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TOEPLITZ OPERATORS ON WEIGHTED HARMONIC BERGMAN SPACES

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ABSTRACT. In this article, we study Toeplitz operators with nonnegative symbols on the $\mathcal{A}_2$-weighted harmonic Bergman space. We characterize the boundedness, compactness, and invertibility of Toeplitz operators with nonnegative symbols on this space.

1. Introduction and preliminaries

For $1 \leq p < \infty$, a nonnegative integrable function ω on the unit disk $\mathbb{D}$, let $L^p(\omega)$ denote the Banach space with norm

$$\|f\|_{L^p(\omega)} := \Big(\int_{\mathbb{D}} \big|f(z)\big|^p \omega(z)\, dA(z)\Big)^{\frac{1}{p}}.$$

The weighted harmonic (resp., analytic) Bergman space $L^p_h(\omega)$ (resp., $L^p_a(\omega)$) is the subspace of $L^p(\omega)$ which consists of harmonic (resp., analytic) functions on $\mathbb{D}$. The goal of this article is to provide a framework to study operator properties (boundedness, compactness, Schatten classes, and invertibility) of Toeplitz operators with nonnegative symbols on $L^2_h(\omega)$.

Weighted analytic function spaces and their Toeplitz operators have captured people's attention for a long time. It is now well known (see [25]) that several results on unweighted Bergman spaces can be extended to the standard weighted

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Bergman space $L_a^2(\omega_\alpha)$, where $\omega_\alpha(z) = (1+\alpha)(1-|z|^2)^\alpha$ and $-1 < \alpha < \infty$. Peláez and Rättyä [20], [21] recently characterized the bounded and Schatten-class Toeplitz operators (induced by a positive Borel measure) on a weighted Bergman space, where the weight is a radial function satisfying the doubling property $\int_r^1 \omega(s)\,ds \leq C\int_{\frac{1+r}{2}}^1 \omega(s)\,ds$. The first results of nonradial weighted Bergman spaces are due to Luecking [15], who investigated the structure of weighted Bergman spaces with Békollé–Bonami weights. Based on Luecking's representation and duality theorems in [15], Chacón [2] and Constantin [5], [6] studied the boundedness and compactness of Toeplitz operators on certain weighted Bergman spaces. In [18], Mitkovski and Wick established a reproducing kernel thesis for operators on Bergman-type spaces, and their definitions include weighted versions of Bergman spaces on more complicated domains.

We will be primarily interested in the weighted harmonic Bergman space $L_h^2(\omega)$. Our choice of the weight ω is motivated by the characterization of the boundedness of P_h acting on $L^2(\omega)$, where P_h is the unweighted harmonic Bergman (orthogonal) projection from $L^2(dA)$ to $L_h^2(dA)$. It is well known that $L_h^2(dA)$ is a reproducing kernel Hilbert space and that

$$P_h f(z) = \int_{\mathbb{D}} f(\lambda)\Big[\frac{1}{(1-\overline{\lambda}z)^2} + \frac{1}{(1-\overline{z}\lambda)^2} - 1\Big]\,dA(\lambda).$$

The harmonic Bergman projection P_h is a Calderón–Zygmund operator on the homogeneous space $(\mathbb{D}, d, dA)$, where d is the Euclidean distance and dA is the Lebesgue measure on $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ is 1. (For the definitions of Calderón–Zygmund operators and homogeneous spaces, we refer the reader to [1].)

The most successful understanding of the (one) weight theory for Calderón–Zygmund operators was spurred by Muckenhoupt's work in the 1970s (see [19]), which led to the $\mathcal{A}_p$ weight and developments of the weighted inequality. We will restrict our attention to the $\mathcal{A}_2$ weight on $(\mathbb{D}, d, dA)$.

Let $0 < \omega \in L^1(\mathbb{D}, dA)$. It is called a *Muckenhoupt* $\mathcal{A}_2$ *weight* if

$$[\omega]_{\mathcal{A}_2} := \sup_{a\in\mathbb{D},0<r<1} \frac{|B(a,r)|_\omega |B(a,r)|_{\omega^{-1}}}{|B(a,r)|^2} < +\infty,$$

where

$$B(a,r) = \big\{z \in \mathbb{D} : d(a,z) = |z-a| < r\big\},$$
$$\big|B(a,r)\big|_\omega = \int_{B(a,r)} \omega(z)\,dA(z),$$

and $|\cdot|$ is the normalized Lebesgue measure on $\mathbb{D}$.

It follows from the remarkable $\mathcal{A}_2$ theorem (see [1], [10]) that P_h *is bounded from* $L^2(\omega)$ *to* $L_h^2(\omega)$ *provided* ω *is a Muckenhoupt* $\mathcal{A}_2$ *weight.* As mentioned above, we will focus on the weighted harmonic Bergman space $L_h^2(\omega)$ with $\omega \in \mathcal{A}_2$. Little is known about this natural function space. However, we will show in Section 2 that $L_h^2(\omega)$ is a reproducing kernel Hilbert space with the reproducing kernel $K_z^\omega(\lambda)$; that is, $f(z) = \langle f, K_z^\omega\rangle_{L^2(\omega)}$ for all f in $L_h^2(\omega)$.

For a positive finite Borel measure ν on $\mathbb{D}$, we densely define the Toeplitz operator T_ν on $L_h^2(\omega)$ by

$$T_\nu f(z) = \langle T_\nu f, K_z^\omega \rangle_{L^2(\omega)} = \int_{\mathbb{D}} f(\lambda)\overline{K_z^\omega(\lambda)}\, d\nu(\lambda) \quad (z \in \mathbb{D}).$$

For a bounded function φ, using the integral representation for the projection operator (from $L^2(\omega)$ to $L_h^2(\omega)$), we can express the Toeplitz operator T_φ (on $L_h^2(\omega)$) as follows:

$$T_\varphi f(z) = \int_{\mathbb{D}} f(\lambda)\overline{K_z^\omega(\lambda)}\varphi(\lambda)\omega(\lambda)\, dA(\lambda) \quad (z \in \mathbb{D}).$$

Although we follow Luecking's methods in [15] and [14] for weighted Bergman spaces, some new difficulties have arisen in the study of the space $L_h^2(\omega)$ and its operators. For instance, harmonic functions do not share many powerful properties with analytic functions. One can easily use the Cauchy formula to estimate the local values of analytic functions. However, because of the tedious remainder, the harmonic version of Cauchy's formula (known as the *Cauchy–Pompeiu formula*) is not valid now. We instead must rely on some known estimates of harmonic functions. In addition, just as with weighted Bergman spaces, one cannot write down an explicit formula for the reproducing kernel of $L_h^2(\omega)$. To overcome this obstacle, we will use the reproducing kernel for the unweighted space L_h^2 to help us study the representation theory of $L_h^2(\omega)$. However, the properties of the reproducing kernel for L_h^2 are much more complicated than those of the Bergman space L_a^2.

Using some properties of harmonic functions and $\mathcal{A}_2$ weights, we establish two different atomic decompositions for functions in $L_h^2(\omega)$ (Theorems 2.6 and 2.13), which extend the representation theorems in [15] to the harmonic case.

In Section 3, we characterize the boundedness, compactness, and Schatten p class of Toeplitz operators T_ν on $L_h^2(\omega)$ by means of the Berezin transform and Carleson measures. We are pleased to mention here that Miao [16] has obtained characterizations for Toeplitz operators with nonnegative symbols to be bounded, compact, and in Schatten classes on the unweighted harmonic Bergman space L_h^2.

Section 4 of this article is devoted to studying the invertibility of Toeplitz operators on the standard weighted harmonic Bergman space $L_h^2(\omega_\alpha)$. Somewhat surprising to us were the results, which illustrate that the invertibility of Toeplitz operators on L_a^2 can imply a reverse Carleson inequality for $L_h^2(\omega_\alpha)$ (see Theorems 4.2 and 4.4). Based on this inequality, we generalize the result on the invertibility of Bergman–Toeplitz operators with nonnegative symbols (see [24]) to the case of $L_h^2(\omega_\alpha)$. As a consequence, we obtain a relationship of the invertibility between Toeplitz operators on $L_a^2(\omega_\alpha)$ and $L_h^2(\omega_\alpha)$ (see Corollary 4.7).

Finally, in Section 5 we establish a reverse Carleson inequality for $L_h^2(\omega)$ with $\omega \in \mathcal{A}_2$. Indeed, we obtain a sufficient condition for $\chi_G\, dA$ to be a reverse Carleson measure for $L_h^2(\omega)$, where G is a measurable set in $\mathbb{D}$ (see Theorem 5.1), which extends Theorem 3.9 for the weighted (analytic) Bergman space in [15] to the harmonic setting.

Throughout the article, positive constants will be denoted by $C, C_0, C_1, \ldots$, which may depend on some fixed numbers and change at each occurrence.

2. The space $L_h^2(\omega)$ and its representation

In this section, we present some elementary structures of $L_h^2(\omega)$ with $\omega \in \mathcal{A}_2$. To study the harmonic Bergman spaces, we need the following important properties of harmonic functions.

Lemma 2.1 ([11, Theorem 1]). *Suppose that f is a harmonic function on the disk $\mathbb{D}$ and $0 < p < \infty$. There exists a positive constant $C = C(p)$ such that for every ball $B(a, r) = \{z \in \mathbb{D} : |z - a| < r\}$ in $\mathbb{D}$,*

$$\left|f(a)\right|^p \leq \frac{C}{|B(a,r)|} \int_{B(a,r)} \left|f(z)\right|^p dA(z).$$

In particular, if $p \geq 1$, then the constant $C \equiv 1$. Using this result one can get the following useful inequalities easily. Given $0 < p < \infty$ and $0 < r < 1$, there exist positive constants $C_1 = C_1(p)$ and $C_2 = C_2(p)$ such that

$$\left|f(a)\right|^p \leq \frac{C_1}{(1-r)^2} \cdot \frac{1}{|D(a,r)|} \int_{D(a,r)} \left|f(z)\right|^p dA(z) \quad (a \in \mathbb{D})$$

and

$$\left|\partial f(a)\right|^p \leq \frac{C_2}{(1-r)^{2+p}} \cdot \frac{1}{|D(a,r)|^{\frac{p+2}{2}}} \int_{D(a,r)} \left|f(z)\right|^p dA(z) \quad (a \in \mathbb{D})$$

for all f harmonic on $\mathbb{D}$, where $\partial f = \frac{\partial f}{\partial z}$.

Remark 2.2. From the above inequalities, it is easy to show that point evaluations are bounded linear functionals on $L_h^2(\omega)$ with $\omega \in \mathcal{A}_2$. As a consequence, $L_h^2(\omega)$ is a reproducing kernel Hilbert space.

Remark 2.3. It is not clear whether $L_h^2(\omega)$ is complete for a general weight. However, if p is an analytic polynomial on $\mathbb{D}$ and $\omega(z) = |p(z)|^2$, Douglas and Wang [8] showed that $L_a^2(\omega)$ is complete, and their proof heavily depends on some particular properties of polynomials.

It is clear that $L_h^2(\omega)$ coincides with its dual space with respect to the $L^2(\omega)$ inner product. The next result illustrates that the dual space of $L_h^2(\omega)$ can be identified with $L_h^2(\omega^{-1})$ via the unweighted inner product, which generalizes Luecking's result for $L_a^2(\omega)$ (see [15, Theorem 2.1]) to $L_h^2(\omega)$.

Lemma 2.4. *Suppose that ω is an $\mathcal{A}_2$ weight. Then the dual space of $L_h^2(\omega)$ can be identified with $L_h^2(\omega^{-1})$. The pairing is given by*

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA(z).$$

Consequently, there exists a bounded, bijective, and linear operator $\mathscr{F}$: $L_h^2(\omega^{-1}) \to (L_h^2(\omega))^*$ such that

$$\mathscr{F}(f)(g) = \langle g, \overline{f} \rangle_{L^2(dA)}$$

for $f \in L_h^2(\omega^{-1})$ and $g \in L_h^2(\omega)$.

Proof. Let ω be an $\mathcal{A}_2$ weight. Recall that the orthogonal projection $P_h : L^2(dA) \to L^2_h(dA)$ is a Calderón–Zygmund operator on $(\mathbb{D}, d, dA)$. Then P_h is bounded on both $L^2(\omega)$ and $L^2(\omega^{-1})$. Thus for each $f \in L^2(\omega)$ and $g \in L^2(\omega^{-1})$, we have

$$\langle P_h f, g\rangle = \langle f, P_h g\rangle.$$

Noting that each $f \in L^2_h(\omega)$ (or $f \in L^2_h(\omega^{-1})$) has the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z^n},$$

we obtain

$$\begin{aligned} P_h f(z) &= \int_{\mathbb{D}} f(\lambda) R_z(\lambda)\, dA(\lambda) \\ &= \lim_{s\to 1^-} \int_{s\mathbb{D}} f(\lambda)\Big[\frac{1}{(1-\overline{z}\lambda)^2} + \frac{1}{(1-\overline{\lambda}z)^2} - 1\Big]\, dA(\lambda) \\ &= f(z) \end{aligned}$$

for $z \in \mathbb{D}$. Then the rest of this proof is exactly the same as the proof of Theorem 2.1 in [15]. We omit the details. □

Let $a \in \mathbb{D}$ and $0 < r < 1$. A pseudohyperbolic disk $D(a, r)$ is defined by

$$D(a,r) = \Big\{ z \in \mathbb{D} : \rho(z,a) = \Big|\frac{z-a}{1-\overline{a}z}\Big| < r \Big\}.$$

We will frequently use the following property of $\mathcal{A}_2$ weights on pseudohyperbolic disks. For the sake of completeness, we include a proof of this fact as follows.

