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NORMED ORLICZ FUNCTION SPACES WHICH CAN BE QUASI-RENORMED WITH EASILY CALCULABLE QUASINORMS

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Abstract. We are interested in the widest possible class of Orlicz functions Φ such that the easily calculable quasinorm $[f]_{\Phi,p} := \|f\|_E\{I_\Phi(\frac{f}{\|f\|_E})\}^{1/p}$ if $f \neq 0$ and $[f]_{\Phi,p} = 0$ if $f = 0$, on the Orlicz space $L^\Phi(\Omega,\Sigma,\mu)$ generated by Φ, is equivalent to the Luxemburg norm $\|\cdot\|_\Phi$. To do this, we use a suitable Δ_2-condition, lower and upper Simonenko indices $p_S^a(\Phi)$ and $q_S^a(\Phi)$ for the generating function Φ, numbers $p \in [1, p_S^a(\Phi)]$ satisfying $q_S^a(\Phi) - p \leq 1$, and an embedding of $L^\Phi(\Omega,\Sigma,\mu)$ into a suitable Köthe function space $E = E(\Omega,\Sigma,\mu)$. We take as E the Lebesgue spaces $L^r(\Omega,\Sigma,\mu)$ with $r \in [1, p_S^l(\Phi)]$, when the measure μ is nonatomic and finite, and the weighted Lebesgue spaces $L^r_\omega(\Omega,\Sigma,\mu)$, with $r \in [1, p_S^a(\Phi)]$ and a suitable weight function ω, when the measure μ is nonatomic infinite but σ-finite. We also use condition ∇_3 if $p_S^a(\Phi) = 1$ and condition ∇^2 if $p_S^a(\Phi) > 1$, proving their necessity in most of the considered cases. Our results seem important for applications of Orlicz function spaces.

1. Introduction and preliminaries

This article is organized as follows. We start with notation and definitions. First, we recall the definitions of the Orlicz function, the Orlicz space, the Luxemburg norm, and the conditions Δ_2, Δ_3, and Δ^2, at infinity, at zero, and on the

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whole of $\mathbb{R}_+$, concerning the growth of the generating Orlicz function. Next, we recall the definitions of Simonenko indices of the Orlicz functions, at infinity and on the whole of $\mathbb{R}_+$, as well as of Matuszewska–Orlicz indices. We also present some auxiliary results concerning the conditions Δ_2, Δ_3, ∇_3, Δ^2, and ∇^2 for the generating Orlicz functions on suitable subintervals of $\mathbb{R}_+$. Finally, we prove crucial auxiliary results concerning the embedding of the Orlicz function spaces into some suitable Lebesgue or weighted Lebesgue function spaces, according to whether the underlying nonatomic measure space is finite or infinite. Thanks to these embedding theorems it was possible to define, on certain Orlicz spaces, some easily calculable quasinorms equivalent to the Luxemburg norm. The construction of the quasinorms and the proofs of their equivalence to the Luxemburg norm are presented in the final part of the article.

We were inspired by an example of easily calculable quasinorm equivalent to the Luxemburg norm in the Orlicz space $L^\Phi(\Omega, \Sigma, \mu)$ generated by the Orlicz function $\Phi(u) = |u|^p \log(e + |u|)$, $p \geq 1$, which was presented by Iwaniec and Verde in [19], as well as by a similar example by Krbec and Schmeisser in [21] for the Orlicz space generated by the Orlicz function $\Phi(u) = |u|^p \log^\alpha(e + |u|)$, with $p \geq 1$ and $\alpha > 0$.

Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$ be the sets of natural numbers, real numbers, and nonnegative reals, respectively. Denote by (Ω, Σ, μ) a positive, complete, and σ-finite nonatomic measure space, and denote by $L^0 = L^0(\Omega, \Sigma, \mu)$ the space of all (equivalence classes of) real-valued and Σ-measurable functions on Ω. A nonnegative, even, and convex function $\Phi : \mathbb{R} \to \mathbb{R}_+$ such that $\Phi(0) = 0$ and Φ is not identically equal to zero is called an *Orlicz function*. We say that an Orlicz function Φ *satisfies* the Δ_2-condition for all $u \in \mathbb{R}$ (at infinity) [at zero] if there is $K > 0$ such that the inequality $\Phi(2u) \leq K\Phi(u)$ holds for all $u \in \mathbb{R}$ (for all $u \in \mathbb{R}$ satisfying $|u| \geq u_0$ with some $u_0 > 0$) [for all $u \in \mathbb{R}$ satisfying $|u| \leq u_0$ with some $u_0 > 0$ such that $\Phi(u_0) > 0$]. Since Φ is even, we then write $\Phi \in \Delta_2(\mathbb{R}_+)$ ($\Phi \in \Delta_2(\infty)$) [$\Phi \in \Delta_2(0)$].

Note that in the case of a finite nonatomic measure space, we can assume without loss of generality that the Orlicz function Φ vanishes only at zero because otherwise we can find a function $\widehat{\Phi}$ vanishing only at zero and equivalent to Φ at infinity that generates the same Orlicz space with an equivalent norm. In the case of an infinite nonatomic measure space, the condition $\Delta_2(\mathbb{R}_+)$ guarantees that the Orlicz function Φ vanishes only at zero while for a finite nonatomic measure space, we need to assume that the function Φ vanishes only at zero, because an Orlicz function Φ can satisfy condition $\Delta_2(\infty)$ even if Φ vanishes outside zero.

For any Orlicz function Φ, we define its function complementary to Φ in the sense of Young by the formula

$$\Phi^*(u) = \sup_{v>0}\big\{|u|v - \Phi(v)\big\}$$

for all $u \in \mathbb{R}$.

Given an Orlicz function Φ, we define on L^0 a convex semimodular (see [3], [20], [23], [26], [28], [29])

$$I_\Phi(x) = \int_\Omega \Phi\big(x(t)\big)\, d\mu.$$

The Orlicz space $L^\Phi = L^\Phi(\Omega, \Sigma, \mu)$ generated by an Orlicz function Φ is defined as

$$L^\Phi = \big\{x \in L^0 : I_\Phi(\lambda x) < +\infty \text{ for some } \lambda > 0\big\}.$$

We will consider Orlicz spaces L^Φ equipped with the Luxemburg norm

$$\|x\|_\Phi = \inf\Big\{\lambda > 0 : I_\Phi\Big(\frac{x}{\lambda}\Big) \le 1\Big\}.$$

We mention that in [8] Orlicz spaces were considered with a family of norms, called the *p-Amemiya norms* ($1 \le p \le \infty$). These norms are equivalent to both the Orlicz norm and the Luxemburg norm. (For more information on Orlicz spaces endowed with p-Amemiya norms and some results about the geometry of these norms, see [4]–[12], [22], [15], and [16].) It is of interest that in some cases the geometry of Orlicz spaces equipped with p-Amemiya norms ($1 < p < \infty$) is better than when they are equipped with the Luxemburg norm or the Orlicz norm.

We say that an Orlicz function Φ satisfies the Δ_3-condition at infinity (at zero) [on $\mathbb{R}$] if Φ is equivalent to the function $|u|\Phi(u)$ at infinity (at zero) [on $\mathbb{R}$]; that is, there exist constants $K, L > 0$, and $u_0 > 0$ such that

$$\Phi(Ku) \le |u|\Phi(u) \le \Phi(Lu)$$

for all $u \in \mathbb{R}$ with $|u| \ge u_0$ (for all $u \in \mathbb{R}$ with $|u| \le u_0$) [for all $u \in \mathbb{R}_+$]. We then write $\Phi \in \Delta_3(\infty)$ ($\Phi \in \Delta_3(0)$) [$\Phi \in \Delta_3(\mathbb{R}_+)$].

Since $|u|\Phi(u)$ is greater than $\Phi(u)$ when $|u| \ge 1 = \Phi(1)$, the Δ_3-condition at infinity just means that, whenever $|u| \ge 1$, we have

$$|u|\Phi(u) \le \Phi(lu),$$

where $l > 0$ is an absolute constant independent of u. Note that we can demand that $l \ge 1$.

Since $|u|\Phi(u)$ is smaller than $\Phi(u)$ when $|u| \le 1 = \Phi(1)$, the Δ_3-condition at zero just means that there exists $k > 0$ such that, for all $u \in \mathbb{R}$ with $|u| \le 1$, we have

$$\Phi(ku) \le |u|\Phi(u).$$

Note that we can demand that $k \le 1$. If an Orlicz function Φ satisfies the Δ_3-condition, then all Orlicz functions equivalent to Φ also enjoy this condition.

We say that an Orlicz function Φ satisfies the Δ^2-condition for all $u \in \mathbb{R}_+$ (at infinity) [at zero] if $\Phi \sim \Phi^2$ for all $u \in \mathbb{R}_+$ (at infinity) [at zero], that is, there exist constants $K, L > 0$ (there exist constants $K, L > 0$ and $u_1 > 0$) [there exist constants $K, L > 0$ and $u_1 > 0$] such that the inequalities

$$\Phi(Ku) \le \Phi^2(u) \le \Phi(Lu) \tag{1.1}$$

hold for all $u > 0$ (for any $u \ge u_1$) [for any $0 < u \le u_1$]. We denote these conditions by $\Delta^2(\mathbb{R}_+)$, $\Delta^2(\infty)$, and $\Delta^2(0)$, respectively. Let us note that in the above conditions it is enough to demand the existence of $0 < K \leq 1$ and $L \geq 1$.

Note also that when $u \geq 1$ and $\Phi(1) = 1$, in order to show that $\Phi \in \Delta^2(\infty)$, we need only check the second inequality in (1.1), because the first inequality is always satisfied with $K = 1$. Similarly, when $0 < u \leq 1$ and $\Phi(1) = 1$, in order to show that $\Phi \in \Delta^2(0)$, we need only to check the first inequality in (1.1), because the second inequality is always satisfied with $L = 1$. (For more information on conditions Δ_3 and Δ^2, we refer readers to [20].)

Throughout this article, we assume without loss of generality that $\Phi(1) = 1$. Indeed, if $\Phi(1) \neq 1$, then we may consider a new Orlicz function Ψ defined by $\Psi(u) = \Phi(au)$ for all $u \in \mathbb{R}_+$, with $a \in (0, +\infty)$ satisfying $\Phi(a) = 1$. Then the Orlicz spaces $L^\Phi(\Omega, \Sigma, \mu)$ and $L^\Psi(\Omega, \Sigma, \mu)$ are isomorphically isometric, namely, $\|\cdot\|_\Psi = a\|\cdot\|_\Phi$.

Let Φ be an Orlicz function vanishing only at zero with the right-hand-side derivative denoted by Φ', and let $\Phi(1) = 1$. The lower and upper Simonenko indices of the Orlicz function Φ (see [25], [30]) are defined by

$$p_S^a(\Phi) = \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)}, \qquad q_S^a(\Phi) = \sup_{t>0} \frac{t\Phi'(t)}{\Phi(t)},$$
$$p_S^l(\Phi) = \inf_{t\geq 1} \frac{t\Phi'(t)}{\Phi(t)}, \qquad q_S^l(\Phi) = \sup_{t\geq 1} \frac{t\Phi'(t)}{\Phi(t)}.$$

Observe that if an Orlicz function Φ satisfies the condition $\Delta_2(\mathbb{R}_+)$ (resp., $\Delta_2(\infty)$), then $1 \leq p_S^a(\Phi) \leq q_S^a(\Phi) < +\infty$ (resp., $1 \leq p_S^l(\Phi) \leq q_S^l(\Phi) < +\infty$).

