphism, where for every $p\in\Gamma, (\mathcal{A}_\delta)_p$ denotes the completion of $\mathcal{A}_\delta/N_{S_{\tilde{p}}}$ with respect to $\|\cdot\|_{S_{\tilde{p}}}$. Therefore, $(\mathcal{A}_\delta)_p$ is a closed subalgebra of the Banach algebra $\mathfrak{B}_p$. By following analogous considerations given in the preceding paragraph, we conclude that an element of $(\mathcal{A}_\delta)_p$ is of the form $\left(\begin{smallmatrix} a+N_p & b+N_p \\ N_p & a+N_p\end{smallmatrix}\right)$, for $a, b\in A$, such that there is a Cauchy net $(a_i)_i \in D(\delta)$ with $a_i+N_p \overset{\|\cdot\|_p}{\to} a+N_p$ and $\delta(a_i)+N_p \overset{\|\cdot\|_p}{\to} b+N_p$.

Following the proof of [15, Theorem 1.3], an involution, denoted by $\sharp$, is defined on $\mathfrak{B}_{S_{\tilde{p}}}$ as follows: $\left(\begin{smallmatrix} b_p & c_p \\ N_p & b_p\end{smallmatrix}\right)^\sharp = \left(\begin{smallmatrix} b_p^* & c_p^* \\ N_p & b_p^*\end{smallmatrix}\right)$. Then, $\mathfrak{B}_{S_{\tilde{p}}}$, endowed with the involution $\sharp$, is a symmetric Banach algebra (see [15, Theorem 1.2(1)]). For an element $\lim_{\|\cdot\|_{S_{\tilde{p}}}}\left(\begin{smallmatrix} a_i+N_p & \delta(a_i)+N_p \\ N_p & a_i+N_p\end{smallmatrix}\right) \in (\mathcal{A}_\delta)_p$, we have that

$$\begin{aligned}
\left(\lim_{\|\cdot\|_{S_{\tilde{p}}}}\begin{pmatrix} a_i+N_p & \delta(a_i)+N_p \\ N_p & a_i+N_p\end{pmatrix}\right)^\sharp &= \begin{pmatrix} \lim_{\|\cdot\|_p}(a_i+N_p) & \lim_{\|\cdot\|_p}(\delta(a_i)+N_p) \\ N_p & \lim_{\|\cdot\|_p}(a_i+N_p)\end{pmatrix}^\sharp\\
&= \begin{pmatrix} \lim_{\|\cdot\|_p}(a_i^*+N_p) & \lim_{\|\cdot\|_p}(\delta(a_i^*)+N_p) \\ N_p & \lim_{\|\cdot\|_p}(a_i^*+N_p)\end{pmatrix}\\
&= \lim_{\|\cdot\|_{S_{\tilde{p}}}}\begin{pmatrix} a_i^*+N_p & \delta(a_i^*)+N_p \\ N_p & a_i^*+N_p\end{pmatrix}.
\end{aligned}$$

Hence $(\mathcal{A}_\delta)_p$ is a closed $\sharp$-subalgebra of the symmetric Banach algebra $\mathfrak{B}_p$, containing the identity $\left(\begin{smallmatrix} 1+N_p & N_p \\ N_p & 1+N_p \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1+N_p & \delta(1)+N_p \\ N_p & 1+N_p \end{smallmatrix}\right)$. Therefore, by [15, Theorem 1.3], $\mathrm{Sp}_{(\mathcal{A}_\delta)_p}(\hat{x}_p) = \mathrm{Sp}_{\mathfrak{B}_p}(\hat{x}_p)$ for every $\hat{x}_p \in (\mathcal{A}_\delta)_p$. So, by [10, Theorem 4.6(2), p. 46], we have that

$$\mathrm{Sp}_{\mathcal{A}_\delta}(\hat{x}) = \bigcup_{p\in\Gamma} \mathrm{Sp}_{(\mathcal{A}_\delta)_p}(\hat{x}_p) = \bigcup_{p\in\Gamma} \mathrm{Sp}_{\mathfrak{B}_p}(\hat{x}_p) = \mathrm{Sp}_{\mathfrak{B}}(\hat{x})$$

for every $\hat{x} = (\hat{x}_p)_p \in \mathcal{A}_\delta = \varprojlim_p (\mathcal{A}_\delta)_p$. The rest of the proof is concluded using an argument in the proof of [15, Theorem 1.3], which we include for the reader's convenience. Let $x \in D(\delta)$ and $\lambda \notin \mathrm{Sp}_A(x)$; that is, $(\lambda 1 - x)^{-1} \in A$. Then the inverse of the element $\lambda\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} x & \delta(x) \\ 0 & x \end{smallmatrix}\right)$ exists in $\mathfrak{B}$, since

$$\left(\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} x & \delta(x) \\ 0 & x \end{pmatrix}\right)^{-1} = \begin{pmatrix} (\lambda 1 - x)^{-1} & (\lambda 1 - x)^{-1}\delta(x)(\lambda 1 - x)^{-1} \\ 0 & (\lambda 1 - x)^{-1} \end{pmatrix}.$$

Therefore, for $\hat{x} = \left(\begin{smallmatrix} x & \delta(x) \\ 0 & x \end{smallmatrix}\right)$, we have that $\lambda \notin \mathrm{Sp}_{\mathfrak{B}}(\hat{x})$. So $\lambda \notin \mathrm{Sp}_{\mathcal{A}_\delta}(\hat{x})$; that is, $(\lambda 1 - x)^{-1} \in D(\delta)$. Hence $\lambda \notin \mathrm{Sp}_{D(\delta)}(x)$. ☐

Remark 4.2. Based on Theorem 4.1 and Proposition 3.18, we have that the domain $D(\delta)$ for any τ-closed $*$-derivation of a commutative pro-C^*-algebra $A[\tau]$ such that $1 \in D(\delta)$ is advertibly complete.

Corollary 4.3. *Let $A[\tau]$ be a pro-C^*-algebra, and let $\delta : D(\delta) \to A$ be a closed $*$-derivation of A such that $1 \in D(\delta)$ and $x \in D(\delta)$. Then, for every analytic function $f \in F_x$, we have that $f(x) \in D(\delta)$.*

Proof. For $\lambda \in \rho_A(x) = \mathbb{C}^* \setminus \sigma_A(x)$, $\lambda \neq \infty$, it is straightforward that $\lambda \notin \mathrm{Sp}_A(x)$. Therefore, by Theorem 4.1, we have that $\lambda \notin \mathrm{Sp}_{D(\delta)}(x)$; that is, $(\lambda 1 - x)^{-1} \in D(\delta)$. Since a pro-$C^*$-algebra is an example of a complete GB*-algebra with jointly continuous multiplication (see [2, p. 95]), the result then follows from Proposition 3.1. ☐

Theorem 4.4. *Let $A[\tau]$ be a complete* GB**-algebra with continuous inversion and jointly continuous multiplication, and let $\delta : D(\delta) \to A$ be a closed $*$-derivation of A such that $1 \in D(\delta)$. Then $\mathrm{Sp}_{D(\delta)}(x) = \mathrm{Sp}_A(x)$ for all $x \in D(\delta)$.*

Proof. The proof is broken up into the following four stages.

(1) On $A \oplus A$, we define the multiplication

$$(a, b) \cdot (x, y) = (ax, ay + bx)$$

for all $a, b, x, y \in A$, and the involution

$$(x, y)^* = (x^*, y^*)$$

for all $x, y \in A$. Observe that $(1, 0)$ is the identity element of $A \oplus A$. If Γ is a family of $*$-seminorms defining the topology τ of A, then we let

$$\tilde{p}(x, y) = p(x) + p(y)$$

for all $x, y \in A, p \in \Gamma$. With respect to the family of $*$-seminorms $(\tilde{p})_{p\in\Gamma}$, describing a topology, say, $\tilde{\tau}$ on $A \oplus A$, it follows that $A \oplus A$ is a complete locally convex $*$-algebra with jointly continuous multiplication with respect to $\tilde{\tau}$.

