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WEIGHTED HERZ SPACE ESTIMATES FOR HAUSDORFF OPERATORS ON THE HEISENBERG GROUP

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Abstract. In this article, we study the Hausdorff operator, defined via a general linear mapping A, on weighted Herz spaces in the setting of the Heisenberg group. Under some assumptions on the mapping A, we establish its sharp boundedness on power-weighted Herz spaces and power-weighted Lebesgue spaces in the Heisenberg group. Our proof is heavily based on the block decomposition of the Herz space, which is quite different from any other function spaces. Our results extend and improve some existing theorems.

1. Introduction

Let $n \geq 2$, and let $\mathbf{R}^n$ be the Euclidean space of dimension n. For a fixed integrable function Φ, Lerner and Liflyand in [16] studied the Hausdorff operator

$$H_{\Phi,A}(f)(x) = \int_{\mathbf{R}^n} \frac{\Phi(y)}{|y|^n} f\big(xA(y)\big)\, dy,$$

where $A(y)$ is an $n \times n$ matrix satisfying $\det A(y) \neq 0$ almost everywhere in the support of Φ. If choosing

$$A(y) = \operatorname{diag}\big[1/|y|, 1/|y|, \ldots, 1/|y|\big],$$

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then one defines $H_{\Phi,A}$ in this special case by

$$H_\Phi(f)(x) = \int_{\mathbf{R}^n} \frac{\Phi(y)}{|y|^n} f\Big(\frac{x}{|y|}\Big)\, dy.$$

In the definition of $H_{\Phi,A}(f)$, for simplicity, one may assume that f is initially in the Schwartz space S. When we establish the boundedness of $H_{\Phi,A}(f)$ for $f \in S$ on a normed (or quasinormed) space X, a standard dense argument together with the Hahn–Banach theorem easily yields the boundedness of $H_{\Phi,A}$ on the whole space X. The two most important function spaces are the Lebesgue space L^p ($p \geq 1$) and the Hardy space H^1, so that the boundedness of H_Φ (even $H_{\Phi,A}$) on L^p and H^1 is well established (see [3], [6], [17], [18], [21], [22], [29], [31], [34]). However, the boundedness of H_Φ on other function spaces was also studied by many authors (see, e.g., [13], [20], [28], [32], and the references therein). We also point to two recent survey papers, by Chen, Fan, and Wang [5] and by Liflyand [19], as good sources for understanding the background and the historical development of this research topic. It is particularly notable that many well-known operators in analysis can be derived from the Hausdorff operator if we choose suitable generating functions Φ (see [5]).

In this article, we will study the boundedness of $H_{\Phi,A}$ on power-weighted (homogeneous) Herz spaces $\dot{K}_p^{\alpha,q}$ with the Heisenberg group as underlying space. The motivation for our research is multifold. In the following, we briefly describe the significance of this subject.

First, from the definition of weighted Herz spaces $\dot{K}_p^{\alpha,q}(\mathbf{R}^n, w)$ (see the next section for the definition), we easily see that when $w = 1$, the space $\dot{K}_q^{\alpha,q}$ coincides with the power-weighted Lebesgue space $L^q(\mathbf{R}^n, |x|^{\alpha q})$. Hence, $\dot{K}_p^{\alpha,q}$ is a natural extension of the weighted Lebesgue space. But this is not all, since the Herz space is not merely a simple upgrade from the Lebesgue space, it is also an important function space uncovered by research in harmonic analysis and its related topics. In 1964, Beurling [2] introduced some fundamental forms of Herz spaces to study certain convolution algebras. About four years later, Herz [9] introduced new versions of the space defined in a slightly different but more convenient setting. Since then, the theory of Herz spaces has been significantly developed, and these spaces have turned out to be quite fundamental in analysis. For instance, they were used by Baernstein and Sawyer [1] to characterize the multipliers on the standard Hardy spaces, and by Lu and Yang [27] in the study of certain partial differential equations. (Readers interested in learning more about these spaces are referred to the papers [15], [25], [26].)