Note that in general (i.e., without the assumption that $\Phi(1) = 1$) we can define the lower and upper Simonenko indices for the Orlicz function Φ in the following way:

$$p_S^l(\Phi) = \inf_{t\geq c} \frac{t\Phi'(t)}{\Phi(t)}, \qquad q_S^l(\Phi) = \sup_{t\geq c} \frac{t\Phi'(t)}{\Phi(t)},$$

where $c > 0$ is such that $\Phi(c) = 1$. Then, defining another (but equivalent) Orlicz function $\widetilde{\Phi}$ by the formula $\widetilde{\Phi}(u) = \Phi(cu)$, by putting $\frac{t}{c} = s$ we get

$$\inf_{t\geq c} \frac{t\Phi'(t)}{\Phi(t)} = \inf_{\frac{t}{c}\geq 1} \frac{\frac{t}{c} c\Phi'(c\frac{t}{c})}{\Phi(c\frac{t}{c})} = \inf_{s\geq 1} \frac{s\widetilde{\Phi}'(s)}{\widetilde{\Phi}(s)},$$

so $p_S^l(\Phi) = p_S^l(\widetilde{\Phi})$ (similarly, $q_S^l(\Phi) = q_S^l(\widetilde{\Phi})$). Let Φ be an Orlicz function satisfying the $\Delta_2(\mathbb{R}_+)$-condition, and let us define

$$h_\Phi(\lambda) = \sup_{t>0} \frac{\Phi(\lambda t)}{\Phi(t)}, \quad \lambda > 0.$$

The numbers

$$i(\Phi) = \lim_{\lambda\to 0_+} \frac{\log h_\Phi(\lambda)}{\log \lambda} = \sup_{0<\lambda<1} \frac{\log h_\Phi(\lambda)}{\log \lambda}$$

and

$$I(\Phi) = \lim_{\lambda\to\infty} \frac{\log h_\Phi(\lambda)}{\log \lambda} = \inf_{1<\lambda<\infty} \frac{\log h_\Phi(\lambda)}{\log \lambda},$$

where log is the natural logarithm, are called the *Matuszewska–Orlicz indices* (see [27]; see also [26]). The existence of the limits follows from the theory of submultiplicative functions (see [1], [2], [14], [24], [27] for details). We always have $1 \le i(\Phi) \le I(\Phi)$ and $i(\Phi) > 1$ if and only if the complementary function Φ^* satisfies the Δ_2-condition.

Theorem 1.1 (see [13, Theorem 1.1]). *Let Φ be an Orlicz function. Then*

$$i(\Phi) = \sup_{\Lambda\sim\Phi} p_S^a(\Lambda) \qquad \textit{and} \qquad I(\Phi) = \inf_{\Lambda\sim\Phi} q_S^a(\Lambda),$$

where $p_S^a(\Lambda)$ and $q_S^a(\Lambda)$ are the Simonenko indices of the Orlicz function Λ vanishing only at zero and satisfying $\Lambda(1) = 1$, and the supremum and infimum are taken over all the functions Λ equivalent to Φ.

The Matuszewska–Orlicz indices enable us to estimate the growth of the function $\Phi \in \Delta_2(\mathbb{R}_+)$ by means of the power functions. Namely, given $\varepsilon > 0$, there is a constant $C > 0$ such that

$$\varPhi(\lambda t) \le C \max(\lambda^{i(\varPhi)-\varepsilon}, \lambda^{I(\varPhi)+\varepsilon})\Phi(t), \quad \lambda, t > 0, \tag{1.2}$$

and

$$\varPhi(\lambda t) \ge C \min(\lambda^{i(\varPhi)-\varepsilon}, \lambda^{I(\varPhi)+\varepsilon})\Phi(t), \quad \lambda, t > 0, \tag{1.3}$$

where C depends on Φ and ε, but is independent of λ and t. Obviously, $\Phi \in \Delta_2(\mathbb{R}_+)$ if and only if $I(\Phi) < \infty$. Let us also note incidentally that (1.2) and (1.3) can be derived from the proof of Lemma 2.7.

We will write $f_1 \overset{l}{\prec} f_2$ ($f_1 \overset{a}{\prec} f_2$) for functions $f_1, f_2 : [0, \infty) \to [0, +\infty)$ if there exist positive constants t_0, b, and c such that $f_1(t) \le bf_2(ct)$ for $t \ge t_0$ (resp., $f_1(t) \le bf_2(ct)$ for $t > 0$). If f_1 and f_2 are convex, then in the above inequalities we can put $b = 1$. Functions f_1 and f_2 are said to be *equivalent* for large $t > 0$ (for all $t > 0$) if $f_1 \overset{l}{\prec} f_2$ and $f_2 \overset{l}{\prec} f_1$ ($f_1 \overset{a}{\prec} f_2$ and $f_2 \overset{a}{\prec} f_1$). We then write $f_1 \sim f_2$ at infinity (resp., $f_1 \sim f_2$ for all $t > 0$). Note also that, in this article, a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *quasi-nondecreasing* (resp., *quasi-nonincreasing*) if there exists a constant $L > 0$ such that, for any $t_1, t_2 \in \mathbb{R}_+$, the inequality $t_1 < t_2$ implies that $f(t_1) \le Lf(t_2)$ (resp., $f(t_1) \ge Lf(t_2)$).

2. Auxiliary results

Denote by P the set of all continuous functions $f : [0, \infty) \to [0, +\infty)$ positive on $(0, \infty)$ and such that

$$f(s) \le \max\left\{1, \frac{s}{t}\right\} f(t) \tag{2.1}$$

for all $s, t > 0$.

Remark 2.1. It is easy to see that (2.1) is equivalent to the conjunction of the following two simple conditions.

(1) If $s \le t$, then $f(s) \le f(t)$; that is, the function f is nondecreasing.
(2) If $0 < t \le s$, then $\frac{f(s)}{s} \le \frac{f(t)}{t}$; that is, the function $\frac{f(s)}{s}$ is nonincreasing.

Let us denote by $\overline{P}$ the subset of the concave functions in P. We will need the following lemma.

Lemma 2.2 ([26, Lemma 14.1, p. 130])**.** *For every $f \in P$ there exist a concave function $\overline{f} \in \overline{P}$ and constants $K, L > 0$ such that*

$$Kf(t) \leq \overline{f}(t) \leq Lf(t)$$

for any $t \geq 0$.

Lemma 2.3. *Let Φ be an Orlicz function vanishing only at zero such that $\Phi(1) = 1$ and $\Phi \in \Delta_2(\mathbb{R}_+)$, and let $p \in [1, \infty)$. Let us denote $f_p(t) = \frac{\Phi(t)}{t^p}$. Then*

(i) *the function f_p is nondecreasing on $\mathbb{R}_+$ if and only if $p \leq p_S^a(\Phi)$;*
(ii) *the function f_p is nonincreasing on $\mathbb{R}_+$ if and only if $p \geq q_S^a(\Phi)$.*

Proof. Note that f_p is nondecreasing (resp., nonincreasing) if and only if $\log f_p$ is nondecreasing (resp., nonincreasing). Since $\log f_p(t) = \log \Phi(t) - p \log t$, then we have $(\log f_p)'(t) = \frac{\Phi'(t)}{\Phi(t)} - \frac{p}{t}$, and $(\log f_p)'(t) \geq 0$ if and only if $p \leq \frac{t\Phi'(t)}{\Phi(t)}$ for any $t > 0$. The last condition is equivalent to $p \leq p_S^a(\Phi)$. (Statement (ii) can be proved similarly.) □

Remark 2.4. In the definitions of the Simonenko indices, without loss of generality we could use the standard derivative $\Phi'(t)$ instead of $\Phi_+^{'}(t)$ or $\Phi_-^{'}(t)$, because the one-sided derivatives of Φ exist and are equal to each other except for at most a countable subset of $\mathbb{R}_+$ (which is of zero Lebesgue measure).

Corollary 2.5. *Let $1 \leq p < \infty$ and $\varepsilon > 0$. Then the function f_p (see Lemma 2.3) is nondecreasing and the function $f_{p+\varepsilon}$ is nonincreasing if and only if $p \leq p_S^a(\Phi)$ and $q_S^a(\Phi) \leq p + \varepsilon$; that is, $p \leq p_S^a(\Phi) \leq q_S^a(\Phi) \leq p + \varepsilon$. Therefore,*

$$\varepsilon = (p + \varepsilon) - p \geq q_S^a(\Phi) - p_S^a(\Phi).$$

In particular, if f_p is nondecreasing and f_{p+1} is nonincreasing, then $q_S^a(\Phi) - p_S^a(\Phi) \leq 1$. Note that if $q_S^a(\Phi) - p \leq \varepsilon$, where $p \leq p_S^a(\Phi)$, then f_p is nondecreasing, $f_{p+\varepsilon}$ is nonincreasing, and we have $q_S^a(\Phi) - p_S^a(\Phi) \leq q_S^a(\Phi) - p \leq \varepsilon$. In particular, we get that when $p = p_S^a(\Phi)$, the function f_p is nondecreasing, and that the function f_{p+1} is nonincreasing if and only if $q_S^a(\Phi) - p_S^a(\Phi) \leq 1$.

Lemma 2.6. *Let Φ be an Orlicz function vanishing only at zero, let $\Phi(1) = 1$, and let $\Phi \in \Delta_2(\mathbb{R}_+)$. Let $x \in L^\Phi$ and $p \geq 1$. The function $\lambda(I_\Phi(\frac{x}{\lambda}))^{\frac{1}{p}}$ is nonincreasing with respect to $\lambda \in (0, \infty)$ if and only if $p \leq p_S^a(\Phi)$, and the function $\lambda(I_\Phi(\frac{x}{\lambda}))^{\frac{1}{p}}$ is nondecreasing with respect to $\lambda \in (0, \infty)$ if and only if $p \geq q_S^a(\Phi)$.*

Proof. Let Φ be an Orlicz function satisfying the assumptions of the lemma, and let $p \geq 1$. We will only show that for any $x \in L^\Phi$ the function $\lambda(I_\Phi(\frac{x}{\lambda}))^{\frac{1}{p}}$ is nonincreasing with respect to $\lambda \in (0, \infty)$ if and only if $p \leq p_S^a(\Phi)$, because the proof of the second statement is similar. By Lemma 2.3, we know that the function $\frac{\Phi(t)}{t^p}$ is nondecreasing on $\mathbb{R}_+$ if and only if $p \leq p_S^a(\Phi)$. Therefore, the function $\lambda^p \Phi(\frac{1}{\lambda}) = \frac{\Phi(\frac{1}{\lambda})}{\frac{1}{\lambda^p}}$ is nonincreasing with respect to $\lambda > 0$ if and only if $p \leq p_S^a(\Phi)$.