Lemma 2.5. *Let $0 < r \le \frac{1}{4}$ and $z \in \mathbb{D}$. If $\xi \in D(z, r)$, then we have*

$$\big|D(z,r)\big|_{\omega} < 8[\omega]_{\mathcal{A}_2} \big|D(\xi,r)\big|_{\omega}.$$

Proof. Observe that $D(z, r) \subset D(\xi, 2r)$. Now it suffices to show the following doubling inequality:

$$\big|D(\xi,2r)\big|_{\omega} < 8[\omega]_{\mathcal{A}_2} \big|D(\xi,r)\big|_{\omega} \quad (\xi \in \mathbb{D}).$$

Since ω is an $\mathcal{A}_2$ weight, we have

$$\frac{|D(\xi,2r)|_{\omega} |D(\xi,2r)|_{\omega^{-1}}}{|D(\xi,2r)|^2} \le [\omega]_{\mathcal{A}_2}.$$

Recall that a pseudohyperbolic disk $D(z, r)$ is a Euclidean disk with center and radius given by

$$\mathfrak{C} = \frac{1-r^2}{1-r^2|z|^2} z, \qquad \mathfrak{R} = \frac{1-|z|^2}{1-r^2|z|^2} r.$$

Combining the above with the Cauchy–Schwarz inequality gives

$$\begin{aligned}
\big|D(\xi,2r)\big|_\omega &\le [\omega]_{\mathcal{A}_2}\frac{|D(\xi,2r)|^2}{|D(\xi,2r)|_{\omega^{-1}}}\\
&\le [\omega]_{\mathcal{A}_2}\frac{|D(\xi,r)|^2}{|D(\xi,r)|_{\omega^{-1}}}\cdot\frac{|D(\xi,2r)|^2}{|D(\xi,r)|^2}\\
&\le 4[\omega]_{\mathcal{A}_2}\big|D(\xi,r)\big|_\omega\cdot\Big(\frac{1-r^2|\xi|^2}{1-4r^2|\xi|^2}\Big)^2\\
&< 8[\omega]_{\mathcal{A}_2}\big|D(\xi,r)\big|_\omega,
\end{aligned}$$

where the last inequality follows from $r \le \frac{1}{4}$. This completes the proof. □

We now turn to the representation theory of the space $L^2_h(\omega)$. These results and their proof strategies are motivated by Luecking's work on weighted Bergman spaces (see [15], [14]).

Before studying the representation theory of $L^2_h(\omega)$, we need to recall the concept of an ϵ-lattice in the unit disk. Let $\epsilon \in (0,1)$. A sequence $\{a_n\}_{n=1}^\infty$ in the unit disk is called an ϵ-*lattice* in the pseudohyperbolic metric if

- $\mathbb{D} = \bigcup_{n=1}^\infty D(a_n,\epsilon)$ and
- $\inf_{n\neq m}\left|\frac{a_n-a_m}{1-\overline{a_n}a_m}\right| \ge \frac{\epsilon}{2}$.

Now we are ready to state the atomic decomposition for $L^2_h(\omega)$.

Theorem 2.6. *Let ω be an $\mathcal{A}_2$ weight. Then there is an ϵ-lattice $\{a_n\}_{n=1}^\infty$ such that for each $f \in L^2_h(\omega)$ we have*

$$f(z) = \sum_{n=1}^\infty c_n\big(1-|a_n|^2\big)^2\big|D(a_n,\epsilon)\big|_\omega^{-\frac{1}{2}}R_{a_n}(z)$$

for some sequence $\{c_n\}$ in $\ell^2(\mathbb{N})$, where

$$R_\lambda(z) = \frac{1}{(1-\overline{z}\lambda)^2} + \frac{1}{(1-\overline{\lambda}z)^2} - 1$$

is the reproducing kernel for L^2_h at $\lambda \in \mathbb{D}$.

Remark 2.7. Lemma 2.2 in [4] gives the following estimate of the module of R_λ: there exists an $r_0 \in (0,\frac{1}{4}]$ such that if $0 < r \le r_0$, then

$$\frac{\frac{1}{2}}{(1-|\lambda|)^2} \le \big|R_\lambda(z)\big| \le \frac{3}{(1-|\lambda|)^2}$$

for all $z \in D(\lambda,r)$. In the rest of this paper, we will use r_0 to denote the constant in this remark.

To prove Theorem 2.6, we need to establish harmonic versions of Luecking's Theorem 4.1 in [15] and Theorem 3.12 in [14].

Theorem 2.8. *Let ω be an $\mathcal{A}_2$ weight. Then there exists an ϵ-lattice $\{a_n\}_{n=1}^\infty$ for some $0 < \epsilon < \frac{1}{16}$ such that*

$$\sum_{n=1}^{\infty} \big|f(a_n)\big|^2 \big|D(a_n, \epsilon)\big|_\omega \approx \|f\|_{L^2(\omega)}^2$$

for all f in $L_h^2(\omega)$. That is, there exist two positive constants C_1 and C_2 such that

$$C_1 \|f\|_{L^2(\omega)}^2 \le \sum_{n=1}^{\infty} \big|f(a_n)\big|^2 \big|D(a_n, \epsilon)\big|_\omega \le C_2 \|f\|_{L^2(\omega)}^2$$

for all f in $L_h^2(\omega)$.

Once Theorem 2.8 is established, we can quickly present the proof of Theorem 2.6 as follows.

Proof of Theorem 2.6. Since both ω and ω^{-1} are $\mathcal{A}_2$ weights, it follows from Theorem 2.8 that we can choose $\epsilon \in (0, \frac{1}{16})$ and an ϵ-lattice $\{a_n\}_{n=1}^\infty$ such that

$$\|g\|_{L^2(\omega^{-1})}^2 \approx \sum_{n=1}^{\infty} \big|g(a_n)\big|^2 \big|D(a_n, \epsilon)\big|_{\omega^{-1}}.$$

By the Cauchy–Schwarz inequality and the definition of $\mathcal{A}_2$ weight,

$$\big|D(a_n, \epsilon)\big|^2 \le \big|D(a_n, \epsilon)\big|_\omega \cdot |D(a_n, \epsilon)|_{\omega^{-1}} \le [\omega]_{\mathcal{A}_2} \big|D(a_n, \epsilon)\big|^2.$$

Therefore,

$$\|g\|_{L^2(\omega^{-1})}^2 \approx \sum_{n=1}^{\infty} \big|g(a_n)\big|^2 \big(1 - |a_n|^2\big)^4 \big|D(a_n, \epsilon)\big|_\omega^{-1}.$$

This implies that the linear operator $\mathscr{L} : L_h^2(\omega^{-1}) \to \ell^2(\mathbb{Z})$ defined by

$$\mathscr{L}(g) := \Big\{g(a_n)\big(1 - |a_n|^2\big)^2 \big|D(a_n, \epsilon)\big|_\omega^{-\frac{1}{2}}\Big\}_{n=1}^\infty$$

is bounded below, and its range is closed. It follows from the closed range theorem that $\mathscr{L}^*$ is surjective.

From the proof of Lemma 2.4, we have

$$g(a_n) = \langle g, R_{a_n} \rangle_{L^2(dA)} \tag{$*$}$$

for each $g \in L_h^2(\omega^{-1})$ and every $n \ge 1$. Let $\{c_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$. Using ($*$), we obtain

$$\mathscr{L}^*\big(\{c_n\}\big)(z) = \sum_{n=1}^{\infty} c_n R_{a_n}(z) \big(1 - |a_n|^2\big)^2 \big|D(a_n, \epsilon)\big|_\omega^{-\frac{1}{2}},$$

which gives the desired result. This completes the proof of Theorem 2.6. ☐

We now turn to the proof of Theorem 2.8. Let $0 < \epsilon < \frac{1}{16}$, and let $\{a_n\}_{n=1}^\infty \subset \mathbb{D}$ be an ϵ-lattice. Define a measure $\mu = \mu_\epsilon$ on $\mathbb{D}$ by

$$\mu(z) = \sum_{n=1}^{\infty} \delta_{a_n}(z) \Big|D\Big(a_n, \frac{\epsilon}{4}\Big)\Big|_\omega, \tag{2.1}$$

where δ_{a_n} is the Dirac measure concentrated at a_n. Indeed, the conclusion of Theorem 2.6 tells us that μ is both a Carleson and reverse Carleson measure for $L^2_h(\omega)$. First, we establish a sufficient condition for a general (positive) measure to be the $L^p_h(\omega)$-Carleson measure, where $0 < p < \infty$.

Proposition 2.9. *Suppose that ν is a positive Borel measure on $\mathbb{D}$. If there exist an $0 < r \leq r_0$ and a constant $C > 0$ independent of $z \in \mathbb{D}$ such that*

$$\nu\big(D(z,r)\big) \leq C\big|D(z,r)\big|_\omega$$

for all $z \in \mathbb{D}$, then ν is a Carleson measure for $L^p_h(\omega)$ $(0 < p < \infty)$; that is, there is a positive constant C_p such that

$$\int_{\mathbb{D}} \big|f(z)\big|^p \, d\nu(z) \leq C_p \|f\|^p_{L^p(\omega)}$$

for $f \in L^p_h(\omega)$. Consequently, μ is a Carleson measure for $L^2_h(\omega)$, that is; there is an absolute constant $C > 0$ such that

$$\int_{\mathbb{D}} \big|f(z)\big|^2 \, d\mu(z) = \sum_{n=1}^{\infty} \big|f(a_n)\big|^2 \Big|D\Big(a_n, \frac{\epsilon}{4}\Big)\Big|_\omega \leq C\|f\|^2_{L^2(\omega)}$$

for all f in $L^2_h(\omega)$.

Proof. Fix an $r \leq r_0$. By Lemma 2.1, we obtain

$$\big|f(z)\big|^{\frac{p}{2}} \leq \frac{C}{|D(z,r)|} \int_{D(z,r)} \big|f(\xi)\big|^{\frac{p}{2}} \, dA(\xi) \quad (z \in \mathbb{D}),$$

where $0 < p < \infty$ and $C = C(p,r)$. The Cauchy–Schwarz inequality and the $\mathcal{A}_2$ condition give us that

$$\big|f(z)\big|^p \leq C^2 [\omega]_{\mathcal{A}_2} \frac{\int_{D(z,r)} |f(\xi)|^p \omega(\xi) \, dA(\xi)}{|D(z,r)|_\omega}.$$

Integrating the above over the unit disk gives

$$\int_{\mathbb{D}} \big|f(z)\big|^p \, d\nu(z) \leq C \int_{\mathbb{D}} \int_{\mathbb{D}} \big|D(z,r)\big|^{-1}_\omega \big|f(\xi)\big|^p \omega(\xi) \chi_{D(\xi,r)}(z) \, dA(\xi) \, d\nu(z),$$

where the constant C depends only on p and r. Noting that $\xi \in D(z,r)$, we have by Lemma 2.5 that

$$\big|D(z,r)\big|_\omega \geq C\big|D(\xi,r)\big|_\omega$$

for some absolute constant C. Therefore,

$$\begin{aligned}
\int_{\mathbb{D}} \big|f(z)\big|^2 \, d\nu(z) &\leq C \int_{\mathbb{D}} \int_{\mathbb{D}} \big|D(z,r)\big|^{-1}_\omega \big|f(\xi)\big|^p \omega(\xi) \chi_{D(\xi,r)}(z) \, dA(\xi) \, d\nu(z) \\
&\leq C \int_{\mathbb{D}} \int_{\mathbb{D}} \big|D(\xi,r)\big|^{-1}_\omega \big|f(\xi)\big|^p \omega(\xi) \chi_{D(\xi,r)}(z) \, d\nu(z) \, dA(\xi).
\end{aligned}$$

Now our hypothesis on ν gives

$$\begin{aligned}\int_{\mathbb{D}} \big|f(z)\big|^p \, d\nu(z) &\le C \int_{\mathbb{D}} \big|D(\xi,r)\big|_{\omega}^{-1} \nu\big(D(\xi,r)\big) \big|f(\xi)\big|^p \omega(\xi)\, dA(\xi) \\ &\le C_1 \int_{\mathbb{D}} \big|f(\xi)\big|^p \omega(\xi)\, dA(\xi),\end{aligned}$$

where the constant $C_1 > 0$ is independent of $f \in L^p_h(\omega)$.

For the second conclusion of this proposition, it is sufficient to show that the inequality

$$\mu\Big(D\Big(a, \frac{1}{4}\Big)\Big) \le C \int_{D(a,\frac{1}{4})} \omega(z)\, dA(z) \quad (a \in \mathbb{D})$$

holds for some absolute constant $C > 0$. Indeed, by the definition of μ, we have

$$\mu\Big(D\Big(a, \frac{1}{4}\Big)\Big) = \sum_{\rho(a_n,a)<\frac{1}{4}} \int_{D(a_n,\frac{\epsilon}{4})} \omega \, dA = \sum_{\rho(a_n,a)<\frac{1}{4}} \Big|D\Big(a_n, \frac{\epsilon}{4}\Big)\Big|_{\omega}.$$

If $\rho(a, a_n) < \frac{1}{4}$, then for each $z \in D(a_n, \frac{\epsilon}{4})$, we have

$$\rho(z, a) \le \rho(z, a_n) + \rho(a_n, a) < \frac{\epsilon}{4} + \frac{1}{4} < \frac{1}{2},$$

to obtain

$$D\Big(a_n, \frac{\epsilon}{4}\Big) \subset D\Big(a, \frac{1}{2}\Big)$$

for every $n \ge 1$ provided that $\rho(a, a_n) < \frac{1}{4}$.