Another reason motivating our study of the Hausdorff operator on the Herz space is that $\dot{K}_p^{\alpha,q}$ has a nice central block decomposition which quite fits the structure of the Hausdorff operator. Recall that the operator $H_\Phi(f)$ is defined via the dilation structure of the Euclidean space, while a dilation acting on a central block b again outputs a central block (up to a constant multiple). With this advantage, we are able to give a better estimate to the Hausdorff operator on $\dot{K}_p^{\alpha,q}$ using the method of block decomposition. Such a method is powerful, and it is quite different from that used for Lebesgue spaces in the existing literature.

Also, the method of block decomposition is not freely adaptable when one studies the Hausdorff operator on Hardy spaces H^p, although it is well known that H^p has a nice atomic characterization (an atom must be a block). We recall that an H^p atom is not necessarily centered at the origin. When the origin does not lie in the support of an atom a (particularly if the center of an atom a is far away from the origin and support of a is in a small interval), it is very hard to control the estimate after a dilation acting on a. This is the main difficulty encountered when studying the Hausdorff operator on H^p for $0 < p < 1$ (see [4]).

Second, in harmonic analysis one wishes to extend the underlying space $\mathbf{R}^n$ to a more general setting. The Heisenberg group $\mathbb{H}^n$ is a noncommutative nilpotent Lie group, with underlying manifold $\mathbf{R}^{2n} \times \mathbf{R}$ and group law

$$x \cdot y = \Big(x_1 + y_1, x_2 + y_2, \dots, x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + 2\sum_{j=1}^{n}(y_j x_{n+j} - x_j y_{n+j})\Big),$$

where $x = (x_1, x_2, \dots, x_{2n+1})$, $y = (y_1, y_2, \dots, y_{2n+1})$. Although the geometric motions on the Heisenberg group $\mathbb{H}^n$ are quite different from those on $\mathbf{R}^n$ due to the loss of interchangeability, we find that $\mathbb{H}^n$ inherits some basic structures of $\mathbf{R}^n$ that are good enough for us to study the Hausdorff operator on $\mathbb{H}^n$. Also, it is known that the Heisenberg group plays significant roles in many branches of mathematics such as representation theory, complex analysis in several variables, harmonic analysis, partial differential equations, and quantum mechanics (see [10], [33] for more details). Thus, an extension of the Hausdorff operator to the Heisenberg group seems quite encouraging.

By definition, the identity element on $\mathbb{H}^n$ is $0 \in \mathbf{R}^{2n+1}$, while the inverse element of x is $-x$. The corresponding Lie algebra is generated by the left-invariant vector fields

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial x_{2n+1}}, \quad j = 1, \dots, n, \\ X_{n+j} &= \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial x_{2n+1}}, \quad j = 1, \dots, n, \\ X_{2n+1} &= \frac{\partial}{\partial x_{2n+1}}. \end{aligned}$$

The only nontrivial commutator relations are

$$[X_j, X_{n+j}] = -4X_{2n+1}, \quad j = 1, \dots, n.$$

The Heisenberg group $\mathbb{H}^n$ is a homogeneous group (see [7]) with dilations

$$\delta_r(x_1, x_2, \dots, x_{2n}, x_{2n+1}) = (rx_1, rx_2, \dots, rx_{2n}, r^2 x_{2n+1}), \quad r > 0.$$

The Haar measure on $\mathbb{H}^n$ coincides with the usual Lebesgue measure on $\mathbf{R}^{2n} \times \mathbf{R}$. We denote the measure of any measurable set $E \subset \mathbb{H}^n$ by $|E|$. Then it is easy to check that

$$\big|\delta_r(E)\big| = r^Q|E|, \qquad d(\delta_r x) = r^Q\, dx,$$

where $Q = 2n + 2$ is called the *homogeneous dimension* of $\mathbb{H}^n$.

The Heisenberg distance derived from the norm

$$|x|_h = \Big[\Big(\sum_{i=1}^{2n} x_i^2\Big)^2 + x_{2n+1}^2\Big]^{\frac{1}{4}},$$

where $x = (x_1, x_2, \ldots, x_{2n}, x_{2n+1})$, is given by

$$d(p,q) = d(q^{-1}p, 0) = |q^{-1}p|_h.$$

This distance d is left-invariant in the sense that $d(p,q)$ remains unchanged when p and q are both left-translated by some fixed vector on $\mathbb{H}^n$. Furthermore, d satisfies the triangular inequality (see p. 320 in [14])