(2) We now show that $\mathrm{Sp}_A(x) = \mathrm{Sp}_{A\oplus A}(x, \delta(x))$ for all $x \in D(\delta)$. Let $x \in D(\delta)$. If z is invertible in A, then it is an easy exercise to show that (z, w) is invertible in $A \oplus A$ for all $w \in A$ and that

$$(z, w)^{-1} = (z^{-1}, -z^{-1}wz^{-1}).$$

Therefore,

$$\begin{aligned} \lambda \notin \mathrm{Sp}_A(x) \quad &\Longleftrightarrow \quad x - \lambda 1 \text{ is invertible in } A \\ &\Longleftrightarrow \quad \big(x - \lambda 1, \delta(x)\big)^{-1} \text{ exists in } A \oplus A \\ &\Longleftrightarrow \quad \big[\big(x, \delta(x)\big) - \lambda(1, 0)\big]^{-1} \text{ exists in } A \oplus A \\ &\Longleftrightarrow \quad \lambda \notin \mathrm{Sp}_{A\oplus A}\big(x, \delta(x)\big). \end{aligned}$$

(3) Next, we show that $\mathrm{Sp}_{A\oplus A}(x, \delta(x)) = \mathrm{Sp}_{G(\delta)}(x, \delta(x))$ for all $x \in D(\delta)$, where $G(\delta)$ is the graph of δ. From the identity

$$(x, y)^{-1} = (x^{-1}, -x^{-1}yx^{-1}),$$

where it is assumed that x, and hence (x, y), is invertible, it follows that $A \oplus A$ has continuous inversion (here, we also use the fact that A has jointly continuous multiplication and continuous inversion). Since $A \oplus A$ is complete with respect to $\tilde{\tau}$, we get that $A \oplus A$ is sequentially complete and hence pseudocomplete (see [1, Proposition 2.6]). Thus, the mere existence of

$$\begin{aligned} &\big((1, 0) + (x, y)^*(x, y)\big)^{-1} \\ &\quad = (1 + x^*x, x^*y + y^*x)^{-1} \\ &\quad = \big((1 + x^*x)^{-1}, -(1 + x^*x)^{-1}(x^*y + y^*x)(1 + x^*x)^{-1}\big) \end{aligned}$$

leads us, by [2, Proposition 2.4], to the information that $A \oplus A$ is symmetric, that is, that the former invertible element is bounded. Furthermore, we show that $(A \oplus A)[\tilde{\tau}]$ is in fact a GB*-algebra. For this, it suffices to show that the respective family $\mathfrak{B}^*_{A\oplus A}$ of subsets of $A \oplus A$ has a greatest member. Toward this end, let $\Omega = (B_1, B_2) \in \mathfrak{B}^*_{A\oplus A}$. Since $(1, 0) \in \Omega$, we have that $1 \in B_1$. Moreover, $\Omega^2 \subset \Omega$ and thus, by induction, $\Omega^n \subset \Omega$ for every $n \in \mathbb{N}$. By induction, we get that

$$(B_1, B_2)^n = \Big(B_1^n, \sum_{k=0}^{n-1} B_1^k B_2 B_1^{n-(k+1)}\Big), \quad \forall n \in \mathbb{N}.$$

Therefore, since $(B_1, B_2)^n \subset (B_1, B_2)$, we have that $\sum_{k=0}^{n-1} B_1^k B_2 B_1^{n-(k+1)} \subset B_2$. Hence, by the latter inclusion, for $x \in B_2$ and for $1 \in B_1$, we take that $nx \in B_2$ for every $n \in \mathbb{N}$. Thus, if there exists a seminorm $p \in \Gamma$ such that $p(x) \neq 0$, we have that $p(nx) = np(x) \to +\infty$ for $n \to +\infty$, which contradicts the boundedness of B_2 (that B_2 is bounded follows from the boundedness of Ω and from the definition of the topology $\tilde{\tau}$ on $A \oplus A$). So $p(x) = 0, \forall p \in \Gamma$. Thus $x = 0$, from which we conclude that $B_2 = \{0\}$. Therefore, we have that $\Omega = (B_1, \{0\})$. From the respective properties of Ω as a member of $\mathfrak{B}^*_{A\oplus A}$, we can easily derive that $B_1 \in \mathfrak{B}^*_A$. Thus, we conclude that $(B_0, \{0\})$ is the greatest member of $\mathfrak{B}^*_{A\oplus A}$, where B_0 is the greatest member of $\mathfrak{B}^*_A$. Hence we deduce that $A \oplus A$ is a GB*-algebra.

Since δ is closed, we have that $G(\delta)$ is a closed subalgebra of $A \oplus A$ containing the identity element $(1, 0)$. Then, by [2, Proposition 2.9], $G(\delta)$ is a GB*-algebra. So $G(\delta)$ has Hermitian involution; that is, $\sigma_{G(\delta)}(x, \delta(x)) \subset \mathbb{R}$ for every self-adjoint element $(x, \delta(x)) \in G(\delta)$ (see [2, Proposition 2.2]). Therefore, $\mathrm{Sp}_{G(\delta)}(x, \delta(x)) \subset \sigma_{G(\delta)}(x, \delta(x)) \subset \mathbb{R}$ for every self-adjoint $(x, \delta(x)) \in G(\delta)$. The result then follows from an argument similar to the proof of [25, Proposition I.4.8].

(4) The map $\theta : D(\delta) \to A, x \mapsto (x, \delta(x))$, is an injective self-adjoint algebra homomorphism of $D(\delta)$ onto its range $G(\delta)$. Therefore,

$$\mathrm{Sp}_{G(\delta)}\big(x, \delta(x)\big) = \mathrm{Sp}_{D(\delta)}(x)$$

for all $x \in D(\delta)$.

By combining stages (2), (3), and (4) above, we get that $\mathrm{Sp}_{D(\delta)}(x) = \mathrm{Sp}_A(x)$ for all $x \in D(\delta)$. □

Corollary 4.5. *Let $A[\tau]$ be a complete* GB**-algebra with continuous inversion and jointly continuous multiplication, and let $\delta : D(\delta) \to A$ be a closed $*$-derivation of A such that $1 \in D(\delta)$ and $x \in D(\delta)$. Then $f(x) \in D(\delta)$ for every $f \in F_x$.*

Proof. The result follows from Theorem 4.4 and Proposition 3.1. □

Remark 4.6.

(1) A pro-C^*-algebra is an example of a complete GB*-algebra with jointly continuous multiplication and continuous inversion. Moreover, every pro-C^*-algebra is in particular a complete locally m-convex algebra. Nevertheless, not every GB*-algebra is m-convex. Hence Theorem 4.4 can be seen as a kind of generalization of Theorem 4.1.

(2) Theorem 4.4 tells us in particular that the algebra $B = \{\left(\begin{smallmatrix} b & c \\ 0 & b \end{smallmatrix}\right) : b, c \in A\}$ is a GB*-algebra. Indeed, the map

$$\phi : B \to A \oplus A : \begin{pmatrix} b & c \\ 0 & b \end{pmatrix} \mapsto (b, c)$$

is an algebraic $*$-isomorphism. Recall that the involution considered on B is $\left(\begin{smallmatrix} b & c \\ 0 & b \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} b^* & c^* \\ 0 & b^* \end{smallmatrix}\right)$. Therefore, since $A \oplus A$ is symmetric, so is B (see also [2, Proposition 2.4]). If we endow B with the locally convex topology inherited from the GB*-topology of $A \oplus A$, that is, the one induced on B by the seminorms $r_{\tilde{p}}(\left(\begin{smallmatrix} b & c \\ 0 & b \end{smallmatrix}\right)) := \tilde{p}(b, c) = p(b) + p(c)$, then B is a GB*-algebra, whose greatest member for its respective family of subsets $\mathfrak{B}_B$ is the set $\phi^{-1}(B_0, 0) = \left(\begin{smallmatrix} B_0 & 0 \\ 0 & B_0 \end{smallmatrix}\right)$.

(3) If we define the topology, involution, and multiplication on $A \oplus A \oplus A$ in an analogous way to that of Theorem 4.4 for $A \oplus A$, it can be proved by using arguments similar to those of stage (3) of the proof of Theorem 4.4 that $A \oplus A \oplus A$ is a GB*-algebra. Hence, by an analogous argument to that of Remark 4.6(2), we have that the algebra $\{\left(\begin{smallmatrix} b & c \\ 0 & d \end{smallmatrix}\right) : b, c, d \in A\}$

is a GB*-algebra. Similarly, it can be shown that the algebra $\{(\begin{smallmatrix} b & 0 \\ c & d \end{smallmatrix}) : b, c, d \in A\}$ is a GB*-algebra. The general question, though, of whether $M_2(A)$ is a GB*-algebra in case A is seems to be unsettled. (For an example of $M_n(A)$ being a GB*-algebra for a particular GB*-algebra A, see [12, Examples 6.2(3)].)