Hence, assuming that $p \le p_S^a(\Phi)$ and taking $a > 0$ and $0 < \lambda_1 < \lambda_2 < \infty$, we obtain

$$\begin{aligned}\lambda_2^p \Phi\Big(\frac{a}{\lambda_2}\Big) &= a^p \frac{\lambda_2^p}{a^p} \Phi\Big(\frac{1}{\frac{\lambda_2}{a}}\Big)\\ &\le a^p \frac{\lambda_1^p}{a^p} \Phi\Big(\frac{1}{\frac{\lambda_1}{a}}\Big)\\ &= \lambda_1^p \Phi\Big(\frac{a}{\lambda_1}\Big);\end{aligned}$$

that is, $\lambda^p \Phi(\frac{a}{\lambda})$ is nonincreasing with respect to λ. Consequently, for $x \in L^\Phi$, $0 < \lambda_1 < \lambda_2 < \infty$, and μ-a.e. $t \in \Omega$, we have $\lambda_2^p \Phi(\frac{x(t)}{\lambda_2}) \le \lambda_1^p \Phi(\frac{x(t)}{\lambda_1})$, and by integrating both sides of the last inequality, we get that $\lambda^p I_\Phi(\frac{x}{\lambda})$ is nonincreasing with respect to $\lambda > 0$ whenever $p \in [1, p_S^a(\Phi)]$. This means that $\lambda (I_\Phi(\frac{x}{\lambda}))^{\frac{1}{p}}$ is also nonincreasing with respect to $\lambda > 0$ for any $x \in L^\Phi$ whenever $p \le p_S^a(\Phi)$.

Assume now that the function $\lambda (I_\Phi(\frac{x}{\lambda}))^{\frac{1}{p}}$ is nonincreasing with respect to $\lambda \in (0, \infty)$ for all $x \in L^\Phi$. Let us define $x = \chi_A$, where $A \in \Sigma$ with $\mu(A) \in (0, \mu(\Omega))$. Then $\lambda^p \Phi(\frac{1}{\lambda})\mu(A)$ is a nonincreasing function of λ, so $\lambda^p \Phi(\frac{1}{\lambda})$ is nonincreasing with respect to $\lambda \in (0, \infty)$. Consequently, the function $\frac{\Phi(u)}{u^p}$ is nondecreasing with respect to $u \in (0, \infty)$, which, by virtue of Lemma 2.3, means that $p \in [1, p_S^a(\Phi)]$. □

Lemma 2.7. *Let Φ be an Orlicz function, let $x \in L^\Phi$, and let $\Phi \in \Delta_2(\mathbb{R}_+)$. Then the function $\lambda (I_\Phi(\frac{x}{\lambda}))^{1/q}$ is quasi-nonincreasing with respect to $\lambda \in (0, \infty)$ if $0 < q < i(\Phi)$ and quasi-nondecreasing with respect to $\lambda \in (0, \infty)$ if $q > I(\Phi)$.*

Proof. We will only present the proof of the fact that the function $\lambda (I_\Phi(\frac{x}{\lambda}))^{\frac{1}{q}}$ is quasi-nonincreasing with respect to $\lambda \in (0, \infty)$ for all $x \in L^\Phi$ if $0 < q < i(\Phi)$, because the proof of the second statement is similar. By virtue of Theorem 1.1, there exists an Orlicz function Ψ such that $p_S^a(\Psi) > i(\Phi) - \delta =: q$ for any $\delta \in (0, i(\Phi))$ and Ψ is equivalent to Φ on $\mathbb{R}_+$, that is, there exist positive constants L_1 and L_2 such that $L_1 \le L_2$ and

$$\Psi(L_1 u) \le \Phi(u) \le \Psi(L_2 u) \tag{2.2}$$

for all $u \in \mathbb{R}_+$. Since $\Phi \in \Delta_2(\mathbb{R}_+)$, by (2.2), we also have $\Psi \in \Delta_2(\mathbb{R}_+)$, whence it follows by (2.2) that there exist positive constants K_1 and K_2 such that $K_1 \le K_2$ and

$$K_1 \Psi(u) \le \Phi(u) \le K_2 \Psi(u) \tag{2.3}$$

for any $u \in \mathbb{R}_+$. From $\inf_{t>0} \frac{t\Psi'(t)}{\Psi(t)} > q$, we get $\frac{\Psi'(t)}{\Psi(t)} > \frac{q}{t}$ for all $t > 0$. Therefore, for all $\lambda \ge 1$ and $u > 0$, we have

$$\begin{aligned}\int_u^{\lambda u} \frac{\Psi'(t)}{\Psi(t)}\, dt &> q \int_u^{\lambda u} \frac{dt}{t},\\ \log \Psi(t)\big|_u^{\lambda u} &> \log t^q \big|_u^{\lambda u},\\ \log \frac{\Psi(\lambda u)}{\Psi(u)} &> \log \frac{\lambda^q u^q}{u^q} = \log \lambda^q,\end{aligned}$$

so $\Psi(\lambda u) > \lambda^q \Psi(u)$ for all $u > 0$ and $\lambda \geq 1$. Next we have

$$\Phi(\lambda u) \geq K_1 \Psi(\lambda u) > K_1 \lambda^q \Psi(u) \geq \frac{K_1}{K_2} \lambda^q \Phi(u). \tag{2.4}$$

In consequence, assuming that $0 < \lambda_1 \leq \lambda_2 < \infty$, we get $0 < 1/\lambda_2 \leq 1/\lambda_1 < \infty$, so by applying inequality (2.4) with $\lambda = \frac{\lambda_2}{\lambda_1}$ and $u = \frac{1}{\lambda_2}$, we obtain

$$\lambda_2^q \Phi\Big(\frac{1}{\lambda_2}\Big) \leq \frac{K_2}{K_1} \lambda_1^q \Phi\Big(\frac{1}{\lambda_1}\Big),$$

which means that the function $\lambda^q \Phi(\frac{1}{\lambda})$ is quasi-nonincreasing. Let us now take arbitrary $a > 0$. Then

$$\begin{aligned} \lambda_2^q \Phi\Big(\frac{a}{\lambda_2}\Big) &= a^q \Big(\frac{\lambda_2}{a}\Big)^q \Phi\Big(\frac{1}{\frac{\lambda_2}{a}}\Big) \\ &\leq L a^q \Big(\frac{\lambda_1}{a}\Big)^q \Phi\Big(\frac{1}{\frac{\lambda_1}{a}}\Big) \\ &= L \lambda_1^q \Phi\Big(\frac{a}{\lambda_1}\Big). \end{aligned}$$

Consequently, there is $L > 0$ such that $\lambda_2^q \Phi(\frac{x(t)}{\lambda_2}) \leq L \lambda_1^q \Phi(\frac{x(t)}{\lambda_1})$ for all $x \in L^\Phi$, $0 < \lambda_1 \leq \lambda_2 < \infty$, and μ-a.e. $t \in \Omega$. By integrating both sides of this inequality over Ω, we obtain $\lambda_2^q I_\Phi(\frac{x}{\lambda_2}) \leq L \lambda_1^q I_\Phi(\frac{x}{\lambda_1})$ for all $x \in L^\Phi$, which means that the function $\lambda^q I_\Phi(\frac{x}{\lambda})$ is quasi-nonincreasing with respect to $\lambda > 0$. Therefore, for any $x \in L^\Phi$, the function $\lambda (I_\Phi(\frac{x}{\lambda}))^{1/q}$ is also quasi-nonincreasing with respect to $\lambda > 0$. □

Lemma 2.8. *For any Orlicz function $\Phi : [0, \infty) \to [0, \infty)$ satisfying the $\Delta_2(\infty)$-condition, there exists an Orlicz function $\widetilde{\Phi}$ equivalent to Φ at infinity such that $\widetilde{\Phi} \in \Delta_2(\mathbb{R}_+)$, $p_S^a(\widetilde{\Phi}) = p \leq p_S^l(\Phi)$, and $q_S^a(\widetilde{\Phi}) = q_S^l(\Phi)$.*

Proof. Without loss of generality, we can assume that $\Phi(1) = 1$. Let $p \in [1, p_S^l(\Phi)]$, and define

$$\widetilde{\Phi}(t) = \begin{cases} t^p & \text{if } t < 1, \\ \Phi(t) & \text{if } t \geq 1. \end{cases}$$

The functions Φ and $\widetilde{\Phi}$ are equal (hence equivalent) at infinity, and $\widetilde{\Phi}$ is an Orlicz function (note that $\widetilde{\Phi}$ is convex because $\widetilde{\Phi}'_-(1) = p \leq p_S^l(\Phi) \leq \frac{\Phi'(1)}{\Phi(1)} = \Phi'(1) = \widetilde{\Phi}'(1)$). Moreover, it is clear that $\widetilde{\Phi} \in \Delta_2(\mathbb{R}_+)$. Since $\inf_{0<t\leq 1} \frac{t\widetilde{\Phi}'(t)}{\widetilde{\Phi}(t)} = \sup_{0<t\leq 1} \frac{t\widetilde{\Phi}'(t)}{\widetilde{\Phi}(t)} = p$, we get

$$p_S^a(\widetilde{\Phi}) = \min\big\{p, p_S^l(\Phi)\big\} = p \leq p_S^l(\Phi) = \inf_{t \geq 1} \frac{t\Phi'(t)}{\Phi(t)}$$

and

$$q_S^a(\widetilde{\Phi}) = \max\big\{p, q_S^l(\Phi)\big\} = q_S^l(\Phi) = \sup_{t \geq 1} \frac{t\Phi'(t)}{\Phi(t)},$$

which finishes our proof. □

Lemma 2.9. *If an Orlicz function Φ satisfies the $\Delta^2(0)$-condition, then $\Phi \in \Delta_3(0)$. If $\Phi \in \Delta^2(\mathbb{R}_+)$, then $\Phi \in \Delta_3(\mathbb{R}_+)$.*

Proof. We can assume without loss of generality that $\Phi(1) = 1$. Let Φ be an Orlicz function such that $\Phi \in \Delta^2(0)$. Then $u\Phi(u) \leq \Phi(u)$ for all $0 < u \leq 1$. Since $\Phi \in \Delta^2(0)$, we can find a constant $S > 0$ such that $\Phi(Su) \leq \Phi^2(u)$ for $0 < u \leq 1$. Moreover, since Φ is convex, we have $\Phi(u) \leq u$ for $0 < u \leq 1$, whence

$$\Phi(Su) \leq \Phi^2(u) = \Phi(u) \cdot \Phi(u) \leq u\Phi(u)$$

for $0 < u \leq 1$, which shows that $\Phi \in \Delta_3(0)$. Since it is known (see [20]) that $\Phi \in \Delta^2(\infty)$ implies that $\Phi \in \Delta_3(\infty)$, we conclude that $\Phi \in \Delta_3(\mathbb{R}_+)$. □

Lemma 2.10 ([20, Theorem 6.1, p. 36])**.** *Assume that the Orlicz function Φ satisfies the $\Delta_3(\infty)$-condition. Then the function Ψ complementary to Φ in the sense of Young satisfies the inequality*

$$k_1 v\Phi^{-1}(k_1 v) \leq \Psi(v) \leq k_2 v\Phi^{-1}(k_2 v)$$

for large values of v, where Φ^{-1} is the inverse function of Φ and k_1, k_2 ($k_1 \leq k_2$) are positive constants independent of v.

Definition 2.11. A function $f : [0, \infty) \to [0, \infty)$ is said to satisfy the $\nabla_3(\infty)$-condition (the $\nabla_3(\mathbb{R}_+)$-condition) if there exist constants $K, L > 0$ such that, for $v \geq 1$ (for $v > 0$), the inequalities $Kf(v) \leq f(f(v)v) \leq Lf(v)$ hold, respectively. We then write $f \in \nabla_3(\infty)$ ($f \in \nabla_3(\mathbb{R}_+)$).