$$d(p,q) \le d(p,x) + d(x,q), \quad p,x,q \in \mathbb{H}^n.$$

For $r > 0$ and $x \in \mathbb{H}^n$, the ball and sphere with center x and radius r on $\mathbb{H}^n$ are given by

$$B(x,r) = \big\{y \in \mathbb{H}^n : d(x,y) < r\big\}$$

and

$$S(x,r) = \big\{y \in \mathbb{H}^n : d(x,y) = r\big\},$$

respectively. We know that

$$\big|B(x,r)\big| = \big|B(0,r)\big| = \Omega_Q r^Q,$$

where

$$\Omega_Q = \frac{2\pi^{n+\frac{1}{2}}\Gamma(\frac{n}{2})}{(n+1)\Gamma(n)\Gamma(\frac{n+1}{2})} \tag{1.1}$$

is the volume of the unit ball $B(0,1)$ on $\mathbb{H}^n$. The area of $S(0,1)$ on $\mathbb{H}^n$ is $\omega_Q = Q\Omega_Q$. (For more details about the Heisenberg group, see [7].)

Now we provide the following definition of Hausdorff operators on the Heisenberg group.

Definition 1.1. Let Φ be a locally integrable function on $\mathbb{H}^n$. The Hausdorff operators on $\mathbb{H}^n$ are defined by

$$T_\Phi f(x) = \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} f(\delta_{|y|_h^{-1}} x)\, dy,$$
$$T_{\Phi,A} f(x) = \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} f\big(A(y)x\big)\, dy,$$

where $A(y)$ is a matrix-valued function, and we assume that $\det A(y) \ne 0$ almost everywhere in the support of Φ.

In the above definition, we note that $T_{\Phi,A} = T_\Phi$ if we choose a special matrix A.

Here and throughout this article, we use the notation $A \preceq B$ to denote that there is a constant $C > 0$ independent of all essential values and variables such that $A \le CB$. We use the notation $A \simeq B$ if there exists a positive constant C independent of all essential values and variables such that $C^{-1}B \le$

$A \le CB$. Also, the class A_p denotes the set of all A_p weights whose definition can be found in the next section. For a matrix M, we will use the norm $\|M\| = \sup_{x\in\mathbb{H}^n, x\neq 0} |Mx|_h/|x|_h$.

Now we are in a position to state our results.

Theorem 1.2. *Let $1 \le q_1, q_2 < \infty, 0 < \alpha_1, \alpha_2 < \infty$, and let $1/q_1 + \alpha_1/Q = 1/q_2 + \alpha_2/Q$. Suppose that $w \in A_1$ with the critical index r_w for the reverse Hölder condition and that $q_1 > q_2 r_w/(r_w - 1)$.*

(i) If $1 \le p < \infty$, then we have, for any $1 < \delta < r_w$,

$$\|T_{\Phi,A} f\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)} \preceq C_1 \|f\|_{\dot{K}^{\alpha_1,p}_{q_1}(\mathbb{H}^n;w)},$$

where

$$C_1 = \int_{\|A^{-1}(y)\|\ge 1} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{1/q_1} \big\|A^{-1}(y)\big\|^{\alpha_1}\, dy + \int_{\|A^{-1}(y)\|<1} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{1/q_1} \big\|A^{-1}(y)\big\|^{\alpha_1 - (Q/q_1+\alpha_1)/\delta}\, dy.$$

(ii) If $0 < p < 1$, then we have, for any $1 < \delta < r_w$ and any $\sigma > (1-p)/p$,

$$\|T_{\Phi,A} f\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)} \preceq C_2 \|f\|_{\dot{K}^{\alpha_1,p}_{q_1}(\mathbb{H}^n;w)},$$

where

$$C_2 = \int_{\|A^{-1}(y)\|\ge 1} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{1/q_1} \big\|A^{-1}(y)\big\|^{\alpha_1} \big(1 + \log_2\big\|A^{-1}(y)\big\|\big)^{\sigma}\, dy + \int_{\|A^{-1}(y)\|<1} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{1/q_1} \big\|A^{-1}(y)\big\|^{\alpha_1 - (Q/q_1+\alpha_1)/\delta} \times \big(1 - \log_2\big\|A^{-1}(y)\big\|\big)^{\sigma}\, dy.$$