Let M be a von Neumann algebra with a faithful semifinite normal trace ϕ, and let $\widetilde{M}$ denote the unital $*$-algebra of ϕ-measurable operators affiliated with M. Then $\widetilde{M}$ is a Fréchet topological $*$-algebra, which is also a GB*-algebra in the sense of Dixon [9, Definition 2.5] (not necessarily locally convex), with respect to the topology of convergence in measure τ_{cm}. With respect to this topology, we note that $\widetilde{M}$ is not necessarily locally convex. Now, $(\widetilde{M}, \tau_{\text{cm}})$ has continuous inversion (see [27]). Observe that $\widetilde{M} \oplus \widetilde{M}$ is a unital topological $*$-algebra with respect to the same multiplication and involution as in stage (1) of the proof of Theorem 4.4.

Corollary 4.7. *Let M be a von Neumann algebra with a faithful semifinite normal trace ϕ such that the topology τ_{cm} on $\widetilde{M}$ is locally convex. Let $\delta : D(\delta) \to \widetilde{M}$ be a closed $*$-derivation of $\widetilde{M}$ with $1 \in D(\delta)$. Then $\mathrm{Sp}_{D(\delta)}(x) = \mathrm{Sp}_{\widetilde{M}}(x)$ for all $x \in D(\delta)$. Furthermore, $f(x) \in D(\delta)$ for every $f \in F_x$.*

In an attempt to answer the question whether dependence on the assumption of continuity of inversion in Theorem 4.4 can be alleviated, we have the following result.

Theorem 4.8. *Let $A[\tau]$ be a complete* GB**-algebra with jointly continuous multiplication, and let $\delta : D(\delta) \to A$ be a closed $*$-derivation of A such that $1 \in D(\delta)$. Then $\sigma_A(x) = \sigma_{D(\delta)}(x)$, for every normal element $x \in D(\delta)$.*

For the proof of Theorem 4.8, we make use of the following result.

Lemma 4.9. *Let A be a* GB**-algebra with jointly continuous multiplication. Consider $A \oplus A$ endowed with topology $\tilde{\tau}$, multiplication, and involution, defined as in the proof of Theorem 4.4. Then $((1,0) + (x,y)^*(x,y))^{-1} \in (A \oplus A)_0$, for every $(x,y) \in A \oplus A$.*

Proof. As noted in the proof of Theorem 4.4,

$$\begin{aligned} \big((1,0) + (x,y)^*(x,y)\big)^{-1} &= \big((1,0) + (x^*x, x^*y + y^*x)\big)^{-1} \\ &= \big((1+x^*x)^{-1}, -(1+x^*x)^{-1}(x^*y + y^*x)(1+x^*x)^{-1}\big). \end{aligned}$$

For simplicity of notation, let us denote throughout the proof

$$a \equiv (1 + x^*x)^{-1}, \qquad b \equiv x^*y + y^*x.$$

We show that the set $S = \{(\frac{1}{2}(a, -aba))^n : n \in \mathbb{N}\}$ is a $\{\tilde{p}\}_{p\in\Gamma}$-bounded subset of $A \oplus A$. Let $p \in \Gamma$ be an arbitrary but fixed seminorm of the family Γ. Since the multiplication on A is jointly continuous, there is a $q \in \Gamma$ such that $p(xyz) \leq q(x)q(y)q(z)$ for all $x, y, z \in A$. Since $\tau \leq \|\cdot\|_{B_0}$ on $A[B_0]$, there is $C_p, C_q > 0$

such that $p(x) \leq C_p\|x\|_{B_0}, q(x) \leq C_q\|x\|_{B_0}$ for every $x \in A[B_0]$. We note that, as can be shown by induction, we have that for every $n \in \mathbb{N}$,

$$(a, -aba)^n = \Big(a^n, -\sum_{k=1}^{n} a^k b a^{n+1-k}\Big).$$

Therefore,

$$\begin{aligned}
\tilde{p}\Big(\Big(\frac{1}{2}(a,-aba)\Big)^n\Big) &= \tilde{p}\Big(\frac{1}{2^n}\Big(a^n, -\sum_{k=1}^{n} a^k b a^{n+1-k}\Big)\Big)\\
&\leq \frac{1}{2^n}\Big[p(a^n) + \sum_{k=1}^{n} q(a^k)q(b)q(a^{n+1-k})\Big]\\
&\leq \frac{1}{2^n}\Big[C_p\|a^n\|_{B_0} + \sum_{k=1}^{n} C_q^2\|a^k\|_{B_0}q(b)\|a^{n+1-k}\|_{B_0}\Big]\\
&\leq \frac{1}{2^n}\Big[C_p + \sum_{k=1}^{n} C_q^2 q(b)\Big]\\
&= \frac{1}{2^n}C_p + \frac{n}{2^n}C_q^2 q(b),
\end{aligned}$$

where the argument of the second-to-last line is deduced from the fact that $a = (1 + x^*x)^{-1} \in B_0$. Thus we have that for every $n \in \mathbb{N}$,

$$\tilde{p}\Big(\Big(\frac{1}{2}(a,-aba)\Big)^n\Big) \leq \frac{1}{2^n}C_p + \frac{n}{2^n}C_q^2 q(b) \leq C_p + C_q^2 q(b).$$

So

$$\sup\Big\{\tilde{p}\Big(\Big[\frac{1}{2}(a,-aba)\Big]^n\Big) : n \in \mathbb{N}\Big\} \leq C_p + C_q^2 q(b) < +\infty.$$

Since the seminorm p was arbitrary, we have that for every seminorm $\tilde{p}$ from the family of the seminorms defining the topology $\tilde{\tau}$ on $A \oplus A$, $\sup\{\tilde{p}(\omega) : \omega \in S\} < +\infty$. Hence the set S is $\{\tilde{p}\}$-bounded, from which the result follows. $\square$

Proof of Theorem 4.8. It is straightforward that $\infty \in \sigma_{D(\delta)}(x)$ if and only if $\infty \in \sigma_A(x)$, for every $x \in D(\delta)$. Thus we proceed to show the result for the finite part of the Allan spectra.

We first show the result for a self-adjoint element $x \in D(\delta)$. We follow the steps of the proof of Theorem 4.4, but we use the Allan spectra. Involution, multiplication, and the topology $\tilde{\tau}$ on $A \oplus A$ are considered to be those of the proof of Theorem 4.4. We note that the self-adjointness of x is only needed in step (3) below.

Step 1: We show that $\sigma_{A\oplus A}(x, \delta(x)) = \sigma_A(x)$ for all $x \in D(\delta)$.

Indeed, on the one hand, if $\lambda \in \sigma_A(x)$, then $\lambda \in \sigma_{A\oplus A}(x, \delta(x))$. For if λ were not in $\sigma_{A\oplus A}(x, \delta(x))$, then $(\lambda(1,0) - (x, \delta(x)))^{-1} \in (A \oplus A)_0$. Therefore, in that case, $(\lambda 1 - x)^{-1}$ exists and $((\lambda 1 - x)^{-1}, (\lambda 1 - x)^{-1}\delta(x)(\lambda 1 - x)^{-1}) = (\lambda(1,0) - (x, \delta(x)))^{-1} \in (A \oplus A)_0$. Hence, by the way the multiplication and the topology $\tilde{\tau}$ are defined, we conclude that $(\lambda 1 - x)^{-1} \in A_0$, which is a contradiction. So $\sigma_A(x) \subset \sigma_{A\oplus A}(x, \delta(x))$.

