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EXTRAPOLATION THEOREMS FOR (p, q)-FACTORABLE OPERATORS

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Abstract. The operator ideal of (p, q)-factorable operators can be characterized as the class of operators that factors through the embedding $L^{q'}(\mu) \hookrightarrow L^p(\mu)$ for a finite measure μ, where $p, q \in [1, \infty)$ are such that $1/p + 1/q \geq 1$. We prove that this operator ideal is included into a Banach operator ideal characterized by means of factorizations through rth and sth power factorable operators, for suitable $r, s \in [1, \infty)$. Thus, they also factor through a positive map $L^s(m_1)^* \to L^r(m_2)$, where m_1 and m_2 are vector measures. We use the properties of the spaces of u-integrable functions with respect to a vector measure and the uth power factorable operators to obtain a characterization of (p, q)-factorable operators and conditions under which a (p, q)-factorable operator is r-summing for $r \in [1, p]$.

1. Introduction

The class of p-factorable operators (denoted by $\mathcal{L}_p$) was introduced by Kwapień in [10], who discovered the following relation between such operators and the p'-dominated operators, that is, $\mathcal{L}_p^* = \mathcal{D}_{p'}$. The equality $\mathcal{L}_{p,q}^* = \mathcal{D}_{q',p'}$, involving the generalization to the (p, q) case of the previous classes, is also known. Moreover, (p, q)-dominated operators can be characterized as the product of p- and q-summing operators, that is, $\mathcal{D}_{p,q} = \Pi_q \circ \Pi_p^{\text{dual}}$ (see [5, Theorem 19.3]). Maurey [12] also studied the class of operators that factor through L^p-spaces of a finite measure, providing an extrapolation theorem for p-summing operators

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which establishes, in one of its versions, that $\Pi_p(X,Y) = \Pi_r(X,Y)$ for every $r \in [1,p]$, provided that $\Pi_p(X,\ell^p) = \Pi_r(X,\ell^p)$ for some $1 \le r < p < \infty$ and that X is a Banach space (see [6, Theorem 3.17]). From this result, it is easy to deduce a corollary for (p,q)-factorable operators. For instance, if $1 < r < p < \infty$, $1 < s < q < \infty$, $\Pi_{r'}(X,\ell^{r'}) = \Pi_{p'}(X,\ell^{r'})$, and $\Pi_{s'}(X,\ell^{s'}) = \Pi_{q'}(X,\ell^{s'})$, then $\mathcal{L}_{p,q}(X,Y) = \mathcal{L}_{r,s}(X,Y)$. Another extrapolation result follows from the Maurey–Rosenthal theorem (see, e.g., [4, Corollaries 2–5]). In this case, if $1/p + 1/q \ge 1$, $r < p$, and $s < q$, and if X and Y are, respectively, r'-convex and s-concave Banach lattices, then any positive (p,q)-factorable operator is (r,s)-factorable.

As far as we know, there is no known connection between p-factorable and pth power factorable operators in the literature. Our aim in this article is to establish such a connection and to obtain applications to the class of (p,q)-factorable operators. To be precise, we provide a number of extrapolation results for (p,q)-factorable operators via a new Banach operator ideal that can be characterized by means of factorizations through L^r-spaces of vector measures. After that, our technique allows us to factor the involved operators through L^r-spaces of scalar measures.

We have organized the paper as follows. After the preliminary Section 2, we introduce the class of operators that factor through pth power factorable operators in Section 3. The class of pth power factorable operators was first defined in [15, Chapter 5] and generalized in [9, Lemma 3.3]. These operators have two main properties. First, every pth power factorable operator factors through an L^p-space of a vector measure (see, e.g., [15, Theorem 4.14], [9], [8]). Second, every pth power factorable operator is rth power factorable for every $r \in [1,p]$ (see [15, Section 2]). These two properties play a key role in our main results. Finally, in Section 4, we prove that this class is in fact a Banach operator ideal. Then we apply several results related to the compactness and convexity-concavity of the operators involved to obtain extrapolation theorems on the Banach ideal of (p,q)-factorable operators.

2. Notation and preliminaries

Throughout the paper, we use standard notation of real Banach spaces and consider only linear and continuous operators. Let X be a Banach space. The unit ball is denoted by B_X, and the topological dual is denoted by X^*. Given a Banach space Y, the Banach space of linear and continuous operators from X into Y with the usual norm is denoted by $\mathcal{L}(X,Y)$, and we understand that $T \in \mathcal{L}(X,Y)$ implies that $T^* \in \mathcal{L}(Y^*,X^*)$. The isometric embedding of a Banach space into its bidual spaces is denoted by $k_X\colon X \hookrightarrow X^{**}$. Let (Ω,Σ,μ) be a positive finite measure space. The space of classes of measurable functions equal almost everywhere with respect to μ is denoted by $L^0(\mu)$. Let $A \in \Sigma$; hence $\chi_A \in L^0(\mu)$ denotes the characteristic function. A *Banach function space* (BFS for short) over μ is a Banach space $Z(\mu)$ continuously embedded into $L^0(\mu)$ and satisfying the following.

(i) (Ideal property). If $g \in Z(\mu)$ and $|f| \le |g|$ ($f \in L^0(\mu)$), then $f \in Z(\mu)$ and $\|f\|_{Z(\mu)} \le \|g\|_{Z(\mu)}$.

(ii) For every $A \in \Sigma$, $\chi_A \in Z(\mu)$.

Observe that $L^\infty(\mu) \subseteq Z(\mu) \subseteq L^1(\mu)$, since μ is assumed to be finite. The Köthe dual space is denoted by $Z(\mu)'$, that is, the BFS of all integral functionals of the topological dual $Z(\mu)^*$.

A BFS $Z(\mu)$ is called *σ-order continuous* or simply *order continuous* (OC for short) if, for every sequence of functions $(f_i)_i \subseteq Z(\mu)^+$ such that $f_i \downarrow 0$, we have $\|f_i\|_{Z(\mu)} \downarrow 0$ (note that σ-OC and OC coincide in BFS; see [15, Remark 2.5]). A BFS $Z(\mu)$ is called *σ-Fatou* or simply *Fatou* if, for every sequence of functions $(f_i)_i \subseteq Z(\mu)^+$ such that $f_i \uparrow f \in L^0(\mu)$ and $\sup_i \|f_i\|_{Z(\mu)} < \infty$, we have $f \in Z(\mu)$ and $\|f_i\|_{Z(\mu)} \uparrow \|f\|_{Z(\mu)}$. The main characterizations of these properties are that a BFS $Z(\mu)$ is OC if and only if $Z(\mu)^* = Z(\mu)'$, and it is Fatou if and only if $Z(\mu)'' = Z(\mu)$. Recall that every BFS is a Banach lattice with the pointwise order. Let X be a Banach lattice, let V be a Banach space, and let $p \in [1,\infty)$. An operator $R \in \mathcal{L}(V,X)$ is called *p-convex* if there exists a constant $K > 0$ such that

$$\Big\| \Big(\sum_{i=1}^n |Rv_i|^p \Big)^{1/p} \Big\|_X \le K \Big(\sum_{i=1}^n \|v_i\|_V^p \Big)^{1/p},$$

for every choice of $v_1, \dots, v_n \in V$. An operator $S \in \mathcal{L}(X,V)$ is called *p-concave* if there exists a constant $K > 0$ such that

$$\Big(\sum_{i=1}^n \|Sx_i\|_V^p \Big)^{1/p} \le K \Big\| \Big(\sum_{i=1}^n |x_i|^p \Big)^{1/p} \Big\|_X,$$

for every choice of $x_1, \dots, x_n \in X$. In the case where $V = X$, and R and S are the identity maps, we say that X is *p-convex* and *p-concave*, respectively. All these definitions and results can be found in [11].

We note the following more specific terminology. Let $0 < p < \infty$, and let $Z(\mu)$ be a BFS. We call the *pth power space* (sometimes called the $(1/p)$*th power space*) of $Z(\mu)$, the space

$$Z(\mu)_{[p]} := \big\{ f \in L^0(\mu) : |f|^{1/p} \in Z(\mu) \big\},$$

equipped with the quasinorm $\|f\|_{Z(\mu)_{[p]}} := \||f|^{1/p}\|_{Z(\mu)}^p$. If $Z(\mu)$ is p-convex, then $Z(\mu)_{[p]}$ is a BFS if and only if the constant of p-convexity is 1 (see [15, Proposition 2.23(iii)]). As a main property, it satisfies the inclusion $Z(\mu) \subseteq Z(\mu)_{[p]} \subseteq Z(\mu)_{[q]}$ for $1 \le p \le q < \infty$, keeping in mind the fact that μ is finite (see [15, Proposition 2.22]). Let $Z(\mu)$ be an OC BFS, let X be a Banach space, and let $1 \le p < \infty$. We say that $T \in \mathcal{L}(Z(\mu),X)$ is *pth power factorable* if there exists a continuous linear operator $T_{[p]}\colon Z(\mu)_{[p]} \to X$ such that $T_{[p]}$ is a linear extension of T, that is, $T = T_{[p]} \circ i_{[p]}$, where $i_{[p]}$ denotes the inclusion $Z(\mu) \subseteq Z(\mu)_{[p]}$ (see [15, Definition 5.1]). Let X and Y be Banach spaces. The class of all pth power factorable operators in $\mathcal{L}(Z(\mu),Y)$ is denoted by $\mathcal{F}_p(Z(\mu),Y)$, and we denote by $\mathcal{F}_q^{\mathrm{dual}}(X,Z(\mu))$ the class of all operators $R \in \mathcal{L}(X,Z(\mu))$ such that $R^* \in \mathcal{F}_q(Z(\mu)^*,X^*)$. We will use the following two characterizations. We say that $S \in \mathcal{L}(Z(\mu),Y^{**})$ is *pth power factorable* if and only if there is some $K > 0$ such that $\|Sf\|_{Y^{**}} \le K\|f\|_{Z(\mu)_{[p]}}$ for all $f \in Z(\mu)$

or, equivalently, $|\langle f, S^* y^{***}\rangle| \le K\|y^{***}\|_{(Y^{**})^*}\|f\|_{Z(\mu)_{[p]}}$ for every $y^{***} \in (Y^{**})^*$ and every $f \in Z(\mu)$. Now assume that $Z(\mu)$ has an OC dual space. Then given $R \in \mathcal{L}(X, Z(\mu))$, we say that R^* is *qth power factorable* if and only if there is some $K > 0$ such that $|\langle Rx, g\rangle| \le K\|x\|_X\|g\|_{(Z(\mu)')_{[q]}}$ for every $x \in X$ and every $g \in Z(\mu)' = Z(\mu)^*$.