Theorem 1.3. *Let $1 \le p < \infty$, $1 \le q_1, q_2 < \infty, -\infty < \alpha_1 < 0, \alpha_2 \in \mathbf{R}$, and let $1/q_1 + \alpha_1/Q = 1/q_2 + \alpha_2/Q$. Suppose that $w \in A_\gamma, 1 \le \gamma < \infty$, with the critical index r_w for the reverse Hölder condition, and suppose that $q_1 > q_2 \gamma r_w/(r_w - 1)$.*

(i) If $1/q_1 + \alpha_1/Q \ge 0$, then for any $1 < \delta < r_w$,

$$\|T_{\Phi,A} f\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)} \preceq C_3 \|f\|_{\dot{K}^{\alpha_1,p}_{q_1}(\mathbb{H}^n;w)},$$

where

$$C_3 = \int_{\|A(y)\|<1} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{\gamma/q_1} \big\|A(y)\big\|^{-\gamma\alpha_1}\, dy + \int_{\|A(y)\|\ge 1} \frac{\Phi(y)}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{\gamma/q_1} \big\|A(y)\big\|^{Q\gamma/q_1 - (Q/q_1+\alpha_1)(\delta-1)/\delta}\, dy.$$

(ii) If $1/q_1 + \alpha_1/Q < 0$, then for any $1 < \delta < r_w$,

$$\|T_{\Phi,A} f\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)} \preceq C_4 \|f\|_{\dot{K}^{\alpha_1,p}_{q_1}(\mathbb{H}^n;w)},$$

where

$$C_4 = \int_{\|A(y)\|<1} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{\gamma/q_1} \big\|A(y)\big\|^{Q\gamma/q_1-(Q/q_1+\alpha_1)(\delta-1)/\delta}\, dy + \int_{\|A(y)\|\geq 1} \frac{\Phi(y)}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{\gamma/q_1} \big\|A(y)\big\|^{-\gamma\alpha_1}\, dy.$$

When the weight is reduced to the power function, we have the following enhanced results.

Theorem 1.4. *Let* $1 \leq q < \infty$.

(*i*) *If* $1 \leq p < \infty$ *and* $0 < \beta < \infty$, *then we have*

$$\|T_{\Phi,A} f\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;|\cdot|_h^\beta)} \preceq C_5 \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;|\cdot|_h^\beta)},$$

where C_5 *is*

$$\begin{cases} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} (\|A^{-1}(y)\|^\beta |\det A^{-1}(y)|)^{1/q} \{1+\log_2(\|A^{-1}(y)\|\|A(y)\|)\}\, dy, & \alpha = 0, \\ \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} (\|A^{-1}(y)\|^\beta |\det A^{-1}(y)|)^{1/q} \|A^{-1}(y)\|^{\alpha(Q+\beta)/Q}\, dy, & \alpha > 0, \\ \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} (\|A^{-1}(y)\|^\beta |\det A^{-1}(y)|)^{1/q} \|A(y)\|^{-\alpha(Q+\beta)/Q}\, dy, & \alpha < 0. \end{cases}$$

(*ii*) *If* $1 \leq p < \infty$ *and* $-Q < \beta \leq 0$, *then we have*

$$\|T_{\Phi,A} f\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;|\cdot|_h^\beta)} \preceq C_6 \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;|\cdot|_h^\beta)},$$

where C_6 *is*

$$\begin{cases} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} (\|A(y)\|^{-\beta} |\det A^{-1}(y)|)^{1/q} \{1+\log_2(\|A^{-1}(y)\|\|A(y)\|)\}\, dy, & \alpha = 0, \\ \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} (\|A(y)\|^{-\beta} |\det A^{-1}(y)|)^{1/q} \|A^{-1}(y)\|^{\alpha(Q+\beta)/Q}\, dy, & \alpha > 0, \\ \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} (\|A(y)\|^{-\beta} |\det A^{-1}(y)|)^{1/q} \|A(y)\|^{-\alpha(Q+\beta)/Q}\, dy, & \alpha < 0. \end{cases}$$

(*iii*) *If* $0 < p < 1$ *and* $-Q < \beta \leq 0 < \alpha < \infty$, *then we have, for any* $\sigma > (1-p)/p$,

$$\|T_{\Phi,A} f\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;|\cdot|_h^\beta)} \preceq C_7 \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;|\cdot|_h^\beta)},$$

where C_7 *is*

$$\int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \big(\|A(y)\|^{-\beta} |\det A^{-1}(y)|\big)^{1/q} \big\|A^{-1}(y)\big\|^{\alpha(Q+\beta)/Q} \big(1 + \big|\log_2 \|A^{-1}(y)\|\big|\big)^\sigma\, dy.$$

In particular, if $\|A^{-1}(y)\|$ and $\|A(y)\|^{-1}$ are comparable, we can obtain the following sharp result.