On the other hand, let $\lambda \notin \sigma_A(x)$. Then $(\lambda 1 - x)^{-1} \in A_0$, and thus there is a nonzero $\mu \in \mathbb{C}$ such that the set $\{(\mu(\lambda 1 - x)^{-1})^n : n \in \mathbb{N}\}$ is $\{p\}_{p\in\Gamma}$-bounded. Hence, for a fixed seminorm p, there exists $M_p > 0$ such that

$$\sup\big\{p\big(\big(\mu(\lambda 1 - x)^{-1}\big)^n\big) : n \in \mathbb{N}\big\} \le M_p.$$

Moreover, for this seminorm $p \in \Gamma$, there exists a seminorm q such that $p(abc) \le q(a)q(b)q(c), a, b, c \in A$, and for the latter seminorm q, there exists $M_q > 0$ such that $\sup\{q((\mu(\lambda 1 - x)^{-1})^n) : n \in \mathbb{N}\} \le M_q$. Following arguments similar to those of Lemma 4.9, we have that for every $n \in \mathbb{N}$,

$$\begin{aligned}
&\tilde{p}\Big(\Big[\frac{\mu}{2}\big((\lambda 1 - x)^{-1}, (\lambda 1 - x)^{-1}\delta(x)(\lambda 1 - x)^{-1}\big)\Big]^n\Big) \\
&\quad = \tilde{p}\Big(\frac{\mu^n}{2^n}\big((\lambda 1 - x)^{-1}\big)^n, \frac{\mu^n}{2^n}\sum_{k=1}^{n}\big((\lambda 1 - x)^{-1}\big)^k\delta(x)\big((\lambda 1 - x)^{-1}\big)^{n+1-k}\Big) \\
&\quad = \tilde{p}\Big(\frac{1}{2^n}\big(\mu(\lambda 1 - x)^{-1}\big)^n, \sum_{k=1}^{n}\big(\mu(\lambda 1 - x)^{-1}\big)^k\frac{1}{2^n\mu}\delta(x)\big(\mu(\lambda 1 - x)^{-1}\big)^{n+1-k}\Big) \\
&\quad \le \frac{1}{2^n}p\big(\big(\mu(\lambda 1 - x)^{-1}\big)^n\big) + \frac{1}{2^n|\mu|}\sum_{k=1}^{n}p\big(\big(\mu(\lambda 1 - x)^{-1}\big)^k\delta(x)\big(\mu(\lambda 1 - x)^{-1}\big)^{n+1-k}\big) \\
&\quad \le \frac{1}{2^n}M_p + \frac{n}{2^n|\mu|}M_q^2 q\big(\delta(x)\big) \\
&\quad \le M_p + \frac{M_q^2}{|\mu|}q\big(\delta(x)\big).
\end{aligned}$$

Therefore, from the string of relations above, we conclude that $(\lambda(1,0) - (x,\delta(x)))^{-1}$ is a bounded element of $A \oplus A$. Hence $\lambda \notin \sigma_{A\oplus A}(x,\delta(x))$. Thus, we have that $\sigma_{A\oplus A}(x,\delta(x)) \subset \sigma_A(x)$.

So, from all the above in step (1), we have that $\sigma_A(x) = \sigma_{A\oplus A}(x,\delta(x))$.

Step 2: We show that $\sigma_{D(\delta)}(x) = \sigma_{G(\delta)}(x,\delta(x))$ for every $x \in D(\delta)$.

On the one hand, if $\lambda \notin \sigma_{G(\delta)}(x,\delta(x))$, then

$$\big(\lambda(1,0) - \big(x,\delta(x)\big)\big)^{-1} = \big((\lambda 1 - x)^{-1}, (\lambda 1 - x)^{-1}\delta(x)(\lambda 1 - x)^{-1}\big) \in G(\delta)_0.$$

Hence we deduce that $(\lambda 1 - x)^{-1} \in D(\delta)_0$, and thus $\lambda \notin \sigma_{D(\delta)}(x)$.

On the other hand, let $\lambda \notin \sigma_{D(\delta)}(x)$. Hence $(\lambda 1 - x)^{-1} \in D(\delta)_0$. From the series of relations of step (1), we have that

$$\big(\lambda(1,0) - \big(x,\delta(x)\big)\big)^{-1} = \big((\lambda 1 - x)^{-1}, (\lambda 1 - x)^{-1}\delta(x)(\lambda 1 - x)^{-1}\big)$$

is a bounded element of $A \oplus A$. The fact that the latter element belongs to $G(\delta)$ is derived from the Leibniz rule as follows:

$$\begin{aligned}
\delta(1) = 0 \quad &\Rightarrow \quad \delta\big((\lambda 1 - x)^{-1}(\lambda 1 - x)\big) = 0 \\
&\Rightarrow \quad \delta\big((\lambda 1 - x)^{-1}\big)(\lambda 1 - x) + (\lambda 1 - x)^{-1}\delta(\lambda 1 - x) = 0 \\
&\Rightarrow \quad \delta\big((\lambda 1 - x)^{-1}\big)(\lambda 1 - x) = -(\lambda 1 - x)^{-1}\delta(\lambda 1 - x) = (\lambda 1 - x)^{-1}\delta(x) \\
&\Rightarrow \quad \delta\big((\lambda 1 - x)^{-1}\big) = (\lambda 1 - x)^{-1}\delta(x)(\lambda 1 - x)^{-1}.
\end{aligned}$$

Step 3: We show that $\sigma_{G(\delta)}(x,\delta(x)) = \sigma_{A\oplus A}(x,\delta(x))$ for every self-adjoint $x \in D(\delta)$. The inclusion $\sigma_{A\oplus A}(x,\delta(x)) \subset \sigma_{G(\delta)}(x,\delta(x))$ is straightforward.

For the inverse inclusion, let $\lambda \notin \sigma_{A\oplus A}(x,\delta(x))$. We first remark that, due to Lemma 4.9, $(A\oplus A)[\tilde{\tau}]$ is a symmetric locally convex $*$-algebra. In addition, as is shown in (3) of the proof of Theorem 4.4, $A\oplus A$ is pseudocomplete and $\mathfrak{B}^*_{A\oplus A}$ has a greatest member. Note that for the latter arguments, continuity of inversion is not needed. Hence we have that $A\oplus A$ is a GB*-algebra. Then $G(\delta)$ is also a GB*-algebra as a closed subalgebra of $A\oplus A$ containing the identity.

We show that $\lambda \notin \sigma_{G(\delta)}(x,\delta(x))$. For this, it suffices to assume that $\lambda \in \mathbb{R}$. This is due to the fact that $G(\delta)$, as a GB*-algebra, has symmetric involution; that is, for the self-adjoint element $(x,\delta(x))$, we have that $\sigma_{G(\delta)}(x,\delta(x)) \subset \mathbb{R}$. Let $\lambda_\epsilon = \lambda + i\epsilon$, where $\epsilon \in \mathbb{R}$. Then $\lambda_\epsilon \in \rho_{G(\delta)}(x,\delta(x)) \subset \rho_{A\oplus A}(x,\delta(x))$ and $\lambda_\epsilon \underset{\epsilon\to 0}{\to} \lambda$, where $\lambda \in \rho_{A\oplus A}(x,\delta(x))$. Then, by [1, Theorem 3.8(3)], we get that $(\lambda_\epsilon(1,0)-(x,\delta(x)))^{-1} \to (\lambda(1,0)-(x,\delta(x)))^{-1}$ for $\epsilon \to 0$ with respect to norm convergence in $(A\oplus A)[(B_0,\{0\})]$, and thus with respect to $\tilde{\tau}$, since $\tilde{\tau} \leq \|\cdot\|_{(B_0,\{0\})}$ on $(A\oplus A)[(B_0,\{0\})]$. Therefore, since $G(\delta)$ is a closed subalgebra of $A\oplus A$, we conclude that $(\lambda(1,0)-(x,\delta(x)))^{-1} \in G(\delta)$. Clearly, $(\lambda(1,0)-(x,\delta(x)))^{-1}$ is a bounded element in $G(\delta)$ since it is a bounded element in $A\oplus A$. So we have that $\lambda \notin \sigma_{G(\delta)}(x,\delta(x))$.

From the three steps above, we conclude that $\sigma_{D(\delta)}(x) = \sigma_{G(\delta)}(x,\delta(x)) = \sigma_{A\oplus A}(x,\delta(x)) = \sigma_A(x)$, for every self-adjoint element $x \in D(\delta)$.