Regarding vector measures, let X be a Banach space, and let $m\colon \Sigma \to X$ be a (countably additive) vector (space-valued) measure. For each $x^* \in X^*$, we define the following scalar measure as $\langle m, x^*\rangle(A) := \langle m(A), x^*\rangle$. The space $L^0(m)$ is the space of $\|m\|$ almost everywhere classes of real functions, where $\|m\|$ is the semi-variation of m. It is said that a function $f \in L^0(m)$ is *m-integrable* if it satisfies the following.

(i) For each $x^* \in X^*$, $f \in L^1(|\langle m, x^*\rangle|)$.
(ii) For each $A \in \Sigma$, there is a unique vector denoted by $\int_A f\,dm \in X$ such that $\langle \int_A f\,dm, x^*\rangle = \int_A f\,d\langle m, x^*\rangle$ for all $x^* \in X^*$.

A *Rybakov measure* for m is any control measure of m, that is, a scalar measure $\mu\colon \Sigma \to [0,\infty)$ such that $\|m\| \ll \mu \ll \|m\|$. Such a measure has the form $|\langle m, x^*\rangle|$ for some $x^* \in X^*$ (see [7, Section IX.2] for more details on these measures). The space of (equivalence classes of) m-integrable functions is denoted by $L^1(m)$, which is a BFS over any Rybakov measure with norm $\|f\|_{L^1(m)} := \sup_{x^* \in B_{X^*}} \int_\Omega |f|\,d|\langle m, x^*\rangle|$. Note that $L^p(m) = L^1(m)_{[1/p]}$ and that $L^1(m) = L^p(m)_{[p]}$. The space $L^p(m)$ is p-convex and $L^p(m) \subseteq L^1(m)$. The integration map associated to a vector measure $m\colon \Sigma \to X$ is denoted by $I_m\colon L^1(m) \to X$, and its restriction to the subspaces $L^p(m)$ is denoted by $I_m^{(p)}$. Observe that, for every vector measure, we have that $\|I_m\| = 1$ (see [15, (3.99)]). If $Z(\mu)$ is an OC BFS, X is a Banach space, and $T \in \mathcal{L}(Z(\mu), X)$, then the expression $m_T(A) := T(\chi_A)$, for $A \in \Sigma$, defines a vector measure such that $|\langle m_T, x^*\rangle| \ll m_T \ll \mu$ for all $x^* \in X^*$. Moreover, we always have the factorization $T\colon Z(\mu) \overset{[i]}{\hookrightarrow} L^1(m_T) \xrightarrow{I_{m_T}} X$, where $[i]$ denotes the inclusion/quotient map (see, e.g., [2]) and satisfies that $\|[i]\| = \|T\|$. We can find all these definitions and results in [7] and [15].

The definition of the operator ideal of (p,q)-factorable operators, denoted by $\mathcal{L}_{p,q}$, can be found in [5, Section 17.10]. Here we give a characterization. Let X and Y be Banach spaces, and let $p, q \in [1,\infty)$ be such that $1/p + 1/q \ge 1$, $T \in \mathcal{L}(X, Y)$ is (p,q)-factorable if there exists a factorization $k_Y \circ T\colon X \xrightarrow{R} L^{q'}(\mu) \overset{I}{\hookrightarrow} L^p(\mu) \xrightarrow{S} Y^{**}$, where μ is a finite measure and I is the natural inclusion. The norm is given by $\alpha_{p,q}(T) := \inf \|R\|\|I\|\|S\|$, where the infimum is taken over all such factorizations. In the case where we include the norm, it is denoted by $[\mathcal{L}_{p,q}, \alpha]$ (see [5, Theorem 8.11]). We have that $\mathcal{L}_{r,s} \subseteq \mathcal{L}_{p,q}$ is satisfied when $1 \le r \le p < \infty$ and $1 \le s \le q < \infty$. The ideal of p-integral operators, denoted by $[\mathcal{I}_p, \iota_p]$, is characterized by the $(p,1)$-factorable operators, that is, $\mathcal{I}_p = \mathcal{L}_{p,1}$. Recall that every p-integrable operator is p-summing, a class which is denoted by $[\Pi_p, \pi_p]$. General information regarding the theory of operator ideals can be found in [5], [6], and [16].

3. Factorization through pth power factorable operators

In this section, we define the class of $\mathcal{F}_{p,q}$-factorable operators and obtain the first characterizations and properties of this class. Note that pth power factorization makes sense only when it is defined over an OC BFS of finite measure, but the codomain is open to be any vector space. We have used this fact to define our class, that is, Banach operators that factor through a BFS that satisfies order continuity and Fatou conditions. Despite the fact that such conditions are restrictive, there are several operator ideals characterized by means of factorizations through BFS's with these conditions. For example, p-nuclear operators factor through an ℓ^p space, p-factorable operators factor through L^p-spaces over finite measures, and certain (q, p)-summing operators factor through Lorentz spaces. The class of operators that factor through an OC and Fatou BFS with OC dual space is clearly an operator ideal with the usual norm. We focus our study on a subclass of this operator ideal, defined by means of conditions of pth power factorization for the operators involved. We will see that this class is in fact a Banach operator ideal that has applications to the study of (p, q)-factorable operators.

Definition 3.1. Let $1 \le p, q < \infty$, let X and Y be Banach spaces, and let $T \in \mathcal{L}(X, Y)$. We say that T is $\mathcal{F}_{p,q}$*-factorable* if there exist a finite measure μ, an OC and Fatou BFS $Z(\mu)$ with OC dual space, and two operators $R \in \mathcal{F}_q^{\mathrm{dual}}(X, Z(\mu))$ and $S \in \mathcal{F}_p(Z(\mu), Y^{**})$ such that $k_Y \circ T = S \circ R$.

From now on, unless otherwise specified, X and Y will be Banach spaces. Let $1 \le p, q < \infty$. We denote by $\mathcal{F}_{p,q}(X, Y)$ the class of all $\mathcal{F}_{p,q}$-factorable operators in $\mathcal{L}(X, Y)$, endowed with the norm (justified in the following section) defined as $\varphi_{p,q}(T) := \inf \|S\|\|R\|$, where the infimum is taken over all operators R and S as in the definition above. Throughout this section, we assume that $[\mathcal{F}_{p,q}, \varphi_{p,q}]$ is a normed operator ideal.

Remark 3.2. Suppose that $T \in \mathcal{L}(X, Y)$ factors through a BFS. Then T is $\mathcal{F}_{1,1}$-factorable since every continuous operator (between the suitable spaces) is 1st power factorable. Observe that we do not need an extension to the 1st power space, so we do not need the order continuity or Fatou conditions. For example, the class of the $\mathcal{F}_{1,1}$-factorable operators includes all the p-factorable and p-integrable operators for every $p \in [1, \infty)$. The Fatou condition is required to obtain a commutative diagram as in the following remark. For instance, $\mathcal{F}_{p,1}$-factorization does not require the BFS in the factorization to be Fatou.

Remark 3.3. From the canonical factorizations of pth power factorable operators (see [15, Section 5.2]), and taking into account the fact that $Z(\mu)$ and $Z(\mu)^*$ are OC and $Z(\mu)$ is Fatou, we deduce the following two factorization schemes:

$$\begin{tikzcd}
X \arrow[rr, "T"] \arrow[d, "((R^*)_{[q]})^* \circ k_X"'] \arrow[drr, "R"] & & Y \arrow[rr, hook, "k_Y"] & & Y^{**} \\
((Z(\mu)')_{[q]})' \arrow[r, hook, "j^*_{[q]}"] & Z(\mu)'' \arrow[r, phantom, "="] & Z(\mu) \arrow[urr, "S"] \arrow[rr, hook, "i_{[p]}"] & & Z(\mu)_{[p]} \arrow[u, "S_{[p]}"']
\end{tikzcd} \tag{3.1}$$

where $j_{[q]}$ and $i_{[p]}$ are continuous inclusions. Observe that the order continuity of $Z(\mu)$ and $Z(\mu)^*$ implies that $Z(\mu)^* = Z(\mu)'$ and also $Z(\mu)^{**} = (Z(\mu)^*)^* = (Z(\mu)')^* = Z(\mu)''$. Thus, Fatou and reflexive are the same for $Z(\mu)$. This allows us to identify the operator R with the composition $R^{**} \circ k_X$. In consequence, the diagram makes sense. Moreover, if $Z(\mu)$ is OC, then so is $Z(\mu)_{[q]}$ (see [15, Lemma 2.21(iii)]). Thus $((Z(\mu)^*)_{[q]})^* = ((Z(\mu)')_{[q]})'$.