Remark 2.12. Note that when Φ is an Orlicz function such that $\Phi(1) = 1$, we always have $\Phi^{-1}(v) \leq \Phi^{-1}(\Phi^{-1}(v)v)$ for $v \geq 1$, so the condition $\Phi^{-1} \in \nabla_3(\infty)$ reduces only to the right inequality in Definition 2.11 for Φ^{-1} in place of f.

Lemma 2.13. *Let Φ be an Orlicz function such that $\Phi(1) = 1$. Then the following conditions are equivalent:*

(1) $\Phi \in \Delta_3(\infty)$, *that is, there exists $K \geq 1$ such that for $u \geq 1$ the inequality $u\Phi(u) \leq \Phi(Ku)$ holds;*
(2) $\Phi^{-1} \in \nabla_3(\infty)$.

Proof. Assume that condition (1) holds, and set $u = \Phi^{-1}(v)$. Then $v \geq 1$ and by virtue of (1) we have $\Phi^{-1}(v)v \leq \Phi(K\Phi^{-1}(v))$, so $\Phi^{-1}(\Phi^{-1}(v)v) \leq K\Phi^{-1}(v)$ for $v \geq 1$, that is, $\Phi^{-1} \in \nabla_3(\infty)$. Now assume that (2) holds. Setting $v = \Phi(u)$, we get $u \geq 1$, and by condition (2) applied to $v = \Phi(u)$ there exists $K \geq 1$ such that $\Phi^{-1}(u\Phi(u)) \leq Ku$, so $u\Phi(u) \leq \Phi(Ku)$ for $u \geq 1$, which ends the proof. □

Lemma 2.14. *Let Φ be an Orlicz function such that $\Phi(1) = 1$. Then the following conditions are equivalent:*

(1) $\Phi \in \Delta_3(\mathbb{R}_+)$, *that is, there exist $K \leq 1$ and $L \geq 1$ such that for $u > 0$ the inequality $\Phi(Ku) \leq u\Phi(u) \leq \Phi(Lu)$ holds;*
(2) $\Phi^{-1} \in \nabla_3(\mathbb{R}_+)$.

Proof. Assume that (1) holds, and set $u = \Phi^{-1}(v)$. Then $v > 0$ and by virtue of (1) we have $\Phi(K\Phi^{-1}(v)) \leq \Phi^{-1}(v)v \leq \Phi(L\Phi^{-1}(v))$, so $K\Phi^{-1}(v) \leq \Phi^{-1}(\Phi^{-1}(v)v) \leq L\Phi^{-1}(v)$ for $v > 0$, that is, $\Phi^{-1} \in \nabla_3(\mathbb{R}_+)$. Now assume that (2) holds. Setting

$v = \Phi(u)$, we get $u > 0$, and by (2) there exist $K \le 1$ and $L \ge 1$ such that $Ku \le \Phi^{-1}(u\Phi(u)) \le Lu$, so $\Phi(Ku) \le u\Phi(u) \le \Phi(Lu)$ for $u > 0$, which ends the proof. □

Definition 2.15. Let $p \in (1, \infty)$. A function $f : [0, \infty) \to [0, \infty)$ is said to satisfy the condition $\Delta^p(0)$ $(\Delta^p(\infty))$ $[\Delta^p(\mathbb{R}_+)]$ if there exist $K, L > 0$ such that $f(Ku) \le f^p(u) \le f(Lu)$ for $0 < u \le 1$ (for $u \ge 1$) [for $u \in \mathbb{R}_+$]. We then write $f \in \Delta^p(0)$ $(f \in \Delta^p(\infty))$ $[f \in \Delta^p(\mathbb{R}_+)]$.

Lemma 2.16. *Let Φ be an Orlicz function such that $\Phi(1) = 1$. The following conditions are equivalent:*

(1) $\Phi \in \Delta^2(\infty)$;
(2) *there exists $p \in (1, \infty)$ such that $\Phi \in \Delta^p(\infty)$;*
(3) *for any $p \in (1, \infty)$, we have $\Phi \in \Delta^p(\infty)$.*

Proof. It is obvious that (3) implies (2).

We show that (1) implies (3). Let $\Phi \in \Delta^2(\infty)$, and let $p \in (1, \infty)$ be arbitrary. Note that if $u \ge 1$, then $\Phi(u) \ge 1$ and the inequality $\Phi(u) \le \Phi^p(u)$ is obvious for any $p \in (1, \infty)$. Therefore, we only need to show that the second inequality in the condition $\Delta^p(\infty)$ holds true. Assume that $1 < p \le 2$. Then, since $\Phi(u) \ge 1$ and $\Phi \in \Delta^2(\infty)$, there exists a constant $K \ge 1$ such that

$$\Phi^p(u) \le \Phi^2(u) \le \Phi(Ku)$$

for $u \ge 1$. If $p > 2$, then there exists the smallest $n \in \mathbb{N}$ such that $p \le 2^n$. Moreover, if $u \ge 1$, then $K^i u \ge 1$ for any $1 \le i \le n-1$. Consequently,

$$\begin{aligned}\Phi^p(u) &\le \Phi^{2^n}(u) = \big[\Phi^2(u)\big]^{2^{n-1}} \\ &\le \big[\Phi(Ku)\big]^{2^{n-1}} \\ &= \big[\Phi^2(Ku)\big]^{2^{n-2}} \le \big[\Phi(K^2u)\big]^{2^{n-2}} \le \cdots \le \Phi\big(K^n(u)\big),\end{aligned}$$

which ends the proof of our implication. In a similar way we can prove that (2) implies (1), so the proof is finished. □

Lemma 2.17. *Let Φ be an Orlicz function such that $\Phi(1) = 1$. The following conditions are equivalent:*

(1) $\Phi \in \Delta^2(0)$,
(2) *there exists $p \in (1, \infty)$ such that $\Phi \in \Delta^p(0)$,*
(3) *for any $p \in (1, \infty)$, we have $\Phi \in \Delta^p(0)$.*

The proof is similar to that of Lemma 2.16.

Lemma 2.18. *Let Φ be an Orlicz function such that $\Phi(1) = 1$. The following conditions are equivalent:*

(1) $\Phi \in \Delta^2(\mathbb{R}_+)$,
(2) *there exists $p \in (1, \infty)$ such that $\Phi \in \Delta^2(\mathbb{R}_+)$,*
(3) *for any $p \in (1, \infty)$, we have $\Phi \in \Delta^p(\mathbb{R}_+)$.*

Proof. The proof is a consequence of Lemmas 2.17, 2.16, and the fact that every function Φ satisfying conditions $\Delta^p(0)$ and $\Delta^p(\infty)$ also satisfies condition $\Delta^p(\mathbb{R}_+)$. □

Definition 2.19. Let $p \in (1,\infty)$. A function $f : [0,\infty) \to [0,\infty)$ is said to satisfy the condition $\nabla^p(\mathbb{R}_+)$ ($\nabla^p(\infty)$) if there exist $K, L > 0$ such that $Kf(v) \le f(v^p) \le Lf(v)$ for $v \in \mathbb{R}_+$ (resp., for $v \ge 1$). We then write $f \in \nabla^p(\mathbb{R}_+)$ (resp., $f \in \nabla^p(\infty)$).

Remark 2.20. Note that the first inequality in the condition $\nabla^p(\infty)$ for the inverse function Φ^{-1} to an Orlicz function Φ such that $\Phi(1) = 1$ always holds because $\Phi^{-1}(v) \le \Phi^{-1}(v^p)$ for $v \ge 1$. Therefore, in order to check that $\Phi^{-1} \in \nabla^p(\infty)$, it is enough to check only that the inequality $\Phi^{-1}(v^p) \le L\Phi^{-1}(v)$ holds for $v \ge 1$ with some $L > 0$ independent of v.

Lemma 2.21. *Let Φ be an Orlicz function such that $\Phi(1) = 1$, and let $p \in (1,\infty)$. Then the following conditions are equivalent:*

(1) $\Phi \in \Delta^p(\mathbb{R}_+)$,
(2) $\Phi^{-1} \in \nabla^p(\mathbb{R}_+)$.

Proof. Let $p \in (1,\infty)$ be arbitrary. Assume first that condition (1) holds. Then putting $u = \Phi^{-1}(v)$ in (1), we obtain

$$\Phi\big(K\Phi^{-1}(v)\big) \le \big(\Phi\big(\Phi^{-1}(v)\big)\big)^p = v^p \le \Phi\big(L\Phi^{-1}(v)\big)$$

for some constants $K \le 1$, $L \ge 1$ independent of v and for all $v \in \mathbb{R}_+$. After composing these inequalities with Φ^{-1}, we obtain

$$K\Phi^{-1}(v) \le \Phi^{-1}(v^p) \le L\Phi^{-1}(v)$$

for all $v \in \mathbb{R}_+$, so condition (2) holds.

Now assume that (2) holds. Putting $v = \Phi(u)$ in (2), we get

$$Ku \le \Phi^{-1}\big(\Phi^p(u)\big) \le Lu$$

for some constants $K \le 1$, $L \ge 1$ independent of u and for all $u \in \mathbb{R}_+$, which is equivalent to

$$\Phi(Ku) \le \Phi^p(u) \le \Phi(Lu)$$

for all $u \in \mathbb{R}_+$, so condition (1) is satisfied. □

Corollary 2.22. *If Φ is an Orlicz function, then conditions $\Phi \in \Delta^2(\mathbb{R}_+)$ and $\Phi^{-1} \in \nabla^2(\mathbb{R}_+)$ are equivalent.*

A result similar to that of Lemma 2.21 is true in the case when $\Phi \in \Delta^p(\infty)$, $p \in (1,\infty)$, which is presented below without proof.

Lemma 2.23. *Let Φ be an Orlicz function such that $\Phi(1) = 1$, and let $p \in (1,\infty)$. Then the following conditions are equivalent:*

(1) $\Phi \in \Delta^p(\infty)$;
(2) $\Phi^{-1} \in \nabla^p(\infty)$.

Corollary 2.24. *For any Orlicz function Φ, conditions $\Phi \in \Delta^2(\infty)$ and $\Phi^{-1} \in \nabla^2(\infty)$ are equivalent.*

Remark 2.25. Since we always have $\Phi^2(u) \leq \Phi(u)$ for $u \leq 1$ and an Orlicz function Φ such that $\Phi(1) = 1$, then assuming that $\Phi \in \Delta^2(\infty)$, we get $\Phi^2(u) \leq \Phi(Ku)$ for $u > 0$ and some constant $K > 0$ independent of u.