Theorem 1.5. *Let* $1 \leq p, q < \infty, -Q < \beta < \infty, \alpha \in \mathbf{R}$, *and let* Φ *be a nonnegative function. Suppose that there is a constant* C *independent of* y *such that* $\|A^{-1}(y)\| \leq C\|A(y)\|^{-1}$ *for all* $y \in \operatorname{supp}(\Phi)$. *Then* $T_{\Phi,A}$ *is bounded on* $\dot{K}_q^{\alpha,p}(\mathbb{H}^n; |\cdot|^\beta)$ *if and only if*

$$\int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} \big\|A^{-1}(y)\big\|^{(Q+\beta)(1/q+\alpha/Q)}\, dy < \infty.$$

In the preceding theorem, letting $p = q$ and $\beta = 0$, we have the following sharp boundedness for Hausdorff operators on Lebesgue spaces.

Corollary 1.6. *Let $1 \le q < \infty, \alpha \in \mathbf{R}$, and let Φ be a nonnegative function. Suppose that there is a constant C independent of y such that $\|A^{-1}(y)\| \le C\|A(y)\|^{-1}$ for all $y \in \operatorname{supp}(\Phi)$. Then $T_{\Phi,A}$ is bounded on $L^q(\mathbb{H}^n; |\cdot|_h^{\alpha})$ if and only if*

$$\int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} \left\|A^{-1}(y)\right\|^{\alpha+Q/q} dy < \infty.$$

Finally in this section, we make the following remarks about our main theorems.

Remark 1.7. Suppose that $A(y) = \operatorname{diag}[1/\lambda_1(y), \ldots, 1/\lambda_{2n}(y), 1/\lambda_{2n+1}(y)]$ with $\lambda_i(y) \neq 0$, for $i = 1, \ldots, 2n+1$. Denote

$$M(y) = \max\left\{\left|\lambda_1(y)\right|, \ldots, \left|\lambda_{2n}(y)\right|, \left|1/\lambda_{2n+1}(y)\right|^{1/2}\right\},$$
$$m(y) = \min\left\{\left|\lambda_1(y)\right|, \ldots, \left|\lambda_{2n}(y)\right|, \left|1/\lambda_{2n+1}(y)\right|^{1/2}\right\}.$$

If there is a constant $C \ge 1$ independent of y such that $M(y) \le Cm(y)$, then it is easy to check that $A(y)$ satisfies the assumptions of Theorem 1.5 and Corollary 1.6.

Remark 1.8. Rechecking the proof of necessity of Theorem 1.5, we find that the necessary condition is also true for all $0 < p, q < \infty$. Therefore, comparing with Theorem 1.4(iii) and Theorem 1.5, we raise an open question: Is the assumption in Theorem 1.4 sharp in the case $0 < p < 1$?

In Section 2, we will introduce some necessary notation and definitions, as well as some known results to be used later in the article. We will prove the main theorems in Section 3.

2. Notation and definitions

We start this section by recalling some standard definitions and notation. The theory of A_p weight was first introduced by Muckenhoupt [30] in a study of weighted L^p boundedness of Hardy–Littlewood maximal functions. (For A_p weights on the Heisenberg group, readers are referred to [8] and [11].) A weight is a nonnegative, locally integrable function on $\mathbb{H}^n$.

Definition 2.1. Let $1 < p < \infty$. We say that a weight $w \in A_p(\mathbb{H}^n)$ if there exists a constant C such that for all balls B,

$$\left(\frac{1}{|B|}\int_B w(x)\, dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-1/(p-1)}\, dx\right)^{p-1} \le C.$$

We say that a weight $w \in A_1(\mathbb{H}^n)$ if there exists a constant C such that for all balls B,

$$\frac{1}{|B|}\int_B w(x)\, dx \le C \operatorname*{ess\,inf}_{x\in B} w(x).$$

We define

$$A_\infty(\mathbb{H}^n) = \bigcup_{1 \le p < \infty} A_p(\mathbb{H}^n).$$

By the standard proofs of Propositions 1.4.1 and 1.4.2 in [24] together with the reverse Hölder inequality on the Heisenberg group in [11], we have the following results.