The factorization scheme above is equivalent to the following one (see [9, Lemma 3.3]):

$$\begin{tikzcd}
& X \arrow[rr, "k_Y \circ T"] \arrow[dl, "I^*_{m_{R^*}} \circ k_X"'] \arrow[dr, "R"] & & Y^{**} & \\
L^1(m_{R^*})^* \arrow[r, hook, "\gamma_q^*"] & L^q(m_{R^*})^* \arrow[r, "{[j]^*}"] & Z(\mu) \arrow[ur, "S"] \arrow[r, "{[i]}"] & L^p(m_S) \arrow[r, hook, "\gamma_p"] & L^1(m_S) \arrow[ul, "I_{m_S}"']
\end{tikzcd} \tag{3.2}$$

where γ_r denotes the canonical inclusion. Recall that every pth power factorable operator is rth power factorable for every $r \in [1, p]$. As a result, T factors through $L^r(m_T)$ for $r \in [1, p]$.

Let us prove that $[\mathcal{F}_{p,q}, \varphi_{p,q}]$ is a Banach operator ideal. The proof is quite standard just following the guidelines of [6, Theorem 5.2].

Theorem 3.4. *Let $1 \le p, q < \infty$, and let X_0, X, Y_0, and Y be Banach spaces. Then we have the following.*

(i) *$\mathcal{F}_{p,q}(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$ containing all the finite-rank operators of $\mathcal{L}(X,Y)$. Moreover, $\varphi_{p,q}$ is a Banach space norm on $\mathcal{F}_{p,q}(X,Y)$, and $\|T\| \le \varphi_{p,q}(T)$ for all $T \in \mathcal{F}_{p,q}(X,Y)$.*

(ii) *The composition of an $\mathcal{F}_{p,q}$-factorable operator with any operator is $\mathcal{F}_{p,q}$-factorable. More formally, if $T \in \mathcal{F}_{p,q}(X,Y)$, $G \in \mathcal{L}(X_0, X)$ and $F \in \mathcal{L}(Y, Y_0)$, then $FTG \in \mathcal{F}_{p,q}(X,Y)$ and $\varphi_{p,q}(FTG) \le \|F\| \varphi_{p,q}(T) \|G\|$.*

Proof. (i) The definition of $\varphi_{p,q}$ ensures that $\|T\| \le \varphi_{p,q}(T)$, for every $T \in \mathcal{F}_{p,q}(X,Y)$. Let $x_0^* \in X^*$ and $y_0 \in Y$. A trivial factorization of the finite-rank operator $F(x) := \langle x, x_0^* \rangle y_0$ ($x \in X$) is given by the operators $S_1 \in \mathcal{L}(\mathbb{R}, Y)$ defined as $S_1(t) := t y_0$ and $R_1 \in \mathcal{L}(X, \mathbb{R})$ defined as $R_1(x) := \langle x, x_0^* \rangle$. It provides the first part of the first statement as follows. Observe that $\mathbb{R}$ can by identified with the BFS $L^s(\nu)$ ($1 < s < \infty$) by just taking Ω as the singleton set with the trivial σ-algebra and having ν be the trivial probability measure. Since $R_1^*(t) = t x_0^*$, we have that such an operator and $k_Y \circ S_1$ (see [15, Lemma 5.4]) are both rth power factorable for every $r \in [1, \infty)$. Clearly $k_Y \circ F = k_Y \circ S_1 \circ R$, and $\varphi_{p,q}(F) \le \|R_1\| \|k_Y \circ S_1\| = \|x_0^*\|_{X^*} \|y_0\|_Y$.

Since axioms of normed space are easy to verify, let us show completeness. It is enough to show that absolutely convergent series converge. Accordingly, let $(T_i)_i \subseteq \mathcal{F}_{p,q}(X,Y)$ be a sequence for which $\sum_i \varphi_{p,q}(T_i) < \infty$. We immediately have $\sum_i \|T_i\| < \infty$, so $\sum_i T_i$ converges, say, to $T \in \mathcal{L}(X,Y)$. We are going to prove that $T \in \mathcal{F}_{p,q}(X,Y)$ and that $\varphi_{p,q}(T) \le \sum_i \varphi_{p,q}(T_i)$.

Let $\varepsilon > 0$. For each $i \ge 1$, we find a finite measure space $(\Omega_i, \Sigma_i, \mu_i)$ such that $\mu_i(\Omega_i) = 1/2^i$, an OC, Fatou BFS $Z_i(\mu_i)$ with OC dual space, so that $\chi_{\Omega_i} \in B_{Z_i(\mu_i)}$, and operators $R_i \colon X \to Z_i(\mu_i)$ and $S_i \colon Z_i(\mu_i) \to Y^{**}$ such that S_i is pth power factorable, R_i^* is qth power factorable, and $k_Y \circ T_i = S_i \circ R_i$. Thus, the operator T_i factors as

$$X \overset{((R_i^*)_{[q]})^*}{\longrightarrow} \big((Z_i(\mu_i)')_{[q]}\big)' \hookrightarrow Z_i(\mu_i)'' = Z_i(\mu_i) \hookrightarrow Z_i(\mu_i)_{[p]} \overset{S_{i[p]}}{\longrightarrow} Y^{**},$$

where R_i and S_i are chosen so that $\|S_{i[p]}\| \le \varphi_{p,q}(T_i) + \varepsilon/2^i$ and $\|R^*_{i\,[q]}\| = 1/2^i$. Hence, $\sum_i \|S_{i[p]}\| \le \sum_i \varphi_{p,q}(T_i) + \varepsilon$ and $\sum_i \|R^*_{i\,[q]}\| = 1$. Observe that by the election of $Z_i(\mu_i)$, we necessarily have $\|S_i\| \le \|S_{i[p]}\|$ and $\|R_i^*\| \le \|R^*_{i\,[q]}\|$ for every $i \ge 1$. Let (Ω, Σ) be the direct sum measurable space of the (Ω_i, Σ_i), assuming that the Ω_i's are pairwise disjoint, and let $\Omega := \bigcup_i \Omega_i$ and $\Sigma := \{A \subset \Omega : A \cap \Omega_i \in \Sigma_i \text{ for all } i\}$. Define a probability measure μ on Σ by specifying that for each j and for each $A_j \in \Sigma_j$, $\mu(A_j) := \mu_j(A_j)$. Then we define the BFS $Z(\mu)$ by means of the norm

$$\|f\|_{Z(\mu)} := \sum_i \frac{\|S_{i[p]}\|^{1/p^2}}{(\sum_j \|S_{j\,[p]}\|)^{1/p^2}} \|f_{|\Omega_i}\|_{Z_i(\mu_i)}, \tag{3.3}$$

where $f \in L^0(\mu)$. To show that $\|\cdot\|_{Z(\mu)}$ is a BFS norm is routine. Let us show that it is an OC and Fatou norm with OC dual norm. First, we claim that $Z(\mu)$ is OC. Let $(f_j)_j$ be a sequence of positive functions in $Z(\mu)$ such that $f_j \downarrow 0$, and define $\alpha_k := \frac{\|S_{k\,[p]}\|^{1/p^2}}{(\sum_j \|S_{j\,[p]}\|)^{1/p^2}}$, for $k \ge 1$. Let $\varepsilon > 0$, and take $j = 1$. Then there exists $i_1 \ge 1$ such that $\sum_{k \ge i_1} \alpha_k \|f_{1|\Omega_k}\|_{Z_k(\mu_k)} < \varepsilon/2$. In addition, by order continuity of $Z_k(\mu_k)$, for each $k \in \{1, \ldots, i_1 - 1\}$ there exists $j_k \ge 1$ such that $\|f_{j|\Omega_k}\|_{Z_k(\mu_k)} < \frac{\varepsilon}{\alpha_k 2^{k+1}}$, for every $j \ge j_k$. Take $j_0 := \max\{j_1, \ldots, j_{i_1-1}\}$; hence $\|f_{j_0|\Omega_k}\|_{Z_k(\mu_k)} < \frac{\varepsilon}{\alpha_k 2^{k+1}}$ for every $k \in \{1, \ldots, i_1 - 1\}$. Moreover, $\|f_{j_0|\Omega_k}\|_{Z_k(\mu_k)} \le \|f_{1|\Omega_k}\|_{Z_k(\mu_k)}$ for every $k \ge 1$. Therefore,

$$\|f_{j_0}\|_{Z(\mu)} = \sum_{1 \le k \le i_1 - 1} \alpha_k \|f_{j_0|\Omega_k}\|_{Z_k(\mu_k)} + \sum_{k \ge i_1} \alpha_k \|f_{j_0|\Omega_k}\|_{Z_k(\mu_k)} < \varepsilon.$$

