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HÖRMANDER-TYPE THEOREMS ON UNIMODULAR MULTIPLIERS AND APPLICATIONS TO MODULATION SPACES

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Abstract. In this article, for the unimodular multipliers $e^{i\mu(D)}$, we establish two Hörmander-type multiplier theorems by assuming conditions on their phase functions μ. As applications, we obtain two multiplier theorems particularly fitting for the modulation spaces, thus allowing us to extend and improve some known results.

1. Introduction

Let T_m be a linear operator defined initially on the Schwartz space $S(\mathbb{R}^n)$ via the Fourier transform

$$\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi), \quad f \in S(\mathbb{R}^n).$$

The operator T_m is called the *Fourier multiplier with symbol m*. An interesting problem to investigate when studying T_m is its boundedness on certain function or distribution spaces. Among the many boundedness criteria found in the literature, two of the most famous are the *Mikhlin multiplier theorem* and the *Hörmander multiplier theorem*. The Mikhlin multiplier theorem says that if the symbol m satisfies the smoothness-decay condition

$$\left|\partial^\alpha m(\xi)\right| \le A_\alpha |\xi|^{-|\alpha|}$$

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for any $\xi \neq 0$ and all multi-indices α with $|\alpha| \leq [n/2]+1$, then the operator T_m is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. The Hörmander multiplier theorem, while its condition looks a little complicated, is a more general criterion. Let $\Psi \in S(\mathbb{R}^n)$ satisfy $\mathrm{supp}\Psi \subset \{\xi \in \mathbb{R}^n : 1 < |\xi| \leq 4\}$ and let $\sum_{j\in\mathbb{Z}} \Psi(\xi/2^j) = 1$ for all $\xi \in \mathbb{R}^n\backslash\{0\}$. For a fixed $s \in \mathbb{R}$, let $(I-\Delta)^{s/2}$ be the Fourier multiplier with symbol $(1+|\xi|^2)^{s/2}$. We denote H^s as the Hilbert space that consists of all $f \in S'$ such that

$$\|f\|_{H^s} = \left\|(I-\Delta)^{s/2} f\right\|_{L^2} < \infty.$$

The Hörmander multiplier theorem (see [14, p. 104]) then states that the operator T_m is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, provided that $m \in L^\infty(\mathbb{R}^n)$ satisfies

$$\sup_{j\in\mathbb{Z}} \left\|m(2^j\cdot)\Psi\right\|_{H^s} < \infty$$

for a fixed $s > n/2$.

Let B_ℓ be the ball centered at the origin with radius ℓ. With this notation, clearly the above condition can be replaced by

$$\sup_{j\in\mathbb{Z}} \left\|m(2^j\cdot)\right\|_{H^s(B_4\backslash B_1)} < \infty.$$

In this article, our attention will be primarily focused on the unimodular Fourier multipliers $e^{i\mu(D)}$ defined via the Fourier transform by

$$\widehat{e^{i\mu(D)}f}(\xi) = e^{i\mu(\xi)}\widehat{f}(\xi).$$

The family of unimodular Fourier multipliers is one of the most notable classes of Fourier multipliers. It arises naturally when studying various physical phenomena and Cauchy problems related to certain partial differential equations (PDEs). For instance, $e^{i|D|}$ is the fundamental operator of the wave equation, and $e^{i|D|^2}$ is the fundamental operator of the Schrödinger equation. However, it is known that, for any $\alpha > 0$, the unimodular Fourier multiplier $e^{i|D|^\alpha}$ is bounded neither on the Lebesgue spaces $L^p(\mathbb{R}^n)$ nor on the Besov spaces $B^s_{p,q}$ for $p \neq 2$, unless $\alpha = 1$ and $n = 1$. On the other hand, in recent years, people have discovered that the operator $e^{i|D|^\alpha}$ is bounded on the modulation spaces $M^s_{p,q}$ for all $1 \leq p,q \leq \infty$, and $s \in \mathbb{R}$ when $0 < \alpha \leq 2$. This newly found result makes spaces $M^s_{p,q}$ very attractive, since we might naturally expect them to play quite a different role from that of Lebesgue spaces and Besov spaces in solving some PDE problems.

Modulation spaces $M^s_{p,q}$ were initially introduced by Feichtinger [11] in 1983 by the short-time Fourier transform (see also [10]). His initial motivation was to use a space different from that of the L^p space to measure the smoothness of a function. Nowadays, these spaces play a significant role in the study of harmonic analysis, PDEs, and time-frequency analysis. In the following, we list a few of these results, among numerous papers. (For the continuity of some operators on modulation spaces, we refer the reader to [1], [6], [9], [16], and [21]–[24] and the references therein. For applications to PDEs in the framework of modulation spaces, see [2], [3], [5], [7], [13], [15], [17], [19], [20], and [26] and the references therein.) Similar to Besov spaces, which are defined based on the dyadic decomposition on the frequency domain (see [25]), the modulation spaces are defined via the decomposition on the frequency domain based on unit squares. To define the modulation spaces, we may use several methods. Below we briefly review a discrete version of the definition.

Take a Schwartz function φ supported in the cube $[-\frac{2}{3}, \frac{2}{3}]^n$ satisfying

$$\sum_{k\in Z^n} \varphi_k(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n,$$

where we denote

$$\varphi_k(\xi) = \varphi(\xi - k) \quad \text{for } k \in \mathbb{Z}^n.$$

For each $k \in \mathbb{Z}^n$, denote a local square projection $\square_k$ on the frequency space via the Fourier transform by

$$\widehat{\square_k f}(\xi) = \varphi_k(\xi)\widehat{f}(\xi).$$

For a triplet $(p, q, s) \in [1, \infty] \times (0, \infty] \times \mathbb{R}$, the modulation space $M^s_{p,q}(\mathbb{R}^n)$ is defined by

$$M^s_{p,q}(\mathbb{R}^n) = \Big\{ f \in S'(\mathbb{R}^n) : \|f\|_{M^s_{p,q}(\mathbb{R}^n)} = \Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\mathrm{sq}} \|\square_k f\|_p^q \Big)^{\frac{1}{q}} < \infty \Big\},$$

where $\langle k\rangle = (1+|k|^2)^{\frac{1}{2}}$ (see [26] for details). We also denote $M_{p,q}(\mathbb{R}^n) = M^0_{p,q}(\mathbb{R}^n)$.

We next recall some known results in the following theorems.

Theorem A ([1, Theorem 1])**.** *Let $1 \le p, q \le \infty$, and $s \in \mathbb{R}$. The Fourier multiplier $e^{i|D|^\alpha}$ is bounded from $M^s_{p,q}(\mathbb{R}^n)$ to $M^s_{p,q}(\mathbb{R}^n)$, provided $0 < \alpha \le 2$.*

Theorem B ([16, Theorem 1.1])**.** *Let $\alpha > 2$, and let μ be a real-valued function of class $C^{[n/2]+3}$ on $\mathbb{R}^n\backslash\{0\}$. Assume that μ satisfies that, for some $\varepsilon > 0$,*

$$\left|\partial^\gamma \mu(\xi)\right| \le A_\gamma |\xi|^{\varepsilon-|\gamma|}, \quad |\gamma| \le [n/2] + 1, \ \textit{when } 0 < |\xi| \le 1,$$

and that

$$\left|\partial^\gamma \mu(\xi)\right| \le A_\gamma |\xi|^{\alpha-2}, \quad 2 \le |\gamma| \le [n/2] + 3, \ \textit{for } |\xi| > 1.$$

Suppose that $1 \le p,\ q \le \infty$ and $s \ge (\alpha - 2)n|1/p - 1/2|$. Then the Fourier multiplier $e^{i\mu(D)}$ is bounded from $M^s_{p,q}(\mathbb{R}^n)$ to $M_{p,q}(\mathbb{R}^n)$.