Now, let x be a normal element in $D(\delta)$, that is, $x^*x = xx^*$. As can be seen in the proof above, one does not require x to be self-adjoint in steps (1) and (2) of the proof. It is for this reason that the only point that needs to be proved is the inclusion $\sigma_{G(\delta)}(x,\delta(x)) \subset \sigma_{A\oplus A}(x,\delta(x))$. So let $\lambda \notin \sigma_{A\oplus A}(x,\delta(x))$. Then $(\lambda(1,0)-(x,\delta(x)))^{-1}$ exists in $A\oplus A$ and is bounded. Therefore, $(\lambda 1-x)^{-1} \in A_0$. Let $\omega = \lambda(1,0)-(x,\delta(x)) = (\lambda 1 - x, -\delta(x))$. It is clear that $\omega,\, \omega^* \in G(\delta)$. We have that

$$\omega^*\omega = \big((\lambda 1-x)^*(\lambda 1-x), -(\lambda 1-x)^*\delta(x) - \delta(x^*)(\lambda 1-x)\big).$$

Note that $\omega^*\omega \in G(\delta)$, as this follows easily from the Leibniz rule and the facts that δ is a $*$-derivation and $\delta(1)=0$. We have that

$$\begin{aligned}
(\omega^*\omega)^{-1} &= \big((\lambda 1-x)^{-1}\big((\lambda 1-x)^{-1}\big)^*, (\lambda 1-x)^{-1}\big((\lambda 1-x)^{-1}\big)^* \\
&\qquad \times \big[(\lambda 1-x)^*\delta(x) + \delta(x^*)(\lambda 1-x)\big](\lambda 1-x)^{-1}\big((\lambda 1-x)^{-1}\big)^*\big) \\
&= \big((\lambda 1-x)^{-1}\big((\lambda 1-x)^{-1}\big)^*, (\lambda 1-x)^{-1}\delta(x)(\lambda 1-x)^{-1}\big((\lambda 1-x)^{-1}\big)^*\big) \\
&\qquad + (\lambda 1-x)^{-1}\big((\lambda 1-x)^{-1}\big)^*\delta(x^*)\big((\lambda 1-x)^{-1}\big)^*.
\end{aligned}$$

For simplicity of notation, we set

$$c = (\lambda 1-x)^{-1}\big((\lambda 1-x)^{-1}\big)^*, \qquad a = (\lambda 1-x)^{-1}, \qquad b = \delta(x).$$

So by the above string of relations, we have that

$$(\omega^*\omega)^{-1} = \big(c, (ab)c + \big((ab)c\big)^*\big).$$

We prove by induction that

$$\big((\omega^*\omega)^{-1}\big)^n = \Big(c^n, \sum_{k=0}^{n-1} c^k(ab)c^{n-k} + \Big(\sum_{k=0}^{n-1} c^k(ab)c^{n-k}\Big)^*\Big).$$

Indeed, the latter relation holds for $n = 1$. If it is true for $n \in \mathbb{N}$, then

$$\begin{aligned}
\big((\omega^*\omega)^{-1}\big)^{n+1} &= \Big(c^n, \sum_{k=0}^{n-1} c^k(ab)c^{n-k} + \Big(\sum_{k=0}^{n-1} c^k(ab)c^{n-k}\Big)^*\Big) \cdot \big(c, (ab)c + \big((ab)c\big)^*\big) \\
&= \Big(c^{n+1}, c^n(ab)c + c^{n+1}(b^*a^*) \\
&\qquad + \sum_{k=0}^{n-1} c^k(ab)c^{n+1-k} + \Big(\sum_{k=0}^{n-1} c^{k+1}(ab)c^{n-k}\Big)^*\Big) \\
&= \Big(c^{n+1}, c^n(ab)c + c^{n+1}(b^*a^*) \\
&\qquad + \sum_{k=0}^{n-1} c^k(ab)c^{n+1-k} + \Big(\sum_{k=1}^{n} c^{k}(ab)c^{n+1-k}\Big)^*\Big) \\
&= \Big(c^{n+1}, \sum_{k=0}^{n} c^k(ab)c^{n+1-k} + \Big(\sum_{k=0}^{n} c^k(ab)c^{n+1-k}\Big)^*\Big),
\end{aligned}$$

where for the above string of relations, we recall that c is a self-adjoint element. Now let $p \in \Gamma_\tau$. From the joint continuity of multiplication, there is a seminorm q such that

$$p(xyzw) \leq q(x)q(y)q(z)q(w)$$

for every $x, y, z, w \in A$. Also, since x is normal, we have that $((\lambda 1 - x)^{-1})^*$ and $(\lambda 1 - x)^{-1}$ commute. Hence, since they are both bounded elements, we have that $c = (\lambda 1 - x)^{-1}((\lambda 1 - x)^{-1})^* \in A_0$ (see [2, p. 92]). Therefore, there is a nonzero $\mu \in \mathbb{C}$ and there are $C_p, C_q > 0$ such that

$$\sup\big\{p\big((\mu c)^n\big) : n \in \mathbb{N}\big\} \leq C_p, \qquad \sup\big\{q\big((\mu c)^n\big) : n \in \mathbb{N}\big\} \leq C_q.$$

For the following relations, we note that since there exists $0 \neq \mu \in \mathbb{C}$ such that $\{(\mu c)^n : n \in \mathbb{N}\}$ is a bounded subset of $A \oplus A$, it is immediate that the set $\{(|\mu| c)^n : n \in \mathbb{N}\}$ is also bounded. Moreover, we can assume without loss of generality that $s(x) = s(x^*)$ and $s(1) = 1$ for every seminorm $s \in \Gamma$ and for every $x \in A$ (see [12, Lemma 5.5]). Therefore, we have that

$$\begin{aligned}
\tilde{p}\Big(\Big[\frac{|\mu|}{2}(\omega^*\omega)^{-1}\Big]^n\Big) &= \tilde{p}\Big(\frac{|\mu|^n}{2^n}c^n, \frac{1}{2^n}\sum_{k=0}^{n-1}\big(|\mu|c\big)^k ab\big(|\mu|c\big)^{n-k} \\
&\qquad + \frac{1}{2^n}\Big(\sum_{k=0}^{n-1}\big(|\mu|c\big)^k ab\big(|\mu|c\big)^{n-k}\Big)^*\Big) \\
&= \frac{1}{2^n}p\big(\big(|\mu|c\big)^n\big) + \frac{1}{2^{n-1}}p\Big(\sum_{k=0}^{n-1}\big(|\mu|c\big)^k ab\big(|\mu|c\big)^{n-k}\Big)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2^n} C_p + \frac{1}{2^{n-1}} \sum_{k=0}^{n-1} q\big((|\mu| c)^k\big) q(a) q(b) q\big((|\mu| c)^{n-k}\big) \\
&= \frac{1}{2^n} C_p + \frac{1}{2^{n-1}} q(a) q(b) q\big((|\mu| c)^n\big) \\
&\quad + \frac{1}{2^{n-1}} \sum_{k=1}^{n-1} q\big((|\mu| c)^k\big) q(a) q(b) q\big((|\mu| c)^{n-k}\big) \\
&\leq C_p + C_q q(a) q(b) + \frac{n-1}{2^{n-1}} C_q^2 q(a) q(b) \\
&\leq C_p + C_q q(a) q(b) + C_q^2 q(a) q(b).
\end{aligned}$$

So, from the above string of relations, we have that there is a positive number $|\mu|$ such that $\sup\{\tilde{p}([\frac{|\mu|}{2}(\omega^*\omega)^{-1}]^n) : n \in \mathbb{N}\} < +\infty$ for every seminorm $\tilde{p}$. Hence we get that $(\omega^*\omega)^{-1} \in (A \oplus A)_0$. Thus $0 \notin \sigma_{A\oplus A}(\omega^*\omega)$. So $0 \notin \sigma_{G(\delta)}(\omega^*\omega)$, since we have already proved coincidence of the Allan spectra for self-adjoint elements. Therefore, we have that $(\omega^*\omega)^{-1} \in G(\delta)_0$. Hence, since $G(\delta)$ is an algebra according to how the multiplication is defined on $A \oplus A$, we have that $\omega^{-1} = (\omega^*\omega)^{-1}\omega^* \in G(\delta)$. Moreover, ω^{-1} is bounded in $G(\delta)$ since it is initially supposed to be bounded in $A \oplus A$. So, we have that $\omega^{-1} = (\lambda(1,0) - (x,\delta)x)^{-1} \in G(\delta)_0$, that is, $\lambda \notin \sigma_{G(\delta)}(x, \delta(x))$. □