Thanks to the ideal property of BFS's, $\|f_j\|_{Z(\mu)} \downarrow 0$. Now, we show that $Z(\mu)'$ is OC. Let $0 < g_j \in Z(\mu)'$ be a sequence such that $g_j \downarrow 0$. Since $Z_k(\mu_k)'$ is OC, for each $k \ge 1$ there exists $j_k \ge 1$ such that $\|g_{j|\Omega_k}\|_{Z_k(\mu_k)'} < \varepsilon/2^{k+1}$, for every $j \ge j_k$. Let $j \ge j_k$. Hence, there exists $f_\varepsilon \in B_{Z(\mu)}$ such that

$$\begin{aligned}\|g_j\|_{Z(\mu)'} &= \sup_{f \in B_{Z(\mu)}} \|f g_j\|_{L^1(\mu)} \le \|f_\varepsilon g_j\|_{L^1(\mu)} + \varepsilon/2 \\ &= \sum_k \|f_{\varepsilon|\Omega_k} g_{j|\Omega_k}\|_{L^1(\mu_k)} + \varepsilon/2 \le \sum_k \|f_{\varepsilon|\Omega_k}\|_{Z(\mu_k)} \|g_{j|\Omega_k}\|_{Z_k(\mu_k)'} + \varepsilon/2 < \varepsilon.\end{aligned}$$

Again, by the ideal property of $Z(\mu)'$, we conclude that $\|g_j\|_{Z(\mu)'} \downarrow 0$. Let us prove that $Z(\mu)$ is Fatou. Let $\varepsilon > 0$ and $f \in Z(\mu)$; thus there exists $h_{\varepsilon,k} \in Z(\mu)'$ such that $\|f_{|\Omega_k}\|_{Z_k(\mu_k)''} \le \|h_{\varepsilon,k|\Omega_k} f_{|\Omega_k}\|_{L^1(\mu_k)} + \varepsilon/2^k$. Taking into account that $\alpha_k \le 1$

and that $Z_k(\mu_k)$ is Fatou for every $k \geq 1$, we obtain

$$\begin{aligned}\|f\|_{Z(\mu)} &= \sum_k \alpha_k \|f_{|\Omega_k}\|_{Z_k(\mu_k)} \leq \sum_k \|f_{|\Omega_k}\|_{Z_k(\mu_k)''} \\ &\leq \sum_k \|h_{\varepsilon,k_{|\Omega_k}} f_{|\Omega_k}\|_{L^1(\mu_k)} + \varepsilon/2^k \\ &\leq \sup_{h\in Z(\mu)'} \sum_k \|h_{|\Omega_k} f_{|\Omega_k}\|_{L^1(\mu_k)} + \varepsilon = \|f\|_{Z(\mu)''} + \varepsilon.\end{aligned}$$

It is time to show the factorization scheme. Define $S \in \mathcal{L}(Z(\mu), Y^{**})$ by $S(f) := \sum_i S_i(f_{|\Omega_i})$, and define $R \in \mathcal{L}(X, Z(\mu))$ by $R(x) := \sum_i R_i(x)\chi_{\Omega_i}$. Clearly, S and R are linear and $\|R\| \leq 1$. Let us check that $k_Y \circ T = S \circ R$:

$$\begin{aligned}S \circ R(x) &= \sum_j S_j\Big(\Big(\sum_i R_i(x)\chi_{\Omega_i}\Big)\Big|_{\Omega_j}\Big) \\ &= \sum_j S_j\big(R_j(x)\big) = \sum_j k_Y \circ T_j(x) = k_Y \circ T(x).\end{aligned}$$

Now we claim that S is pth power factorable. Let $f \in Z(\mu)$. From (3.3), we get

$$\|f_{|\Omega_i}\|_{Z(\mu)_{[p]}} = \frac{\|S_{i[p]}\|^{1/p}}{(\sum_j \|S_{j_{[p]}}\|)^{1/p}} \big\||f_{|\Omega_i}|^{1/p}\big\|^p_{Z_i(\mu_i)}. \tag{3.4}$$

Therefore, we have that

$$\begin{aligned}\sum_i \|f_{|\Omega_i}\|_{Z(\mu)_{[p]}} &= \sum_i \frac{\|S_{i[p]}\|^{1/p}}{(\sum_j \|S_{j_{[p]}}\|)^{1/p}} \big\||f_{|\Omega_i}|^{1/p}\big\|^p_{Z_i(\mu_i)} \\ &\leq \Big(\sum_i \frac{\|S_{i[p]}\|^{1/p^2}}{(\sum_j \|S_{j_{[p]}}\|)^{1/p^2}} \big\||f_{|\Omega_i}|^{1/p}\big\|_{Z_i(\mu_i)}\Big)^p \\ &= \Big\|\sum_i |f_{|\Omega_i}|^{1/p}\Big\|^p_{Z(\mu)} = \big\||f|^{1/p}\big\|^p_{Z(\mu)} = \|f\|_{Z(\mu)_{[p]}}.\end{aligned} \tag{3.5}$$

Finally, thanks to (3.4), Hölder's inequality, and (3.5), we obtain

$$\begin{aligned}\|Sf\|_{Y^{**}} &= \Big\|\sum_i S_i(f_{|\Omega_i})\Big\|_{Y^{**}} \leq \sum_i \big\|S_i(f_{|\Omega_i})\big\|_{Y^{**}} \\ &\leq \sum_i \|S_{i[p]}\| \big\||f_{|\Omega_i}|^{1/p}\big\|^p_{Z_i(\mu_i)} \\ &= \Big(\sum_j \|S_{j_{[p]}}\|\Big)^{1/p} \cdot \sum_i \|S_{i[p]}\|^{1/p'} \|f_{|\Omega_i}\|_{Z(\mu)_{[p]}} \\ &\leq \sum_j \|S_{j_{[p]}}\| \cdot \Big(\sum_i \|f_{|\Omega_i}\|^p_{Z(\mu)_{[p]}}\Big)^{1/p} \\ &\leq \sum_j \|S_{j_{[p]}}\| \cdot \sum_i \|f_{|\Omega_i}\|_{Z(\mu)_{[p]}} \leq \sum_j \|S_{j_{[p]}}\| \cdot \|f\|_{Z(\mu)_{[p]}}.\end{aligned}$$

This means that S is pth power factorable and $\|S\| \leq \|S_{[p]}\| \leq \sum_j \|S_{j\,[p]}\|$. Now we prove that R^* is qth power factorable. First note that $R^*(g) = \sum_i R_i^*(g|_{\Omega_i})$. Taking into account Hölder's inequality, we proceed as follows:

$$\begin{aligned}
\big\|R^*(g)\big\|_{X^*} &= \Big\|\sum_i R_i^*(g_{|\Omega_i})\Big\|_{X^*} = \sum_i \big\|R_i^*(g_{|\Omega_i})\big\|_{X^*} \\
&\leq \sum_i \|R^*_{i\,[q]}\| \|g_{|\Omega_i}\|_{(Z_i(\mu_i)')_{[q]}} \\
&\leq \Big(\sum_i \|R^*_{i\,[q]}\|^{q'}\Big)^{1/q'} \Big(\sum_i \|g_{|\Omega_i}\|^q_{(Z_i(\mu_i)')_{[q]}}\Big)^{1/q} \\
&= D\Big(\sum_i \|g_{|\Omega_i}\|^q_{(Z_i(\mu_i)')_{[q]}}\Big)^{1/q} \\
&= D\Big(\sum_i \Big(\sup_{f\in B_{Z_i(\mu_i)}} \big\|f_{|\Omega_i}|g_{|\Omega_i}|^{1/q}\big\|_{L^1(\mu_i)}\Big)^{q^2}\Big)^{1/q} \\
&\leq D\Big(\sup_{f\in B_{Z(\mu)}} \sum_i \big\|f_{|\Omega_i}|g_{|\Omega_i}|^{1/q}\big\|_{L^1(\mu_i)}\Big)^q = D\|g\|_{(Z(\mu)')_{[q]}}.
\end{aligned}$$

Then R^* is qth power factorable and $\|R^*\| \leq \|(R^*)_{[q]}\| \leq D = (2^{q'}-1)^{-1}$. In conclusion, T is $\mathcal{F}_{p,q}$-factorable, with

$$\varphi_{p,q}(T) \leq \|S\|\|R\| = \|S\|\|R^*\| \leq \sum_i \|R^*_{i\,[q]}\| \sum_i \|S_{i[p]}\| \leq \sum_i \varphi_{p,q}(T_i) + \varepsilon.$$

Since ε is arbitrary, we also have that $\varphi_{p,q}(T) \leq \sum_i \varphi_{p,q}(T_i)$.

The ideal property (ii) simply takes into account the fact that pth power factorable operators have the left ideal property (see [15, Lemma 5.4]); that is, the composition on the left-hand side with another operator of Banach spaces is pth power factorable. So, the factorization

$$X_0 \xrightarrow{G} X \xrightarrow{R} Z(\mu) \xrightarrow{S} Y^{**} \xrightarrow{F^{**}} Y_0^{**}$$

produces the corresponding extensions through the qth power space of $Z(\mu)'$: $G^* \circ R^*_{[q]} \circ j_{[q]}$, and through the pth power space of $Z(\mu)$: $F^{**} \circ S_{[p]} \circ i_{[p]}$. Therefore,

$$\varphi_{p,q}(\mathrm{GTF}) \leq \|G^* \circ R^*\| \cdot \|F^{**} \circ S\| \leq \|G\| \cdot \|R\| \cdot \|S\| \cdot \|F\|.$$

The arbitrary factorization $S \circ R$ implies that $\varphi_{p,q}(\mathrm{GTF}) \leq \|G\| \cdot \varphi_{p,q}(T) \cdot \|F\|$. □

The following proposition is an immediate consequence of the definition. It establishes the first inclusion property for $\mathcal{F}_{p,q}$-factorable operators.