By checking the proof of Theorem A, it is not difficult to see that the conclusion of the theorem can extend to symbols of the form $m(\xi) = e^{i\mu(\xi)}$, where the μ's are positively homogeneous functions of degree $\alpha \in [0, 2]$, smooth away from the origin. More precisely, the conclusions of Theorem A and Theorem B are essentially applicable to a class of unimodular multipliers $e^{i\mu(D)}$ whose phase functions are $\mu(\xi) = |\xi|^\alpha \Omega(\frac{\xi}{|\xi|})$, where the functions Ω are defined on the unit sphere and satisfy some smoothness conditions.

The aim of this article is twofold. We will establish a Hörmander multiplier theorem on $L^p(\mathbb{R}^n)$ for the unimodular multipliers $e^{i\mu(D)}$ by assuming the condition merely on the phase functions μ; and we will establish a Hörmander-type multiplier theorem for $e^{i\mu(D)}$ specifically working to the modulation spaces. In the following, we give a more detailed description of our plan.

We first study the L^p boundedness of $e^{i\mu(D)}$ by using only the assumption on $\|\mu(2^{-j}\cdot)\|_{H^L}$ (see (1.1)). It will be more convenient if one can establish some boundedness criteria for $e^{i\mu(D)}$ based directly on the smoothness and decay rate of the phase functions μ. This observation motivates us to formulate the following two theorems, by virtue of the special structure of unimodular multipliers.

Theorem 1. *Given $L > n/2$, assume that μ satisfies*

$$\sup_{j\in Z} \|\mu_j\|_{H^L(B_4\setminus B_1)} < \infty, \tag{1.1}$$

where $\mu_j(\xi) = \mu(2^{-j}\xi)$. Then $e^{i\mu(D)}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and

$$\|e^{i\mu(D)} f\|_{L^p(\mathbb{R}^n)} \preceq \sup_{j\in Z}\big(1 + \|\mu_j\|^{L+1}_{H^L(B_4\setminus B_1)}\big)\|f\|_{L^p(\mathbb{R}^n)}.$$

On the other hand, if μ satisfies the stronger condition

$$\sum_{j\in Z} \|\mu_j\|_{H^L(B_4\setminus B_1)} < \infty, \tag{1.2}$$

then $e^{i\mu(D)}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$, and

$$\|e^{i\mu(D)} f\|_{L^p(\mathbb{R}^n)} \le C\Big(1 + \sum_{j\in Z} \|\mu_j\|_{H^L(B_4\setminus B_1)}\Big)^{L+1} \|f\|_{L^p(\mathbb{R}^n)}.$$

Theorem 2. *For some $L > n/2$, if μ is supported on B_1 and satisfies*

$$\|\nabla^2\mu\|_{H^{L-2}(B_1)} < \infty, \tag{1.3}$$

then $e^{i\mu(D)}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$. More precisely, we have

$$\|e^{i\mu(D)} f\|_{L^p} \le C\big(1 + \big\|\nabla^2\mu\|_{H^{L-2}(B_1)}\big)^{L+1}\|f\big\|_{L^p}.$$

We will use the Hörmander multiplier theorem to prove Theorem 1 in the case $1 < p < \infty$. When $p = 1$, the Hörmander multiplier theorem is not applicable. The L^1 boundedness of an operator T_m is equivalent to $m \in \mathcal{F}B$ (the inverse Fourier transform of m is a bounded Borel measure). However, we will invoke Young's inequality to achieve the target. Thus, to obtain the L^1 boundedness of $e^{i\mu(D)}$, essentially we try to control the $\mathcal{F}L^1$-norm of $e^{i\mu} - 1$. Note that $\mathcal{F}L^1$ is a Banach algebra. It is possible to take another approach (see [12]). In fact, we may prove that for all $p \ge 1$,

$$\|e^{i\mu(D)} f\|_{L^p} \le \big(\|e^{i\mu} - 1\|_{\mathcal{F}L^1} + 1\big)\|f\|_{L^p} \le e^{c\|\mu\|_{\mathcal{F}L^1}}\|f\|_{L^p}$$

for some $c > 0$. Thus the condition $\|e^{i\mu}-1\|_{\mathcal{F}L^1} < \infty$ is weaker than the condition $(1+\sum_{j\in Z}\|\mu_j\|_{H^L(B_4\setminus B_1)})^{L+1} < \infty$ in Theorem 1. But our aim, for the convenience of application, is to establish sufficient conditions for $e^{i\mu(D)}$ to verify their boundedness on L^p by merely checking the regularity conditions of μ. Also, a bright point appears in Theorem 2. When μ is a compact supported function, we find that we can use the weaker condition $\|\nabla^2\mu\|_{H^{L-2}} < \infty$ instead of $\|\mu\|_{H^L} < \infty$ to obtain the L^p boundedness of $e^{i\mu(D)}$, where $L > n/2$ is optimal in the Hörmander multiplier theorem.

On the other hand, Theorem A and Theorem B are Mikhlin-type multiplier theorems on the modulation spaces. We observe that the phase functions in the theorems are essentially the function $\mu(\xi) = |\xi|^\beta \Omega(\frac{\xi}{|\xi|})$, where Ω satisfies certain smoothness conditions on the unit sphere. Therefore, it will be more interesting and handy if we establish some Hörmander-type theorems for unimodular Fourier multipliers on the modulation spaces to fit the special structure of $M^s_{p,q}(\mathbb{R}^n)$. This observation inspires us to establish the following two Hörmander-type theorems on $M^s_{p,q}(\mathbb{R}^n)$.

Theorem 3. *Fix any $L > n/2$. If the phase function μ satisfies*

(1) $\sup_{j\geq -n} \|\mu_j\|_{H^L(B_4\setminus B_1)} \leq A$, *and*
(2) $\sup_{|y|>n} \|\nabla^2\mu\|_{H^{L-2}(B(y,2\sqrt{n}))} \leq A$,

then $e^{i\mu(D)}$ is bounded on $M^s_{p,q}(\mathbb{R}^n)$ for all $(p,q,s) \in (1,\infty)\times(0,\infty)\times\mathbb{R}$, and

$$\|e^{i\mu(D)} f\|_{M^s_{p,q}(\mathbb{R}^n)} \leq C(1+A)^{L+1}\|f\|_{M^s_{p,q}(\mathbb{R}^n)}. \tag{1.4}$$

Additionally, if condition (1) is replaced by

$$\sum_{j\geq -n} \|\mu_j\|_{H^L(B_4\setminus B_1)} < A,$$

then (1.4) holds for all $1 \leq p \leq \infty$, $0 < q < \infty$, and $s \in \mathbb{R}$.

Theorem 4. *Let $N_0 = [n/2] + 1$. If*

(1) $\sup_{j\geq -n} \|\mu_j\|_{H^{N_0}(B_4\setminus B_1)} \leq C$, *and*
(2) $\|\nabla^2\mu\|_{H^{N_0-2}(B(y,2\sqrt{n}))} \leq C|y|^\delta$ *for any $|y| > n$, where $\delta > 0$,*

then $e^{i\mu(D)}$ is bounded from $M^{s+\delta n|\frac{1}{p}-\frac{1}{2}|}_{p,q}(\mathbb{R}^n)$ to $M^s_{p,q}(\mathbb{R}^n)$ when $1 < p < \infty, 0 < q < \infty$, and $s \in \mathbb{R}$. Precisely, we have

$$\|e^{i\mu(D)} f\|_{M^s_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{M^{s+\delta n|\frac{1}{p}-\frac{1}{2}|}_{p,q}(\mathbb{R}^n)}$$

for all $(p,q,s) \in (1,\infty)\times(0,\infty)\times\mathbb{R}$.

Furthermore, if condition (1) is replaced by

$$\sum_{j\geq -n} \|\mu_j\|_{H^{N_0}(B_4\setminus B_1)} \leq C,$$

then, in the conclusion, the range of p can be enlarged to $1 \leq p \leq \infty$.

Before proceeding, we offer the following comments as a way to clarify the comparison of our theorems to some known results.