Remark 4.10. If A is a complete GB*-algebra with continuous inversion and jointly continuous multiplication, and $\delta : D(\delta) \to A$ is a closed $*$-derivation of A with $1 \in D(\delta)$, then by Theorem 4.4, we have that $\mathrm{Sp}_{D(\delta)}(x) = \mathrm{Sp}_A(x)$ for every $x \in D(\delta)$. The latter equality tells us in particular that if $x \in D(\delta)$ is invertible in A, then $x^{-1} \in D(\delta)$. So, by the assumed continuity of inversion for A, we conclude that $D(\delta)$ is a locally convex algebra with continuous inversion. Hence, by [1, Theorem 4.1], we get that $\mathrm{Sp}_{D(\delta)}(x) \subset \sigma_{D(\delta)}(x) \subset \mathrm{cl}(\mathrm{Sp}_{D(\delta)}(x))$. Furthermore, since A is pseudocomplete and has continuous inversion, it follows from [1, Theorem 4.1] that $\mathrm{cl}(\mathrm{Sp}_A(x)) = \sigma_A(x)$. Then, by the two previous relations and the fact that $\mathrm{Sp}_{D(\delta)}(x) = \mathrm{Sp}_A(x)$, we get that $\sigma_{D(\delta)}(x) \subset \mathrm{cl}(\mathrm{Sp}_A(x)) = \sigma_A(x)$. The other inclusion, $\sigma_A(x) \subset \sigma_{D(\delta)}(x)$, is immediate. Thus we have that $\sigma_{D(\delta)}(x) = \sigma_A(x)$, that is, the result of Theorem 4.8.

Nevertheless, by abandoning the hypothesis of continuity of inversion we are driven to shift our attention to the Allan spectra. This comes as a consequence of the fact that in the proof of Theorem 4.8, we make use of [1, Theorem 3.8(3)] and for the application of the latter result, we need to work with a bounded resolvent (see the proof of [1, Theorem 3.8(3)]).

Based on Theorem 4.8, the following corollary is immediate.

Corollary 4.11. *Let A be a commutative complete* GB**-algebra with jointly continuous multiplication, and let $\delta : D(\delta) \to A$ be a closed $*$-derivation with $1 \in D(\delta)$. Then $\sigma_A(x) = \sigma_{D(\delta)}(x)$ for all $x \in D(\delta)$.*

Corollary 4.12. *Let A be a commutative complete* GB**-algebra with jointly continuous multiplication, and let $\delta : D(\delta) \to A$ be a closed $*$-derivation with $1 \in D(\delta)$. Then, for every $f \in F_x$, we have that $f(x) \in D(\delta)$.*

Proof. The result follows from Theorem 4.8 and Proposition 3.1. □

Remark 4.13. As can be seen by Example 3.24, condition $1 \in D(\delta)$ cannot be dropped from hypothesis of Corollary 4.12.

Corollary 4.14 below is a converse to Theorem 3.6, answering the question raised in Remark 3.7(1).

Corollary 4.14. *Let $A[\tau]$ be a complete* GB**-algebra with jointly continuous multiplication. Let $\delta : D(\delta) \to A$ be a closed $*$-derivation of A. Then the following are equivalent:*

(i) *$1 \in D(\delta)$ and the norm-closed unit ball of $\overline{A[B_0] \cap D(\delta)}^{\|\cdot\|_{B_0}}$ is τ-closed, and*

(ii) $\overline{A[B_0] \cap D(\delta)}^{\|\cdot\|_{B_0}} = A[B_0]$.

Proof. (i) $\Rightarrow$ (ii) Let $x \in D(\delta)$ be self-adjoint. Then $x_n = x(1 + \frac{1}{n}x^2)^{-1} \to x$ with respect to the topology τ, and $x_n \in A[B_0]$ (see Proposition 2.4). Also, $n \notin \sigma_A(-x^2)$, since $(n1 + x^2)^{-1} \in A[B_0] \subset A_0$. Therefore, by Theorem 4.8, we have that $n \notin \sigma_{D(\delta)}(-x^2)$. So $(1 + \frac{1}{n}x^2)^{-1} \in D(\delta)$. Therefore, since $x \in D(\delta)$, we get that $x_n \in D(\delta)$ for all $n \in \mathbb{N}$. So $x \in \overline{A[B_0] \cap D(\delta)}^{\tau}$. Since $D(\delta)$ is a $*$-algebra and all elements of $D(\delta)$ are linear combinations of self-adjoint elements of $D(\delta)$, we get that $D(\delta) \subseteq \overline{A[B_0] \cap D(\delta)}^{\tau}$. Therefore, since $\overline{D(\delta)}^{\tau} = A$, we get that $A \subseteq \overline{A[B_0] \cap D(\delta)}^{\tau}$. Hence it follows that $\overline{A[B_0] \cap D(\delta)}^{\tau} = A$, and therefore

$$\overline{\overline{A[B_0] \cap D(\delta)}^{\|\cdot\|_{B_0}}}^{\tau} = A.$$

Let

$$C = \overline{A[B_0] \cap D(\delta)}^{\|\cdot\|_{B_0}}.$$

Since $D(\delta)$ is a $*$-subalgebra of A, and $A[B_0]$ is a C^*-algebra with respect to $\|\cdot\|_{B_0}$, we get that C is a C^*-algebra with respect to $\|\cdot\|_{B_0}$.

Since $A[\tau]$ is complete, it follows that $A[\tau]$ is also the τ-completion of C. Furthermore, $\tau|_C \leq \|\cdot\|_{B_0}$. By [4, Theorem 2.2], $A[\tau]$ is a GB*-algebra over B_τ, where B_τ is the τ-closure of the norm-closed unit ball $\mathcal{U}(C)$ of C.

Since $\mathcal{U}(C)$ is τ-closed, $\mathcal{U}(C) = B_\tau$ and $C = A[B_\tau]$. Thus $A[\tau]$ is also a GB*-algebra over $\mathcal{U}(C)$ (not only B_0). Furthermore, by the proof of [4, Theorem 2.2], B_τ is the largest member of the collection $\mathcal{B}^*$ associated with $A[\tau]$, which is viewed here as the τ-completion of C. Now B_0 is the largest member of $\mathcal{B}^*$ corresponding to $A[\tau]$. Therefore $B_0 = B_\tau = \mathcal{U}(C)$, and so $A[B_0] = A[B_\tau] = C$, that is,

$$\overline{A[B_0] \cap D(\delta)}^{\|\cdot\|_{B_0}} = A[B_0].$$

(ii) $\Rightarrow$ (i) By Theorem 3.6, we have that $1 \in D(\delta)$. Furthermore, the relation $\overline{A[B_0] \cap D(\delta)}^{\|\cdot\|_{B_0}} = A[B_0]$ clearly gives us that $B_0 = \mathcal{U}(C)$. Since B_0 is τ-closed, we get that $\mathcal{U}(C)$ is τ-closed. □

An example of a GB*-algebra which is not a C^*-algebra and which satisfies the hypothesis of Corollary 4.14 is given in Example 3.20.

5. Generators of one-parameter groups of automorphisms of GB*-algebras

In this section, we characterize those $*$-derivations of GB*-algebras which are infinitesimal generators of one-parameter groups of $*$-automorphisms of the GB*-algebra, thereby also extending some corresponding results for C^*-algebras, as given in [23, Section 3.4].

Let $X[\tau]$ be a sequentially complete locally convex space, and let $(T_t)_{t\geq 0}$ be a one-parameter family of continuous linear operators from X into itself. We say that (T_t) is *equicontinuous* with respect to a family of seminorms $(p_\alpha)_{\alpha\in I}$ on X which define the topology τ if, for every seminorm $p_\alpha, \alpha \in I$, there exists a seminorm $p_\beta, \beta \in I$ such that $p_\alpha(T_t(x)) \leq p_\beta(x)$ for all $x \in X$ and $t \geq 0$ (see [32, p. 234]).

The above family $(T_t)_{t\geq 0}$ of linear operators on X will be called a C_0-*semigroup* of linear operators if the following hold for all $x \in X$ (see [32, Definition, p. 234]):

(1) $T_{t+s}x = T_tT_sx$, for $t, s \geq 0$;
(2) $T_0x = x$;
(3) $\lim_{t\to t_0} T_tx = T_{t_0}x$, for every $t_0 \geq 0$.