Proposition 3.5. *Let $1 \leq r \leq p < \infty$ and $1 \leq s \leq q < \infty$. Let X and Y be Banach spaces. Then $\mathcal{F}_{p,q}(X,Y) \subseteq \mathcal{F}_{r,s}(X,Y)$, and $\varphi_{r,s}(T) \leq \varphi_{p,q}(T)$ for every $T \in \mathcal{F}_{p,q}(X,Y)$.*

Proof. We just prove the norm equality. Let $\varepsilon > 0$ and $T \in \mathcal{F}_{p,q}(X,Y)$. Hence there exist a finite measure μ, a Fatou and OC BFS with OC dual space $Z(\mu)$, and two operators $R \in \mathcal{F}_q^{\mathrm{dual}}(X, Z(\mu))$ and $S \in \mathcal{F}_p(Z(\mu), Y^{**})$ such that $k_Y \circ T = S \circ R$. On the one hand, we know that $S = S_{[r]} \circ i_{[r]} = S_{[p]} \circ i_{[p]}$ and also that

$R^* = R^*_{[s]} \circ j_{[s]} = R^*_{[q]} \circ j_{[q]}$. On the other hand, from the diagram above, we can choose R and S so that

$$\varphi_{p,q}(T) + \varepsilon \geq \|R\|\|S\| = \|R^*_{[q]} \circ j_{[q]}\|\|S_{[p]} \circ i_{[p]}\| = \|R^*_{[s]} \circ j_{[s]}\|\|S_{[r]} \circ i_{[r]}\| \geq \varphi_{r,s}(T),$$

obtaining the inequality from the arbitrary choice of $\varepsilon > 0$. □

As we have seen in the preliminary section, several characterizations of pth power factorable operators exist. We present below some of these characterizations in order to provide some new ones for $\mathcal{F}_{p,q}$-factorable operators.

Proposition 3.6. *Let $1 \leq p, q < \infty$, let X and Y be Banach spaces, and let $T \in \mathcal{L}(X, Y)$. Then the following statements are equivalent.*

(i) $T \in \mathcal{F}_{p,q}(X, Y)$.

(ii) *There exist a finite measure μ, an OC and Fatou BFS $Z(\mu)$, with OC dual space, and two operators $F \in \mathcal{L}(X, ((Z(\mu)')_{[q]})')$ and $G \in \mathcal{L}(Z(\mu)_{[p]}, Y^{**})$ such that $k_Y \circ T = G \circ i_{[p]} \circ (j_{[q]})^* \circ F$, where $i_{[p]}$ and $j_{[q]}$ are the inclusions into the pth and qth power spaces of $Z(\mu)$ and $Z(\mu)'$, respectively.*

(iii) *There exist a finite measure μ, an OC and Fatou BFS $Z(\mu)$, with OC dual space, and two operators $R \in \mathcal{L}(X, Z(\mu))$ and $S \in \mathcal{L}(Z(\mu), Y^{**})$ such that $k_Y \circ T = S \circ R$ and the diagram*

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "F"'] & Y \arrow[r, hook, "k_Y"] & Y^{**} \\ L^q(m_{R^*})^* \arrow[r, "H"] & L^p(m_S) \arrow[ur, "G"'] & \end{tikzcd}$$

commutes, where F and G are bounded operators, $H = [i] \circ [j]^*$, and $[i]$ and $[j]$ denote the inclusion/quotient maps.

Proof. (i) ⇒ (ii): This is clear from diagram (3.1).

(ii) ⇒ (iii): We use the characterization of pth power factorable operator given in [9, Lemma 3.3] to obtain the factorization

$$T\colon X \xrightarrow{I^*_{m_{R^*}} \circ k_X} L^q(m_{R^*})^* \xrightarrow{[j]^*} Z(\mu) \xrightarrow{[i]} L^p(m_S) \xrightarrow{I_{m_S}} Y^{**},$$

where $[j]$ and $[i]$ are not inclusions necessarily.

(iii) ⇒ (i): By hypothesis, it follows that $Z(\mu) \overset{[i]}{\hookrightarrow} L^p(m_S)$ and $Z(\mu)' \overset{[i]}{\hookrightarrow} L^q(m_{R^*})$. Hence, the characterization in [9, Lemma 3.3] again implies (i). □

Note that characterization (ii) of the above proposition coincides with the characterization of (p, q)-factorable operators when we take $Z(\mu) := L^c(\mu)$ for a suitable $c \in [1, \infty)$ (see [5, Theorem 18.11]). We study this fact in the next section.

We now proceed to prove that, with additional conditions, there is an interpretation of the $\mathcal{F}_{p,q}$-factorable operators in terms of interpolation.

Theorem 3.7. *Let $1 \leq p, q < \infty$, let X and Y be Banach spaces such that Y and Y^* are reflexive, and let $T \in \mathcal{L}(X, Y)$. Then the following two statements are equivalent.*

(i) $T \in \mathcal{F}_{p,q}(X, Y)$.

(ii) *There exist a finite measure μ, an OC and Fatou BFS $Z(\mu)$, with OC dual space, a constant $K > 0$, and operators $R \in \mathcal{L}(X, Z(\mu))$ and $S \in \mathcal{L}(Z(\mu), Y^{**})$ such that $K_Y \circ T = S \circ R$ and*

$$\big|\langle Tx, y^*\rangle\big| \le K\big(\|x\|_X \big\|(S^* \circ k_{Y^*})y^*\big\|_{(Z(\mu)')_{[q]}}\big)^\theta \big(\|y^*\|_{Y^*}\|Rx\|_{Z(\mu)_{[p]}}\big)^{1-\theta},$$

for every $x \in X$, $y^* \in Y^*$ and every $\theta \in (0,1)$.

Then (i) $\Rightarrow$ (ii).

If in (ii) we have that $R(X)$ is dense in $Z(\mu)$ and $S^* \circ k_{Y^*}(Y^*)$ is dense in $Z(\mu)'$, then (ii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii): Let $x \in X$ and $y^* \in Y^*$. From the factorization $k_Y \circ T = S \circ R$, the characterization of sth power factorable operators given in Section 2, and taking into account the fact that $y^* = (k_Y^* \circ k_{Y^*})y^*$, by reflexivity of Y and Y^*, we get

$$\begin{aligned}
\big|\langle Tx, y^*\rangle\big| &= \big|\big\langle Tx, (k_Y^* \circ k_{Y^*})y^*\big\rangle\big| = \big|\langle k_Y \circ Tx, k_{Y^*}y^*\rangle\big| \\
&= \big|\big\langle Rx, (S^* \circ k_{Y^*})y^*\big\rangle\big|^\theta \big|\big\langle Rx, S^*(k_{Y^*}y^*)\big\rangle\big|^{1-\theta} \\
&\le K\big(\|x\|_X \big\|(S^* \circ k_{Y^*})y^*\big\|_{(Z(\mu)')_{[q]}}\big)^\theta \big(\|y^*\|_{Y^*}\|Rx\|_{Z(\mu)_{[p]}}\big)^{1-\theta},
\end{aligned}$$

for some $K \ge (\|R^*_{[q]}\|)^\theta (\|S_{[p]}\|\|k_{Y^*}\|)^{1-\theta}$ and for every $\theta \in (0,1)$.

(ii) $\Rightarrow$ (i): This follows from the next property of the exponential map. We claim that $0 \le a \le b^\theta \cdot c^{1-\theta}$ for every $\theta \in (0,1)$ implies that $a \le b$ and $a \le c$. Assume that $b \le c$, thus $a \le c$. Moreover, $b^\theta \cdot c^{1-\theta}$ is a decreasing function of θ. Hence, by continuity of such a function, we have that $a \le \inf\{b^\theta \cdot c^{1-\theta} : \theta \in (0,1)\} = b$. Analogously, we get the same result assuming that $c \le b$. By applying this property, the characterization given in Section 2, and the density hypothesis, we obtain our result. Let $\varepsilon > 0$, and fix $f \in Z(\mu)'$ and $x^* \in X^*$. By density of $(S^* \circ k_{Y^*})(Y^*)$, there exists $y_0^* \in Y^*$ such that $\|f - (S^*(k_{Y^*}))y_0^*\|_{Z(\mu)'} < \varepsilon/2$. Thus, there exists some $K_0 > 0$ such that

$$\begin{aligned}
\big|\langle Rx, f\rangle\big| &\le \big|\big\langle Rx, f - (S^* \circ k_{Y^*})y_0^*\big\rangle\big| + \big|\big\langle Rx, (S^* \circ k_*)y_0^*\big\rangle\big| \\
&\le K_0\|x\|_X\big(\varepsilon/2 + \big\|(S^* \circ k_{Y^*})y_0^*\big\|_{(Z(\mu)')_{[q]}}\big) \\
&\le K_0\|x\|_X\big(\varepsilon/2 + \big\|(S^* \circ k_{Y^*})y_0^* - f\big\|_{(Z(\mu)')_{[q]}} + \|f\|_{(Z(\mu)')_{[q]}}\big) \\
&\le K_0\|x\|_X\big(\varepsilon + \|f\|_{(Z(\mu)')_{[q]}}\big).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary for a fixed $f \in Z(\mu)'$, we have that R^* is qth power factorable. To show that S is pth power factorable, we just take into account the fact that $(k_Y^* \circ k_{Y^*})y^* = y^* \in Y^* = Y^{***}$ and proceed in an analogous way. $\square$