Lemma 2.26. *If Φ is an Orlicz function satisfying the $\Delta_3(\infty)$-condition, then there is a constant $l \in (0, \infty)$ such that $\Phi^*(lv) \leq v\Phi^{-1}(v)$ for $v \geq 0$, where Φ^* denotes the function complementary to Φ in the sense of Young.*

Proof. By condition $\Delta_3(\infty)$ for Φ, there exists $k \in (1, \infty)$ such that

$$u\Phi(u) \leq \Phi(ku) \tag{2.5}$$

for $u \geq 1$. Since $u\Phi(u) \leq \Phi(u) \leq \Phi(ku)$ for $u \in [0, 1]$, we get by (2.5) that

$$u\Phi(u) \leq \Phi(ku) \tag{2.6}$$

for $u \geq 0$. Let us define the function

$$\Phi_1(u) = \int_0^{|u|} \Phi(t)\, dt.$$

Then, by (2.6), we have

$$\Phi_1(u) = \int_0^{u} \Phi(t)\, dt \leq u\Phi(u) \leq \Phi(ku)$$

for $u \geq 0$, whence we deduce that there exists $l \in (0, 1)$ such that

$$\Phi^*(lv) \leq \Phi_1^*(v) \tag{2.7}$$

for $v \geq 0$, where Φ^* and Φ_1^* are the functions complementary to Φ and Φ_1, respectively. Of course,

$$\Phi_1^*(v) = \int_0^{v} \Phi^{-1}(t)\, dt \leq v\Phi^{-1}(v)$$

for $v \geq 0$, which together with (2.7) gives $\Phi^*(lv) \leq v\Phi^{-1}(v)$ for $v \geq 0$, finishing the proof. □

Corollary 2.27. *Assuming that M is an Orlicz function of the form $M(t) = t\varrho(t)$, where ϱ^{-1} is an Orlicz function satisfying the $\Delta_3(\infty)$-condition, then from Lemma 2.26 we conclude that there is a constant $l \in (0, 1)$ such that $(\varrho^{-1})^*(lv) \leq v\varrho(v)$ for all $v \geq 0$. In consequence, taking into account that $[(\varrho^{-1})^*]^* = \varrho^{-1}$, there exists a constant $B \in [1, \infty)$ such that $M^*(v) \leq \varrho^{-1}(Bv)$ for all $v \geq 0$.*

Remark 2.28. Note that for an Orlicz function Φ such that $\Phi(1) = 1$ we have $\Phi^2(u) \leq \Phi(u) \leq \Phi(ku)$ for all $u \in [0, 1]$ and any $k \geq 1$, which means that $\Phi^{-1}(v^2) \leq k\Phi^{-1}(v)$ for all $v \in [0, 1]$ and any $k \geq 1$. Therefore, in the case of a nonatomic infinite measure space, assuming that $\Phi^{-1} \in \nabla^2(\infty)$, we even have $\Phi^{-1}(v^2) \leq k\Phi^{-1}(v)$ for all $v \geq 0$ and some $k \geq 1$. In a similar way, in the case of a nonatomic infinite measure space, assuming $\Phi^{-1} \in \nabla_3(\infty)$, we get $\Phi^{-1}(v\Phi^{-1}(v)) \leq L\Phi^{-1}(v)$ for all $v \geq 0$ and some $L \geq 1$.

We will also use the following lemma and corollary to prove our main results.

Lemma 2.29. *Let Φ be an Orlicz function, let (Ω,Σ,μ) be a nonatomic infinite but σ-finite measure space, and let $\omega(t)=\sum_{n=1}^{\infty}[2^n(1+\mu(T_n))]^{-1}\chi_{T_n}(t)$, where $\{T_n\}_{n=1}^{\infty}$ is a sequence of Σ-measurable sets of positive and finite measure such that $\bigcup_{n=1}^{\infty}T_n=\Omega$. Then for any $r\in[1,p_S^l(\Phi)]$ we have $L^\Phi(\Omega,\Sigma,\mu)\subseteq L^r_\omega(\Omega,\Sigma,\mu)$ and there exists a constant $K>0$ such that*

$$\|x\|_{L^r_\omega}=\Big(\int_\Omega\big|x(t)\big|^r\omega(t)\,d\mu(t)\Big)^{\frac{1}{r}}\le K\|x\|_\Phi$$

for any $x\in L^\Phi(\Omega,\Sigma,\mu)$.

Proof. We can assume without loss of generality that $\Phi(1)=1$. By the proof of Lemma 2.7, we know that, given any $r\in[1,p_S^l(\Phi)]$, there exists a constant $L_r>0$ such that for all $u\in[1,\infty)$, we have

$$|u|^r\le L_r\Phi(u). \tag{2.8}$$

Let us define the Musielak–Orlicz function (for the definition of such functions, see [17], [28], and the last section of [3])

$$\Psi(t,u)=|u|^r\omega(t),\quad \forall u\in\mathbb{R},t\in\Omega.$$

Then for any $x\in L^\Psi(\Omega,\Sigma,\mu)$ (the Musielak–Orlicz space generated by the Musielak–Orlicz function Ψ over the measure space (Ω,Σ,μ)), we have

$$\begin{aligned}\|x\|_\Psi&=\inf\Big\{\lambda>0:\int_\Omega\Big|\frac{x(t)}{\lambda}\Big|^r\omega(t)\,d\mu(t)\le 1\Big\}\\&=\inf\Big\{\lambda>0:\int_\Omega\big|x(t)\big|^r\omega(t)\,d\mu(t)\le\lambda^r\Big\}\\&=\inf\Big\{\lambda>0:\Big(\int_\Omega\big|x(t)\big|^r\omega(t)\,d\mu(t)\Big)^{1/r}\le\lambda\Big\}\\&=\Big(\int_\Omega\big|x(t)\big|^r\omega(t)\,d\mu(t)\Big)^{1/r}=\|x\|_{L^r_\omega}.\end{aligned}$$

Let us note that condition (2.8) for $u\in[1,\infty)$ implies that, for μ-a.e. $t\in\Omega$ and all $u\in\mathbb{R}$, the inequality

$$\Psi(t,u)\le L_r\Phi(u)\omega(t)+\omega(t)$$

holds. Therefore, for arbitrary $x\in L^\Phi(\Omega,\Sigma,\mu)$ we have

$$\begin{aligned}I_\Psi\Big(\frac{x}{\|x\|_\Phi}\Big)&\le L_r\int_\Omega\Phi\Big(\frac{x(t)}{\|x\|_\Phi}\Big)\omega(t)\,d\mu(t)+\int_\Omega\omega(t)\,d\mu(t)\\&=L_r\sum_{n=1}^{\infty}\big[2^n\big(1+\mu(T_n)\big)\big]^{-1}\int_{T_n}\Phi\Big(\frac{x(t)}{\|x\|_\Phi}\Big)\,d\mu(t)+\int_\Omega\omega(t)\,d\mu(t)\\&\le L_r\sum_{n=1}^{\infty}\big[2^n\big(1+\mu(T_n)\big)\big]^{-1}+1\le L_r+1.\end{aligned}$$

Hence, by convexity of I_Ψ, we get $I_\Psi(\frac{x}{(1+L_r)\|x\|_\Phi}) \leq 1$, whence

$$\|x\|_{L^r_\omega} = \|x\|_\Psi \leq (1+L_r)\|x\|_\Phi,$$

which finishes the proof. □

Corollary 2.30. *If $\mu(\Omega) < \infty$, then by (2.8) we get $L^\Phi(\Omega, \Sigma, \mu) \subseteq L^r(\Omega, \Sigma, \mu) = L^r_{\chi_\Omega}(\Omega, \Sigma, \mu)$. So, setting $\omega(t) = \chi_\Omega(t)$ in the last lemma, we obtain*

$$\|x\|_{L^r} \leq K\|x\|_\Phi, \quad \forall x \in L^\Phi(\Omega, \Sigma, \mu)$$

where $K = L_r + \mu(\Omega)$, with L_r satisfying condition (2.8).

3. Main results

We now give one of the most important results of this article.

Theorem 3.1. *Assume that (Ω, Σ, μ) is a nonatomic finite measure space and that Φ is an Orlicz function. Let $p \in [1, p^l_S(\Phi)]$, and let $q(\Phi) := q^l_S(\Phi)$ satisfy the inequality $q(\Phi) - p \leq 1$. Define $\varrho(t) := \frac{\Phi(t)}{t^p}$. Then the following assertions hold.*

(1) *If $\Phi \in \Delta_2(\infty)$, then there exist positive constants K and L such that for any $f \in L^\Phi$, we have*

$$K[f]_{\widetilde{\Phi},p} \leq \|f\|_\Phi \leq L[f]_{\widetilde{\Phi},p}, \tag{3.1}$$

where $[f]_{\widetilde{\Phi},p}$ is a quasinorm in L^Φ equal to zero if $f = 0$ and for $f \in L^\Phi\backslash\{0\}$ defined by the formula

$$[f]_{\widetilde{\Phi},p} := \begin{cases} \|f\|_{L^r} I_{\widetilde{\Phi}}(\frac{f}{\|f\|_{L^r}}) & \text{if } p = 1 \text{ and } \varrho \in \nabla_3(\infty), \\ \|f\|_{L^r} (I_{\widetilde{\Phi}}(\frac{f}{\|f\|_{L^r}}))^{\frac{1}{p}} & \text{if } p > 1 \text{ and } \varrho \in \nabla^2(\infty), \end{cases}$$

where $r \in [1, p^l_S(\Phi)]$ and $\widetilde{\Phi}$ is the function from Lemma 2.8 with $p^a_S(\widetilde{\Phi}) = p \leq p^l_S(\Phi)$, $q^a_S(\widetilde{\Phi}) = q(\Phi)$, and $\widetilde{\Phi} \in \Delta_2(\mathbb{R}_+)$.

(2) *If $\Phi \in \Delta_2(\mathbb{R}_+)$, then there exist positive constants K and L such that for any $f \in L^\Phi$, we have*

$$K[f]_{\Phi,p} \leq \|f\|_\Phi \leq L[f]_{\Phi,p} \tag{3.2}$$

with the quasinorm $[\,\cdot\,]_{\Phi,p}$ defined as in statement (1) with $r \in [1, p^l_S(\Phi)]$ and with Φ instead of $\widetilde{\Phi}$.

Proof. The proof proceeds in the following way. First, using Lemma 2.6 and Corollary 2.30 we get the lower estimate of $[\,\cdot\,]_{\widetilde{\Phi},p}$ by $\|\cdot\|_{\widetilde{\Phi}}$ for an Orlicz function $\widetilde{\Phi}$ from Lemma 2.8 with $p^a_S(\widetilde{\Phi}) = p \leq p^l_S(\Phi)$, $q^a_S(\widetilde{\Phi}) = q(\Phi)$, $\widetilde{\Phi} \in \Delta_2(\mathbb{R}_+)$ equivalent to Φ. In order to prove the upper estimate of $[\,\cdot\,]_{\widetilde{\Phi},p}$ by $\|\cdot\|_\Phi$, we apply Lemmas 2.2 and 2.3, Corollary 2.5, and Lemma 2.8 (which is possible by the assumption that $g^a_S(\widetilde{\Phi}) - p \leq 1$), as well as the condition $\varrho \in \nabla_3(\infty)$ if $p = 1$ or the condition $\varrho \in \nabla^2(\infty)$ if $p > 1$, along with Lemmas 2.10 and 2.13 and Corollary 2.24.