The following result is a generalization of the Hille–Yosida theorem to sequentially complete locally convex spaces. In this regard, we observe that if T is the infinitesimal generator of a C_0-semigroup of linear operators on X, then, for every $n \in \mathbb{N}$, we have that $(n1 - T)^{-1}$ exists and is a continuous linear operator from X into itself (see [32, Theorem 1, p. 240]).

Theorem 5.1 ([32, p. 246])**.** *Let $X[\tau]$ be a sequentially complete locally convex space. Let T be a densely defined linear operator on X. Then T is the infinitesimal generator of an equicontinuous semigroup of class C_0 with respect to $(p_\alpha)_\alpha$ if and only if the operators $\{(1 - \frac{1}{n}T)^{-m}\}$ are equicontinuous in $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ with respect to $(p_\alpha)_\alpha$, where $(p_\alpha)_\alpha$ is a defining family of seminorms for the topology τ on X.*

If $\{\alpha_t : t \geq 0\}$ is a C_0-semigroup of automorphisms of a GB*-algebra $A[\tau]$, then it can be extended to a group of automorphisms of A. Since, for every $t \geq 0$, α_t is an invertible linear operator on A whose inverse is an automorphism, we can define α_{-t} to be α_t^{-1} for every $t \geq 0$.

From this and Theorem 5.1, we can deduce the following theorem.

Theorem 5.2. *Let $A[\tau]$ be a sequentially complete* GB*-*algebra, and let $\delta : D(\delta) \to A$ be a $*$-derivation. Then δ is the generator of an equicontinuous one-parameter automorphism group of A with respect to $(p_\alpha)_\alpha$ if and only if $\{(1 - \frac{1}{n}\delta)^{-m} : n \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}\}$ is equicontinuous with respect to $(p_\alpha)_\alpha$, where $(p_\alpha)_\alpha$ is a defining family of seminorms for the topology τ on A.*

With respect to this theorem, we note that every sequentially complete locally convex algebra is pseudocomplete (see [1, Proposition 2.6]). Hence, in view of Definition 2.3, a sequentially complete GB*-algebra is actually a symmetric sequentially complete locally convex $*$-algebra such that the collection $\mathcal{B}^*$ has a greatest member.

If $A[\|\cdot\|]$ is a C^*-algebra, and δ is a $*$-derivation of A, then the closure $\overline{\delta}$ of δ is a generator of a one-parameter group of $*$-automorphisms of A if and only if $(1\pm\delta)(D(\delta))$ is dense in A, and for every self-adjoint $x \in D(\delta)$, there exists a state f_x of A such that $|f_x(x)| = \|x\|$ and $f_x(\delta(x)) = 0$ (see [23, Proposition 3.4.4]). This result is the Lumer–Phillips theorem (see [32, p. 250]) put in the context of C^*-algebras, and it would be interesting to know if one can extend this result to GB*-algebras. It would therefore be interesting to know if the Lumer–Phillips theorem for Banach spaces extends to complete locally convex spaces. In order to do this, we first have to extend the notion of dissipative operator to the setting of locally convex spaces.

A densely defined operator $T : D(T) \to X$ on a Banach space X is called *dissipative* if for every $x \in D(T)$, there exists $f_x \in X^*$ such that $f_x(x) = \|x\|$, $|f_x(y)| \leq \|y\|$ for all $y \in X$, and $\mathrm{Re}(f_x(Tx)) \leq 0$. Clearly, the notion of dissipativeness of an operator depends on the norm defining the topology of X. Furthermore, the above definition coincides with that given in [19, Definition 1.2].

We now generalize the concept of dissipative operator to the setting of a general complete locally convex space. Let $X[\tau]$ be a complete locally convex space, and let $T : D(T) \to X$ be a densely defined linear operator. Let $(p_\alpha)_\alpha$ denote a family of seminorms on X defining the topology τ. We say that T is *dissipative* if for every $x \in D(T)$ and every seminorm p_α, there exists $f_x^\alpha \in X^*$ such that $f_x^\alpha(x) = p_\alpha(x)$, $|f_x^\alpha(y)| \leq p_\alpha(y)$ for all $y \in X$ and $\mathrm{Re}(f_x^\alpha(Tx)) \leq 0$.

It is clear that if X is a Banach space, then this definition coincides with that of dissipative operator for Banach spaces given above. As with the normed case, the above definition of dissipative operator depends on the family of seminorms $(p_\alpha)_\alpha$ on X defining the topology τ. The following result gives one implication of the Lumer–Phillips theorem for locally convex spaces.

Proposition 5.3. *Let $X[\tau]$ be a complete locally convex space, and let T be a generator of an equicontinuous C_0-semigroup of linear operators on X with respect to a family of seminorms $(p_\alpha)_\alpha$ defining the topology τ. Then T is dissipative with respect to a defining family of seminorms for τ, and $\mathcal{R}(I-T) = X$.*

Proof. It follows from [32, p. 240] that $\mathcal{R}(I-T) = X$. Let $(p_\alpha)_\alpha$ be a family of seminorms on X defining the topology τ and with respect to which T is the generator of an equicontinuous C_0-semigroup of linear operators on X, $\{T_t\}_{t\geq 0}$ say. For each α, let $p'_\alpha(x) = \sup_{t\geq 0} p_\alpha(T_t(x)), x \in X$. Then, $(p'_\alpha)_\alpha$ is a family of seminorms, equivalent to $(p_\alpha)_\alpha$, which define the topology τ and such that $p'_\alpha(T_t(x)) \leq p'_\alpha(x)$ for all $x \in X, t \geq 0$ and for all α. With respect to the seminorms $(p'_\alpha)_\alpha$, $X = \varprojlim_\alpha \overline{X_\alpha}$, where $\overline{X_\alpha}$ is the Banach space completion of the normed space $X_\alpha = X/N_\alpha$ with respect to the norm $\widetilde{p'_\alpha}(x + N_\alpha) = p'_\alpha(x)$ for all $x \in X$, where $N_\alpha = \{x \in X : p'_\alpha(x) = 0\}$. By [3, Corollary 4.5 or Theorem 2.5], we have that T can be expressed as an inverse limit of densely defined operators T_α on $\overline{X_\alpha}$ which are generators of a C_0-semigroup of contraction operators on $\overline{X_\alpha}$ and where $T_\alpha(x + N_\alpha) = T(x) + N_\alpha, x \in D(T)$. By the Lumer–Phillips theorem for Banach spaces, there exists, for each $x \in D(T)$ and each α, a $g_x^\alpha \in \overline{X_\alpha}^*$ such that $g_x^\alpha(x + N_\alpha) = \widetilde{p'_\alpha}(x + N_\alpha)$, $|g_x^\alpha(y + N_\alpha)| \leq \widetilde{p'_\alpha}(y + N_\alpha)$ for all $y \in X$, and

$\mathrm{Re}[g_x^\alpha(T_\alpha(x+N_\alpha))] \leq 0$. Now let $f_x^\alpha(y) = g_x^\alpha(y+N_\alpha)$ for all $y \in X$. Then

$$\left|f_x^\alpha(y)\right| = \left|g_x^\alpha(y+N_\alpha)\right| \leq \widetilde{p'_\alpha}(y+N_\alpha) = p'_\alpha(y)$$

for all $y \in X$. Also,

$$f_x^\alpha(x) = g_x^\alpha(x+N_\alpha) = \widetilde{p'_\alpha}(x+N_\alpha) = p'_\alpha(x).$$

Now $\mathrm{Re}(f_x^\alpha(Tx)) = \mathrm{Re}[g_x^\alpha(T_\alpha(x+N_\alpha))] \leq 0$, for all α. Therefore, T is dissipative with respect to the family $(p'_\alpha)_\alpha$. □

What now about the reverse implication? If one again is to try and work with inverse limits, then one would have to start off by writing T as an inverse limit of operators, and then apply the results in [3] and the Lumer–Phillips theorem for Banach spaces. Now one cannot in general express T as an inverse limit of operators in order to start the argument, and therefore an alternative argument to that of inverse limits must be found.

We now give a converse of Proposition 5.3 above. Namely, we prove the following result.