Since $D(a_n, \frac{\epsilon}{4}) \cap D(a_m, \frac{\epsilon}{4}) = \varnothing$ for $n \ne m$, we obtain

$$\bigcup_{\rho(a_n,a)<\frac{1}{4}} D\Big(a_n, \frac{\epsilon}{4}\Big) \subset D\Big(a, \frac{1}{2}\Big)$$

and

$$\mu\Big(D\Big(a, \frac{1}{4}\Big)\Big) \le \Big|D\Big(a, \frac{1}{2}\Big)\Big|_{\omega} \le C\Big|D\Big(a, \frac{1}{4}\Big)\Big|_{\omega}$$

for every $a \in \mathbb{D}$, where the constant $C > 0$ (independent of a) comes from Lemma 2.5. This completes the proof of Proposition 2.9. $\square$

In order to finish the proof of Theorem 2.8, we need to show that there is an $\epsilon \in (0, \frac{1}{16})$ such that $\mu = \mu_\epsilon$ is a reverse Carleson measure for $L^2_h(\omega)$. More precisely, we will prove the following proposition.

Proposition 2.10. *There exists an $\epsilon \in (0, \frac{1}{16})$ such that*

$$\int_{\mathbb{D}} \big|f(z)\big|^2 \, d\mu(z) = \sum_{n=1}^{\infty} \big|f(a_n)\big|^2 \big|D(a_n, \epsilon)\big|_{\omega} \ge C\|f\|^2_{L^2(\omega)}$$

for all f in $L^2_h(\omega)$, where $C > 0$ is an absolute constant.

The rest of this section is devoted to the proof of the above *reverse Carleson inequality.* To do so, we need to prove the following two lemmas related to harmonic functions, which extend the results in [15] and [14] for weighted Bergman spaces to the present situation.

Lemma 2.11. *Let f be a harmonic function on $\mathbb{D}$, and let $\epsilon \in (0, \frac{1}{16})$. Then there exists a constant $C_1 > 0$ (independent of z, ϵ, and f) such that*

$$\big|f(z) - f(0)\big| \leq C_1 \epsilon \int_{D(0,\frac{1}{4})} \big|f(\lambda)\big| \, dA(\lambda)$$

when $|z| < \epsilon$. As a consequence, there exists a constant $C_2 > 0$ (independent of z, ϵ, and f) such that

$$\big|f(w) - f(\xi)\big|^2 \leq \frac{C_2 \epsilon^2}{|D(\xi, \frac{1}{4})|_\omega} \int_{D(\xi,\frac{1}{4})} \big|f(\lambda)\big|^2 \omega(\lambda) \, dA(\lambda)$$

when $\xi \in D(w, \epsilon)$.

Proof. Observe that

$$f(z) - f(0) = \int_0^1 \frac{\partial}{\partial t} \big[f(tz)\big] \, dt = \int_0^1 \big[\nabla f(tz) \cdot z\big] \, dt$$

for $|z| < \epsilon < \frac{1}{16}$. Thus we have

$$\big|f(z) - f(0)\big| \leq \sup_{|\xi| \leq \epsilon} \big|\nabla f(\xi)\big| \cdot |z|.$$

Recall that

$$\begin{aligned} |\nabla f|^2 &= 2\big(|\partial f|^2 + |\overline{\partial} f|^2\big) = 2\big(|\partial f|^2 + |\partial \overline{f}|^2\big) \\ &\leq \big[\sqrt{2}\big(|\partial f| + |\partial \overline{f}|\big)\big]^2, \end{aligned}$$

where $\overline{\partial} f = \frac{\partial f}{\partial \overline{z}}$. By Lemma 2.1, there is an absolute constant $C > 0$ such that

$$\big|\nabla f(\xi)\big| \leq \frac{2\sqrt{2} C}{(1 - \frac{1}{16})^3} \cdot \Big(\frac{1 - (\frac{1}{16})^2 |\xi|^2}{\frac{1}{16}(1 - |\xi|^2)}\Big)^3 \int_{D(\xi,\frac{1}{16})} \big|f(\lambda)\big| \, dA(\lambda).$$

Note that $|\xi| \leq \epsilon < \frac{1}{16}$. If $\lambda \in D(\xi, \frac{1}{16})$, then

$$|\lambda| < |\xi| + \frac{1}{16}|1 - \overline{\lambda}\xi| < \frac{1}{4},$$

to get $D(\xi, \frac{1}{16}) \subset D(0, \frac{1}{4})$ and

$$\big|\nabla f(\xi)\big| \leq C_1 \int_{D(0,\frac{1}{4})} \big|f(\lambda)\big| \, dA(\lambda)$$

for all $|\xi| \leq \epsilon$, where the constant C_1 is independent of z, ξ, and ϵ. Therefore,

$$\big|f(z) - f(0)\big| \leq \sup_{|\xi| \leq \epsilon} \big|\nabla f(\xi)\big| \cdot |z| \leq C_1 |z| \int_{D(0,\frac{1}{4})} \big|f(\lambda)\big| \, dA(\lambda)$$

for $|z| < \epsilon$ with $\epsilon \in (0, \frac{1}{16})$. This proves the first part of the lemma.

Let φ_ξ be the Möbius map. Then $f\circ\varphi_\xi$ is harmonic on $\mathbb{D}$. By changing variables, we have that

$$\left|f\big(\varphi_\xi(z)\big)-f(\xi)\right| \le \frac{C_3\epsilon}{|D(\xi,\frac{1}{4})|}\int_{D(\xi,\frac{1}{4})}\left|f(\lambda)\right|dA(\lambda)$$

for some absolute constant $C_3>0$. The Cauchy–Schwarz inequality gives

$$\left|f\big(\varphi_\xi(z)\big)-f(\xi)\right|^2 \le \frac{C_3^2[\omega]_{\mathcal{A}_2}\epsilon^2}{|D(\xi,\frac{1}{4})|_\omega}\int_{D(\xi,\frac{1}{4})}\left|f(\lambda)\right|^2\omega(\lambda)\,dA(\lambda).$$

Let $w=\varphi_\xi(z)$. We have $|\varphi_\xi(w)|=|z|<\epsilon$ and

$$\left|f(w)-f(\xi)\right|^2 \le \frac{C_2\epsilon^2}{|D(\xi,\frac{1}{4})|_\omega}\int_{D(\xi,\frac{1}{4})}\left|f(\lambda)\right|^2\omega(\lambda)\,dA(\lambda)$$

if $\xi\in D(w,\epsilon)$, as desired. □

Lemma 2.12. *Let f be a harmonic function, and let $\epsilon\in(0,\frac{1}{16})$. Let μ be the measure defined in (2.1). Then there exists a constant $C>0$ (independent of ϵ) such that*

$$\int_{\mathbb{D}}\int_{\mathbb{D}}\chi_\epsilon(z,\xi)\left|D(\xi,\epsilon)\right|_\omega^{-1}\left|f(z)-f(\xi)\right|^2\omega(z)\,dA(z)\,d\mu(\xi)\le C\epsilon^2\|f\|^2_{L^2(\omega)}$$

for all $f\in L^2_h(\omega)$, where

$$\chi_\epsilon(z,\xi)=\begin{cases}1 & \text{if } z\in D(\xi,\epsilon),\\ 0 & \text{otherwise.}\end{cases}$$

Proof. By Lemma 2.11, we have

$$\begin{aligned}&\chi_\epsilon(z,\xi)\left|f(z)-f(\xi)\right|^2\frac{\omega(z)}{|D(\xi,\epsilon)|_\omega}\\ &\quad\le\Big(\frac{C\epsilon^2}{|D(\xi,\frac{1}{4})|_\omega}\int_{D(\xi,\frac{1}{4})}\left|f(\lambda)\right|^2\omega(\lambda)\,dA(\lambda)\Big)\frac{\chi_\epsilon(z,\xi)\omega(z)}{|D(\xi,\epsilon)|_\omega}.\end{aligned}$$

Integrating over $z\in\mathbb{D}$ on both sides gives

$$\int_{\mathbb{D}}\chi_\epsilon(z,\xi)\left|f(z)-f(\xi)\right|^2\frac{\omega(z)}{|D(\xi,\epsilon)|_\omega}\,dA(z)\le\frac{C\epsilon^2}{|D(\xi,\frac{1}{4})|_\omega}\int_{D(\xi,\frac{1}{4})}|f|^2\omega\,dA.$$

Now integrating with respect to $d\mu(\xi)$ and noting that

$$\chi_{D(\xi,\frac{1}{4})}(\lambda)=\chi_{D(\lambda,\frac{1}{4})}(\xi),$$

we have

$$\begin{aligned}&\int_{\mathbb{D}}\frac{C\epsilon^2}{|D(\xi,\frac{1}{4})|_\omega}\int_{D(\xi,\frac{1}{4})}\left|f(\lambda)\right|^2\omega(\lambda)\,dA(\lambda)\,d\mu(\xi)\\ &\quad=C\epsilon^2\int_{\mathbb{D}}\Big(\int_{\mathbb{D}}\frac{\chi_{D(\lambda,\frac{1}{4})}(\xi)}{|D(\xi,\frac{1}{4})|_\omega}\,d\mu(\xi)\Big)\left|f(\lambda)\right|^2\omega(\lambda)\,dA(\lambda).\end{aligned}$$

Since $\lambda \in D(\xi, \frac{1}{4})$, Lemma 2.5 tells us that there is an absolute constant C such that

$$\Big|D\Big(\xi, \frac{1}{4}\Big)\Big|_\omega \geq C\Big|D\Big(\lambda, \frac{1}{4}\Big)\Big|_\omega.$$

Thus we obtain

$$\int_{\mathbb{D}} \frac{\chi_{D(\lambda,\frac{1}{4})}(\xi)}{|D(\xi,\frac{1}{4})|_\omega}\, d\mu(\xi) \leq C^{-1}\frac{\mu(D(\lambda,\frac{1}{4}))}{|D(\lambda,\frac{1}{4})|_\omega}.$$

By Lemma 2.9, we have

$$\mu\Big(D\Big(\lambda, \frac{1}{4}\Big)\Big)\cdot\Big|D\Big(\lambda, \frac{1}{4}\Big)\Big|_\omega^{-1} \leq C_1$$

for some constant $C_1 > 0$ (independent of ϵ), completing the proof. □

Now we are ready to prove the reverse Carleson inequality in Proposition 2.10.

Proof of Proposition 2.10. Recall that μ satisfies the condition

$$\mu\Big(D\Big(a, \frac{1}{4}\Big)\Big) \leq C\Big|D\Big(a, \frac{1}{4}\Big)\Big|_\omega$$

for all $a \in \mathbb{D}$. Applying Lemma 2.12 to $\epsilon \in (0, \frac{1}{16})$, we have

$$\Big[\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{\chi_\epsilon(z,\xi)}{|D(\xi,\epsilon)|_\omega}\big|f(z)-f(\xi)\big|^2\omega(z)\,dA(z)\,d\mu(\xi)\Big]^{\frac{1}{2}} \leq C\epsilon\|f\|_{L^2(\omega)}.$$

The triangle inequality gives

$$I - J \leq C\epsilon\|f\|_{L^2(\omega)},$$

where

$$I := \Big[\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{\chi_\epsilon(z,\xi)}{|D(\xi,\epsilon)|_\omega}\big|f(z)\big|^2\omega(z)\,dA(z)\,d\mu(\xi)\Big]^{\frac{1}{2}}$$

and

$$J := \Big[\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{\chi_\epsilon(z,\xi)}{|D(\xi,\epsilon)|_\omega}\big|f(\xi)\big|^2\omega(z)\,dA(z)\,d\mu(\xi)\Big]^{\frac{1}{2}}.$$

For the first integral I, we note that for each $z \in \mathbb{D}$,

$$\int_{\mathbb{D}} \frac{\chi_\epsilon(z,\xi)}{|D(\xi,\epsilon)|_\omega}\,d\mu(\xi) = \int_{D(z,\epsilon)} \frac{d\mu(\xi)}{|D(\xi,\epsilon)|_\omega} \geq C_1\frac{\mu(D(z,\epsilon))}{|D(z,\epsilon)|_\omega},$$

where C_1 is an absolute constant.

Since $\mathbb{D} = \bigcup_{n=1}^\infty D(a_n,\epsilon)$, we can select $D(a_j,\epsilon)$ such that $z \in D(a_j,\epsilon)$ for each $z \in \mathbb{D}$. Applying Lemma 2.5, we get

$$\begin{aligned}\mu\big(D(z,\epsilon)\big) &= \sum_{n=1}^\infty \delta_{a_n}\big(D(z,\epsilon)\big)\int_{D(a_n,\frac{\epsilon}{4})}\omega\,dA \\ &\geq \Big|D\Big(a_j, \frac{\epsilon}{4}\Big)\Big|_\omega \geq C_2\big|D(a_j,\epsilon)\big|_\omega \geq C_3\big|D(z,\epsilon)\big|_\omega,\end{aligned}$$

where C_2 and C_3 are absolute positive constants. Therefore, we have

$$\int_{\mathbb{D}} \frac{\chi_\epsilon(z,\xi)}{|D(\xi,\epsilon)|_\omega}\,d\mu(\xi) \geq \widetilde{C}$$

for some absolute constant $\widetilde{C} > 0$, to obtain

$$I \geq \widetilde{C}\|f\|_{L^2(\omega)}.$$

For the second integral J, we observe that

$$\int_{\mathbb{D}} \chi_\epsilon(z,\xi)\omega(z)\,dA(z) = \big|D(\xi,\epsilon)\big|_\omega,$$

to get

$$J = \Big(\int_{\mathbb{D}} \big|f(\xi)\big|^2\,d\mu(\xi)\Big)^{\frac{1}{2}}.$$

Thus we have

$$\widetilde{C}\|f\|_{L^2(\omega)} - \Big(\int_{\mathbb{D}} \big|f(\xi)\big|^2\,d\mu(\xi)\Big)^{\frac{1}{2}} \leq I - J \leq C\epsilon\|f\|_{L^2(\omega)}.$$

Equivalently,

$$(\widetilde{C} - C\epsilon)\|f\|_{L^2(\omega)} \leq \Big(\int_{\mathbb{D}} \big|f(\xi)\big|^2\,d\mu(\xi)\Big)^{\frac{1}{2}}$$

for each $0 < \epsilon < \frac{1}{16}$.