Proposition 2.2. *(i) We have $A_p(\mathbb{H}^n) \subsetneq A_q(\mathbb{H}^n)$, for $1 \le p < q < \infty$.*

(ii) If $w \in A_p(\mathbb{H}^n)$, $1 < p < \infty$, then there is an $\varepsilon > 0$ such that $p - \varepsilon > 1$ and $w \in A_{p-\varepsilon}(\mathbb{H}^n)$.

A close relation to $A_\infty(\mathbb{H}^n)$ is the reverse Hölder condition. If there exist $r > 1$ and a fixed constant C such that

$$\Big(\frac{1}{|B|}\int_B w(x)^r\,dx\Big)^{1/r} \le \frac{C}{|B|}\int_B w(x)\,dx$$

for all balls $B \subset \mathbb{H}^n$, then we say that w satisfies the *reverse Hölder condition of order* r and we write $w \in \mathrm{RH}_r(\mathbb{H}^n)$. According to Theorem 19 and Corollary 21 in [12], $w \in A_\infty(\mathbb{H}^n)$ if and only if there exists some $r > 1$ such that $w \in \mathrm{RH}_r(\mathbb{H}^n)$. Moreover, if $w \in \mathrm{RH}_r(\mathbb{H}^n), r > 1$, then $w \in \mathrm{RH}_{r+\epsilon}(\mathbb{H}^n)$ for some $\epsilon > 0$. We thus write $r_w \equiv \sup\{r > 1 : w \in \mathrm{RH}_r(\mathbb{H}^n)\}$ to denote the *critical index of w for the reverse Hölder condition.*

An important example of $A_p(\mathbb{H}^n)$ weight is the power function $|x|_h^\alpha$. By proofs similar to those of Propositions 1.4.3 and 1.4.4 in [24], we obtain the following properties of power weights.

Proposition 2.3. *Let $x \in \mathbb{H}^n$. Then*

(i) $|x|_h^\alpha \in A_1(\mathbb{H}^n)$ if and only if $-Q < \alpha \le 0$;

(ii) $|x|_h^\alpha \in A_p(\mathbb{H}^n)$, $1 < p < \infty$, if and only if $-Q < \alpha < Q(p-1)$.

We will denote by q_w the *critical index* for w, that is, the infimum of all the q's such that w satisfies the condition A_q. From Proposition 2.2, we see that unless $q_w = 1$, w is never an A_{q_w} weight. Also by Propositions 2.2 and 2.3, we can obtain that if $0 < \alpha < \infty$, then

$$|x|_h^\alpha \in \bigcap_{\frac{Q+\alpha}{Q}<p<\infty} A_p, \tag{2.1}$$

where $(Q+\alpha)/Q$ is the critical index of $|x|_h^\alpha$.

For any $w \in A_\infty(\mathbb{H}^n)$ and any Lebesgue measurable set E, write $w(E) = \int_E w(x)\,dx$. We have the following standard characterization of A_p weights.

Proposition 2.4. *If $w \in A_p(\mathbb{H}^n)$, $1 \le p < \infty$, then for any $f \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$ and any ball $B \subset \mathbb{H}^n$,*

$$\frac{1}{|B|}\int_B \big|f(x)\big|\,dx \le C\Big(\frac{1}{w(B)}\int_B \big|f(x)\big|^p w(x)\,dx\Big)^{1/p}.$$

Proof. When $p = 1$, by definition of $A_1(\mathbb{H}^n)$, we have

$$\begin{aligned}\frac{w(B)}{|B|}\int_B \big|f(x)\big|\,dx &= \Big(\frac{1}{|B|}\int_B w(x)\,dx\Big)\Big(\int_B \big|f(x)\big|\,dx\Big)\\ &\le C\operatorname*{essinf}_{x\in B} w(x)\Big(\int_B \big|f(x)\big|\,dx\Big)\\ &\le C\int_B \big|f(x)\big|w(x)\,dx.\end{aligned}$$