Theorem 5.4. *Let $X[\tau]$ be a complete locally convex space, and let $T : D(T) \to X$ be a densely defined linear operator which is dissipative with respect to a family of seminorms $(p_\alpha)_\alpha$ defining the topology τ, and such that $\mathcal{R}(\lambda I - T) = X$ for all $\lambda > 0$. Then T is a generator of an equicontinuous C_0-semigroup of linear operators on X with respect to $(p_\alpha)_\alpha$.*

Proof. Let $x \in X$, and let $(p_\alpha)_\alpha$ be a family of seminorms defining the topology τ on X. Since T is dissipative with respect to $(p_\alpha)_\alpha$, there exists for every α and every $x \in D(T)$ an $f_x^\alpha \in X^*$ such that $f_x^\alpha(x) = p_\alpha(x)$, $|f_x^\alpha(y)| \leq p_\alpha(y)$ for all $y \in X$ and $\mathrm{Re}(f_x^\alpha(Tx)) \leq 0$. Let $\lambda > 0$. Then

$$\begin{aligned}
\lambda p_\alpha(x) &= \lambda f_x^\alpha(x) \\
&\leq \lambda f_x^\alpha(x) - \mathrm{Re}\big(f_x^\alpha(Tx)\big) \\
&= \mathrm{Re}\big(\lambda f_x^\alpha(x)\big) - \mathrm{Re}\big(f_x^\alpha(Tx)\big) \\
&= \mathrm{Re}\big(f_x^\alpha(\lambda x)\big) - \mathrm{Re}\big(f_x^\alpha(Tx)\big) \\
&= \mathrm{Re}\big[f_x^\alpha\big((\lambda I - T)x\big)\big] \\
&\leq \big|f_x^\alpha\big((\lambda I - T)x\big)\big| \\
&\leq p_\alpha\big((\lambda I - T)x\big),
\end{aligned}$$

that is, $0 \leq \lambda p_\alpha(x) \leq p_\alpha((\lambda I - T)x)$ for all $x \in D(T)$, for all α, and for all $\lambda > 0$. Therefore, if $(\lambda I - T)x = 0$, then $p_\alpha((\lambda I - T)x) = 0$ for all α, and consequently, $p_\alpha(x) = 0$ for all α. Therefore $x = 0$, and hence

$$(\lambda I - T)^{-1} : \mathcal{R}(\lambda I - T) = X \to X$$

exists for all $\lambda > 0$. From the above inequality, it follows that

$$\lambda p_\alpha\big((\lambda I - T)^{-1}x\big) \leq p_\alpha\big((\lambda I - T)\big[(\lambda I - T)^{-1}x\big]\big) = p_\alpha(x)$$

for all $\lambda > 0$, for all $x \in X$, and for all α. Therefore,

$$\lambda p_\alpha\big((\lambda I - T)^{-n}x\big) = \lambda p_\alpha\big((\lambda I - T)^{-1}\big[(\lambda I - T)^{1-n}x\big]\big) \le p_\alpha\big((\lambda I - T)^{1-n}x\big)$$

for all $\lambda > 0$, for all $x \in X$, for all $n \in \mathbb{N}$, and for all α. Hence

$$\begin{aligned} p_\alpha\big((\lambda I - T)^{-n}x\big) &\le \frac{1}{\lambda} p_\alpha\big((\lambda I - T)^{1-n}x\big) \\ &\le \frac{1}{\lambda^2} p_\alpha\big((\lambda I - T)^{2-n}x\big) \\ &\le \frac{1}{\lambda^n} p_\alpha\big((\lambda I - T)^{n-n}x\big) \\ &= \frac{1}{\lambda^n} p_\alpha(x) \end{aligned}$$

for all $n \in \mathbb{N}$, $x \in X$, and α. Hence

$$p_\alpha\Big(\frac{1}{\lambda^n}\Big(I - \frac{1}{\lambda}T\Big)^{-n} x\Big) = p_\alpha\big((\lambda I - T)^{-n}x\big) \le \frac{1}{\lambda^n} p_\alpha(x)$$

for all $x \in X$, for all α, and $n \in \mathbb{N}$. Therefore,

$$\sup_{n \in \mathbb{N}} p_\alpha\Big(\Big(I - \frac{1}{\lambda}T\Big)^{-n} x\Big) \le p_\alpha(x)$$

for all $x \in X$, for all α, and for all $\lambda > 0$. So we get that $\{(I - \frac{1}{m}T)^{-n} : n, m \in \mathbb{N}\}$ is equicontinuous with respect to $(p_\alpha)_\alpha$. By Theorem 5.1, that is, a generalized Hille–Yosida theorem, we get that T is the generator of an equicontinuous C_0-semigroup of linear operators on X with respect to $(p_\alpha)_\alpha$. □

In the case where X is a Banach space, the proof of the Lumer–Phillips theorem, given in [19] and [32], shows that if $\mathcal{R}(I - T) = X$, then $\mathcal{R}(\lambda I - T) = X$ for all $\lambda > 0$. This makes use of the Neumann power series and an analytic continuation argument. It would be interesting to see if this argument, or a variation thereof, holds for the general complete locally convex case. Nevertheless, in light of this remark, we see that Theorem 5.4 generalizes the Lumer–Phillips theorem for Banach spaces to complete locally convex spaces.

If $X[\tau]$ is a GB*-algebra, and T above is an unbounded $*$-derivation of X, then it would be interesting to know if the continuous linear functionals f_x^α above are positive linear functionals. That this is so for C^*-algebras is well known (see [23, Proposition 3.4.4]). In the following proposition, we answer this question for pro-C^*-algebras.

Proposition 5.5. *Let $A[\tau]$ be a pro-C^*-algebra, and let $\delta : D(\delta) \to A$ be a generator of a one-parameter group $(\alpha_t)_{t\in\mathbb{R}}$ of $*$-automorphisms of A. Let $0 \le x \in D(\delta)$. If $(p_\alpha)_\alpha$ is a family of C^*-seminorms defining the topology τ on A, then, for each α, there exists a (not necessarily nonzero) continuous positive linear functional f_x^α on A such that $f_x^\alpha(\delta(x)) \le 0$ and $f_x^\alpha(x) = p_\alpha(x)$.*

Proof. Let $x \ge 0$ with $x \in D(\delta)$, and let δ be as in the hypothesis. Then $x_\alpha = x + N_\alpha \in D(\delta_\alpha) = D(\delta)/N_\alpha$ for all α and $\delta = \varprojlim_\alpha \delta_\alpha$. As in the proof of

Proposition 5.3, we have that

$$g_x^\alpha(x+N_\alpha) = \widetilde{p_\alpha}(x+N_\alpha),$$
$$g_x^\alpha(y+N_\alpha) \leq \widetilde{p_\alpha}(y+N_\alpha)$$

for all $y \in A$ and $\mathrm{Re}[g_x^\alpha(\delta_\alpha x_\alpha)] \leq 0$. If $x \notin N_\alpha$, then $\|g_x^\alpha\|_\alpha = 1$. So

$$\left|g_x^\alpha(x+N_\alpha)\right| = \widetilde{p_\alpha}(x+N_\alpha) = \|g_x^\alpha\|_\alpha \widetilde{p_\alpha}(x+N_\alpha).$$

By a standard result for C^*-algebras, and remembering that A/N_α is a C^*-algebra for every α, we get that g_x^α is a positive linear functional on A/N_α.

Now if $x \in N_\alpha$, then

$$g_x^\alpha(x+N_\alpha) = \widetilde{p_\alpha}(x+N_\alpha) = 0,$$

and therefore

$$\left|g_x^\alpha(x+N_\alpha)\right| = \widetilde{p_\alpha}(x+N_\alpha) = \|g_x^\alpha\|_\alpha \widetilde{p_\alpha}(x+N_\alpha).$$

Therefore, as in the above, g_x^α is a positive linear functional on A/N_α. This holds for every α and every $0 \leq x \in D(\delta)$. Therefore f_x^α, defined as in the proof of Proposition 5.3, is a positive linear functional on A for every α. By the proof of the same proposition, we get that

$$f_x^\alpha\big(\delta(x)\big) = \mathrm{Re}\big[f_x^\alpha\big(\delta(x)\big)\big] \leq 0.$$

This completes the proof. $\square$

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