Since $C, \widetilde{C}$ are both independent of ϵ, we can choose

$$0 < \epsilon < \min\Big\{\frac{1}{16}, \frac{\widetilde{C}}{2C}, r_0\Big\}$$

such that

$$\|f\|^2_{L^2(\omega)} \leq \frac{1}{(\widetilde{C} - C\epsilon)^2}\int_{\mathbb{D}} \big|f(\xi)\big|^2\,d\mu(\xi).$$

Recalling the definition of μ, we conclude that

$$\|f\|^2_{L^2(\omega)} \leq \frac{4}{\widetilde{C}^2}\sum_{n=1}^{\infty}\big|f(a_n)\big|^2\Big|D\Big(a_n, \frac{\epsilon}{4}\Big)\Big|_\omega \leq \frac{4}{\widetilde{C}^2}\sum_{n=1}^{\infty}\big|f(a_n)\big|^2\big|D(a_n,\epsilon)\big|_\omega.$$

This completes the proof. □

The proof of Theorem 2.6 implies the following result immediately.

Theorem 2.13. *Suppose that ω satisfies the $\mathcal{A}_2$ condition. Then there is an ϵ-lattice $\{a_n\}_{n=1}^\infty \subset \mathbb{D}$ such that each $f \in L^2_h(\omega)$ has the form*

$$f(z) = \sum_{n=1}^{\infty} c_n K^\omega_{a_n}(z)\big|D(a_n,\epsilon)\big|^{\frac{1}{2}}_\omega$$

for some sequence $\{c_n\}$ in $\ell^2(\mathbb{N})$, where $K^\omega_{a_n}$ is the reproducing kernel for $L^2_h(\omega)$.

Proof. We consider the linear map $\mathscr{S} : L_h^2(\omega) \to \ell^2(\mathbb{N})$:

$$\mathscr{S} f = \big\{f(a_n)\big|D(a_n,\epsilon)\big|_\omega^{\frac{1}{2}}\big\}_{n=1}^\infty.$$

Propositions 2.9 and 2.10 imply that $\mathscr{S}^* : \ell^2(\mathbb{N}) \to L_h^2(\omega)$ is surjective and that

$$\Big\langle \mathscr{S}^*\big(\{c_n\}\big), f\Big\rangle_{L^2(\omega)} = \Big\langle \sum_{n=1}^\infty c_n K_{a_n}^\omega(z)\big|D(a_n,\epsilon)\big|_\omega^{\frac{1}{2}}, f\Big\rangle_{L^2(\omega)}$$

for $\{c_n\} \in \ell^2(\mathbb{N})$ and $f \in L_h^2(\omega)$. Therefore,

$$\mathscr{S}^*\big(\{c_n\}\big)(z) = \sum_{n=1}^\infty c_n K_{a_n}^\omega(z)\big|D(a_n,\epsilon)\big|_\omega^{\frac{1}{2}}.$$

This completes the proof of this theorem. □

3. Boundedness and compactness of T_ν on $L_h^2(\omega)$

Recall that the Toeplitz operator T_ν initially defined on a dense subspace of $L_h^2(\omega)$ is given by

$$T_\nu f(z) = \int_{\mathbb{D}} f(\lambda)\overline{K_z^\omega(\lambda)}\, d\nu(\lambda) \quad (z \in \mathbb{D}).$$

In this section, we will characterize the boundedness and compactness of T_ν on $L_h^2(\omega)$ via the Berezin transform and Carleson measures for $L_h^2(\omega)$. First, we define the Berezin transform $\widetilde{\nu}$ of ν as

$$\widetilde{\nu}(z) = \frac{1}{\|R_z\|_{L^2(\omega)}^2}\int_{\mathbb{D}}\big|R_z(\lambda)\big|^2\, d\nu(\lambda),$$

where

$$R_z(\lambda) = \frac{1}{(1-\overline{\lambda}z)^2} + \frac{1}{(1-\overline{z}\lambda)^2} - 1$$

is the reproducing kernel for L_h^2. The first main result of this section is Theorem 3.1.

Theorem 3.1. *Let ν be a positive finite Borel measure on $\mathbb{D}$, and let ω be an $\mathcal{A}_2$ weight. The following conditions are equivalent:*

(1) *T_ν extends to a bounded linear operator on $L_h^2(\omega)$;*

(2) *ν is a Carleson measure for $L_h^2(\omega)$;*

(3) *there exist an $0 < r \le r_0$ and a constant $C > 0$ independent of $z \in \mathbb{D}$ such that*

$$\nu\big(D(z,r)\big) \le C\big|D(z,r)\big|_\omega$$

for all $z \in \mathbb{D}$;

(4) *the Berezin transform $\widetilde{\nu}$ is bounded.*

To prove Theorem 3.1, we need the following useful lemma.

Lemma 3.2. *Let $\omega \in \mathcal{A}_2$. If $0 < r \le r_0$, then there is a constant $C = C(r)$ such that*

$$\frac{|D(\lambda,r)|_\omega}{2(1-|\lambda|)^4} \le \|R_\lambda\|^2_{L^2(\omega)} \le C\frac{|D(\lambda,r)|_\omega}{(1-|\lambda|)^4}$$

for all $\lambda \in \mathbb{D}$.

Proof. Let $\lambda \in \mathbb{D}$. By Remark 2.7, there exists an $r_0 \in (0, \frac{1}{4}]$ such that if $0 < r \le r_0$. Then

$$\frac{\frac{1}{2}}{(1-|\lambda|)^2} \le \left|R_\lambda(z)\right| \le \frac{3}{(1-|\lambda|)^2}$$

for all $z \in D(\lambda, r)$. It follows that for each $r \in (0, r_0]$, we have

$$\begin{aligned}\|R_\lambda\|^2_{L^2(\omega)} &\ge \int_{D(\lambda,r)} \left|R_\lambda(z)\right|^2 \omega(z)\, dA(z)\\ &\ge \frac{1}{4}\int_{D(\lambda,r)} \frac{\omega(z)}{(1-|\lambda|)^4}\, dA(z) = \frac{|D(\lambda,r)|_\omega}{4(1-|\lambda|)^4}.\end{aligned}$$

Now we turn to the proof of the other inequality. Using

$$|z-\lambda| < r\big(1-|\lambda|\big) < r|1-\overline{z}\lambda| \quad (z, \lambda \in \mathbb{D}),$$

we have

$$S(\lambda, r) := \big\{z \in \mathbb{D} : |z-\lambda| < r\big(1-|\lambda|\big)\big\} \subset D(\lambda, r).$$

Thus, we have by Lemma 2.1 in [6] that

$$C_1\frac{|S(\lambda,r)|_\omega}{(1-|\lambda|)^4} \le \|K_\lambda\|^2_{L^2(\omega)} \le C_2\frac{|S(\lambda,r)|_\omega}{(1-|\lambda|)^4} \le C_2\frac{|D(\lambda,r)|_\omega}{(1-|\lambda|)^4}$$

for some positive constants $C_1 = C_1(r)$ and $C_2 = C_2(r)$, where $K_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)^2}$ is the reproducing kernel for L^2_a at λ. Recalling that

$$R_\lambda(z) = 2\operatorname{Re}\big(K_\lambda(z)\big) - 1,$$

we have

$$\|R_\lambda\|^2_{L^2(\omega)} \le \frac{4C|D(\lambda,r)|\omega}{(1-|\lambda|)^4} + 2\|\omega\|_{L^1}.$$

Consequently, to complete the proof we only need to show that there is a constant C_3 depending only on r such that

$$\|\omega\|_{L^1} \le C_3\frac{|D(\lambda,r)|_\omega}{(1-|\lambda|)^4}$$

for every $\lambda \in \mathbb{D}$. Indeed, we may assume that $\|\omega\|_{L^1} = 1$. Then it is easy to see that

$$C\big(1-|\lambda|\big)^2 \le \big|D(\lambda,r)\big|$$

for some constant $C = C(r)$. Thus we have

$$C\big(1-|\lambda|\big)^2 \le \big|D(\lambda,r)\big| \le \big|D(\lambda,r)\big|^{\frac{1}{2}}_\omega \cdot \|\omega^{-1}\|^{\frac{1}{2}}_{L^1},$$

to get

$$\frac{|D(\lambda, r)|_\omega}{(1-|\lambda|)^4} \geq C_3,$$

which completes the proof of Lemma 3.2. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. We will show that (4) ⇒ (3) ⇒ (2) ⇒ (1) ⇒ (4).

(4) ⇒ (3): Note that there exists an $r \in (0, r_0]$ such that

$$\frac{\frac{1}{2}}{(1-|z|)^2} \leq \left|R_z(\lambda)\right| \leq \frac{3}{(1-|z|)^2}$$

for $\lambda \in D(z,r)$. From the definition of $\widetilde{\nu}$, there is a constant $C > 0$ such that

$$\frac{1}{\|R_z\|^2_{L^2(\omega)}} \int_{D(z,r)} \left|R_z(\lambda)\right|^2 d\nu(\lambda) \leq \widetilde{\nu}(z) \leq C$$

for all $z \in \mathbb{D}$. Combining these with Lemma 3.2 gives us that

$$\nu\big(D(z,r)\big) \leq C\big|D(z,r)\big|_\omega \quad (z \in \mathbb{D})$$

for some positive constant $C = C(r)$.

Proposition 2.9 gives (3) ⇒ (2), so we only need to show that (2) ⇒ (1). Assume that ν is a Carleson measure. By condition (2) and Lemma 3.2, we obtain condition (3). Now Proposition 2.9 implies that ν is also a Carleson measure for $L^1_h(\omega)$. Then for f, g—two functions harmonic on a neighborhood of $\overline{\mathbb{D}}$ (which, by Theorem 2.6, are dense in $L^2_h(\omega)$)—we have

$$\left|\langle T_\nu f, g\rangle\right| \leq \int_{\mathbb{D}} \left|f(z)g(z)\right| d\nu(z) \leq C\|fg\|_{L^1(\omega)} \leq C\|f\|_{L^2(\omega)}\|g\|_{L^2(\omega)},$$

to get that T_ν is bounded.

(1) ⇒ (4): Suppose that T_ν is bounded on $L^2_h(\omega)$. We consider the partial sum $\sigma_N = \sum_{n=1}^N t_n K^\omega_{a_n}$, where $N \geq 1$, $\{t_n\}$ are complex numbers, and $\{a_n\} \subset \mathbb{D}$. Direct calculations show that

$$\|\sigma_N\|_{L^2(\nu)} \leq C\|\sigma_N\|_{L^2(\omega)}$$

for some constant $C > 0$. This implies that if $\lim_{N\to\infty} \|\sigma_N - g\|_{L^2(\omega)} = 0$ for some $g \in L^2_h(\omega)$, then

$$\lim_{N\to\infty} \langle f, \sigma_N\rangle_{L^2(\nu)} = \langle f, g\rangle_{L^2(\nu)}$$

for every $f \in L^2_h(\omega)$. Applying Theorem 2.13, we obtain

$$\langle T_\nu f, g\rangle_{L^2(\omega)} = \langle f, g\rangle_{L^2(\nu)}$$

for every $f, g \in L^2_h(\omega)$. In particular, we have

$$\langle T_\nu R_z, R_z\rangle_{L^2(\omega)} = \langle R_z, R_z\rangle_{L^2(\nu)} = \widetilde{\nu}(z)\|R_z\|^2_{L^2(\omega)}$$

to get $\widetilde{\nu}(z) \leq \|T_\nu\|$ for all $z \in \mathbb{D}$. The proof of Theorem 3.1 is now complete. □

From the above theorem, it is natural to characterize the compactness of T_ν via the vanishing Carleson measure. In fact, we will characterize the compact Toeplitz operator with positive measure as the symbol via not only the vanishing Carleson measure (for the $\mathcal{A}_2$-weighted harmonic Bergman space) but also the Berezin transform.

Theorem 3.3. *Let ν be a positive finite Borel measure on $\mathbb{D}$, and let $\omega \in \mathcal{A}_2$. The following conditions are equivalent:*

(1) T_ν is compact on $L^2_h(\omega)$.

(2) ν is a vanishing Carleson measure for $L^2_h(\omega)$; that is,

$$\lim_{|z|\to 1^-} \frac{\nu(D(z,r))}{|D(z,r)|_\omega} = 0$$

for some $r \in (0,1)$.

(3) The Berezin transform satisfies that $\lim_{|z|\to 1^-} \widetilde{\nu}(z) = 0$.

Proof. We will show that (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1): To prove (1), we only need to show that the inclusion operator $i : L^2_h(\omega) \to L^2(\nu)$ is compact; that is,

$$\lim_{n\to\infty} \int_{\mathbb{D}} \big|f_n(z)\big|^2 \, d\nu(z) = 0$$

whenever $\|f_n\|_{L^2(\omega)} \to 0$ $(n \to \infty)$, where $\{f_n\}_{n=1}^\infty$ is a bounded sequence in $L^2_h(\omega)$ which converges to zero uniformly on each compact subset of $\mathbb{D}$.