The *dual operator ideal* of an operator ideal $[\mathcal{U}, \omega]$ is defined as follows. Let $T \in \mathcal{L}(X, Y)$. We say that $T \in \mathcal{U}^{\mathrm{dual}}(X, Y)$ if and only if $T^* \in \mathcal{U}(Y^*, X^*)$ and has norm $\omega^{\mathrm{dual}}(T) := \omega(T^*)$. This operator ideal is denoted by $[\mathcal{U}, \omega]^{\mathrm{dual}}$ (see, e.g., [16, Section 4.4]). It is easy to show that $\mathcal{L}_{p,q} = \mathcal{L}_{q,p}^{\mathrm{dual}}$ (see, e.g., [16, Section 19.1.4]). The class of $\mathcal{F}_{p,q}$-factorable operators sometimes satisfies the same property.

Proposition 3.8. *Let $1 \le p, q < \infty$. Let X and Y be Banach spaces such that Y is reflexive. Then $\mathcal{F}_{p,q}(X,Y) = \mathcal{F}^{\mathrm{dual}}_{q,p}(X,Y)$.*

Proof. Let X and Y be Banach spaces, and let $T \in \mathcal{F}_{p,q}(X,Y)$. Then there exist a finite measure μ and an OC and Fatou BFS $Z(\mu)$, with OC dual space, such that $k_{X^*} \circ T^* \colon Y^* = (Y^{**})^* \xrightarrow{S^*} Z(\mu)' \xrightarrow{R^*} X^* \xhookrightarrow{k_{X^*}} (X^*)^{**}$ for some operators R and S, where $k_{X^*} \circ R^*$ is qth power factorable. Now, taking into account the fact that Y is reflexive and $Z(\mu)$ is so (thanks to the order continuity and Fatou properties), we have that $S^{**} = S$, hence it is pth power factorable. This means that $T^* \in \mathcal{F}_{q,p}(Y^*, X^*)$. The conditions on $Z(\mu)'$ also hold, because it is OC and the dual $Z(\mu)'' = Z(\mu)$ is so by hypothesis. Moreover, $Z(\mu)'$ is also Fatou since $(Z(\mu)')'' = (Z(\mu)'')' = Z(\mu)'$, and $Z(\mu)''$ is also OC since it coincides with $Z(\mu)$. With this and by the same process, we obtain the other inclusion. The equality of norms is trivially fulfilled. □

Remark 3.9. As we have seen, there are several situations where we need reflexivity properties of the range or its dual. This is due to the definition of $\mathcal{F}_{p,q}$-factorable operator, in which we have included a factorization through an operator $S \in \mathcal{F}_p(Z(\mu), Y^{**})$. This bidual as a range space has a main role in the following section, in which we relate this class with the class of (p,q)-factorable operators.

To finish the section, we construct an example related to the Hardy operator and based on the example shown in [9, Section 3].

Example 3.10. Let $\Omega := [0,1]$, and let (Ω, Σ, μ) be a measure space so that $dx \ll \mu$. Let $m \colon \Sigma \to E$ be a Banach space-valued measure such that $dx \ll \langle m, x_0^* \rangle \ll dx$, for some $x_0^* \in B_{E^*}$, and let the Radon–Nikodým derivative be $\frac{d\langle m, x_0^*\rangle}{dx} \ge 1$. For example, if $E := L^1[0,1]$ and $m(A) = \chi_A$, we have that $\langle m, \chi_{[0,1]} \rangle = dx$ and so $d\langle m, x_0^* \rangle / dx = 1$, for $x_0^* := \chi_{[0,1]} \in B_{L^\infty[0,1]}$. Let $1 \le u \le r < \infty$, let $X(\mu)$ be a BFS, and let $h \in L^u(\Omega)^{X(\mu)} := \{ f \in L^0(\mu) : f \cdot L^u(\Omega) \subseteq X(\mu)\}$, the space of multiplication operators, which makes sense since $dx \ll \mu$. Also, it is a Banach space endowed with the norm $\|h\|_{L^u(\Omega)^{X(\mu)}} := \sup_{f \in B_{L^u(\Omega)}} \|hf\|_{X(\mu)}$.

We define the operator $G \colon L^r(m) \to X(\mu)$ as

$$Gf(y) := h(y) \int_\Omega f \cdot \chi_{[0,y]} \, dx.$$

Note that the operator G is the Hardy operator when $h(y) = 1/y$ and $X(\mu)$ is chosen suitably. Let $v \in [u, r]$. We claim that G is $\mathcal{F}_{p,q}$-factorable for $p \in [1, v/u]$ and $q \in [1, v']$. We have the factorization

$$G \colon L^r(m) \xrightarrow{V} L^v(\Omega) \xrightarrow{M_h} X(\mu),$$

where $Vf(y) := \int_\Omega f \cdot \chi_{[0,y]} \, dx$ denotes the Volterra operator and M_h is the multiplication operator. The operator M_h is pth power factorable for $p \in [1, v/u]$. Thus, by the definition of h,

$$\|M_h f\|_{X(\mu)} \le \|h\|_{L^u(\Omega)^{X(\mu)}} \|f\|_{L^u(\Omega)} \le K \|f\|_{L^v(\Omega)_{[p]}},$$

for some $K > 0$. Now, let us check that V is continuous. Since $v \leq r$ and $\frac{d\langle m,x_0^*\rangle}{dx} \geq 1$, we have that

$$\|Vf\|_{L^v(\Omega)} \leq \|f\|_{L^v(\Omega)} \leq C\Big(\int_\Omega |f|^r \frac{d\langle m,x_0^*\rangle}{dx}\,dx\Big)^{1/r}$$
$$= C\Big(\int_\Omega |f|^r \, d\langle m,x_0^*\rangle\Big)^{1/r} \leq C\|f\|_{L^r(m)},$$

for some $C > 0$. The adjoint operator of V is defined as $V^*g(y) = \int_\Omega g \cdot \chi_{[y,1]}\,dx$. Let $e_0^* \in B_{E^*}$ such that $\mu_0 := |\langle m, e^*\rangle|$ is a Rybakov measure of m. As $L^r(m)$ is OC, we have that $L^r(m)^* = L^r(m)' = L^r(m)^{L^1(\mu_0)}$. Taking into account the fact that $L^r(m) \subseteq L^1(m)$ and Fubini's theorem, we obtain

$$\begin{aligned}
\|V^*g\|_{L^r(m)^*} &= \sup_{f\in B_{L^r(m)}} \|f\cdot V^*g\|_{L^1(\mu_0)} \leq \sup_{f\in B_{L^r(m)}}\sup_{e^*\in B_{E^*}} \int_\Omega f\cdot V^*g\,d\big|\langle m,e^*\rangle\big| \\
&\leq \sup_{f\in B_{L^r(m)}}\sup_{e^*\in B_{E^*}} \int_\Omega \big|f(y)\big| \int_\Omega \big|g(x)\big|\,dx\,d\big|\langle m,e^*\rangle\big|(y) \\
&= \sup_{f\in B_{L^r(m)}} \int_\Omega \big|g(x)\big| \sup_{e^*\in B_{E^*}} \int_\Omega \big|f(y)\big|\,d\big|\langle m,e^*\rangle\big|(y)\,dx \\
&= \sup_{f\in B_{L^r(m)}} \int_\Omega \big|g(x)\big|\|f\|_{L^1(m)}\,dx \leq \|g\|_{L^1(\Omega)} \leq D\|g\|_{L^{v'}(\Omega)_{[q]}},
\end{aligned}$$

for some $D > 0$ and any $q \in [1, v']$. This means that V^* is qth power factorable.

4. Extrapolation theorems

In the previous section, we presented the Banach operator ideal of $\mathcal{F}_{p,q}$-factorable operators. We now show that every $\mathcal{F}_{p,q}$-factorable operator between finite-dimensional Banach spaces is 1-summing. Moreover, with a condition on the indexes, it is not hard to show that there is continuous inclusion from the operator ideal of (p,q)-factorable operators into the operator ideal of $\mathcal{F}_{p,q}$-factorable operators. Thanks to this fact, we show a number of extrapolation results and also characterize (p,q)-factorable operators.

From the characterization of (p,q)-factorable operators by means of the embedding $L^{q'}(\mu) \hookrightarrow L^p(\mu)$, it is not hard to show the first inclusion relation between such operators and the $\mathcal{F}_{p,q}$-factorable operators.