As mentioned before, a Hörmander-type multiplier theorem is "better" than a Mikhlin-type multiplier theorem since a symbol satisfying the condition in the latter must satisfy the condition in the former, but it is not always true vice versa. Hence, Theorems 3 and 4 encompass a wider scope than that of Theorems A and B. Actually, if $n = 2, 3$ and $\beta > \frac{3}{2}$, then one can use Theorem 3 to check that the unimodular multiplier $e^{i\mu(D)}$, with $\mu(\xi) = |\sin|\xi||^\beta$, is bounded on $M^s_{p,q}(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$, $1 < q < \infty$, and $s \in \mathbb{R}$. But this conclusion cannot be derived from Theorem A or B.

Regarding the special case $\mu(\xi) = |\xi|^\beta \Omega(\frac{\xi}{|\xi|})$, the boundedness of $e^{i\mu(D)}$ on the spaces $M^s_{p,q}(\mathbb{R}^n)$ when $0 < \beta \le 2$ and $1 < p < \infty$ needs a smoothness condition $\Omega \in C^{[\frac{n}{2}]+3}(S^{n-1})$ in Theorems A and B. But in Theorem 3 we assume a weaker regularity condition $\Omega \in C^L(S^{n-1})$, $(L > n/2)$, to achieve the same conclusion. We also observe that Theorem 4 represents a substantial improvement over the main theorem in [16]. For the special case $\mu(\xi) = |\xi|^\beta \Omega(\frac{\xi}{|\xi|})$, where the Hessian determinant of μ is nonzero at some point ξ_0, the index $\delta n|\frac{1}{p} - \frac{1}{2}|$ is optimal.

Additionally, we compare our results to a recent paper by Zhao, Chen, Fan, and Guo [27], in which the authors also studied the boundedness of unimodular Fourier multipliers on the modulation spaces. However, to obtain the main results (see, e.g., [27, Theorems 4.8, 4.9, 4.10]), their proofs are based on obtaining estimates of lower and upper bounds for the operator norms of $e^{i\mu(D)}$. To this end, additional conditions, such as homogeneity or nondegeneracy, are assumed on the phase functions μ. These assumptions, however, are not needed in Theorems 3 and 4. Also, we assume less regularity on μ in our results.

Similar to the characterization of multipliers in Lebesgue spaces, the $M_{1,1}$ boundedness of T_m is equivalent to $m \in M_{1,\infty}$. We may directly consider the modulation space norm corresponding to $e^{i\mu(D)}$ (e.g., $\|\mathcal{F}^{-1}(e^{i\mu} - 1)\|_{M_{1,\infty}}$) to establish the $M_{1,1}$ boundedness of $e^{i\mu(D)}$. However, we will not pursue this approach for various technical reasons. More importantly, as we emphasized before, the aim of this article is to establish some sufficient boundedness criteria that can be easily verified by checking the regularity conditions merely on μ.

This article is organized as follows. In Section 2, we will prove some necessary lemmas. The proof of Theorem 1, in which we use the Hörmander multiplier theorem to obtain the criterion of L^p boundedness of $e^{i\mu(D)}$ for $1 < p < \infty$, will be presented in Section 3. The computation is rather technical when L is a noninteger in the process of executing the proof. When $p = 1$, we will decompose the multiplier into two multipliers so that Young's inequality can be applied. In Section 4, we will prove Theorem 2. Again, since the Hörmander multiplier theorem does not work on the L^1 boundedness, we will not only apply the Hörmander multiplier theorem but also invoke Young's inequality. In order to obtain a weaker regularity condition in Theorem 2, we will construct an auxiliary phase function $\mu_{\vec{A},b}(\xi)$ and use the elliptic estimate for the Neumann problem. Theorems 3 and 4 are deduced from Theorems 1 and 2. Their proofs will be presented in Section 5 and Section 6, respectively. Recently, there have been many investigations in the inverse direction (see, e.g., [8], [18]), where sharp continuity estimates in L^p are deduced from results in modulation spaces.

In this article, C always denotes a positive constant that is independent of all essential variables. We use the notation $A \simeq B$ to mean that there are two positive constants c_1 and c_2 independent of all essential variables such that $c_1 A \le B \le c_2 A$.

2. Some lemmas

For the Besov spaces $B^s_{p,q}$, it is well known that $H^s = B^s_{2,2}$. For any $(p,q,s) \in (1,\infty)\times[1,\infty)\times(0,1)$, we may use the following equivalent norm (see [26, pp. 17–18]) of $B^s_{p,q}$:

$$\|f\|_{\Lambda^s_{p,q}} \approx \|f\|_{B^s_{p,q}},$$

where

$$\|f\|_{\Lambda^s_{p,q}} = \|f\|_{L^p} + \Big(\int_{\mathbb{R}^n} |h|^{-\mathrm{sq}}\Big(\int_{\mathbb{R}^n}\big|f(x+h)-f(x)\big|^p\,dx\Big)^{\frac{q}{p}}\frac{dh}{|h|^n}\Big)^{\frac{1}{q}}.$$

The following lemma will play a pivotal role in the proof of our main theorems.

Lemma 5. *Let $\mu \in H^L$ be supported in the unit ball B_1. For any $L \in (n/2, n/2+1)$, there exists a constant C, which depends on the choice of L, such that*

$$\|e^{i\mu}-1\|_{H^L} \le C\big(\|\mu\|_{H^L} + \|\mu\|^{L+1}_{H^L}\big).$$

Proof. For a multi-index α with $|\alpha|>0$, a direct computation involving an induction argument yields that $\nabla^\alpha(e^{i\mu}-1)$ can be represented as

$$\nabla^\alpha(e^{i\mu}-1) = \sum_{\sum_{i=1}^n k_i|\alpha_i|=|\alpha|} C(k_1,\alpha_1,\dots,k_n,\alpha_n)\prod_{i=1}^n\Big(\frac{\partial^{\alpha_i}\mu}{\partial x^{\alpha_i}}\Big)^{k_i} e^{i\mu} = f_\alpha e^{i\mu}. \quad (2.1)$$

□

Here in the expression

$$f_\alpha = \sum_{\sum_{i=1}^n k_i|\alpha_i|=|\alpha|} C(k_1,\alpha_1,\dots,k_n,\alpha_n)\prod_{i=1}^n\Big(\frac{\partial^{\alpha_i}\mu}{\partial x^{\alpha_i}}\Big)^{k_i},$$

each α_i $(i=1,\dots,n)$ is a multi-index which may be equal to zero, and $C(k_1,\alpha_1,\dots,k_n,\alpha_n)$ are constants depending on $\alpha_1,\dots,\alpha_n$ and $k_1,\dots,k_n$. We formulate the required estimate for f_α as a separate lemma in the following.

Lemma 6. *Let L be a positive number, and let k be the integer satisfying $L-1<k\le L$. For any α with $|\alpha|\le k\le n/2$, there exists a constant C depending on $\alpha_1,\dots,\alpha_n$ and $k_1,\dots,k_n$ such that*

$$\|f_\alpha\|_{H^{L-k}} \le C\|\mu\|_{H^L}\big(\|\mu\|^{k-1}_{H^L}+1\big).$$

Proof. By the definition of f_α, a direct computation gives that

$$\begin{aligned}
&\big|f_\alpha(x)-f_\alpha(y)\big|\\
&= \Big|\sum_{\sum_{i=1}^n k_i|\alpha_i|=|\alpha|} C_{k_1,\alpha_1,\dots,k_n,\alpha_n}\Big(\prod_{i=1}^n\Big(\frac{\partial^{\alpha_i}\mu}{\partial x^{\alpha_i}}\Big)^{k_i}(x) - \prod_{i=1}^n\Big(\frac{\partial^{\alpha_i}\mu}{\partial x^{\alpha_i}}\Big)^{k_i}(y)\Big)\Big|\\
&\le C\sum_{\sum_{i=1}^n k_i|\alpha_i|=|\alpha|}\Big|\prod_{i=1}^n\Big(\frac{\partial^{\alpha_i}\mu}{\partial x^{\alpha_i}}\Big)^{k_i}(x) - \prod_{i=1}^n\Big(\frac{\partial^{\alpha_i}\mu}{\partial x^{\alpha_i}}\Big)^{k_i}(y)\Big|\\
&\le C\sum_{\sum_{i=1}^n k_i|\alpha_i|=|\alpha|}\ \sum_{j,k_j>0}\Big|\Big(\frac{\partial^{\alpha_j}\mu}{\partial x^{\alpha_j}}\Big)^{k_j}(x) - \Big(\frac{\partial^{\alpha_j}\mu}{\partial x^{\alpha_j}}\Big)^{k_j}(y)\Big|
\end{aligned}$$