From the proof of Proposition 2.9, there exists a positive constant $C = C(r)$ such that

$$\begin{aligned}
\int_{\mathbb{D}} \big|f_n(z)\big|^2 \, d\nu(z) &\le C \int_{\mathbb{D}} \big|D(\xi,r)\big|_\omega^{-1} \nu\big(D(\xi,r)\big) \big|f_n(\xi)\big|^2 \omega(\xi) \, dA(\xi) \\
&= C\Big(\int_{|\xi|\le s} + \int_{|\xi|>s}\Big) \frac{\nu(D(\xi,r))}{|D(\xi,r)|_\omega} \big|f_n(\xi)\big|^2 \omega(\xi) \, dA(\xi),
\end{aligned}$$

where $s \in (0,1)$. Using the assumption in condition (2), we can choose an $s_0 \in (0,1)$ to make the second integral as small as we like. Fix s_0. It is easy to show that the first integral converges to zero, since $f_n \to 0$ $(n \to \infty)$ uniformly on compact subsets. This proves (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3): Observe that

$$\widetilde{\nu}(z) = \Big\langle T_\nu \frac{R_z}{\|R_z\|_{L^2(\omega)}}, \frac{R_z}{\|R_z\|_{L^2(\omega)}} \Big\rangle_{L^2(\omega)} \le \Big\| T_\nu\Big(\frac{R_z}{\|R_z\|_{L^2(\omega)}}\Big)\Big\|_{L^2(\omega)}.$$

So, it is sufficient to show that $\frac{R_z}{\|R_z\|_{L^2(\omega)}}$ converges to zero weakly in $L^2_h(\omega)$ as $|z| \to 1^-$. Noting that $\frac{R_z}{\|R_z\|_{L^2(\omega)}}$ is a unit vector in $L^2_h(\omega)$, we only need to show that it converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|z| \to 1^-$. Observe that Lemma 3.2 implies that there exists a positive constant $C = C(r_0)$ such that

$$\Big|\frac{R_z(\lambda)}{\|R_z\|_{L^2(\omega)}}\Big|^2 \le C\big|R_z(\lambda)\big|^2 \int_{D(z,r_0)} \omega^{-1} \, dA.$$

Now we conclude that $\frac{R_z}{\|R_z\|_{L^2(\omega)}}$ converges to zero uniformly (as $|z| \to 1^-$) on each disk $|\lambda| \leq s < 1$, since $|D(z, r_0)| \to 0$ as $|z| \to 1^-$, and that $\omega^{-1} \in L^1(dA)$.

(3) $\Rightarrow$ (2): By the definition of $\widetilde{\nu}$ and Lemma 3.2, there exists a constant $C = C(r_0) > 0$ such that

$$\widetilde{\nu}(z) = \frac{1}{\|R_z\|^2_{L^2(\omega)}} \int_{\mathbb{D}} \left|R_z(\lambda)\right|^2 d\nu(\lambda) \geq \frac{C}{4} \frac{\nu(D(z, r_0))}{|D(z, r_0)|_\omega}.$$

Thus we have $\lim_{|z|\to 1^-} \frac{\nu(D(z,r_0))}{|D(z,r_0)|_\omega} = 0$ if $\lim_{|z|\to 1^-} \widetilde{\nu}(z) = 0$, to complete the proof. □

In the rest of this section, we will consider the special class of compact Toeplitz operators. We will give a characterization of ν for T_ν to be in the Schatten class $\mathcal{S}^p$ $(p \geq 1)$. The following theorem is the third main result in Section 3.

Theorem 3.4. *Let ν be a positive finite Borel measure on $\mathbb{D}$, and let $\omega \in \mathcal{A}_2$. Then for $1 \leq p < \infty$, $T_\nu \in \mathcal{S}^p$ if and only if*

$$\sum_{n=1}^{\infty} \Big(\frac{\nu(D(a_n, \epsilon))}{|D(a_n, \epsilon)|_\omega} \Big)^p < +\infty,$$

where $(\{a_n\}_{n=1}^{\infty}, \epsilon)$ is the ϵ-lattice obtained by Theorem 2.6.

In order to prove the above result, we need one more lemma, which is a straightforward consequence of Lemmas 2.1 and 3.2.

Lemma 3.5. *Let $\omega \in \mathcal{A}_2$ and $0 < r \leq r_0$. There exists a constant $C = C(r) > 0$ such that*

$$C^{-1} \leq K_z^\omega(z) \cdot \big|D(z, r)\big|_\omega \leq C$$

for $z \in \mathbb{D}$, where K_z^ω is the reproducing kernel for $L_h^2(\omega)$.

Proof. Note that $K_\lambda^\omega(\lambda) = \|K_\lambda^\omega\|^2_{L^2(\omega)}$ for each $\lambda \in \mathbb{D}$. Applying Lemma 2.1 to the function $K_\lambda^\omega(z)$, we get a constant C depending only on r such that

$$\left|K_\lambda^\omega(z)\right|^2 \leq \frac{C\|K_\lambda^\omega\|^2_{L^2(\omega)}}{|D(z, r)|_\omega} = \frac{CK_\lambda^\omega(\lambda)}{|D(z, r)|_\omega}.$$

Taking $\lambda = z$, we get the inequality on the right-hand side in Lemma 3.5.

For the reverse inequality, note that for each $z \in \mathbb{D}$,

$$\begin{aligned} \frac{1}{(1 - |z|)^2} &\leq \frac{2}{(1 - |z|^2)^2} - 1 = R_z(z) = \langle R_z, K_z^\omega \rangle_{L^2(\omega)} \\ &\leq \|R_z\|_{L^2(\omega)} \cdot \|K_z^\omega\|_{L^2(\omega)} \leq C \frac{|D(z, r)|_\omega^{\frac{1}{2}}}{(1 - |z|)^2} \cdot \sqrt{K_z^\omega(z)}, \end{aligned}$$

where the constant C comes from Lemma 3.2. This finishes the proof. □

We are ready to prove Theorem 3.4. The method of its proof is quite standard.

Proof of Theorem 3.4. Suppose that $\sum_{n=1}^{\infty}(\frac{\nu(D(a_n,\epsilon))}{|D(a_n,\epsilon)|_\omega})^p < +\infty$. We consider the ϵ-lattice $\{a_n\}_{n=1}^{\infty}$ given by Theorem 2.6. Recall that $\epsilon < r_0$ (see the proof of Proposition 2.10). For an arbitrary orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of $L_h^2(\omega)$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\langle T_\nu e_n, e_n\rangle &= \sum_{n=1}^{\infty}\int_{\mathbb{D}}\big|e_n(z)\big|^2\,d\nu(z) = \int_{\mathbb{D}} K_z^\omega(z)\,d\nu(z)\\ &\le \sum_{n=1}^{\infty}\int_{D(a_n,\epsilon)} K_z^\omega(z)\,d\nu(z)\\ &\le C\sum_{n=1}^{\infty}\int_{D(a_n,\epsilon)}\big|D(z,\epsilon)\big|_\omega^{-1}\,d\nu(z),\end{aligned}$$

where the constant C comes from Lemma 3.5. Note that $\rho(z,a_n) < \epsilon$ for every $n \ge 1$. By Lemma 2.5 and its proof, we can choose a constant $C_1 = C_1(\epsilon)$ such that

$$\sum_{n=1}^{\infty}\langle T_\nu e_n, e_n\rangle \le C_1\sum_{n=1}^{\infty}\frac{\nu(D(a_n,\epsilon))}{|D(a_n,\epsilon)|_\omega}.$$

This shows that $T_\nu \in \mathcal{S}^1$.

On the other hand, if $\sup_{n\ge1}\frac{\nu(D(a_n,\epsilon))}{|D(a_n,\epsilon)|_\omega} < +\infty$, then by the proof Theorem 7.4 in [25] (or the proof of (3) $\Rightarrow$ (2) in Theorem 3.1), we deduce that T_ν is bounded on $L_h^2(\omega)$; that is, $T_\nu \in \mathcal{S}^\infty$. Now applying the interpolation theorem for the Schatten classes (see [25, Theorem 2.6] if needed), we obtain that $T_\nu \in \mathcal{S}^p$ for each $p \in (1,+\infty)$ if

$$\sum_{n=1}^{\infty}\Big(\frac{\nu(D(a_n,\epsilon))}{|D(a_n,\epsilon)|_\omega}\Big)^p < +\infty.$$

Conversely, we assume that $T_\nu \in \mathcal{S}^p$ for $1 \le p < \infty$. We recall by Theorem 2.6 that each $f \in L_h^2(\omega)$ has the form

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} c_n\big(1-|a_n|\big)^2 R_{a_n}(z)\big|D(a_n,\epsilon)\big|_\omega^{-\frac{1}{2}}\\ &:= \sum_{n=1}^{\infty} c_n h_n(z),\end{aligned}$$

where $\{c_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$. The Cauchy–Schwarz inequality shows that

$$\|f\|_{L^2(\omega)}^2 \le \Big(\sum_{n=1}^{\infty}|c_n|^2\Big)\cdot\Big(\sum_{n=1}^{\infty}\frac{(1-|a_n|)^4}{|D(a_n,\epsilon)|_\omega}\|R_{a_n}\|_{L^2(\omega)}^2\Big).$$

Since $\epsilon < r_0$, we have by Lemma 3.2 that

$$\|f\|_{L^2(\omega)}^2 \le C\sum_{n=1}^{\infty}|c_n|^2,$$

where the constant $C > 0$ depends only on ϵ.

Fix an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $L^2_h(\omega)$, and define a linear operator $\mathscr{A}$ on $L^2_h(\omega)$ by

$$\mathscr{A}\Big(\sum_{n=1}^{\infty} c_n e_n\Big) = \sum_{n=1}^{\infty} c_n h_n.$$

Then $\mathscr{A}$ is a bounded surjective linear operator on $L^2_h(\omega)$. Thus $\mathscr{A}^*$ is well defined on $L^2_h(\omega)$ and $\mathscr{A}^* T_\nu \mathscr{A} \in \mathcal{S}^p$, so that

$$\sum_{n=1}^{\infty} \big|\langle \mathscr{A}^* T_\nu \mathscr{A} e_n, e_n\rangle_{L^2(\omega)}\big|^p < +\infty.$$

On the other hand,

$$\begin{aligned}
\sum_{n=1}^{\infty} \big|\langle \mathscr{A}^* T_\nu \mathscr{A} e_n, e_n\rangle_{L^2(\omega)}\big|^p &= \sum_{n=1}^{\infty} \big|\langle T_\nu \mathscr{A} e_n, \mathscr{A} e_n\rangle_{L^2(\omega)}\big|^p \\
&= \sum_{n=1}^{\infty} \big|\langle T_\nu h_n, h_n\rangle_{L^2(\omega)}\big|^p \quad \text{(by the definition of } \mathscr{A}) \\
&= \sum_{n=1}^{\infty} \Big(\int_{\mathbb{D}} \big|h_n(z)\big|^2 \, d\nu(z)\Big)^p \\
&\geq \sum_{n=1}^{\infty} \Big(\int_{D(a_n,\epsilon)} \big|h_n(z)\big|^2 \, d\nu(z)\Big)^p.
\end{aligned}$$

Recalling that

$$\big|h_n(z)\big|^2 = \frac{(1-|a_n|)^4 |R_{a_n}(z)|^2}{|D(a_n,\epsilon)|_\omega},$$

we have by Lemma 3.2 that

$$\big|h_n(z)\big|^2 \geq \frac{1}{4|D(a_n,\epsilon)|_\omega}$$

for each $n \geq 1$ if $\rho(z, a_n) < \epsilon < r_0$. Therefore,

$$\sum_{n=1}^{\infty} \big|\langle \mathscr{A}^* T_\nu \mathscr{A} e_n, e_n\rangle_{L^2(\omega)}\big|^p \geq 4^{-p} \sum_{n=1}^{\infty} \Big(\frac{\nu(D(a_n,\epsilon))}{|D(a_n,\epsilon)|_\omega}\Big)^p.$$

This completes the proof of Theorem 3.4. □

4. Invertibility of Toeplitz operators on $L^2_h(\omega_\alpha)$

A fundamental and interesting problem is to determine when a Toeplitz operator is invertible on the Hardy or Bergman space (see [7]). In this section, we study the invertibility problem of Toeplitz operators on the standard weighted harmonic Bergman space $L^2_h(\omega_\alpha)$ with $\omega_\alpha = (1+\alpha)(1-|z|^2)^\alpha, \alpha > -1$. Recall that the reproducing kernel for $L^2_h(\omega_\alpha)$ is given by

$$R^\alpha_z(\lambda) = K^\alpha_z(\lambda) + \overline{K^\alpha_z(\lambda)} - 1 \quad (z, \lambda \in \mathbb{D}),$$

where

$$K_z^\alpha(\lambda) = \frac{1}{(1-\overline{z}\lambda)^{2+\alpha}}$$

is the reproducing kernel for the weighted Bergman space $L_a^2(\omega_\alpha)$ (see [22] if needed).

For the (unweighted) Bergman space setting, the second author and Zheng provided a necessary and sufficient condition for the Toeplitz operators with nonnegative symbols to be invertible on L_a^2 (see [24]). The main tool used in [24] is Luecking's result in [12, Main Theorem] on the reverse Carleson measure for the Bergman space, which also holds for the harmonic Bergman space. More precisely, Luecking established the following results.

Lemma 4.1 ([13, Theorem 1])**.** *Suppose that G is a measurable subset of $\mathbb{D}$. Then the following are equivalent.*

(i) There exists a $\delta \in (0,1)$ such that

$$|G \cap K| \geq \delta |\mathbb{D} \cap K|$$

for every ball K whose center lies on $\partial\mathbb{D}$.

(ii) $\chi_G\, dA$ is a reverse Carleson measure for $L_h^2(\omega_\alpha)$. That is, there is a constant $C > 0$ such that

$$\int_{\mathbb{D}} \left|f(z)\right|^2 \omega_\alpha(z)\, dA(z) \leq C \int_G \left|f(z)\right|^2 \omega_\alpha(z)\, dA(z)$$

for all $f \in L_h^2(\omega_\alpha)$.

Motivated by the ideas and techniques used in [24], we are able to characterize the invertibility of the Toeplitz operator T_φ ($\varphi \geq 0$) on $L_h^2(\omega_\alpha)$ in terms of the reverse Carleson measure and the Berezin transform.