Therefore,

$$\frac{1}{|B|}\int_B \big|f(x)\big|\,dx \le \frac{C}{w(B)}\int_B \big|f(x)\big|w(x)\,dx.$$

When $1 < p < \infty$, by the Hölder inequality,

$$\begin{aligned}\frac{1}{|B|}\int_B \big|f(x)\big|\,dx &\le \Big(\frac{1}{|B|}\int_B \big|f(x)\big|^p w(x)\,dx\Big)^{1/p}\Big(\frac{1}{|B|}\int_B w(x)^{1-p'}\,dx\Big)^{1-1/p}\\ &\le C\Big(\frac{1}{|B|}\int_B \big|f(x)\big|^p w(x)\,dx\Big)^{1/p}\Big(\frac{1}{|B|}\int_B w(x)\,dx\Big)^{-1/p}\\ &\le C\Big(\frac{1}{w(B)}\int_B \big|f(x)\big|^p w(x)\,dx\Big)^{1/p}.\end{aligned}$$

The proposition is proved. □

Proposition 2.5. *Let $w \in A_p \cap \mathrm{RH}_r$, with $p \ge 1$ and $r > 1$. Then there exist constants $C_1, C_2 > 0$ such that*

$$C_1\Big(\frac{|E|}{|B|}\Big)^p \le \frac{w(E)}{w(B)} \le C_2\Big(\frac{|E|}{|B|}\Big)^{(r-1)/r}$$

for any measurable subset E of a ball B. Especially, for any $\lambda > 1$,

$$w\big(B(x_0,\lambda R)\big) \le C\lambda^{Qp} w\big(B(x_0,R)\big).$$

Proof. The first inequality can be easily deduced by taking $f(x) = \chi_E(x)$ in Proposition 2.4.

For the second one, since $w \in A_p \cap \mathrm{RH}_r$, using Hölder's inequality, we have

$$\begin{aligned}\int_E w(x)\,dx &\le \Big(\int_E w(x)^r\,dx\Big)^{1/r}|E|^{1-1/r}\\ &\le \Big(\frac{1}{|B|}\int_B w(x)^r\,dx\Big)^{1/r}|B|^{1/r}|E|^{1-1/r}\\ &\le C\Big(\frac{|E|}{|B|}\Big)^{(r-1)/r}\int_B w(x)\,dx.\end{aligned}$$

This proves the proposition. □

Given a weight function w on $\mathbb{H}^n$, for any measurable set $E \subset \mathbb{H}^n$, as usual we denote by $L^p(E;w)$ the weighted Lebesgue space of all functions satisfying

$$\|f\|_{L^p(E;w)} = \Big(\int_E \big|f(x)\big|^p w(x)\,dx\Big)^{1/p} < \infty.$$

We denote $L^\infty(\mathbb{H}^n; w) = L^\infty(\mathbb{H}^n)$ and $\|f\|_{L^\infty(\mathbb{H}^n;w)} = \|f\|_{L^\infty(\mathbb{H}^n)}$ for $p = \infty$.

Let $B_k = \{x \in \mathbb{H}^n : |x|_h < 2^k\}$, $D_k = B_k \setminus B_{k-1}$ for $k \in \mathbf{Z}$.

Definition 2.6 ([23], Definition 1.1)**.** Suppose that $\alpha \in \mathbf{R}, 0 < p, q < \infty$. Let w be a weight on $\mathbb{H}^n$. The homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(\mathbb{H}^n; w)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{H}^n; w) = \left\{ f \in L^q_{\text{loc}}\big(\mathbb{H}^n \setminus \{0\}; w\big) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;w)} < \infty \right\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;w)} = \left\{ \sum_{k=-\infty}^{+\infty} w(B_k)^{\alpha p/Q} \|f\|^p_{L^q(D_k,w)} \right\}^{1/p}.$$

In general, the spaces $\dot{K}_q^{\alpha,p}(\mathbb{H}^n; w)$ are quasi-Banach spaces. When $1 \le p, q < \infty$, then $\dot{K}_q^{\alpha,p}(\mathbb{H}^n; w)$ are Banach spaces. If $w = 1$, we denote the Herz spaces by $\dot{K}_q^{\alpha,p}(\mathbb{H}^n)$. Obviously, $\dot{K}_p^{\alpha,p}(\mathbb{H}^n) = L^p(\mathbb{H}^n; |\cdot|_h^{\alpha p})$ for all $\alpha \in \mathbf{R}$. Therefore, Herz spaces are a natural generalization of Lebesgue spaces with power weights.