$$
\begin{aligned}
&\times \prod_{i\neq j}^{n}\Big(\Big|\frac{\partial^{\alpha_i}\mu}{\partial x^{\alpha_i}}(x)\Big|^{k_i}+\Big|\frac{\partial^{\alpha_i}\mu}{\partial x^{\alpha_i}}(y)\Big|^{k_i}\Big)\\
&\leq C\sum_{\sum k_i|\alpha_i|=|\alpha|}\sum_{j,k_j>0}\Big|\frac{\partial^{\alpha_j}\mu}{\partial x^{\alpha_j}}(x)-\frac{\partial^{\alpha_j}\mu}{\partial x^{\alpha_j}}(y)\Big|\Big(\Big|\frac{\partial^{\alpha_j}\mu}{\partial x^{\alpha_j}}(x)\Big|+\Big|\frac{\partial^{\alpha_j}\mu}{\partial x^{\alpha_j}}(y)\Big|\Big)^{k_j-1}\\
&\times \prod_{i\neq j}^{n}\Big(\Big|\frac{\partial^{\alpha_i}\mu}{\partial x^{\alpha_i}}(x)\Big|+\Big|\frac{\partial^{\alpha_i}\mu}{\partial x^{\alpha_i}}(y)\Big|\Big)^{k_i}\\
&\leq C\sum_{\sum_{i=1}^n k_i|\alpha_i|=|\alpha|}\sum_{j,k_j>0}\big|\nabla^{\alpha_j}\mu(x)-\nabla^{\alpha_j}\mu(y)\big|\\
&\times \prod_{i=1}^{n}\big(\big|\nabla^{\alpha_i}\mu(x)\big|+\big|\nabla^{\alpha_i}\mu(y)\big|\big)^{k_i-\delta_j^i}.
\end{aligned}
$$

Take $\tilde{p}_j$, p_i such that

$$
\begin{aligned}
&\frac{1}{\tilde{p}_j}=\max\Big\{0,\frac{2}{n}\Big(|\alpha_j|+\frac{n}{2}-L\Big)\Big\},\\
&\frac{1}{p_i}=\begin{cases}0, & k_i\leq\delta_i^j \text{ or } |\alpha_i|+\frac{n}{2}\leq L,\\ \frac{2(k_i-\delta_i^j)}{n}(|\alpha_j|+\frac{n}{2}-L), & k_i>\delta_i^j \text{ and } |\alpha_i|+\frac{n}{2}>L.\end{cases}
\end{aligned}
$$

In the above notation, $\delta_i^j=1$ if $i=j$ and $\delta_i^j=0$ if $i\neq j$. It is easy to check that, when $k_j>0$ and $\sum_{i=1}^n k_i|\alpha_i|=|\alpha|$, we have

$$
\begin{aligned}
\frac{1}{\tilde{p}_j}+\sum_{i=1}^{n}\frac{1}{p_i}&=\max\Big\{0,\frac{2}{n}\Big(|\alpha_j|+\frac{n}{2}-L\Big)\Big\}\\
&\quad+\frac{2}{n}\sum_{k_i>\delta_i^j,|\alpha_i|+\frac{n}{2}>L}(k_i-\delta_i^j)\Big(|\alpha_i|+\frac{n}{2}-L\Big)\\
&=\frac{2}{n}\sum_{k_i>\delta_i^j,|\alpha_i|+\frac{n}{2}>L}k_i\Big(|\alpha_i|+\frac{n}{2}-L\Big)\\
&\leq\frac{2}{n}\sum_{i=1}^{n}k_i|\alpha_i|-\Big(\frac{2L}{n}-1\Big)\sum_{k_i>\delta_i^j,|\alpha_i|+\frac{n}{2}>L}k_i\\
&\leq\frac{2}{n}|\alpha|\leq 1.
\end{aligned}
$$

For any $h\in\mathbb{R}^n$, using the support condition $\operatorname{supp}\mu\subset B_1$ and Hölder's inequality, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\big|f_\alpha(x+h)-f_\alpha(x)\big|^2\,dx\\
&\quad\leq C\int_{\mathbb{R}^n}\sum_{\sum_{i=1}^n k_i|\alpha_i|=|\alpha|}\sum_{k_j>0}\big|\nabla^{\alpha_j}\mu(x+h)-\nabla^{\alpha_j}\mu(x)\big|^2
\end{aligned}
$$

$$
\begin{aligned}
&\times \prod_{i=1}^{n} \big(\big|\nabla^{\alpha_i}\mu(x+h)\big| + \big|\nabla^{\alpha_i}\mu(x)\big|\big)^{2(k_i-\delta_j^i)}\,dx \\
&\le C\int_{\mathbb{R}^n} \sum_{\sum_{i=1}^n k_i|\alpha_i|=|\alpha|}\ \sum_{k_j>0} \big|\nabla^{\alpha_j}\mu(x\pm h)-\nabla^{\alpha_j}\mu(x)\big|^2 \prod_{i=1}^{n}\big|\nabla^{\alpha_i}\mu(x)\big|^{2(k_i-\delta_j^i)}\,dx \\
&\le C \sum_{\sum_{i=1}^n k_i|\alpha_i|=|\alpha|}\ \sum_{k_j>0} \big\|\big|\nabla^{\alpha_j}\mu(\cdot\pm h)-\nabla^{\alpha_j}\mu(\cdot)\big|^2\big\|_{\tilde p_j} \prod_{i=1}^{n}\big\|\big|\nabla^{\alpha_i}\mu(\cdot)\big|^{2(k_i-\delta_j^i)}\big\|_{p_i} \\
&= C \sum_{\sum_{i=1}^n k_i|\alpha_i|=|\alpha|}\ \sum_{k_j>0} \big\|\nabla^{\alpha_j}\mu(\cdot\pm h)-\nabla^{\alpha_j}\mu(\cdot)\big\|_{2\tilde p_j}^2 \prod_{i=1}^{n}\|\nabla^{\alpha_i}\mu\|_{2(k_i-\delta_j^i)p_i}^{2(k_i-\delta_j^i)}. \qquad (2.2)
\end{aligned}
$$

By the Sobolev embedding theorem and the selection of p_i, we know that $\|\nabla^{\alpha_i}\mu\|_{2(k_i-\delta_j^i)p_i} \le C\|\mu\|_{H^L}$. So, by using an equivalent norm $\|\cdot\|_{\Lambda^s_{p,q}}$ of the Besov space H^{L-k} and the fact $L-k<1$, we get

$$
\begin{aligned}
&\|f_\alpha\|_{H^{L-k}} \\
&\simeq \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big|f_\alpha(x+h)-f_\alpha(x)\big|^2 |h|^{n+2(L-k)}\,dx\,dh\Big)^{\frac12} \\
&\le C\Big(\int_{\mathbb{R}^n} \sum_{\sum_{i=1}^n k_i|\alpha_i|=|\alpha|}\ \sum_{k_j>0} \big\|\nabla^{\alpha_j}\mu(\cdot\pm h)-\nabla^{\alpha_j}\mu(\cdot)\big\|_{2\tilde p_j}^2 \\
&\qquad \times \prod_{i=1}^{n}\|\nabla^{\alpha_i}\mu\|_{2(k_i-\delta_j^i)p_i}^{2(k_i-\delta_j^i)}\,dx|h|^{n+2(L-k)}\,dh\Big)^{\frac12} \\
&\le C\Big(\int_{\mathbb{R}^n} \sum_{\sum_{i=1}^n k_i|\alpha_i|=|\alpha|}\ \sum_{k_j>0} \big\|\nabla^{\alpha_j}\mu(\cdot\pm h)-\nabla^{\alpha_j}\mu(\cdot)\big\|_{2\tilde p_j}^2 \\
&\qquad \times \|\mu\|_{H^s}^{2\sum_{i=1}^n k_i-2}\,dx|h|^{n+2(L-k)}\,dh\Big)^{\frac12} \\
&\simeq C \sum_{\sum_{i=1}^n k_i|\alpha_i|=|\alpha|}\ \sum_{k_j>0} \|\nabla^{\alpha_j}\mu\|_{B^{s-k}_{2\tilde p_j,2}}\|\mu\|_{H^s}^{\sum_{i=1}^n k_i-1}. \qquad (2.3)
\end{aligned}
$$