Theorem 4.2. *Let $\varphi \geq 0$ be in $L^\infty(\omega_\alpha)$. The following conditions are equivalent.*

(1) *The Toeplitz operator T_φ is invertible on $L_h^2(\omega_\alpha)$.*

(2) *The Berezin transform $\widetilde{\varphi}$ is invertible in $L^\infty(\omega_\alpha)$, where*

$$\widetilde{\varphi}(z) := \frac{1}{\|R_z^\alpha\|^2_{L^2(\omega_\alpha)}} \int_{\mathbb{D}} \varphi(\lambda)\left|R_\alpha(z,\lambda)\right|^2 \omega_\alpha(\lambda)\, dA(\lambda)$$

and

$$\|R_z^\alpha\|^2_{L^2(\omega_\alpha)} = R_z^\alpha(z) = \frac{2}{(1-|z|^2)^{2+\alpha}} - 1.$$

(3) *There exists $r > 0$ such that $\chi_G\, dA$ is a reverse Carleson measure for $L_h^2(\omega_\alpha)$, where $G := \{z \in \mathbb{D} : \varphi(z) > r\}$.*

(4) *There exists a constant $C > 0$ such that*

$$\int_{\mathbb{D}} \left|f(z)\right|^2 \varphi(z)\omega_\alpha(z)\, dA(z) \geq C \int_{\mathbb{D}} \left|f(z)\right|^2 \omega_\alpha(z)\, dA(z)$$

for $f \in L_h^2(\omega_\alpha)$.

Before giving the proof of the main theorem of this section, we need another lemma, which was proved in [13, Lemma 3], [17, Theorem 9], and [23, Theorem C].

Lemma 4.3. *Suppose that the ball K has radius $0 < t < 1$ and center $u = (1,0) \in \mathbb{R}^2$. Let f be the harmonic function*

$$f(\lambda) = f_s(\lambda) := \sqrt{1+\alpha} R^{\alpha}_{z_0}(\lambda)\big(1-|z_0|^2\big)^{\frac{2+\alpha}{2}},$$

where $z_0 = (1-st)u$, $0 < s < 1$. Then for each $\epsilon > 0$, there exist $s = s(\epsilon)$ and a positive constant $C = C(\epsilon)$ (independent of K) such that

$$\int_{B\setminus K} \big|f(\lambda)\big|^2 \big(1-|\lambda|\big)^{\alpha}\, dA(\lambda) < \epsilon$$

and

$$\int_{G\cap K} \big|f(\lambda)\big|^2 \big(1-|\lambda|\big)^{\alpha}\, dA(\lambda) \le C\Big(\frac{|G\cap K|}{|\mathbb{D}\cap K|}\Big)^{\beta},$$

where

$$\beta = \begin{cases} 1 & \text{if } 0 \le \alpha < \infty, \\ 1-\frac{1}{\gamma} & \text{if } -1 < \alpha < 0, \end{cases}$$

and γ is a number in $(1, -\frac{1}{\alpha})$ if $-1 < \alpha < 0$.

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. We will show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). Without loss of generality, we may assume that $0 \le \varphi \le 1$.

(1) $\Rightarrow$ (2): This is trivial.

(2) $\Rightarrow$ (3): Suppose that $\widetilde{\varphi}$ is bounded below by some positive constant δ. By Lemma 4.1, it is sufficient to show that there exists a $\delta' \in (0,1)$ such that

$$|G\cap K| \ge \delta' |\mathbb{D}\cap K|$$

for all balls K whose centers lie on $\partial\mathbb{D}$.

Since $\omega_\alpha\, dA$ is a rotation-invariant measure, we may assume without loss of generality that K has its center at the point $(1,0)$. It is also clear that we only need to prove the inequality for a sufficiently small radius t, say, $t < 1$. Now we consider the subset

$$G = \Big\{\lambda \in \mathbb{D} : \varphi(\lambda) > \frac{\delta}{4}\Big\}.$$

For each $z \in \mathbb{D}$,

$$\begin{aligned} \delta \le \widetilde{\varphi}(z) &= \frac{1}{\|R^{\alpha}_z\|^2_{L^2(\omega_\alpha)}} \int_{\mathbb{D}} \varphi(\lambda)\big|R^{\alpha}_z(\lambda)\big|^2 \omega_\alpha(\lambda)\, dA(\lambda) \\ &= \frac{1}{\|R^{\alpha}_z\|^2_{L^2(\omega_\alpha)}} \Big(\int_G + \int_{\mathbb{D}\setminus G}\Big) \varphi(\lambda)\big|R^{\alpha}_z(\lambda)\big|^2 \omega_\alpha(\lambda)\, dA(\lambda) \\ &\le \big(1-|z|^2\big)^{2+\alpha} \int_G \varphi(\lambda)\big|R^{\alpha}_z(\lambda)\big|^2 \omega_\alpha(\lambda)\, dA(\lambda) + \frac{\delta}{4}. \end{aligned}$$

Let L_z be the following integral:

$$L_z = \frac{1}{\|R^{\alpha}_z\|^2_{L^2(\omega_\alpha)}} \int_{G\cap K} \varphi(\lambda)\big|R^{\alpha}_z(\lambda)\big|^2 \omega_\alpha(\lambda)\, dA(\lambda).$$

Then for each $z \in \mathbb{D}$, we have

$$\begin{aligned}
L_z &= \frac{1}{\|R_z^\alpha\|^2_{L^2(\omega_\alpha)}}\Big(\int_G - \int_{G\setminus K}\Big)\varphi(\lambda)\big|R_z^\alpha(\lambda)\big|^2\omega_\alpha(\lambda)\,dA(\lambda)\\
&\geq \frac{1}{2}\big(1-|z|^2\big)^{2+\alpha}\Big(\int_G - \int_{G\setminus K}\Big)\varphi(\lambda)\big|R_z^\alpha(\lambda)\big|^2\omega_\alpha(\lambda)\,dA(\lambda)\\
&\geq \frac{3\delta}{8} - \frac{1}{2}\big(1-|z|^2\big)^{2+\alpha}\int_{G\setminus K}\varphi(\lambda)\big|R_z^\alpha(\lambda)\big|^2\omega_\alpha(\lambda)\,dA(\lambda)\\
&\geq \frac{3\delta}{8} - \frac{1}{2}\big(1-|z|^2\big)^{2+\alpha}\int_{G\setminus K}\big|R_z^\alpha(\lambda)\big|^2\omega_\alpha(\lambda)\,dA(\lambda) \quad (\text{using } 0 \leq \varphi \leq 1).
\end{aligned}$$

For the δ above, Lemma 4.3 guarantees that we can select $z_0 \in \mathbb{D}$ to define a function f (as the one in Lemma 4.3), which satisfies the following three inequalities:

$$\begin{aligned}
L_{z_0} &\geq \frac{3\delta}{8} - \frac{1}{2}\big(1-|z_0|^2\big)^{2+\alpha}\int_{G\setminus K}\big|R_{z_0}^\alpha(\lambda)\big|^2\omega_\alpha(\lambda)\,dA(\lambda)\\
&= \frac{3\delta}{8} - \frac{1}{2}\int_{G\setminus K}\big|f(\lambda)\big|^2\big(1-|\lambda|^2\big)^\alpha\,dA(\lambda),
\end{aligned}$$

$$\int_{G\setminus K}\big|f(\lambda)\big|^2\big(1-|\lambda|^2\big)^\alpha\,dA(\lambda) < \frac{\delta}{4},$$

and

$$\int_{G\cap K}\big|f(\lambda)\big|^2\big(1-|\lambda|\big)^\alpha\,dA(\lambda) \leq C\Big(\frac{|G\cap K|}{|\mathbb{D}\cap K|}\Big)^\beta,$$

where the constant C depends only on δ and α. Therefore,

$$\begin{aligned}
\frac{\delta}{4} \leq L_{z_0} &\leq \big(1-|z_0|^2\big)^{2+\alpha}\int_{K\cap G}\big|R_{z_0}^\alpha(\lambda)\big|^2\omega_\alpha(\lambda)\,dA(\lambda)\\
&= \int_{G\cap K}\big|f(\lambda)\big|^2\big(1-|\lambda|^2\big)^\alpha\,dA(\lambda)\\
&\leq C\Big(\frac{|G\cap K|}{|\mathbb{D}\cap K|}\Big)^\beta.
\end{aligned}$$

Now we get (3) by Lemma 4.1.

(3) $\Rightarrow$ (4): Observe that

$$\int_{\mathbb{D}}|f|^2\varphi\omega_\alpha\,dA > r\int_G|f|^2\omega_\alpha\,dA \geq \frac{r}{C}\int_{\mathbb{D}}|f|^2\omega_\alpha\,dA,$$

where the last inequality follows from the definition of the reverse Carleson measure.

(4) $\Rightarrow$ (1): Using the same arguments as in the proof of Corollary 3 in [12], we obtain that $\|I - T_\varphi\| < 1$, which implies that T_φ is invertible on $L_h^2(\omega_\alpha)$. This completes the whole proof of Theorem 4.2. $\square$

Let $\mathbb{T}_\varphi$ denote the Toeplitz operator with symbol φ on the weighted Bergman space $L_a^2(\omega_\alpha)$. Combining the main result in [12] and the techniques used in the proof of Theorem 4.2, we can generalize Theorem 3.2 in [24] to the case of standard weighted Bergman spaces.

Theorem 4.4. *Let $\varphi \geq 0$ be in $L^\infty(\omega_\alpha)$. Then the following are equivalent.*

(1) *The Toeplitz operator $\mathbb{T}_\varphi$ is invertible on $L_a^2(\omega_\alpha)$.*

(2) *The Berezin transform $\widehat{\varphi}$ is invertible in $L^\infty(\omega_\alpha)$, where*

$$\widehat{\varphi}(z) := \frac{1}{\|K_z^\alpha\|_{L^2(\omega_\alpha)}^2} \int_{\mathbb{D}} \varphi(\lambda) \left|K_z^\alpha(\lambda)\right|^2 \omega_\alpha(\lambda)\, dA(\lambda).$$

(3) *There exists $r > 0$ such that $\chi_G\, dA$ is a reverse Carleson measure for $L_a^2(\omega_\alpha)$, where $G := \{z \in \mathbb{D} : \varphi(z) > r\}$.*

(4) *There exists a $\delta \in (0,1)$ such that*

$$|G \cap K| \geq \delta |\mathbb{D} \cap K|$$

for every ball K whose center lies on $\partial\mathbb{D}$.

(5) *There exists a constant $C > 0$ such that*

$$\int_{\mathbb{D}} \left|f(z)\right|^2 \varphi(z) \omega_\alpha(z)\, dA(z) \geq C \int_{\mathbb{D}} \left|f(z)\right|^2 \omega_\alpha(z)\, dA(z)$$

for $f \in L_a^2(\omega_\alpha)$.

Proof. From the proof of Theorem 4.2, it is sufficient to show that one can replace the harmonic function f defined in Lemma 4.3 by a suitable analytic function g. Indeed, we construct the desired function g as follows. Suppose that K has radius $0 < t < 1$ and center $u = (1,0) \in \mathbb{R}^2$. Define

$$g(\lambda) = \sqrt{\alpha + 1} K_{z_0}^\alpha(\lambda) \left(1 - |z_0|^2\right)^{\frac{2+\alpha}{2}},$$

where $z_0 = (1 - st)u$, $0 < s < 1$. Then it is not hard to check that both inequalities in Lemma 4.3 hold for g. Now the rest of the proof follows exactly that of Theorem 4.2. $\square$

Note that $\mathbb{T}_\varphi \geq 0$ on $L_a^2(\omega_\alpha)$ does not imply that $\varphi \geq 0$. In view of this fact and the above theorem, it is natural to ask the following question.

Question 4.5. Is $\mathbb{T}_\varphi$ invertible on the weighted Bergman space $L_a^2(\omega_\alpha)$ if $\mathbb{T}_\varphi \geq 0$ and the Berezin transform $\widehat{\varphi}$ is invertible in $L^\infty(\omega_\alpha)$?

To study the above question, we only need to consider the case of $\alpha = 0$ in $L_a^2(\omega_\alpha)$, that is, the classical Bergman space L_a^2. We give a negative answer to Question 4.5 by the following proposition.

Proposition 4.6. *Let $\varphi(z) = |z|^2 + a|z| + b$ ($z \in \mathbb{D}$). There exist a and b such that the Bergman Toeplitz operator $\mathbb{T}_\varphi \geq 0$ and the Berezin transform $\widehat{\varphi}$ is invertible in $L^\infty(\mathbb{D})$, but $\mathbb{T}_\varphi$ is not invertible on L_a^2.*

Proof. Let $a = -1.51$ and $b = \frac{381}{700}$. By Proposition 3.4 in [24], the eigenvalues of $\mathbb{T}_\varphi$ are given by

$$\lambda_n = \frac{2n+2}{2n+4} + a\frac{2n+2}{2n+3} + b \quad (n \geq 0).$$

Then it is easy to check that $\lambda_2 = 0$, and so $\mathbb{T}_\varphi$ is not invertible on L_a^2. Moreover, one can show that $\mathbb{T}_\varphi$ is positive since

$$\min_{n\geq 0} \lambda_n = \lambda_2 = 0.$$

It remains only to prove that $\widehat{\varphi}$ is bounded below. Using the calculations in Proposition 3.4 of [24] again, we obtain

$$\begin{aligned} \widehat{\varphi}(z) &= \Big[2 - \frac{1}{x^2} - \frac{(1-x^2)^2}{x^4}\log(1-x^2)\Big] \\ &\quad - \frac{151}{200}\Big[3 - \frac{1}{x^2} + \frac{(1-x^2)^2}{2x^3}\log\frac{1+x}{1-x}\Big] + \frac{381}{700} \\ &:= F(x) \quad \big(x = |z| \in [0,1]\big). \end{aligned}$$

Now one can show

$$\inf_{z\in\mathbb{D}} \widehat{\varphi}(z) \geq \min_{x\in[0,1]} F(x) \geq \frac{9}{1000} > 0$$

by adopting some techniques from the proof of Proposition 3.4 in [24]. However, this inequality implies that $\widehat{\varphi}$ is invertible in $L^\infty(\mathbb{D})$. $\square$

Based on Theorem 4.4, we can establish a relationship of the invertibility between Toeplitz operators (with nonnegative symbols) on $L_a^2(\omega_\alpha)$ and $L_h^2(\omega_\alpha)$.