Let $p \in [1, p_S^l(\Phi)]$, and let $\widetilde{\Phi}$ be the function from Lemma 2.8 with $p_S^a(\widetilde{\Phi}) = p$. By Lemma 2.6 and the fact that $\widetilde{\Phi} \in \Delta_2(\mathbb{R}_+)$ (note that $\widetilde{\Phi}(1) = 1$), we get for all $f \in L_{\widetilde{\Phi}} \backslash \{0\}$ and for the constant $K > 1$ from Corollary 2.30 that

$$\begin{aligned}\|f\|_{\widetilde{\Phi}} &= \|f\|_{\widetilde{\Phi}} \Big(I_{\widetilde{\Phi}} \Big(\frac{f}{\|f\|_{\widetilde{\Phi}}} \Big) \Big)^{\frac{1}{p}} \\ &\le K^{-1} \|f\|_{L^r} \Big(I_{\widetilde{\Phi}} \Big(\frac{Kf}{\|f\|_{L^r}} \Big) \Big)^{\frac{1}{p}} \\ &\le N \|f\|_{L^r} \Big(I_{\widetilde{\Phi}} \Big(\frac{f}{\|f\|_{L^r}} \Big) \Big)^{\frac{1}{p}} \\ &= N[f]_{\widetilde{\Phi},p}\end{aligned}$$

for some absolute constant $N > 0$ independent of f. Hence, by equivalence of the norms $\|\cdot\|_\Phi$ and $\|\cdot\|_{\widetilde{\Phi}}$, we get $\|f\|_\Phi \le M[f]_{\widetilde{\Phi},p}$ for any $f \in L^\Phi \backslash \{0\}$ and for some absolute constant $M > 0$ independent of f, which ends the proof of the upper estimate of the Luxemburg norm $\|f\|_\Phi$ by the quasinorm $[f]_{\widetilde{\Phi},p}$.

Now we will find the lower estimate in (3.1). Obviously, we can write the Orlicz function Φ in the form

$$\Phi(t) = t^p \cdot \varrho(t), \tag{3.3}$$

where $\varrho(t) := \frac{\Phi(t)}{t^p}$. In the remainder of the proof we assume that $\varrho \in \nabla^2(\infty)$ or, equivalently, that $\varrho^{-1} \in \Delta^2(\infty)$ (see Corollary 2.24). Note that by virtue of Lemmas 2.2–2.3 and Remark 2.1, the assumption

$$q_S^a(\widetilde{\Phi}) - p_S^a(\widetilde{\Phi}) = q(\Phi) - p \le 1 \tag{3.4}$$

guarantees the existence of a concave function equivalent at infinity to the function ϱ, so that ϱ^{-1} is equivalent at infinity to a convex function. Indeed, by Lemma 2.3 applied to the function $\widetilde{\varrho}(t) := \frac{\widetilde{\Phi}(t)}{t^p}$, where $p_S^a(\widetilde{\Phi}) = p \le p_S^l(\Phi)$, we conclude that $\widetilde{\varrho}$ is nondecreasing and that $\frac{\widetilde{\varrho}(t)}{t} = \frac{\widetilde{\Phi}(t)}{t^{p+1}}$ is nonincreasing because of (3.4), that is, $q_S^a(\widetilde{\Phi}) \le p + 1 = p_S^a(\widetilde{\Phi}) + 1$. Therefore, by virtue of Remark 2.1, we can apply Lemma 2.2, obtaining the existence of a concave function $\overline{\varrho}$ equivalent to $\widetilde{\varrho}$. Since $\Phi \sim \widetilde{\Phi}$ at infinity, we get $\varrho \sim \widetilde{\varrho}$ at infinity, and since $\overline{\varrho} \sim \widetilde{\varrho}$, so ϱ is equivalent to the concave function $\overline{\varrho}$ at infinity.

We will show that $[f]_{\Phi,p} \le C$ for $f \in S(L^\Phi)$ and some absolute constant $C > 0$ independent of f. Since $\Phi(t) = t^p \cdot \varrho(t)$, by the definition of the quasinorm $[\,\cdot\,]_{\Phi,p}$, where $p \in [1, p_S^l(\Phi)]$, it is enough to show that

$$\|f\|_{L^r} \Big(\int_\Omega \Big(\frac{|f(t)|}{\|f\|_{L^r}} \Big)^p \varrho\Big(\frac{|f(t)|}{\|f\|_{L^r}} \Big)\, d\mu(t) \Big)^{\frac{1}{p}} = \Big(\int_\Omega \big|f(t)\big|^p \varrho\Big(\frac{|f(t)|}{\|f\|_{L^r}} \Big)\, d\mu(t) \Big)^{\frac{1}{p}} \le F,$$

where $F > 0$ is some absolute constant independent of f or, equivalently, that

$$\int_\Omega \big|f(t)\big|^p \varrho\Big(\frac{|f(t)|}{\|f\|_{L^r}} \Big)\, d\mu(t) \le D,$$

where $D > 0$ is some absolute constant independent of f. Below in both cases $p = 1$ and $p > 1$ we will use the $\nabla_3(\infty)$-condition for the function ϱ (equivalently, by Lemma 2.13, the $\Delta_3(\infty)$-condition for the function ϱ^{-1}), which is possible because the $\Delta^2(\infty)$-condition for the function ϱ^{-1} (equivalently, condition $\nabla^2(\infty)$ for the function ϱ) implies the $\Delta_3(\infty)$-condition for the function ϱ^{-1} (see [20]; equivalently, the $\nabla_3(\infty)$-condition for ϱ). By virtue of Theorem 6.1 from [20] (see also Lemma 2.10 in this article), we get that the function $(\varrho^{-1})^*$ complementary to ϱ^{-1} in the sense of Young is equivalent to $M(t) := t\varrho(t)$ at infinity. Consequently, the function M^* complementary to M is equivalent to ϱ^{-1} at infinity; that is, there exist constants $A, B > 0$ such that

$$\varrho^{-1}(At) \le M^*(t) \le \varrho^{-1}(Bt) \tag{3.5}$$

for large values of $t > 0$, so also for $t \ge 1$. Then, by Young's inequality, for any $f \in S(L^\Phi)$ we get

$$\begin{aligned}
&\int_\Omega |f(t)|^p \varrho\Big(\frac{|f(t)|}{\|f\|_{L^r}}\Big)\, d\mu(t)\\
&\quad = B\int_\Omega \big(|f(t)|\big)^p \frac{1}{B}\varrho\Big(\frac{|f(t)|}{\|f\|_{L^r}}\Big)\, d\mu(t)\\
&\quad \le B\Big[\int_\Omega M\big(|f(t)|^p\big)\, d\mu(t) + \int_\Omega M^*\Big(\frac{1}{B}\varrho\Big(\frac{|f(t)|}{\|f\|_{L^r}}\Big)\Big)\, d\mu(t)\Big],
\end{aligned} \tag{3.6}$$

where $B > 0$ is the constant from the second inequality in (3.5). Defining the set

$$E = \Big\{t \in \Omega : \frac{|f(t)|}{\|f\|_{L^r}} \ge 1\Big\},$$

noting that $\varrho(1) = \frac{\Phi(1)}{1} = 1$, and applying Corollary 2.30, we can continue the upper estimate in (3.6) as

$$\begin{aligned}
&\int_\Omega \big(|f(t)|\big)^p \varrho\Big(\frac{|f(t)|}{\|f\|_{L^r}}\Big)\, d\mu(t)\\
&\quad \le B\Big[\int_\Omega M\big(|f(t)|^p\big)\, d\mu(t) + \int_{\Omega\setminus E} M^*\Big(\frac{1}{B}\varrho\Big(\frac{|f(t)|}{\|f\|_{L^r}}\Big)\Big)\, d\mu(t)\\
&\qquad + \int_E \varrho^{-1}\circ\varrho\Big(\frac{|f(t)|}{\|f\|_{L^r}}\Big)\, d\mu(t)\Big]\\
&\quad \le B\Big[\int_\Omega \big(|f(t)|\big)^p \varrho\big(\big(|f(t)|\big)^p\big)\, d\mu(t) + \int_{\Omega\setminus E} M^*\Big(\frac{\varrho(1)}{B}\Big)\, d\mu(t)\\
&\qquad + \|f\|_{L^r}^{-1}\|f\|_1\Big]\\
&\quad \le B\int_\Omega \big(|f(t)|\big)^p \varrho\big(\big(|f(t)|\big)^p\big)\, d\mu(t) + BM^*\Big(\frac{1}{B}\Big)\mu(\Omega)\\
&\qquad + B\|f\|_{L^r}^{-1} k\|f\|_{L^r} \quad (k > 0)\\
&\quad \le C + C\int_\Omega \big(|f(t)|\big)^p \varrho\big(\big(|f(t)|\big)^p\big)\, d\mu(t),
\end{aligned} \tag{3.7}$$

with $C > 0$ being some absolute constant independent of f. We consider two cases.

Case 1. Let $p=1$. Since $\|f\|_\Phi=1$ and $\Phi\in\Delta_2(\infty)$, we obtain that $I_\Phi(f)=1$ and consequently

$$\int_\Omega |f(t)|\varrho\Big(\frac{|f(t)|}{\|f\|_{L^r}}\Big)\,d\mu(t)\le C+C\int_\Omega |f(t)|\varrho\big(|f(t)|\big)\,d\mu(t)=C+CI_\Phi(f)=2C,$$

which ends the proof of the theorem in this case.

Case 2. Assume that $p>1$. Then $\int_\Omega(|f(t)|)^p\varrho(|f(t)|)\,d\mu(t)=I_\Phi(f)=1$, so defining the set

$$F=\big\{t\in\Omega:\big|f(t)\big|\ge 1\big\}$$

and using the fact that $\varrho\in\nabla^2(\infty)$ (equivalently, $\varrho\in\nabla^p(\infty)$), we continue the upper estimate of (3.7) as

$$\begin{aligned}
&\int_\Omega \big(|f(t)|\big)^p\varrho\Big(\frac{|f(t)|}{\|f\|_{L^r}}\Big)\,d\mu(t)\\
&\quad\le C+C\int_\Omega\big(|f(t)|\big)^p\varrho\big(\big(|f(t)|\big)^p\big)\,d\mu(t)\\
&\quad\le C+C\int_F\big(|f(t)|\big)^p\varrho\big(|f(t)|^p\big)\,d\mu(t)\\
&\qquad+C\int_{\Omega\setminus F}\big(|f(t)|\big)^p\varrho\big(|f(t)|^p\big)\,d\mu(t)\\
&\quad\le C+\widetilde{C}\int_F\big(|f(t)|\big)^p\varrho\big(|f(t)|\big)\,d\mu(t)+C\int_{\Omega\setminus F}\varrho(1)\,d\mu(t)\\
&\quad\le C+C\mu(\Omega)+\widetilde{C}\int_F\big(|f(t)|\big)^p\varrho\big(|f(t)|\big)\,d\mu(t)\\
&\quad\le C_3\int_\Omega\big(|f(t)|\big)^p\varrho\big(|f(t)|\big)\,d\mu(t)=C_3,
\end{aligned}$$

with some absolute constant $C_3>0$ independent of f. Hence if $\|f\|_\Phi=1$, then

$$[f]_{\Phi,p}\le C_3.$$

Thus $[\frac{f}{\|f\|_\Phi}]_{\Phi,p}\le C_3\|\frac{f}{\|f\|_\Phi}\|_\Phi$ for $f\in L^\Phi\setminus\{0\}$, so $[f]_{\Phi,p}\le C_3\|f\|_\Phi$ for all $f\in L^\Phi$.

Finally, since the functional $[\,\cdot\,]_{\Phi,p}$ is positively homogeneous, by virtue of (3.1), it is evident that $[\,\cdot\,]_{\widetilde{\Phi},p}$ is a quasinorm on the space L^Φ.

In the proof of the upper estimate in (3.2) under the assumption that $\Phi\in\Delta_2(\mathbb{R}_+)$, we work with the original function Φ, obtaining directly that $\|f\|_\Phi\le M[f]_{\Phi,p}$ for all $f\in L^\Phi$. The proof of the lower estimate in (3.2) runs in the same way as that of (3.1). □

Remark 3.2. Note that if $p_S^l(\Phi)=1$ and $q_S^l(\Phi)\le p_S^l(\Phi)+1=2$ or if $p_S^l(\Phi)>1$ and $q_S^l(\Phi)=p_S^l(\Phi)+1$, the power p generating the quasinorm $[\,\cdot\,]_{\Phi,p}$ is unique and equals $p_S^l(\Phi)$.