Proposition 4.1. *Let $1 \leq p,q,r,s < \infty$ such that $1/(pr)+1/(qs) \geq 1$. Let X and Y be Banach spaces. Then $\mathcal{L}_{p,q}(X,Y) \subseteq \mathcal{F}_{r,s}(X,Y)$ and $\varphi_{r,s} \leq \alpha_{p,q}$. In particular, if $1/p^2 + 1/q^2 \geq 1$, then $\mathcal{L}_{p,q}(X,Y) \subseteq \mathcal{F}_{p,q}(X,Y)$.*

Proof. Let $T \in \mathcal{L}_{p,q}(X,Y)$. Then there exists a probability measure μ such that

$$T\colon X \xrightarrow{R_0} L^{q'}(\mu) \overset{I}{\hookrightarrow} L^p(\mu) \xrightarrow{S_0} Y^{**},$$

where I is the canonical inclusion. Choosing any $c \in [pr, (qs)']$, we have the factorization for such an inclusion map

$$I\colon L^{q'}(\mu) \overset{I_1}{\hookrightarrow} L^{(c'/s)'}(\mu) \overset{I_2}{\hookrightarrow} L^c(\mu) \overset{I_3}{\hookrightarrow} L^{c/r}(\mu) \overset{I_4}{\hookrightarrow} L^p(\mu).$$

Since $L^{u/v}(\mu) = L^u(\mu)_{[v]}$, we have that $R := I_2 \circ I_1 \circ R_0 \in \mathcal{F}_s^{\mathrm{dual}}(X, L^c(\mu))$ and $S := S_0 \circ I_4 \circ I_3 \in \mathcal{F}_r(L^c(\mu), Y^{**})$ (see [15, Lemma 5.4]). Let $\varepsilon > 0$, and choose R_0 and S_0 so that

$$\alpha_{p,q}(T) + \varepsilon \geq \|S_0\|\|I\|\|R_0\| = \|S_0\|\|I_4 \circ I_3\|\|I_2 \circ I_1\|\|R_0\| \geq \varphi_{r,s}(T).$$

The equality of the norms of the inclusion maps is given by the choice of μ as a probability measure. □

With this inclusion property and conditions for compactness of the operators involved, we can obtain some extrapolation results. For example, when the operator is over finite-dimensional Banach spaces, such operators are actually absolutely summing.

Proposition 4.2. *Let $1 \leq p, q < \infty$ be such that $1/p + 1/q \geq 1$, and let E and F be finite-dimensional spaces. Then $\mathcal{F}_{p,q}(E, F) \subseteq \Pi_1(E, F)$ and $\pi_1 \leq \varphi_{p,q}$.*

Proof. Let $T \in \mathcal{F}_{p,q}(E, F)$. Hence there exist a BFS $Z(\mu)$ and two operators $R \in \mathcal{F}_q^{\mathrm{dual}}(E, Z(\mu))$ and $S \in \mathcal{F}_p(Z(\mu), F)$ such that $T = S \circ R$. Observe that the vector measures m_{R^*} and m_S take values in the finite-dimensional spaces E^* and F, respectively. This implies that $L^q(m_{R^*})^* = L^{q'}(|m_{R^*}|)$ and $L^p(m_S) = L^p(|m_S|)$ order isomorphically (see, e.g., [15, Remark 3.17]). In consequence, taking into account diagram (3.2) and Maurey's factorization theorem (see, e.g., [5, Theorem 18.9]), there exists a probability measure μ_0 such that T factors through the embedding $L^\infty(\mu_0) \hookrightarrow L^1(\mu_0)$; that is, $T \in \mathcal{L}_{1,1}(E, F) = \mathcal{I}_1(E, F) \subseteq \Pi_1(E, F)$. Let $\varepsilon > 0$, and take R and S as above, that is, that satisfies (3.2) and also

$$\begin{aligned} \varphi_{p,q}(T) + \varepsilon \geq \varphi_{1,1}(T) + \varepsilon &\geq \|S\|\|R\| = \big\|[i]\big\|\big\|[j]^*\big\| \\ &= \|I_{m_S}\|\big\|[i]\big\|\big\|[j]^*\big\|\|I^*_{m_{R^*}}\| = \|I_{m_S}\|\|F\|\|I\|\|G\|\|I^*_{m_{R^*}}\| \\ &\geq \|I_{m_S} \circ F\|\|I\|\|G \circ I^*_{m_{R^*}}\| \geq \alpha_{1,1}(T) = \iota_1(T) \geq \pi_1(T), \end{aligned} \tag{4.1}$$

where $L^\infty(|m_{R^*}|) \xrightarrow{G} L^\infty(\mu_0) \overset{I}{\hookrightarrow} L^1(\mu_0) \xrightarrow{F} L^1(|m_S|)$. Recall (see Proposition 3.5, [5, Theorem 18.9]) that $\|[i]\| = \|S\|$, $\|[j]\| = \|R\|$, and $\|I_{m_S}\| = \|I_{m_{R^*}}\| = 1$. □

There are some other conditions that imply compactness of the operators involved.

Theorem 4.3. *Let $1 \leq p, q < \infty$ and $1 < r, s < \infty$ be such that $1/(pr) + 1/(qs) \geq 1$. Then $\mathcal{L}_{p,q}(\ell^t, \ell^w) \subseteq \mathcal{L}_{u,v}(\ell^t, \ell^w)$ for every $w \in [1, 2)$, $t \in (2, \infty]$, $u \in (1, r]$, and $v \in (1, s]$.*

Proof. Let $T \in \mathcal{L}_{p,q}(\ell^t, \ell^w)$. By virtue of Proposition 4.1, T is $\mathcal{F}_{r,s}$-factorable. Keep in mind (3.2), which establishes the factorization of $k_Y \circ T$ through a positive operator $L^v(m_R)^* \to L^u(m_S)$, for $u \in (1, r]$ and $v \in (1, s]$. Thanks to [15, Lemma 3.53(v)], we can ensure that the ranges of the vector measures m_S and m_{R^*} are relatively compact. Now, [15, Proposition 3.56] says that this is equivalent to the fact that the restrictions of the integration maps $I^{(v)}_{m_{R^*}}$ and $I^{(u)}_{m_S}$ are compact. Finally, the proof of [13, Theorem 1] gives the isomorphic identities $L^u(m_S) = L^u(|m_S|)$ and $L^v(m_{R^*}) = L^v(|m_{R^*}|)$. That is, R factors through $L^v(m_{R^*})^* = L^{v'}(|m_{R^*}|)$ for every $v \in (1, s]$, and S factors through

$L^u(m_S) = L^u(|m_S|)$ for every $u \in (1,r]$. In conclusion, $k_Y \circ T = S \circ R$ factors through a positive map $[i] \circ [j]^* \colon L^{v'}(|m_{R^*}|) \to L^u(|m_S|)$ (see Proposition 3.6(iii)); that is, $T \in \mathcal{L}_{u,v}(\ell^t, \ell^w)$ (see [5, Theorem 18.11]). The proof of the norm inequality is as in (4.1), taking into account Propositions 3.5 and 4.1. □

Remark 4.4. Once we have diagram (3.2), there are several possible ways that lead us to an $(r,1)$-factorable, and thus r-summing, operator. One direction we can follow, studied above, consists of finding conditions that ensure that $L^1(m_{R^*})$ is an abstract Lebesgue (AL) space (see, e.g., [3], [13]–[15]). Another direction we can take is oriented to finding factorizations through L^p-spaces of a finite scalar measure. For example, we can study if the vector measures involved, m_S and m_{R^*}, have σ-finite variation (see, e.g., [1]). Let $T \in \mathcal{L}_{p,q}(X,Y)$. Hence there is a probability measure μ such that $k_Y \circ T \colon X \to Y^{**}$ factors through the natural embedding $I \colon L^{q'}(\mu) \hookrightarrow L^p(\mu)$. From the proof of Proposition 4.1, we have that such embedding is $\mathcal{F}_{r,s}$-factorable, provided that $1/(pr) + 1/(qs) \geq 1$. Moreover, we know that the canonical embeddings $F := (I_2 \circ I_1)^*$ and $G := I_4 \circ I_3$ have finite variation (where the I_j's are defined in the proof of this proposition). Nevertheless, despite the fact that we cannot ensure that such embeddings are compact, in the case where the Radon–Nikodým derivatives $d|m_F|/d\mu \in L^{(c'/r)'}(\mu)$ and $d|m_G|/d\mu \in L^{(c/s)'}(\mu)$, by virtue of [15, Proposition 5.13], we can factor this embedding as

$$I \colon L^{q'}(\mu) \to L^s(m_F)^* \subseteq L^{s'}\big(|m_F|\big) \to L^c(\mu) \to L^r\big(|m_G|\big) \subseteq L^r(m_G) \to L^p(\mu),$$

and so we obtain an (r,s)-factorable operator.