Corollary 4.7. *Let φ be a nonnegative bounded measurable function on $\mathbb{D}$. The following four conditions are equivalent:*
(1) *$\mathbb{T}_\varphi$ is invertible on $L_a^2(\omega_\alpha)$;*
(2) *T_φ is invertible on $L_h^2(\omega_\alpha)$;*
(3) *$\widehat{\varphi}$ is invertible in $L^\infty(\omega_\alpha)$;*
(4) *$\widetilde{\varphi}$ is invertible in $L^\infty(\omega_\alpha)$.*

5. A reverse Carleson inequality for $L_h^2(\omega)$

In the previous section, we studied the invertibility problem of Toeplitz operators via reverse Carleson measures for the standard weighted harmonic Bergman spaces. In this section, we establish a sufficient condition for $\chi_G\, dA$ to be a reverse Carleson measure for $L_h^2(\omega)$, where $\omega \in \mathcal{A}_2$ and G is a measurable subset in $\mathbb{D}$.

For $a \in \mathbb{D}$, $0 < r < 1$, recall that

$$S(a,r) = \big\{z \in \mathbb{D} : |z-a| < r\big(1-|a|\big)\big\}.$$

The main result in this section is Theorem 5.1, which is a harmonic version of Theorem 3.9 in [15].

Theorem 5.1. *Suppose that $G \subset \mathbb{D}$ and that ω satisfies the $\mathcal{A}_2$ condition. If there exist $\delta \in (0,1)$ and $r \in (0,1)$ such that for all $a \in \mathbb{D}$,*

$$\big|G \cap S(a,r)\big| \geq \delta\big|S(a,r)\big|,$$

then there exists a positive constant $C = C(r,\delta)$ such that

$$\int_{\mathbb{D}} |f(z)|^2\omega(z)\,dA(z) \leq C\int_G |f(z)|^2\omega(z)\,dA(z)$$

for all $f \in L^2_h(\omega)$.

To prove the above theorem, we will adopt some ideas and techniques from [15]. First, we need to introduce a new (weight) function ω^* and discuss some properties of ω^*. In the rest of this section, we use r and δ to denote the numbers provided in Theorem 5.1.

Now we define a positive function ω^* on the open unit disk as follows:

$$\omega^*(z) = \omega^*_r(z) := \frac{|S(z,r)|_\omega}{|S(z,r)|}.$$

It is clear that $\omega^* \in L^1(dA)$, and so ω^* is a weight. Moreover, ω^* has the following important property.

Lemma 5.2. *Let $z \in \mathbb{D}$. Then there exist constants C_1 and C_2 depending only on r such that*

$$C_1\omega^*(a) \leq \omega^*(z) \leq C_2\omega^*(a)$$

for all $a \in S(z,r)$. Consequently, we have

$$\int_{\mathbb{D}} \frac{\omega^*(a)}{|S(a,r)|}\chi_{S(a,r)}(z)\,dA(a) \leq C_3\omega^*(z) \quad (z \in \mathbb{D}),$$

where $C_3 = C_3(r)$ is a constant.

Proof. By Lemma 2.2 in [6], there exists a positive constant C depending only on r such that

$$C^{-1}\big|S(a,r)\big|_\omega \leq \big|S(z,r)\big|_\omega \leq C\big|S(a,r)\big|_\omega.$$

Moreover, it is well known that $|S(z,r)|$ is equivalent to $|S(a,r)|$ (with constants independent of a and z) if $a \in S(z,r)$. This gives the first conclusion of the lemma. Based on this result, we have

$$\begin{aligned}
\int_{\mathbb{D}} \frac{\omega^*(a)}{|S(a,r)|}\chi_{S(a,r)}(z)\,dA(a) &\leq C\int_{\mathbb{D}} \frac{\omega^*(z)}{|S(a,r)|}\chi_{S(a,r)}(z)\,dA(a) \quad \big(\text{using } z \in S(a,r)\big)\\
&\leq C\int_{\mathbb{D}} \frac{\omega^*(z)}{|S(a,r)|}\chi_{D(a,r)}(z)\,dA(a)\\
&= C\int_{\mathbb{D}} \frac{\omega^*(z)}{|S(a,r)|}\chi_{D(z,r)}(a)\,dA(a)\\
&= C\int_{D(z,r)} \frac{\omega^*(z)}{|S(a,r)|}\,dA(a)\\
&\leq C_3\omega^*(z),
\end{aligned}$$

where the second inequality follows from $S(a,r)\subset D(a,r)$, and C_3 depends only on r, as desired. $\square$

Another property of ω^* is given by the following inequality, which will be used to estimate the integral of $|f|^2\omega$ over the subset G.

Lemma 5.3. *Let ω be an $\mathcal{A}_2$ weight. Then there exists a constant $C>0$ depending only on r such that*

$$\|f\|^2_{L^2(\omega)}\le C\|f\|^2_{L^2(\omega^*)}$$

for all $f\in L^2_h(\omega)$.

Proof. Using the definition of ω^*, we have

$$\begin{aligned}\|f\|^2_{L^2(\omega^*)}&=\int_{\mathbb{D}}\big|f(z)\big|^2\omega^*(z)\,dA(z)\\&=\int_{\mathbb{D}}\omega(\xi)\Big(\int_{\mathbb{D}}\big|f(z)\big|^2\frac{\chi_{S(z,r)}(\xi)}{|S(z,r)|}\,dA(z)\Big)\,dA(\xi).\end{aligned}$$

We next deal with the bracketed expression. Observing that

$$S\Big(\xi,\frac{r}{2(1+r)}\Big)\subset\big\{z\in\mathbb{D}:|z-\xi|<r\big(1-|z|\big)\big\},$$

we obtain

$$\begin{aligned}\int_{\mathbb{D}}\big|f(z)\big|^2\frac{\chi_{S(z,r)}(\xi)}{|S(z,r)|}\,dA(z)&=\int_{\{z\in\mathbb{D}:|z-\xi|<r(1-|z|)\}}\frac{|f(z)|^2}{|S(z,r)|}\,dA(z)\\&\ge\int_{S(\xi,\frac{r}{2(1+r)})}\frac{|f(z)|^2}{|S(z,r)|}\,dA(z)\\&\ge\frac{C}{|S(\xi,r)|}\int_{S(\xi,\frac{r}{2(1+r)})}\big|f(z)\big|^2\,dA(z)\\&=C\Big[\frac{1}{|S(\xi,\frac{r}{2(1+r)})|}\int_{S(\xi,\frac{r}{2(1+r)})}|f|^2\,dA\Big]\cdot\frac{|S(\xi,\frac{r}{2(1+r)})|}{|S(\xi,r)|}\\&\ge\frac{C}{16}\big|f(\xi)\big|^2,\end{aligned}$$

where the second inequality follows from $z\in S(\xi,\frac{r}{2(1+r)})$ and the last inequality follows form the subharmonicity of $|f|^2$ (see Lemma 2.1). Thus we get

$$\|f\|^2_{L^2(\omega^*)}\ge\frac{C}{16}\int_{\mathbb{D}}\big|f(\xi)\big|^2\omega(\xi)\,dA(\xi)=\frac{C}{16}\|f\|^2_{L^2(\omega)}$$

to complete the proof of Lemma 5.3. $\square$

In order to finish the proof of the main theorem in this section, the following two key lemmas are also needed.

Proposition 5.4. *Let G be the subset which satisfies the assumption in Theorem 5.1. For $\eta \in (0,1)$, we define a subset F as the following:*

$$F := \big\{z \in \mathbb{D} : \omega(z) \geq \eta\omega^*(z)\big\}.$$

Then one can choose η (depending only on δ and r) sufficiently small such that

$$\big|F \cap S(a,r)\big| \geq \Big(1 - \frac{\delta}{2}\Big)\big|S(a,r)\big|$$

and

$$\big|G \cap S(a,r) \cap F\big| \geq \frac{\delta}{2}\big|S(a,r)\big|$$

for all $a \in \mathbb{D}$.

Proof. First, we claim that for each $\delta' \in (0,1)$, there exists $\eta' = \eta'(\delta') > 0$ such that

$$\big|\big\{z \in S(a,r) : \omega(z) < \eta'\omega^*(a)\big\}\big| < \delta'\big|S(a,r)\big|$$

for all $a \in \mathbb{D}$.

Indeed, for each $\kappa \in (0,1)$ and $a \in \mathbb{D}$, we have

$$\begin{aligned}
&\big|\big\{z \in S(a,r) : \omega(z) < \kappa\omega^*(a)\big\}\big| \cdot \frac{1}{\kappa\omega^*(a)} \\
&\quad< \int_{\{z\in S(a,r):\omega(z)<\kappa\omega^*(a)\}} \frac{1}{\omega(z)}\, dA(z) \\
&\quad\leq \big|S(a,r)\big|_{\omega^{-1}} \leq [\omega]_{\mathcal{A}_2}\big|S(a,r)\big|^2 \cdot \big|S(a,r)\big|_{\omega}^{-1},
\end{aligned}$$

to obtain

$$\big|\big\{z \in S(a,r) : \omega(z) < \kappa\omega^*(a)\big\}\big| < \big([\omega]_{\mathcal{A}_2}\kappa\big)\big|S(a,r)\big|$$

for all $a \in \mathbb{D}$ and $\kappa \in (0,1)$. By this inequality, we can choose any $0 < \eta' \leq \frac{\delta'}{[\omega]_{\mathcal{A}_2}}$ to finish the proof of the claim.

Lemma 5.2 guarantees that there is a constant $C = C(r)$ such that

$$\big\{z \in S(a,r) : \omega(z) < C\tau\omega^*(z)\big\} \subset \big\{z \in S(a,r) : \omega(z) < \tau\omega^*(a)\big\} \quad (a \in \mathbb{D})$$

for every $\tau \in (0,1)$. By the claim and its proof, there exists a $\tau = \tau(\delta) < \frac{1}{C}$ such that

$$\big|\big\{z \in S(a,r) : \omega(z) < \tau\omega^*(a)\big\}\big| < \frac{\delta}{2}\big|S(a,r)\big| \quad (a \in \mathbb{D}).$$

Therefore, we can take $\eta = \eta(\delta, r) = C\tau < 1$ such that

$$\big|\big\{z \in S(a,r) : \omega(z) < \eta\omega^*(z)\big\}\big| < \frac{\delta}{2}\big|S(a,r)\big| \quad (a \in \mathbb{D}).$$

We define the subset F by the η chosen above, so that

$$\big|F \cap S(a,r)\big| = \big|\big\{z \in S(a,r) : \omega(z) \geq \eta\omega^*(z)\big\}\big| \geq \Big(1 - \frac{\delta}{2}\Big)\big|S(a,r)\big|$$

for all $a \in \mathbb{D}$.

By the assumption

$$\big|G \cap S(a,r)\big| \geq \delta\big|S(a,r)\big|,$$

we have

$$\begin{aligned}\delta|S(a,r)| &\le \big|G\cap S(a,r)\big|\\ &= \big|\big[G\cap S(a,r)\cap F\big]\cup\big[G\cap S(a,r)\cap(\mathbb{D}\setminus F)\big]\big|\\ &\le \big|G\cap S(a,r)\cap F\big| + \big|S(a,r)\cap(\mathbb{D}\setminus F)\big|\\ &= \big|G\cap S(a,r)\cap F\big| + \big|S(a,r)\big| - \big|S(a,r)\cap F\big|\\ &\le \big|G\cap S(a,r)\cap F\big| + \big|S(a,r)\big| - \Big(1-\frac{\delta}{2}\Big)\big|S(a,r)\big|,\end{aligned}$$

to obtain

$$\big|G\cap S(a,r)\cap F\big| \ge \frac{\delta}{2}\big|S(a,r)\big|$$

for all $a\in\mathbb{D}$, as desired. □

Lemma 5.5. *If G_0 is a measurable subset of $\mathbb{D}$ that satisfies*

$$\big|G_0\cap S(a,r)\big| \ge \delta_0\big|S(a,r)\big| \quad (a\in\mathbb{D})$$

for some $\delta_0>0$, then there exists a constant $C=C(\delta_0,r)>0$ such that

$$\int_{\mathbb{D}}\big|f(z)\big|^2\omega^*(z)\,dA(z) \le C\int_{G_0}\big|f(z)\big|^2\omega^*(z)\,dA(z)$$

for all $f\in L^2_h(\omega^*)$.

Because the proof of the above lemma is long and requires a number of technical lemmas, we will prove it at the end of this section. With this lemma, we are going to prove Theorem 5.1.

Proof of Theorem 5.1. By Proposition 5.4 and Lemma 5.5, we have

$$\int_{\mathbb{D}}\big|f(z)\big|^2\omega^*(z)\,dA(z) \le C_1\int_{G\cap F}\big|f(z)\big|^2\omega^*(z)\,dA(z) \le C_1\eta^{-1}\int_G|f|^2\omega\,dA$$

for all $f\in L^2_h(\omega^*)$, where C_1 is a constant depending only on r and where $\eta=\eta(\delta,r)<1$ is chosen by Proposition 5.4.