By the choice of $\tilde p_j$, the embedding theorem between Besov spaces here gives $H^L = B^L_{2,2} \subset B^{L-k+\alpha_j}_{2\tilde p_j,2}$. Hence (2.3) means that

$$
\begin{aligned}
&\|f_\alpha\|_{H^{L-k}} \\
&\le C \sum_{\sum_{i=1}^n k_i|\alpha_i|=|\alpha|}\ \sum_{k_j>0} \|\nabla^{\alpha_j}\mu\|_{B^{L-k}_{2\tilde p_j,2}}\|\mu\|_{H^L}^{\sum_{i=1}^n k_i-1} \\
&\le C \sum_{\sum_{i=1}^n k_i|\alpha_i|=|\alpha|}\ \sum_{k_j>0} \|\mu\|_{B^{L-k+\alpha_j}_{2\tilde p_j,2}}\big(\|\mu\|_{H^L}^{|\alpha|-1}+1\big) \\
&\le C\|\mu\|_{H^L}\big(\|\mu\|_{H^L}^{k-1}+1\big).
\end{aligned}
$$

This proves Lemma 6. □

We return to the proof of Lemma 5. By Lemma 6 and (2.2), we have that

$$\begin{aligned}\|e^{i\mu}-1\|_{H^L} &\simeq \|e^{i\mu}-1\|_2 + \sum_{0<|\alpha|\le k} \big\|\nabla^\alpha (e^{i\mu}-1)\big\|_{H^{L-k}} \\ &\le C\|\mu\|_2 + \sum_{0<|\alpha|\le k} \|f_\alpha e^{i\mu}\|_{H^{L-k}}.\end{aligned}$$

Here, without loss of generality, we always consider the case $n>1$ since the case $n=1$ is easier. It gives $p=\frac{n}{2(L-k)}>n/2\ge 1$. As $L-k<1$, by an equivalent norm of the space H^{L-k} (see [25]) and using Hölder's inequality, we know that

$$\begin{aligned}&\|f_\alpha e^{i\mu}\|_{H^{L-k}} \\ &\simeq \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big|f_\alpha(x+h)e^{i\mu(x+h)} - f_\alpha(x)e^{i\mu(x)}\big|^2 |h|^{-n-2(L-k)}\,dx\,dh\Big)^{\frac12} \\ &\le C\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\big|\big(f_\alpha(x+h)-f(x)\big)e^{i\mu(x+h)}\big|^2 \\ &\quad + \big|f_\alpha(x)(e^{i\mu(x+h)}-e^{i\mu(x)})\big|^2\big)|h|^{-n-2(L-k)}\,dx\,dh\Big)^{\frac12} \\ &\le C\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\big|f_\alpha(x+h)-f(x)\big|^2 \\ &\quad + \big|f_\alpha(x)\big|\big|\mu(x+h)-\mu(x)\big|^2\big)\,dx|h|^{-n-2(L-k)}\,dh\Big)^{\frac12} \\ &\le C\Big(\|f_\alpha\|_{H^{L-k}} \\ &\quad + \Big(\int_{\mathbb{R}^n}\|f_\alpha\|_{2p'}^2\Big(\int_{R^n}\big|\mu(x+h)-\mu(x)\big|^{2p}\Big)\,dx\Big)^{\frac1p}|h|^{-n-2(L-k)}\,dh\Big)^{\frac12} \\ &\simeq C\big(\|f_\alpha\|_{H^{L-k}} + \|f_\alpha\|_{2p'}\|\mu\|_{B^{L-k}_{2p,2}}\big). \end{aligned} \tag{2.4}$$

Since $L-k-n/2\ge -\frac{n}{2p'}$ and $L-n/2\ge L-k-\frac{n}{p}$, the embedding theorem between Besov spaces yields that

$$\|f_\alpha\|_{2p'}\le C\|f_\alpha\|_{H^{L-k}};\qquad \|\mu\|_{B^{L-k}_{2p,2}}\le \|\mu\|_{H^L}.$$

Thus, (2.4) and Lemma 6 imply that

$$\begin{aligned}\|f_\alpha e^{i\mu}\|_{H^{L-k}} &\le C\big(\|f_\alpha\|_{H^{L-k}} + \|f_\alpha\|_{2p'}\|\mu\|_{B^{L-k}_{2p,2}}\big) \\ &\le C\big(\|f_\alpha\|_{H^{L-k}} + \|f_\alpha\|_{H^{L-k}}\|\mu\|_{H^L}\big) \\ &\le C\big(\|\mu\|_{H^L} + \|\mu\|_{H^L}^k\big)\big(1+\|\mu\|_{H^L}\big) \\ &\le C\big(\|\mu\|_{H^L} + \|\mu\|_{H^L}^{k+1}\big).\end{aligned}$$

Lemma 5 now is derived from (2.1) and (2.4). Precisely,

$$\begin{aligned}\|e^{i\mu}-1\|_{H^L} &\le \|\mu\|_2 + \sum_{0<|\alpha|\le k}\|f_\alpha e^{i\mu}\|_{H^{L-k}} \\ &\le C\big(\|\mu\|_{H^L} + \|\mu\|_{H^L}^{k+1}\big) \le C\big(\|\mu\|_{H^L} + \|\mu\|_{H^L}^{L+1}\big).\end{aligned}$$

Remark. If $L = [n/2] + 1$, then by Lemma 6, one can obtain

$$\|e^{i\mu} - 1\|_{H^L} \le C\big(\|\mu\|_{H^L} + \|\mu\|_{H^L}^L\big).$$

But when L is a noninteger, we are not able to prove the same estimate. In the proof of (2.4), if we use the inequality $|e^{i\mu(x+h)} - e^{i\mu(x)}|^2 \le C|\mu(x+h) - \mu(x)|^{2q}$, where $L - k < q \le 1$, then we have

$$\|e^{i\mu} - 1\|_{H^L} \le C_q\big(\|\mu\|_{H^L} + \|\mu\|_{H^L}^{k+q}\big),$$

which is an improvement of Lemma 5.