We conclude the article with a characterization of the $\mathcal{L}_{p,q}$ operator ideal. Let $1 \leq p,q,r,s < \infty$ be such that $1/pr + 1/qs \geq 1$ and $T \in \mathcal{L}(X,Y)$. Observe that $1/u^2 + 1 > 1$ for every $1 \leq u \leq r$. From Propositions 3.5 and 4.1, we have that

$$\pi_u(T) \leq \iota_u(T) = \alpha_{u,1}(T) \overset{(\bullet)}{\geq} \varphi_{u,1}(T) \leq \varphi_{r,s}(T) \leq \alpha_{p,q}(T).$$

For this reason, we are interested in finding conditions that allow us to reverse the inequality $(\bullet)$. As we noted in the preliminary section, despite $Z(\mu)$ being a BFS, its pth power space $Z(\mu)_{[p]}$ may be a quasi-BFS; hence the following definition is a *seminorm*, instead of a norm: $\|f\|_{b,Z(\mu)_{[p]}} := \sup\{|\langle f, \xi\rangle| : \xi \in B_{(Z(\mu)_{[p]})^*}\}$, where $f \in L^0(\mu)$. Let $1 \leq p < \infty$ and $0 < q < \infty$, let $Z(\mu)$ be a BFS, and let X be a Banach space. An operator $T \in \mathcal{L}(Z(\mu), X)$ is *bidual (p,q)-power-concave* if there is a constant $K > 0$ such that

$$\sum_{j=1}^n \|Tf_j\|_X^{q/p} \leq K \Big\| \sum_{j=1}^n |f_j|^{q/p} \Big\|_{b,Z(\mu)_{[q]}},$$

for every choice of $f_1, \ldots, f_n \in Z(\mu)$ and $n \in \mathbb{N}$ (see [15, Definition 6.1]). Let $1 \leq p,q < \infty$ and $1 \leq r,s \leq \infty$, and let X and Y be Banach spaces. Let us denote by $\mathcal{F}^{r,s}_{p,q}(X,Y)$ the set of all operators $T \in \mathcal{L}(X,Y)$ such that there exists an OC and Fatou BFS $Z(\mu)$, with OC dual space which is r-convex and s-concave, so that $k_Y \circ T = S \circ R$ for some operators $R \in \mathcal{F}^{\mathrm{dual}}_q(X, Z(\mu))$ and $S \in \mathcal{F}_p(Z(\mu), Y^{**})$.

Obviously, $\mathcal{F}_{p,q}(X,Y)=\mathcal{F}_{p,q}^{1,\infty}(X,Y)$. Let us denote $\mathcal{F}_{p,q}^{r}(X,Y):=\mathcal{F}_{p,q}^{r,r}(X,Y)$ and consider this subclass with the norm $\varphi_{p,q}$.

Theorem 4.5. *Let $1\leq p,q,r,s<\infty$ be such that $1/(pr)+1/(qs)=1$. Let X and Y be Banach spaces, and let $T\in\mathcal{L}(X,Y)$. Then the following statements are equivalent:*

(i) $T\in\mathcal{L}_{r,s}(X,Y)$,
(ii) $T\in\mathcal{F}_{p,q}^{pr}(X,Y)$.

In this case, $\alpha_{r,s}(T)=\varphi_{p,q}(T)$.

Proof. (i) $\Rightarrow$ (ii): This follows from the proof of Proposition 4.1, by just taking $c=pr$ and $Z(\mu)=L^c(\mu)$, which is c-concave and c-convex. Moreover, $\varphi_{p,q}(T)\leq\alpha_{r,s}(T)$.

(ii) $\Rightarrow$ (i): Define $c:=pr$. Hence $c'=qs$, thus $r=c/p$ and $s=c'/q$. Keep in mind that $k_Y\circ T\colon X\xrightarrow{R}Z(\mu)\xrightarrow{S}Y^{**}$, where $Z(\mu)$ is c-concave and c-convex, and S is pth power factorable. Thanks to [15, Proposition 6.2(iii)], S is bidual (p,c)-power-concave, which implies, by [15, Theorem 6.27(iii)], that $S\colon Z(\mu)\overset{M_{\chi\Omega}}{\hookrightarrow}L^c(h\,d\mu)\overset{\gamma_p}{\hookrightarrow}L^{c/p}(h\,d\mu)\xrightarrow{(S_0)_{[p]}}Y^{**}$, for some $0<h\in L^1(\mu)$, where $M_{\chi\Omega}$ denotes the multiplication operator and γ_p denotes the canonical inclusion map. Analogously, $R^*\colon Z(\mu)'\overset{M_{\chi\Omega}}{\hookrightarrow}L^{c'}(g\,d\mu)\overset{\gamma_q}{\hookrightarrow}L^{c'/q}(g\,d\mu)\xrightarrow{(R_0)_{[q]}}X^*$, where $0<g\in L^1(\mu)$. Let us choose μ so that $\|\gamma_p\|\leq1$ and $\|\gamma_q\|\leq1$. Taking duals in the second factorization and combining this with the first one, we obtain that $k_Y\circ T$ factors through the positive map $U\colon L^{s'}(g\,d\mu)\to L^r(h\,d\mu)$ defined by $U:=\gamma_p\circ M_{\chi\Omega}\circ M_{\chi\Omega}^*\circ\gamma_q^*$. By virtue of [5, Corollary 18.10], U factors as $U\colon L^{s'}(g\,d\mu)\xrightarrow{U_1}L^{s'}(\mu_0)\overset{I}{\hookrightarrow}L^r(\mu_0)\xrightarrow{U_2}L^r(h\,d\mu)$, so that $\alpha_{r,s}(U)=\|U\|=\|U_2\|\|I\|\|U_1\|$. Thus, we conclude that $T\in\mathcal{L}_{r,s}(X,Y)$.

In order to show the norm inequality, we describe our arguments for the pth power operator S; the same will be valid for R^*. From [15, Theorem 6.27(iii)], one can easily check that $\|S\|=\|(S_0)_{[p]}\|$, just by testing with simple functions and taking into account that $\|\gamma_p\circ M_{\chi\Omega}\|\leq1$, by the election of μ. That is, given $\varepsilon>0$, there exist $f\in B_{L^{c/p}(hd\mu)}$ and a suitable simple function $s_f\in B_{Z(\mu)}$ such that

$$\begin{aligned}\big\|(S_0)_{[p]}\big\|-\varepsilon/2&\leq\big\|(S_0)_{[p]}(f)\big\|_{Y^{**}}\\&\leq\big\|(S_0)_{[p]}\big(\gamma_p\circ M_{\chi\Omega}(s_f)\big)\big\|_{Y^{**}}+\big\|(S_0)_{[p]}\big(f-\gamma_p\circ M_{\chi\Omega}(s_f)\big)\big\|_{Y^{**}}\\&\leq\big\|S(s_f)\big\|_{Y^{**}}+\varepsilon/2\leq\|S\|+\varepsilon/2.\end{aligned}$$

Thus, $\|S\|\geq\|(S_0)_{[p]}\|\|\gamma_p\circ M_{\chi\Omega}\|$. An analogous result is obtained for R^*; that is, $\|R\|=\|R^*\|\geq\|M_{\chi\Omega}^*\circ\gamma_q^*\|\|(R_0)_{[q]}^*\circ k_X\|$. Let $\varepsilon>0$. Then choose $Z(\mu)$, R, and S such that

$$\begin{aligned}\varphi_{p,q}(T)+\varepsilon&\geq\|S\|\|R\|\geq\big\|(S_0)_{[p]}\big\|\|\gamma_p\circ M_{\chi\Omega}\|\|M_{\chi\Omega}^*\circ\gamma_q^*\|\big\|(R_0)_{[q]}^*\circ k_X\big\|\\&\geq\big\|(S_0)_{[p]}\big\|\alpha_{r,s}(\gamma_p\circ M_{\chi\Omega}\circ M_{\chi\Omega}^*\circ\gamma_q^*)\big\|(R_0)_{[q]}^*\circ k_X\big\|\geq\alpha_{r,s}(T).\end{aligned}$$

The arbitrariness of ε brings us to the result. $\square$

Corollary 4.6. *Let $1 \leq u \leq p < \infty$ and $1 \leq v \leq q < \infty$ be such that $1/pu + 1/qv \geq 1$. Let $r \in [u,p]$ and $s \in [v,q]$ be such that $1/(rt) + 1/(sw) = 1$, and let X and Y be Banach spaces such that $\mathcal{F}^c_{u,v}(X,Y) \subseteq \mathcal{F}^{rt}_{t,w}(X,Y)$ for some $c \in [pu,(qv)']$. Then $\mathcal{L}_{p,q}(X,Y) \subseteq \mathcal{L}_{r,s}(X,Y)$.*

The proof is a simple application of Proposition 4.1 and Theorem 4.5. After some computations, it is possible to obtain indexes that satisfy the conditions of this corollary.

Example 4.7. Let us take the following values in the corollary above: $p = 12; u = 4; q = s = \frac{49}{48}; v = w = \frac{2500}{2499}; r = 7; t = \frac{625}{91} \simeq 6.7$. Therefore, we have: $pu = 48; qv = sw = \frac{625}{612}; rt = (sw)' = (qv)' = \frac{625}{13}$. Thus, $1/(pu) + 1/(qv) \simeq 1.000033 \geq 1$ and $1/(rt) + 1/(sw) = 1$. Let X and Y be Banach spaces, and let Y_0 a finite-dimensional subspace of Y. From the proof of Theorem 3.4(i), every finite-rank operator is $\mathcal{F}_{t,w}$-factorable for every $t, w \in [1,\infty)$; hence it is clear that $\mathcal{F}^c_{u,v}(X,Y_0) \subseteq \mathcal{F}^{rt}_{t,w}(X,Y_0)$. The corollary above implies that, for every finite-dimensional subspace $Y_0 \subseteq Y$, $\mathcal{L}_{12,\frac{49}{48}}(X,Y_0) \subseteq \mathcal{L}_{7,\frac{49}{48}}(X,Y_0)$.

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