Remark 3.3. Note that by virtue of Lemma 2.23, the assumption $\varrho\in\nabla^2(\infty)$ in Theorem 3.1 may be replaced by $\varrho^{-1}\in\Delta^2(\infty)$.

Remark 3.4. If for an Orlicz function Φ we have $p^a_S(\Phi) = q^a_S(\Phi)$, then denoting this common value by p the function Φ turns out to be equal to $|u|^p$. Hence for $f \in L^\Phi \backslash \{0\}$, we get directly

$$\begin{aligned}
[f]_{\widetilde{\Phi},p} &= \|f\|_{L^r} \Big(I_{\widetilde{\Phi}} \Big(\frac{f}{\|f\|_{L^r}} \Big) \Big)^{\frac{1}{p}} \\
&= \|f\|_{L^r} \Big(\int_\Omega \frac{(f(t))^p}{\|f\|^p_{L^r}} \, d\mu(t) \Big)^{\frac{1}{p}} \\
&= \Big(\int_\Omega \big(f(t)\big)^p \, d\mu(t) \Big)^{\frac{1}{p}} = \|f\|_{L^p},
\end{aligned}$$

meaning that in this case our quasinorm is actually the L^p-norm.

Now we will show the necessity of condition $\nabla_3(\infty)$ or $\nabla^p(\infty)$ on the generating Orlicz function Φ in Theorem 3.1.

Theorem 3.5. *If μ is nonatomic and finite, $p^l_S(\Phi) = 1 = \Phi(1)$, and $\varrho \notin \nabla_3(\infty)$, where Φ and ϱ are the functions defined in Theorem 3.1, then there is no absolute constant $C > 0$ independent of f such that the estimate*

$$\|f\|_{L^1} I_\Phi\Big(\frac{f}{\|f\|_{L^1}} \Big) =: [f]_{\Phi,1} \le C \|f\|_\Phi \tag{3.8}$$

holds for $f \in L^\Phi \backslash \{0\}$, which means that when $p^l_S(\Phi) = 1$, the condition $\varrho \in \nabla_3(\infty)$ is necessary for the upper estimate of the quasinorm in Theorem 3.1.

Proof. Let $p^l_S(\Phi) = 1$. If $\varrho \notin \nabla_3(\infty)$, then (see Definition 2.11) for any sequence $\{b_k\}$ of positive numbers with $b_k \nearrow \infty$ one can find a nondecreasing sequence $\{t_k\}$ of positive numbers such that $\varrho(t_k) \min\{1, \mu(\Omega)\} \ge 1$ and

$$\varrho\big(\varrho(t_k) t_k\big) > b_k \varrho(t_k). \tag{3.9}$$

Define $f_k(t) = t_k \chi_{A_k}(t)$ for $A_k \in \Sigma$ such that

$$I_\Phi(f_k) = \int_{A_k} f_k(t) \varrho\big(f_k(t)\big) \, d\mu(t) = t_k \varrho(t_k) \mu(A_k) = 1, \tag{3.10}$$

which implies that $\|f_k\|_\Phi = 1$ for all $k \in \mathbb{N}$. Applying (3.10), we obtain

$$\begin{aligned}
[f_k]_{\Phi,1} &= \|f_k\|_{L^1} I_\Phi \Big(\frac{f_k}{\|f_k\|_{L^1}} \Big) \\
&= \int_{A_k} f_k(t) \varrho \Big(\frac{f_k(t)}{\|f_k\|_{L^1}} \Big) \, d\mu(t) \\
&= \int_{A_k} t_k \varrho \Big(\frac{t_k}{t_k \mu(A_k)} \Big) \, d\mu(t) \\
&= t_k \varrho \Big(\frac{1}{\mu(A_k)} \Big) \mu(A_k) \\
&= t_k \varrho\big(t_k \varrho(t_k) \big) \mu(A_k),
\end{aligned}$$

whence, by (3.9), we have for all $k \in \mathbb{N}$

$$[f_k]_{\Phi,1} = t_k \varrho\big(t_k \varrho(t_k)\big)\mu(A_k) > t_k b_k \varrho(t_k)\mu(A_k) = b_k.$$

Since $b_k \nearrow \infty$, the proof is finished. $\square$

Theorem 3.6. *If μ is nonatomic and finite, $\Phi(1) = 1$, and $\varrho \notin \nabla^2(\infty)$, where Φ and ϱ are the functions defined in Theorem 3.1, then for any r and p with $1 \le r < p \le p_S^l(\Phi)$, there is no absolute constant $C > 0$ independent of f such that the estimate*

$$\|f\|_{L^r}\Big(I_\Phi\Big(\frac{f}{\|f\|_{L^r}}\Big)\Big)^{\frac{1}{p}} =: [f]_{\Phi,p} \le C\|f\|_\Phi \tag{3.11}$$

holds for $f \in L^\Phi \backslash \{0\}$, which means that when $1 \le r < p \le p_S^l(\Phi)$, the condition $\varrho \in \nabla^2(\infty)$ is necessary for the upper estimate of the quasinorm in Theorem 3.1.

Proof. Take r and p such that $1 \le r < p \le p_S^l(\Phi)$, and assume that $\varrho \notin \nabla^2(\infty)$. Then $\varrho \notin \nabla^{\frac{p}{r}}(\infty)$, and hence for any sequence $\{b_k\}$ of positive numbers with $b_k \nearrow \infty$ one can find a nondecreasing sequence $\{t_k\}$ of positive numbers such that $\varrho(t_k)\min\{1, \mu(\Omega)\} \ge 1$ and

$$\varrho(t_k^{\frac{p}{r}}) > b_k \varrho(t_k). \tag{3.12}$$

Define $f_k(t) = t_k \chi_{A_k}(t)$ for $A_k \in \Sigma$ such that

$$I_\Phi(f_k) = \int_{A_k} \big(f_k(t)\big)^p \varrho\big(f_k(t)\big)\, d\mu(t) = t_k^p \varrho(t_k)\mu(A_k) = 1, \tag{3.13}$$

which implies that $\|f_k\|_\Phi = 1$ for all $k \in \mathbb{N}$. On the other hand, applying (3.12) and (3.13), we obtain

$$\begin{aligned}
[f_k]_{\Phi,p}^p &= I_\Phi\Big(\frac{f_k}{\|f_k\|_{L^r}}\Big)\|f_k\|_{L^r}^p \\
&= \int_{A_k} \big(f_k(t)\big)^p \varrho\Big(\frac{f_k(t)}{\|f_k\|_{L^r}}\Big)\, d\mu(t) \\
&= \int_{A_k} t_k^p \varrho\Big(\frac{t_k}{t_k(\mu(A_k))^{1/r}}\Big)\, d\mu(t) \\
&= t_k^p \varrho\Big(\frac{1}{(\mu(A_k))^{1/r}}\Big)\mu(A_k) \\
&= t_k^p \varrho\Big(t_k^{\frac{p}{r}}\big(\varrho(t_k)\big)^{\frac{1}{r}}\Big)\mu(A_k) \\
&> t_k^p \varrho(t_k^{\frac{p}{r}})\mu(A_k),
\end{aligned}$$

whence by $t_k \ge 1$ for any $k \in \mathbb{N}$, we have

$$[f_k]_{\Phi,p}^p > t_k^p b_k \varrho(t_k)\mu(A_k) = b_k$$

for all $k \in \mathbb{N}$. Now the assumption that $b_k \nearrow \infty$ finishes the proof. $\square$

Example 3.7. Let (Ω, Σ, μ) be a nonatomic finite measure space. Consider the function $\Phi(t) = \frac{|t|^p}{\log_a(a+1)} \log_a(a + |t|)$, where $a \in [e-1, \infty)$ and $p \in (1, \infty)$. We will show what the quasinorm $[\,\cdot\,]_{\Phi,p}$ defined in Theorem 3.1 and equivalent to the Luxemburg norm $\|\cdot\|_\Phi$ looks like in this case. Note that $\Phi(1) = 1$, $\Phi \in \Delta_2(\mathbb{R}_+)$ (because $\limsup_{t\to\infty} \frac{\Phi(2t)}{\Phi(t)} < \infty$, and $\limsup_{t\to 0} \frac{\Phi(2t)}{\Phi(t)} < \infty$) and $\frac{t\Phi'(t)}{\Phi(t)} = p + \frac{t}{(a+t)\log(a+t)}$ for any $t > 0$, where $\log(a+t)$ is the natural logarithm of $a+t$. We will find the upper estimate of $q_S^l(\Phi)$ and calculate $p_S^l(\Phi)$. Note that the function $g(t) := p + \frac{t}{(a+t)\log(a+t)}$ can have its local extremum only at $t \in \mathbb{R}$ satisfying $\log(a+t) = \frac{t}{a}$. Let $t_0 > 0$ be a positive solution of this equation. Then for $a \geq e - 1$, we have

$$\begin{aligned} g(t_0) &= p + \frac{t_0}{(a+t_0)\log(a+t_0)} \\ &= p + \frac{t_0}{(a+t_0)\frac{t_0}{a}} = p + \frac{a}{a+t_0} > p. \end{aligned}$$

We know from L'Hôpital's rule that $\lim_{t\to\infty} \frac{t}{(a+t)\log(a+t)} = 0$. Therefore

$$\begin{aligned} q_S^l(\Phi) &:= \sup_{t\geq 1} \frac{t\Phi'(t)}{\Phi(t)} \\ &= \sup_{t\geq 1}\Big(p + \frac{t}{(a+t)\log(a+t)}\Big) \\ &= \max\Big\{p + \frac{1}{(1+a)\log(1+a)}, p, p + \frac{a}{a+t_0}\Big\} \\ &\leq \max\Big\{p + \frac{1}{(1+a)\log(1+a)}, p + \frac{a}{a+1}\Big\} \\ &\leq \max\Big\{p + \frac{1}{e}, p + \frac{a}{a+1}\Big\} = p + \frac{a}{a+1} < p + 1 \end{aligned}$$

because $\frac{a}{a+1} \geq \frac{e-1}{e} > \frac{1}{e}$. Moreover,

$$\inf_{t\geq 1} \frac{t\Phi'(t)}{\Phi(t)} = \min\Big\{p + \frac{1}{(1+a)\log(1+a)}, p, p + \frac{a}{a+t_0}\Big\} = p,$$

so $p_S^l(\Phi) = p$. Consequently, $q_S^l(\Phi) - p < p + 1 - p = 1$ and we can apply Theorem 3.1, by which we obtain that the norm $\|f\|_\Phi$ is equivalent to the quasinorm

$$\begin{aligned} [f]_{\Phi,s} &= \|f\|_{L^r}\Big(I_\Phi\Big(\frac{f}{\|f\|_{L^r}}\Big)\Big)^{1/s} \\ &= \|f\|_{L^r}\Big(\int_\Omega \frac{1}{\log(a+1)}\Big(\Big|\frac{f(x)}{\|f\|_{L^r}}\Big|\Big)^s \log\Big(a + \frac{|f(x)|}{\|f\|_{L^r}}\Big)\,d\mu(x)\Big)^{1/s} \\ &= \Big(\int_\Omega \frac{|f(x)|^s}{\log(a+1)} \log\Big(a + \frac{|f(x)|}{\|f\|_{L^r}}\Big)\Big)^{1/s} \end{aligned}$$

with $r \in [1, p_S^l(\Phi)]$ and $s \in [1, p_S^l(\Phi)]$ satisfying $q_S^l(\Phi) - s < 1$.