From Lemma 5.3, it is clear that

$$\int_{\mathbb{D}}\big|f(z)\big|^2\omega(z)\,dA(z) \le C_1\eta^{-1}\int_G\big|f(z)\big|^2\omega(z)\,dA(z)$$

for all $f\in L^2_h(\omega)$, which gives the desired inequality in Theorem 5.1. □

Now we turn to the proof of Lemma 5.5. Before giving the proof, we need to introduce some notation and prove three technical lemmas.

Let $0<\theta<\frac{1}{2}$. We define the subset

$$E_\theta(a)=E_\theta(f,a):=\big\{z\in S(a,r):\big|f(z)\big|>\theta\big|f(a)\big|\big\}$$

and the operator

$$B_\theta f(a):=\frac{1}{|E_\theta(a)|}\int_{E_\theta(a)}\big|f(z)\big|^2\,dA(z) \quad (a\in\mathbb{D}).$$

Clearly,

$$\frac{1}{|S(a,r)\backslash E_\theta(a)|}\int_{S(a,r)\backslash E_\theta(a)}|f|^2\,dA \le \theta^2\big|f(a)\big|^2 < \frac{1}{|E_\theta(a)|}\int_{E_\theta(a)}|f|^2\,dA,$$

and thus we have

$$B_\theta f(a) \ge \frac{1}{|S(a,r)|}\int_{S(a,r)}\big|f(z)\big|^2\,dA(z) \quad (a\in\mathbb{D}).$$

For $\epsilon\in(0,1)$, we consider the following two subsets, which will be very useful in establishing our main result. Define

$$A = A_\epsilon := \Big\{a\in\mathbb{D} : \big|f(a)\big|^2 \le \frac{\epsilon}{|S(a,r)|}\int_{S(a,r)}\big|f(z)\big|^2\,dA(z)\Big\}$$

and

$$B = B_\epsilon := \big\{a\in\mathbb{D} : \big|f(a)\big|^2 \le \epsilon^2 B_\theta f(a)\big\}.$$

The following inequality gives a useful estimate for the Lebesgue measure of the set $\{z\in S(a,r) : |f(z)| > \theta|f(a)|\}$, where f is a harmonic function.

Lemma 5.6. *Fix $\epsilon\in(0,1)$. For any $\delta'\in(0,1)$, there exists $\theta\in(0,\frac{1}{2})$ such that*

$$\big|\big\{z\in S(a,r) : \big|f(z)\big| > \theta\big|f(a)\big|\big\}\big| > \Big(1-\frac{\delta'}{2}\Big)\big|S(a,r)\big|$$

for every f harmonic on $\mathbb{D}$, satisfying

$$\big|f(a)\big|^2 > \frac{\epsilon^2}{|S(a,r)|}\int_{S(a,r)}\big|f(z)\big|^2\,dA(z) \quad (a\in\mathbb{D}).$$

Proof. See the proof of Lemma 2 in [13]. □

The next lemma provides an estimate for the integral of $|f|^2\omega^*$ over the set A, which can be proved easily by combining the definition of A and the second conclusion of Lemma 5.2.

Lemma 5.7. *Let $\epsilon\in(0,1)$. Then there exists a constant C (independent of ϵ) such that*

$$\int_A\big|f(z)\big|^2\omega^*(z)\,dA(z) \le C\epsilon\int_{\mathbb{D}}\big|f(z)\big|^2\omega^*(z)\,dA(z)$$

for all $f\in L^2_h(\omega^*)$.

The proof of Lemma 5.5 requires the following inequality.

Lemma 5.8. *Let $\epsilon\in(0,1)$. Then there exists a constant $C=C(r)$ such that*

$$\int_B\big|f(z)\big|^2\omega^*(z)\,dA(z) \le C\epsilon\int_{\mathbb{D}}\big|f(z)\big|^2\omega^*(z)\,dA(z)$$

for all $f\in L^2_h(\omega^*)$.

Proof. Observe that

$$\int_B \big|f(z)\big|^2 \omega^*(z)\,dA(z) = \int_{B\cap A} \big|f(z)\big|^2 \omega^*(z)\,dA(z) + \int_{B\setminus A} \big|f(z)\big|^2 \omega^*(z)\,dA(z)$$
$$\leq \int_A \big|f(z)\big|^2 \omega^*(z)\,dA(z) + \int_{B\setminus A} \big|f(z)\big|^2 \omega^*(z)\,dA(z).$$

Based on Lemma 5.7, it is sufficient to show that the following inequality holds for some constant $C = C(r)$:

$$J := \int_{B\setminus A} \big|f(z)\big|^2 \omega^*(z)\,dA(z) \leq C\epsilon \int_{\mathbb{D}} \big|f(z)\big|^2 \omega^*(z)\,dA(z).$$

Recall that for each $a \in B$, we have

$$\big|f(a)\big|^2 \leq \frac{\epsilon^2}{|E_\theta(a)|} \int_{E_\theta(a)} \big|f(z)\big|^2\,dA(z).$$

From the above inequality, we have

$$\begin{aligned} J &= \int_{B\setminus A} \big|f(a)\big|^2 \omega^*(a)\,dA(a) \\ &\leq \epsilon^2 \int_{\mathbb{D}} \Big(\int_{B\setminus A} \frac{\omega^*(a)}{|E_\theta(a)|} \chi_{E_\theta(a)}(z)\,dA(a) \Big) \big|f(z)\big|^2\,dA(z) \\ &\leq \epsilon^2 \int_{\mathbb{D}} \big|f(z)\big|^2 \Big(\int_{B\setminus A} \omega^*(a) \frac{\chi_{S(a,r)}(z)}{|E_\theta(a)|}\,dA(a) \Big)\,dA(z). \end{aligned}$$

The last inequality follows from the fact that $E_\theta(a) \subset S(a, r)$.

To finish the proof of this lemma, we need to verify the following claim.

Claim. There is a positive constant $C = C(r)$ such that

$$\big|E_\theta(a)\big| \geq C\epsilon \big|S(a,r)\big| \qquad \text{or} \qquad \big|E_\theta(a)\big| \geq C\big|S(a,r)\big|$$

for each $a \notin A$.

If the above claim is true, then we get

$$\int_{B\setminus A} \omega^*(a) \frac{\chi_{S(a,r)}(z)}{|E_\theta(a)|}\,dA(a) \leq C^{-1}\epsilon \int_{B\setminus A} \omega^*(a) \frac{\chi_{S(a,r)}(z)}{|S(a,r)|}\,dA(a).$$

Using Lemma 5.2 again, we have

$$\int_{B\setminus A} \omega^*(a) \frac{\chi_{S(a,r)}(z)}{|S(a,r)|}\,dA(a) \leq C_1 \omega^*(z),$$

where the constant C_1 depends only on r. From the definition of J, we obtain

$$J \leq C\epsilon \int_{\mathbb{D}} \big|f(z)\big|^2 \omega^*(z)\,dA(z)$$

for some positive constant $C = C(r)$.

Now we verify the claim. For each $a \notin A$, we have

$$\begin{aligned}\big|f(a)\big|^2 &> \frac{\epsilon}{|S(a,r)|}\int_{S(a,r)}\big|f(z)\big|^2\,dA(z)\\ &= \frac{\epsilon}{r^2(1-|a|)^2}\int_{S(a,r)}\big|f(z)\big|^2\,dA(z).\end{aligned}$$

Using a change of variables, $z = a + r(1-|a|)\lambda$, gives

$$\big|f(a)\big|^2 > \epsilon\int_{\mathbb{D}}\big|f\big(a + r\big(1-|a|\big)\lambda\big)\big|^2\,dA(\lambda).$$

Let $g(\lambda) = f(a + r(1-|a|)\lambda)$. Then g is also harmonic on $\mathbb{D}$ and

$$\big|g(0)\big|^2 > \epsilon\int_{\mathbb{D}}\big|g(\lambda)\big|^2\,dA(\lambda).$$

Applying Lemma 2.11 to g, we get a constant $C_0 = C_0(r)$ such that

$$\big|g(z) - g(0)\big| \le C_0|z|\int_{D(0,\frac{r}{4})}\big|g(\lambda)\big|\,dA(\lambda) \le C_0|z|\int_{\mathbb{D}}|g|\,dA$$

whenever $|z| \le \frac{r}{16}$. The Cauchy–Schwarz inequality gives that

$$\big|g(z) - g(0)\big| \le C_0|z|\Big(\int_{\mathbb{D}}\big|g(\lambda)\big|^2\,dA(\lambda)\Big)^{\frac{1}{2}} \le C_0\epsilon^{-\frac{1}{2}}\big|g(0)\big|\cdot|z|$$

provided that $|z| \le \frac{r}{16}$.

If

$$|z| < \min\Big\{\frac{r}{16}, \frac{\epsilon^{\frac{1}{2}}}{2C_0}\Big\},$$

then

$$\big|g(z)\big| \ge \big|g(0)\big| - \big|g(z) - g(0)\big| \ge \frac{|g(0)|}{2}.$$

Since $0 < \theta < \frac{1}{2}$, we have

$$\big|g(z)\big| > \theta\big|g(0)\big| \quad \text{for } |z| < \min\Big\{\frac{r}{16}, \frac{\epsilon^{\frac{1}{2}}}{2C_0}\Big\}.$$

This means that

$$B\Big(0, \frac{r}{16}\Big) \subset \big\{z \in \mathbb{D} : \big|g(z)\big| > \theta\big|g(0)\big|\big\}$$

or

$$B\Big(0, \frac{\epsilon^{\frac{1}{2}}}{2C_0}\Big) \subset \big\{z \in \mathbb{D} : \big|g(z)\big| > \theta\big|g(0)\big|\big\}.$$

On the other hand, observe that

$$\begin{aligned}
\big|E_\theta(a)\big| &= \int_{\{z\in S(a,r):|f(z)|>\theta|f(a)|\}} dA(z)\\
&= \int_{\{|\frac{z-a}{r(1-|a|)}|<1:|f(z)|>\theta|f(a)|\}} dA(z)\\
&= \big|S(a,r)\big| \int_{\{|\lambda|<1:|f(a+r(1-|a|)\lambda)|>\theta|f(a)|\}} dA(\lambda)\\
&= \big|S(a,r)\big| \int_{\{|\lambda|<1:|g(\lambda)|>\theta|g(0)|\}} dA(\lambda)\\
&= \big|S(a,r)\big| \cdot \big|\big\{\lambda\in\mathbb{D} : \big|g(\lambda)\big| > \theta\big|g(0)\big|\big\}\big|,
\end{aligned}$$

to obtain

$$\big|E_\theta(a)\big| \geq \Big(\frac{r}{16}\Big)^2 \big|S(a,r)\big|$$

or

$$\big|E_\theta(a)\big| \geq \frac{\epsilon}{4C_0^2}\big|S(a,r)\big|.$$

This gives the proof of the claim, and the proof of Lemma 5.8 is completed.

□

We are now ready to prove Lemma 5.5.

Proof of Lemma 5.5. Suppose that $|G_0 \cap S(a,r)| \geq \delta_0 |S(a,r)|$. From Lemmas 5.7 and 5.8, we can choose ϵ small enough so that

$$\int_{\mathbb{D}} \big|f(z)\big|^2 \omega^*(z)\, dA(z) < 2\int_{\mathbb{D}\setminus B} \big|f(z)\big|^2 \omega^*(z)\, dA(z).$$

On the other hand, if $a \notin B$, then

$$\big|f(a)\big|^2 > \epsilon^2 B_\theta f(a) \geq \frac{\epsilon^2}{|S(a,r)|}\int_{S(a,r)} \big|f(z)\big|^2\, dA(z).$$

For the δ_0 above, we apply Lemma 5.6 to choose a $\theta \in (0, \frac{1}{2})$ such that

$$\big|\big\{z \in S(a,r) : \big|f(z)\big| > \theta\big|f(a)\big|\big\}\big| > \Big(1 - \frac{\delta_0}{2}\Big)\big|S(a,r)\big|.$$

Since $|G_0 \cap S(a,r)| \geq \delta_0|S(a,r)|$, we have

$$\big|\big\{z \in S(a,r)\cap G_0 : \big|f(z)\big| > \theta\big|f(a)\big|\big\}\big| > \frac{\delta_0}{2}\big|S(a,r)\big|$$

to get

$$\frac{1}{|S(a,r)|}\int_{S(a,r)\cap G_0} \big|f(z)\big|^2\, dA(z) > \frac{\theta^2\delta_0}{2}\big|f(a)\big|^2 \quad (a \notin B).$$

Multiplying the above inequality by $\omega^*(a)$ and integrating over $\mathbb{D}\backslash B$ gives

$$\begin{aligned}\frac{\theta^2\delta_0}{2}\int_{\mathbb{D}\backslash B}\omega^*(a)\big|f(a)\big|^2\,dA(a) &< \int_{\mathbb{D}\backslash B}\frac{\omega^*(a)}{|S(a,r)|}\int_{S(a,r)\cap G_0}|f|^2\,dA\,dA(a)\\ &= \int_{G_0}|f|^2\Big(\int_{\mathbb{D}\backslash B}\frac{\omega^*(a)\chi_{S(a,r)}(z)}{|S(a,r)|}\,dA(a)\Big)\,dA\\ &\le C\int_{G_0}\big|f(z)\big|^2\omega^*(z)\,dA(z),\end{aligned}$$

where the last inequality follows from Lemma 5.2 and where C depends only on r.

Therefore,

$$\int_{G_0}\big|f(z)\big|^2\omega^*(z)\,dA(z) > \frac{\delta_0\theta^2}{4C}\int_{\mathbb{D}}\big|f(z)\big|^2\omega^*(z)\,dA(z)$$

for all $f\in L^2_h(\omega^*)$, to complete the proof of Lemma 5.5. □

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