Lemma 7 ([26, p. 37])**.** *Let $L > n/2$ and $f \in H^L(\mathbb{R}^n)$. We have*

$$\|\hat{f}\|_{L^1(\mathbb{R}^n)} \le C\|f\|_{H^L(\mathbb{R}^n)}.$$

This lemma is a direct consequence of the Plancherel equality and the definition of H^L. In fact, by the Plancherel equality, Hölder's inequality, and the fact $L > n/2$, one knows that

$$\begin{aligned}\|\hat{f}\|_{L^1(R^n)} &= \int_{\mathbb{R}^n} \big(1+|\xi|^2\big)^{-\frac{L}{2}} \big(1+|\xi|^2\big)^{\frac{L}{2}} \big|\hat{f}(\xi)\big|\, d\xi \\ &\le \Big(\int_{\mathbb{R}^n} \big(1+|\xi|^2\big)^{-L} d\xi\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n} \big(1+|\xi|^2\big)^{L} \big|\hat{f}(\xi)\big|^2\, d\xi\Big)^{\frac{1}{2}} \\ &= C\|f\|_{H^L(\mathbb{R}^n)}.\end{aligned}$$

3. Proof of Theorem 1

The first part of Theorem 1 is a direct corollary of the Hörmander multiplier theorem and Lemma 5. For any $L > n/2$, take $l < L$ such that $n/2 < l < n/2+1$. By the Hörmander multiplier theorem, for $1 < p < \infty$, we have that

$$\|e^{i\mu(D)}\|_{L^p - L^p} \le C \sup_{j \in Z} \|e^{i\mu_j}\|_{H^l(B_4 \setminus B_1)}.$$

On the other hand, for any $j \in \mathbb{Z}$, we can extend μ_j from $B_4 \setminus B_1$ to B_6 in the sense that $\tilde{\mu}_j$ is supported in B_6 and $\tilde{\mu}_j(\xi) = \mu_j(\xi)$ for $\xi \in B_4 \setminus B_1$. Additionally, we may choose $\tilde{\mu}_j$ such that

$$\|\tilde{\mu}_j\|_{H^l(B_6)} \le C\|\mu_j\|_{H^l(B_4 \setminus B_1)}.$$

Hence, from Lemma 5, we see that

$$\begin{aligned}\|e^{i\mu(D)}\|_{L^p - L^p} &\le C \sup_{j \in Z} \|e^{i\tilde{\mu}_j}\|_{H^l(B_6)} \le C\big(1 + \sup_{j \in Z} \|e^{i\tilde{\mu}_j} - 1\|_{H^l}\big) \\ &\le C\big(1 + \sup_{j \in Z}\big(\|\tilde{\mu}_j\|_{H^l} + \|\tilde{\mu}_j\|_{H^l}^{l+1}\big)\big) \le C \sup_{j \in Z}\big(1 + \|\mu_j\|_{H^l(B_4 \setminus B_1)}\big)^{l+1} \\ &\le C \sup_{j \in Z}\big(1 + \|\mu_j\|_{H^l(B_4 \setminus B_1)}\big)^{l+1} \le C \sup_{j \in Z}\big(1 + \|\mu_j\|_{H^L(B_4 \setminus B_1)}\big)^{L+1}.\end{aligned}$$

We thus complete the proof of the first part of Theorem 1.

To prove the second part of Theorem 1, we take a function $\Phi \in C_0^\infty(B_2)$ satisfying $\Phi(x) = 1$ for $x \in B_1$, and set $\psi_j(x) = \Phi(2^{j-1}x) - \Phi(2^j x)$. By the definition, we have that $\mathrm{supp}(\psi_0) \subset B_4 \setminus B_1$, $\psi_j(x) = \psi_0(2^j x)$, and

$$\sum_{j\in Z} \psi_j(x) \equiv 1 \quad \text{for all } x \neq 0.$$

For $1 \le p \le \infty$, first we have

$$\begin{aligned}\|e^{i\mu(D)}f\|_{L^p} &= \big\|(e^{i\mu}\hat{f})^\vee\big\|_{L^p} \le \big\|\big((e^{i\mu}-1)\hat{f}\big)^\vee\big\|_{L^p} + \|f\|_{L^p}\\ &\le \big\|(e^{i\mu}-1)^\vee * f\big\|_{L^p} + \|f\|_{L^p},\end{aligned}$$

where $g^\vee$ denotes the inverse Fourier transform of a function g. Thus, Young's inequality yields that

$$\begin{aligned}\|e^{i\mu(D)}f\|_{L^p} &\le \big(\big\|(e^{i\mu}-1)^\vee\big\|_{L^1} + 1\big)\|f\|_{L^p}\\ &\le \Big(\sum_{j\in Z}\big\|\big[\psi_j(e^{i\mu}-1)\big]^\vee\big\|_{L^1} + 1\Big)\|f\|_{L^p}.\end{aligned}$$

Next, an easy scaling argument gives that

$$\|e^{i\mu(D)}f\|_{L^p} \le \Big(\sum_{j\in Z}\big\|\big[\psi_0(e^{i\mu_j}-1)\big]^\vee\big\|_{L^1} + 1\Big)\|f\|_{L^p}.$$

Similar to the proof for the first part, for any $j \in \mathbb{Z}$, extend $\mu_j\chi_{B_4\setminus B_1}$ to $\tilde{\mu}_j$ such that $\tilde{\mu}_j$ is supported in B_6 and

$$\|\tilde{\mu}_j\|_{H^L} \le C\|\mu_j\|_{H^L(B_4\setminus B_1)}.$$

Since ψ_0 is supported in $B_4 \setminus B_1$, by Lemmas 7 and 5, we have that

$$\begin{aligned}\big\|\big[\psi_0(e^{i\mu_j}-1)\big]^\vee\big\|_{L^1} &= \|\check{\psi}_0\|_{L^1}\big\|(e^{i\tilde{\mu}_j}-1)^\vee\big\|_{L^1}\\ &\le C\|e^{i\tilde{\mu}_j}-1\|_{H^L}\\ &\le C\big(\|\tilde{\mu}_j\|_{H^L} + \|\tilde{\mu}_j\|_{H^L}^{L+1}\big)\\ &\le C\big(\|\mu_j\|_{H^L(B_4\setminus B_1)} + \|\mu_j\|_{H^L(B_4\setminus B_1)}^{L+1}\big).\end{aligned}$$

This completes the proof of Theorem 1. □

4. Proof of Theorem 2

We begin the proof by introducing an auxiliary phase function

$$\mu_{\vec{A},b}(\xi) = \mu(\xi) - \vec{A}\xi - b = \mu(\xi) - \sum_{i=1}^n a_i\xi_i - b,$$

where b is a real number and $\vec{A} = (a_1, a_2, \dots, a_n)$ is a vector in $\mathbb{R}^n$, and they are to be chosen. Take a function $\varphi \in C_0^\infty(B_2)$ satisfying $\varphi(x) = 1$ for $x \in B_1$. As μ is supported in B_1, checking the Fourier transform we have that

$$\begin{aligned}(e^{i\mu(D)}f)(x) &= \big[(e^{i\mu}-1)\hat{f}\big]^\vee(x) + f(x)\\ &= \big[(e^{i\mu}-1)\varphi\hat{f}\big]^\vee(x) + f(x).\end{aligned}$$

Thus, with the definition of $\mu_{\vec{A},b}$, we obtain that

$$\begin{aligned}(e^{i\mu(D)}f)(x) &= (e^{i\mu}\varphi\hat{f})^{\vee}(x) + f(x) - \check{\varphi} * f(x)\\ &= e^{ib}(e^{i\mu_{\vec{A},b}}\varphi\hat{f})^{\vee}(x+\vec{A}) + f(x) - \check{\varphi} * f(x).\end{aligned}$$

For any $\vec{A}, b$ and $1 \le p \le \infty$, Young's inequality yields that

$$\begin{aligned}\|e^{i\mu(D)}f\|_{L^p} &\le \big\{\big\|(e^{i\mu_{\vec{A},b}}\varphi)^{\vee}\big\|_{L^1} + C\big\}\|f\|_{L^p}\\ &\le \big\{\big\|\big((e^{i\mu_{\vec{A},b}}-1)\varphi\big)^{\vee}\big\|_1 + C\big\}\|f\|_{L^p}.\end{aligned}$$

Again, we may construct a function $\tilde{\mu}_{\vec{A},b}$ supported in B_4 that satisfies $\tilde{\mu}_{\vec{A},b}(\xi) = \mu_{\vec{A},b}(\xi)$ for $\xi \in B_2$ and

$$\|\tilde{\mu}_{\vec{A},b}\|_{H^L} \le C\|\mu_{\vec{A},b}\|_{H^L(B_2)}. \tag{4.1}$$