Theorem 3.8. *Let (Ω, Σ, μ) be an infinite σ-finite nonatomic measure space, and let Φ be an Orlicz function satisfying the $\Delta_2(\mathbb{R}_+)$-condition and such that $\Phi(1) = 1$. Let $p \in [1, p_S^a(\Phi)]$ satisfy $q_S^a(\Phi) - p \leq 1$. Define $\varrho(t) := \frac{\Phi(t)}{t^p}$. Then there exist positive constants K and L such that for all $f \in L^\Phi$ we have*

$$K[f]_{\Phi,p} \leq \|f\|_\Phi \leq L[f]_{\Phi,p}, \tag{3.14}$$

where $[f]_{\Phi,p}$ is a quasinorm equal to zero if $f = 0$ and for $f \in L^\Phi \setminus \{0\}$ defined by

$$[f]_{\Phi,p} := \begin{cases} \|f\|_{L^r_\omega} I_\Phi(\frac{f}{\|f\|_{L^r_\omega}}) & \text{if } p = 1 \text{ and } \varrho \in \nabla_3(\infty), \\ \|f\|_{L^r_\omega} (I_\Phi(\frac{f}{\|f\|_{L^r_\omega}}))^{\frac{1}{p}} & \text{if } p > 1 \text{ and } \varrho \in \nabla^2(\infty), \end{cases}$$

where $\|\cdot\|_{L^r_\omega}$, $r \in [1, p_S^a(\Phi)]$, is the norm on the weighted space $L^r_\omega(\Omega, \Sigma, \mu)$ considered in Lemma 2.29.

Proof. The proof of Theorem 3.8 is similar to the proof of Theorem 3.1. The upper estimate for the Luxemburg norm is obtained by using Lemmas 2.6 and 2.29, as well as the assumption $\Phi \in \Delta_2(\mathbb{R}_+)$. Indeed, assuming $p \in [1, p_S^a(\Phi)]$, for all $f \in L_\Phi \setminus \{0\}$ and a constant $K \geq 1$ from Lemma 2.29, which is also true for $r \in [1, p_S^a(\Phi)]$, because $p_S^a(\Phi) \leq p_S^l(\Phi)$, we get

$$\begin{aligned} \|f\|_\Phi &= \|f\|_\Phi \Big(I_\Phi\Big(\frac{f}{\|f\|_\Phi}\Big)\Big)^{\frac{1}{p}} \\ &\leq K^{-1} \|f\|_{L^r} \Big(I_\Phi\Big(\frac{Kf}{\|f\|_{L^r}}\Big)\Big)^{\frac{1}{p}} \\ &\leq N \|f\|_{L^r} \Big(I_\Phi\Big(\frac{f}{\|f\|_{L^r}}\Big)\Big)^{\frac{1}{p}} = N[f]_{\Phi,p}, \end{aligned}$$

where $N > 0$ is an absolute constant independent of f, and this is the upper estimate of $\|f\|_\Phi$.

On the other hand, by applying Corollary 2.27 and taking into account Remark 2.28, the lower estimate of the Luxemburg norm can be obtained in a similar (and even easier) way as in the proof of Theorem 3.1. $\square$

Remark 3.9. Note that if $p_S^a(\Phi) = 1$ and $q_S^a(\Phi) \leq p_S^a(\Phi) + 1 = 2$ or if $p_S^a(\Phi) > 1$ and $q_S^a(\Phi) = p_S^a(\Phi) + 1$, the power p generating our quasinorm $[\cdot]_{\Phi,p}$ is unique and equals $p_S^a(\Phi)$.

Theorem 3.10. *If μ is nonatomic and infinite, $p_S^a(\Phi) = 1 = \Phi(1)$, and $\varrho \notin \nabla_3(\infty)$, where Φ and ϱ are the functions defined in Theorem 3.8, then for ω defined as in Lemma 2.29, there is no absolute constant $C > 0$ independent of f such that the estimate*

$$\|f\|_{L^1_\omega} I_\Phi\Big(\frac{f}{\|f\|_{L^1_\omega}}\Big) =: [f]_{\Phi,1} \leq C\|f\|_\Phi \tag{3.15}$$

holds for all $f \in L^\Phi \setminus \{0\}$, which means that when $p_S^a(\Phi) = 1$, the condition $\varrho \in \nabla_3(\infty)$ is necessary for the upper estimate of the quasinorm in Theorem 3.8.

Proof. Take $p_S^a(\Phi) = 1$. Assume that $\varrho \notin \nabla_3(\infty)$. Then for any sequence $\{b_k\}$ of positive numbers with $b_k \nearrow \infty$, one can find a nondecreasing sequence $\{t_k\}$ of positive numbers such that $\varrho(t_k) \geq 1$ and

$$\varrho\big(\varrho(t_k)t_k\big) > b_k \varrho(t_k). \tag{3.16}$$

Define $f_k(t) = t_k \chi_{A_k}(t)$ for $A_k \in \Sigma$ such that

$$I_\Phi(f_k) = t_k \varrho(t_k)\mu(A_k) = 1,$$

which gives that $\|f_k\|_\Phi = 1$ for all $k \in \mathbb{N}$. Since

$$\omega(t) = \sum_{n=1}^{\infty} \big[2^n\big(1 + \mu(T_n)\big)\big]^{-1} \chi_{T_n}(t) \leq 1$$

for μ-a.e. $t \in \Omega$, where $\{T_n\}_{n=1}^\infty$ is a sequence of Σ-measurable sets of positive and finite measure such that $\bigcup_{n=1}^\infty T_n = \Omega$, then

$$\begin{aligned} I_\Phi^\omega(f_k) &= \int_{A_k} f_k(t)\varrho\big(f_k(t)\big)\omega(t)\, d\mu(t) \\ &= t_k\varrho(t_k)\|\chi_{A_k}\|_{L^1_\omega} \leq t_k\varrho(t_k)\mu(A_k) = I_\Phi(f_k) = 1. \end{aligned}$$

Hence

$$\|\chi_{A_k}\|_{L^1_\omega} \leq \frac{1}{t_k\varrho(t_k)} \tag{3.17}$$

for all $k \in \mathbb{N}$. Applying (3.16) and (3.17), as well as the fact that ϱ is nondecreasing, we obtain

$$\begin{aligned} [f_k]_{\Phi,1} &= \|f_k\|_{L^1_\omega} I_\Phi\Big(\frac{f_k}{\|f_k\|_{L^1_\omega}}\Big) \\ &= \int_{A_k} f_k(t)\varrho\Big(\frac{f_k(t)}{\|f_k\|_{L^1_\omega}}\Big)\, d\mu(t) = \int_{A_k} t_k \varrho\Big(\frac{t_k}{t_k\|\chi_{A_k}\|_{L^1_\omega}}\Big)\, d\mu(t) \\ &= t_k\varrho\Big(\frac{1}{\|\chi_{A_k}\|_{L^1_\omega}}\Big)\mu(A_k) \geq t_k\varrho\big(t_k\varrho(t_k)\big)\mu(A_k) \\ &> t_k b_k \varrho(t_k)\mu(A_k) = b_k \nearrow \infty, \end{aligned}$$

which ends the proof. □

Theorem 3.11. *If Φ and ϱ are the functions defined in Theorem 3.8, $\varrho \notin \nabla^2(\infty)$, and μ is a nonatomic and infinite measure, then for any r and p with $1 \leq r < p \leq p_S^a(\Phi)$ and ω defined as in Lemma 2.29, there is no absolute constant $C > 0$ independent of f such that*

$$\|f\|_{L^r_\omega}\Big(I_\Phi\Big(\frac{f}{\|f\|_{L^r_\omega}}\Big)\Big)^{\frac{1}{p}} =: [f]_{\Phi,p} \leq C\|f\|_\Phi \tag{3.18}$$

holds for all $f \in L^\Phi\backslash\{0\}$, which means that when $1 \leq r < p \leq p_S^a(\Phi)$, the condition $\varrho \in \nabla^2(\infty)$ is necessary for the upper estimate of the quasinorm in Theorem 3.8.

Proof. Let the assumptions about Φ, ρ, r, and p be satisfied, and assume that $\varrho \notin \nabla^2(\infty)$. Then $\varrho \notin \nabla^{\frac{p}{r}}(\infty)$, and so for any sequence $\{b_k\}$ of positive numbers with $b_k \nearrow \infty$ one can find a nondecreasing sequence $\{t_k\}$ of positive numbers such that

$$\varrho(t_k) \geq 1 \qquad \text{and} \qquad \varrho(t_k^{\frac{p}{r}}) > b_k \varrho(t_k). \tag{3.19}$$

Define $f_k(t) = t_k \chi_{A_k}(t)$ for $A_k \in \Sigma$ such that

$$I_\Phi(f_k) = t_k^p \varrho(t_k)\mu(A_k) = 1.$$

Hence $\|f_k\|_\Phi = 1$ for all $k \in \mathbb{N}$. Since $\omega(t) \leq 1$ μ-a.e. $t \in \Omega$, where ω is as in Lemma 2.29, we obtain

$$\begin{aligned} I_\Phi^\omega(f_k) &= \int_{A_k} \big(f_k(t)\big)^p \varrho\big(f_k(t)\big)\omega(t)\, d\mu(t) \\ &= t_k^p \varrho(t_k)\|\omega\chi_{A_k}\|_{L^1} \leq t_k^p \varrho(t_k)\mu(A_k) = I_\Phi(f_k) = 1; \end{aligned}$$

that is,

$$\|\chi_{A_k}\|_{L_\omega^r}^r = \|\omega\chi_{A_k}\|_{L^1} \leq \frac{1}{t_k^p \varrho(t_k)}. \tag{3.20}$$

Applying (3.20), the fact that ϱ is nondecreasing on $\mathbb{R}_+$ when $p \leq p_S^a(\Phi)$, and (3.19), we obtain

$$\begin{aligned} [f_k]_{\Phi,p}^p &= \|f_k\|_{L_\omega^r}^p I_\Phi\Big(\frac{f_k}{\|f_k\|_{L_\omega^r}}\Big) \\ &= \int_{A_k} \big(f_k(t)\big)^p \varrho\Big(\frac{f_k(t)}{\|f_k\|_{L_\omega^r}}\Big)\, d\mu(t) \\ &= \int_{A_k} t_k^p \varrho\Big(\frac{t_k}{t_k\|\chi_{A_k}\|_{L_\omega^r}}\Big)\, d\mu(t) \\ &= t_k^p \varrho\Big(\frac{1}{\|\chi_{A_k}\|_{L_\omega^r}}\Big)\mu(A_k) \\ &\geq t_k^p \varrho\big(t_k^{\frac{p}{r}}\big(\varrho(t_k)\big)^{\frac{1}{r}}\big)\mu(A_k) \\ &\geq t_k^p \varrho(t_k^{\frac{p}{r}})\mu(A_k) > t_k^p b_k \varrho(t_k)\mu(A_k) = b_k \nearrow \infty, \end{aligned}$$

which ends the proof. $\square$

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