Now, Lemma 5 implies that, for any f satisfying $\|f\|_{L^p} = 1$,

$$\begin{aligned}\|e^{i\mu(D)}f\|_{L^p} &\le \big\|(e^{i\tilde{\mu}_{\vec{A},b}}-1)^{\vee} * \check{\varphi}\big\|_1 + C\\ &\le C\big(1 + \big\|(e^{i\tilde{\mu}_{\vec{A},b}}-1)^{\vee}\big\|_1\big) \le C\big(1 + \|e^{i\tilde{\mu}_{\vec{A},b}}-1\|_{H^L}\big)\\ &\le C\big(1 + \|\tilde{\mu}_{\vec{A},b}\|_{H^L}\big)^{L+1} \le C\big(1 + \|\mu_{\vec{A},b}\|_{H^L(B_2)}\big)^{L+1}.\end{aligned}$$

To complete the proof of the theorem, it remains to check that

$$\inf_{\vec{A},b}\|\mu_{\vec{A},b}\|_{H^L(B_2)} \le C\|\nabla^2\mu\|_{H^{L-2}(B_2)}. \tag{4.2}$$

Choose $\vec{A}, b$ such that

$$\int_{B_2}\mu_{\vec{A},b}\,dx = \int_{B_2}\partial_j\mu_{\vec{A},b}\,dx = 0, \quad j = 1, 2, \ldots, n.$$

We note that (4.2) is a direct consequence of Poincare's inequality when $L \ge 2$. To prove (4.2) for a general $L \in \mathbb{R}$, we need to invoke the following elliptic estimate for the Neumann problem.

Lemma 8 ([25, p. 233])**.** *Let u be the solution of the Neumann problem*

$$\begin{cases}\triangle u = g, & x \in B_1,\\ \frac{\partial u}{\partial n} = 0, & x \in \partial B_1.\end{cases}$$

If

$$\int_{B_1} g = 0,$$

then for any $L \in \mathbb{R}$ one has

$$\|\nabla u\|_{H^{L-1}(B_1)} \le C\|u\|_{H^L(B_1)} \le C\|g\|_{H^{L-2}(B_1)}.$$

This lemma is a special case of a result in [25, p. 233]. Clearly, in the lemma one can replace B_1 by B_2. By an argument of duality, there exists a function $g \in C^\infty$ with $\|g\|_{H^{-L}} = 1$ such that

$$\|\mu_{\vec{A},b}\|_{H^L(B_2)} \le 2\|\mu_{\vec{A},b}g\|_{L^1}.$$

Set

$$a = \frac{1}{|B_2|}\int_{B_2} g \quad \text{and} \quad \tilde{g} = g - a.$$

Then

$$\|\tilde{g}\|_{H^{-L}} \le C.$$

Let u be the solution of the Neumann problem

$$\begin{cases} \triangle u = \tilde{g}, & x \in B_2, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial B_2. \end{cases}$$

By the above lemma, one obtains

$$\|u\|_{H^{2-L}(B_2)} \le C\|\tilde{g}\|_{H^{-L}(B_2)} \le C.$$

Recall that $\int_{B_2} \mu_{\vec{A},b}\, dx = 0$. Using the duality, we now have

$$\begin{aligned} \|\mu_{\vec{A},b}\|_{H^L(B_2)} &= 2\Big|\int_{B_2} \mu_{\vec{A},b}\tilde{g}\, dx\Big| = 2\Big|\int_{B_2} \mu_{\vec{A},b}\triangle u\, dx\Big| \\ &= 2\Big|\int_{B_2} \nabla\mu_{\vec{A},b}\nabla u\, dx\Big|. \end{aligned}$$

Hölder's inequality thus gives

$$\begin{aligned} \|\mu_{\vec{A},b}\|_{H^L(B_2)} &\le C\|\nabla\mu_{\vec{A},b}\|_{H^{L-1}(B_2)}\|\nabla u\|_{H^{-L+1}(B_2)} \\ &\le C\|\nabla\mu_{\vec{A},b}\|_{H^{L-1}(B_2)}\|\tilde{g}\|_{H^{-L}(B_2)} \le C\|\nabla\mu_{\vec{A},b}\|_{H^{L-1}(B_2)}. \end{aligned}$$

A similar argument gives

$$\|\nabla\mu_{\vec{A},b}\|_{H^{L-1}(B_2)} \le C\|\nabla^2\mu_{\vec{A},b}\|_{H^{L-2}(B_2)} = C\|\nabla^2\mu\|_{H^{L-2}(B_2)}.$$

Combining all estimates, we finally obtain

$$\|\mu_{\vec{A},b}\|_{H^L(B_2)} \le C\|\nabla\mu_{\vec{A},b}\|_{H^{L-1}(B_2)} \le C\|\nabla^2\mu\|_{H^{L-2}(B_2)},$$

where C is a constant independent of $\vec{A}$ and b. This completes the proof of Theorem 2. □

5. Proof of Theorem 3

Theorem 3 is a direct corollary of Theorems 1 and 2. The definition of the modulation spaces gives

$$\|e^{i\mu(D)}f\|_{M^s_{p,q}(R^n)} = \Big(\sum_{k\in Z^n}\langle k\rangle^{\mathrm{sq}}\|\Box_k e^{i\mu(D)}f\|^q_{L^p}\Big)^{\frac{1}{q}},$$

where

$$\Box_k e^{i\mu(D)}f = \varphi_k * e^{i\mu(D)}f.$$

First, by using the definition of the local projections $\Box_k$ and checking the Fourier transform, it is easy to see that

$$\Box_k = \sum_{|j|\le 1}\Box_{k+j}\Box_k$$

and that

$$\sum_{k\in\mathbb{Z}^n} \Box_k = I_{\mathrm{id}},$$

where I_{id} is the identity operator.

With these identities, if $k = 0$, we have

$$\Box_0 e^{i\mu(D)} f = \Box_0 e^{i\mu(D)} \sum_{|j|\le 1} \Box_j(f).$$

Now we show that $\Box_0 e^{i\mu(D)}$ is an L^p multiplier. In fact, following the proof for (1) in Theorem 1, to prove that $\Box_0 e^{i\mu(D)}$ is an L^p multiplier, we need to check that, for $L > n/2$,

$$\sup_{j\in Z} \big\|\mu(2^{-j}\cdot)\varphi(2^{-j}\cdot)\big\|_{H^L(B_4\setminus B_1)} < \infty.$$

Note that $\varphi(2^{-j}\cdot)$ is supported on $B_{\sqrt{n}2^j}$. Since $\sqrt{n}2^j < 1$ when $j < -\log_2(\sqrt{n}+2)$, we have

$$\big\|\mu(2^{-j}\cdot)\varphi(2^{-j}\cdot)\big\|_{H^L(B_4\setminus B_1)} = 0$$

whenever $j < -\frac{1}{2}\log_2 n$. For $j \ge -\frac{1}{2}\log_2 n$, it is easy to check that

$$\big\|\mu(2^{-j}\cdot)\varphi(2^{-j}\cdot)\big\|_{H^L(B_4\setminus B_1)} \preceq \big\|\mu(2^{-j}\cdot)\big\|_{H^L(B_4\setminus B_1)}.$$

Thus by Theorem 1, Theorem 3(1) yields that, for $1 < p < \infty$,

$$\|\Box_0 e^{i\mu(D)} f\|_{L^p} \preceq \sum_{|j|\le 1} \big\|\Box_j(f)\big\|_{L^p}.$$

Similarly, we can show that for, $1 < p < \infty$,

$$\|\Box_k e^{i\mu(D)} f\|_{L^p} \preceq \sum_{|j|\le 1} \big\|\Box_{j+k}(f)\big\|_{L^p},$$

when $|k| = 1, 2, \ldots, n$.

We continue to investigate the terms $\Box_k e^{i\mu(D)} f$, for $|k| > n$. For any $k \in \mathbb{Z}^n$ with $|k| > n$, again we have

$$\Box_k e^{i\mu(D)} f = \Box_k e^{i\mu(D)} \sum_{|j|\le 1} \Box_{k+j}(f).$$

Recalling the support condition of $\Box_k$ in the frequency domain, following the proof of Theorem 2, we may assume that μ is supported on the ball $B_{2\sqrt{n}} + k$. Thus, following the same proof as in Theorem 2, we know that, for $L > n/2$, Theorem 3(2) implies that for $|k| > n$ and $1 \le p \le \infty$,

$$\|\Box_k e^{i\mu(D)} f\|_{L^p} \preceq \sum_{|j|\le 1} \big\|\Box_{j+k}(f)\big\|_{L^p}.$$

Combining the estimates of $\|\Box_k e^{i\mu(D)} f\|_{L^p}$ for all k, we have that, for $1 < p < \infty$,

$$\Big(\sum_{k\in Z^n} \langle k\rangle^{\mathrm{sq}} \|\Box_k e^{i\mu(D)} f\|_{L^p}^q\Big)^{\frac{1}{q}} \preceq \Big(\sum_{k\in Z^n} \langle k\rangle^{\mathrm{sq}} \sum_{|j|\le 1} \big\|\Box_{k+j}(f)\big\|_{L^p}^q\Big)^{\frac{1}{q}}$$
$$\preceq \Big(\sum_{k\in Z^n} \langle k\rangle^{\mathrm{sq}} \big\|\Box_{k+j}(f)\big\|_{L^p}^q\Big)^{\frac{1}{q}}.$$

Furthermore, if Theorem 3(1) is replaced by

$$\sum_{j\ge -n} \|\mu_j\|_{H^L(B_4\backslash B_1)} < A,$$

then by Theorem 1, we have that, for all $k \in \mathbb{Z}^n$ and $1 \le p \le \infty$,

$$\|\Box_k e^{i\mu(D)} f\|_{L^p} \preceq \sum_{|j|\le |k|+1} \big\|\Box_j(f)\big\|_{L^p}.$$

This inequality clearly yields

$$\|e^{i\mu(D)} f\|_{M^s_{p,q}} \le C\|f\|_{M^s_{p,q}}$$

for $1 \le p \le \infty$. This completes the proof of Theorem 3. □

6. Proof of Theorem 4

As we see in the proof of Theorem 3, Theorem 4(1) and Theorem 1 together yield that, for all $1 < p < \infty$,

$$\|\Box_0 e^{i\mu(D)} f\|_p \preceq \|\Box_0 f\|_p.$$

On the other hand, for any term $\Box_k e^{i\mu(D)} f$, where $|k| \ge 1$, choose $\tilde{\phi}_k \in C_0^\infty(B(k,2))$ satisfying $\tilde{\phi}_k = 1$ on the support of $\widehat{\varphi_k}$. Also, let $\tilde{\mu}_k$ be an extension of $\mu\chi_{B_{(k,1)}}$ as we did before. A direct computation yields that for all $1 \le p \le \infty$,

$$\begin{aligned}
\|\Box_k e^{i\mu(D)} f\|_p &= \big\|\Box_k (e^{i\mu}\tilde{\phi}_k \hat{f})^\vee\big\|_p \\
&\le \big\|(e^{i\mu}\tilde{\phi}_k)^\vee * (\Box_k f)\big\|_p \\
&\le \big\|(e^{i\mu}\tilde{\phi}_k)^\vee\big\|_1 \|\Box_k f\|_p \\
&\le C\big(1 + \big\|\big[\tilde{\phi}_k(e^{i\mu} - 1)\big]^\vee\big\|_1\big)\|\Box_k f\|_p \\
&= C\big(1 + \big\|\tilde{\phi}_k\big(|k|^{-\delta}\cdot\big)^\vee * (e^{i\tilde{\mu}_k(|k|^{-\delta}\cdot)} - 1)^\vee\big\|_1\big)\|\Box_k f\|_p \\
&= C\big(1 + \big\|(e^{i\tilde{\mu}_k(|k|^{-\delta}\cdot)} - 1)^\vee\big\|_1\big)\|\Box_k f\|_p \\
&= C\big(1 + \|e^{i\tilde{\mu}_k(|k|^{-\delta}\cdot)} - 1\|_{H^{N_0}}\big)\|\Box_k f\|_p.
\end{aligned}$$

From Lemma 5, we know that

$$\|e^{i\tilde{\mu}_k(|k|^{-\delta}\cdot)} - 1\|_{H^{N_0}} = \sum_{|\alpha|\le N_0} \big\|\nabla^\alpha(e^{i\tilde{\mu}_k(|k|^{-\delta}\cdot)} - 1)\big\|_2$$

$$
\begin{aligned}
&= \sum_{|\alpha|\le N_0} |k|^{-\delta(|\alpha|-\frac{n}{2})} \big\|\nabla^\alpha (e^{i\tilde{\mu}_k}-1)\big\|_2 \\
&\le \sum_{|\alpha|\le N_0} |k|^{-\delta(|\alpha|-\frac{n}{2})} \big(1+\|\tilde{\mu}_k\|_{H^{N_0}}\big)^{|\alpha|} \\
&\le \sum_{|\alpha|\le N_0} |k|^{-\delta(|\alpha|-\frac{n}{2})} \big(1+\|\mu\|_{H^{N_0}(B(k,1))}\big)^{|\alpha|} \\
&\le \sum_{|\alpha|\le N_0} |k|^{-\delta(|\alpha|-\frac{n}{2})} |k|^{\delta|\alpha|} \\
&= C|k|^{\frac{n\delta}{2}}.
\end{aligned}
$$

Thus, for any $k \in Z^n$ with $|k| \ge 1$ and $p = 1, \infty$, we have

$$\|\Box_k e^{i\mu(D)} f\|_{L^p} \le C|k|^{\frac{n\delta}{2}} \|\Box_k f\|_{L^p} \preceq C|k|^{\frac{n\delta}{2}} \|f\|_{L^p}.$$

On the other hand, it is easy to see that, with the Plancherel equality, we have

$$\|\Box_k e^{i\mu(D)} f\|_{L^2} \preceq \|f\|_{L^2}.$$

Thus, an interpolation (see [4]) yields that for all $1 \le p \le \infty$ and $|k| \ge 1$,

$$\|\Box_k e^{i\mu(D)} f\|_{L^p} \le C|k|^{n\delta|1/p-1/2|} \|f\|_{L^p}.$$

As before, we know that for any $|k| \ge 1$,

$$
\begin{aligned}
\|\Box_k e^{i\mu(D)} f\|_{L^p} &\le \sum_{|j|\le 1} \|\Box_k e^{i\mu(D)} \Box_{k+j} f\|_{L^p} \\
&\preceq C|k|^{n\delta|1/p-1/2|} \sum_{|j|\le 1} \|\Box_{k+j} f\|_{L^p}.
\end{aligned}
$$

Combining the last estimate with the estimate obtained for $\|\Box_0 e^{i\mu(D)} f\|_{L^p}$, we have that for $1 < p < \infty$,

$$
\begin{aligned}
\|e^{i\mu(D)} f\|_{M^s_{p,q}} &= \Big(\sum_{k\in Z^n} \big(1+|k|\big)^{\mathrm{sq}} \|\Box_k e^{i\mu(D)} f\|_p^q\Big)^{\frac{1}{q}} \\
&\preceq C\Big(\sum_{k\in Z^n} \big(1+|k|\big)^{(s+n\delta|1/p-1/2|)q} \|\varphi_k * f\|_p^q\Big)^{\frac{1}{q}} \\
&= C\|f\|_{M^{s+n\delta|1/p-1/2|}_{p,q}}.
\end{aligned}
$$

Finally, if Theorem 4(1) is replaced by

$$\sum_{j\ge -n} \|\mu_j\|_{H^L(B_4\setminus B_1)} < A,$$

then we also have, for $1 \le p \le \infty$,

$$\|\Box_0 e^{i\mu(D)} f\|_{L^p} \preceq \sum_{|j|\le 1} \|\Box_j f\|_{L^p}.$$

This completes the proof of Theorem 4. □

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