Definition 2.7. Let $0 < \alpha < \infty$ and $1 \le q < \infty$. A function $b(x)$ on $\mathbb{H}^n$ is said to be a *central* $(\alpha, q; w)$*-block* if it satisfies

$$\operatorname{supp}(b) \subset B(0, r) \qquad \text{and} \qquad \|b\|_{L^q(\mathbb{H}^n;w)} \le w\big(B(0,r)\big)^{-\alpha/Q}.$$

The following decomposition theorem shows that the central blocks are the "building blocks" of Herz spaces.

Proposition 2.8. *Let $0 < \alpha < \infty$, $0 < p < \infty$, $1 \le q < \infty$, and let $w \in A_1(\mathbb{H}^n)$. Then $f \in \dot{K}_q^{\alpha,p}(\mathbb{H}^n; w)$ if and only if*

$$f = \sum_{k=-\infty}^{+\infty} \lambda_k b_k,$$

where $\sum_{k=-\infty}^{+\infty} |\lambda_k|^p < \infty$, and each b_k is a central $(\alpha, q; w)$-block with the support in B_k. Moreover,

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;w)} \simeq \inf\left\{ \Big(\sum_{k=-\infty}^{+\infty} |\lambda_k|^p \Big) \right\}^{1/p},$$

where the infimum is taken over all decompositions of f as above.

We omit the proof here as the procedure is the same as that of Theorem 1.1 in [25].

3. Proof of the main theorems

3.1. **Proof of Theorem 1.2.** According to Proposition 2.8, any $f \in \dot{K}_{q_1}^{\alpha_1,p}(\mathbb{H}^n; w)$ has a block decomposition

$$f = \sum_{k=-\infty}^{\infty} \lambda_k b_k,$$

where $(\sum_{k=-\infty}^{\infty}|\lambda_k|^p)^{1/p}\preceq\|f\|_{\dot{K}_{q_1}^{\alpha_1,p}(\mathbb{H}^n;w)}$, and each b_k is a central $(\alpha_1,q_1;w)$-block with the support in B_k.

By definition,

$$\big|T_{\Phi,A}f(x)\big|\leq\sum_{k=-\infty}^{\infty}|\lambda_k|\int_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}\big|b_k\big(A(y)x\big)\big|\,dy.$$

Denote

$$\widetilde{H}(b_k)(x)=\int_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}\big|b_k\big(A(y)x\big)\big|\,dy.$$

To prove the theorem, it suffices to show that

$$\widetilde{H}(b_k)=\sum_{j=-\infty}^{\infty}\mu_{kj}b_{kj},$$

where each b_{kj} is a central $(\alpha_2,q_2;w)$-block with the support in B_{k+j}, and

$$\Big(\sum_{j=-\infty}^{\infty}|\mu_{kj}|^p\Big)^{1/p}\preceq C_1$$

uniformly for $k\in\mathbf{Z}$.

We rewrite

$$\widetilde{H}(b_k)(x)=\sum_{j=-\infty}^{\infty}\int_{2^{j-1}\leq\|A^{-1}(y)\|<2^j}\frac{|\Phi(y)|}{|y|_h^Q}\big|b_k\big(A(y)x\big)\big|\,dy:=\sum_{j=-\infty}^{\infty}g_{kj}(x).\qquad(3.1)$$

Now we will check that each $g_{kj}(x)$ is a central $(\alpha_2,q_2;w)$-block multiplied by a factor. First, we claim that

$$\operatorname{supp}g_{kj}=\operatorname{supp}b_k\big(A(y)x\big)\subset B\big(0,2^k\big\|A^{-1}(y)\big\|\big)\subset B_{k+j}.\qquad(3.2)$$

In fact, if $|x|_h\geq 2^k\|A^{-1}(y)\|$, then

$$\begin{aligned}\big|A(y)x\big|_h&=\Big(\frac{|A^{-1}(y)A(y)x|_h}{|A(y)x|_h}\Big)^{-1}|x|_h\geq\Big(\sup_{z\neq0}\frac{|A^{-1}(y)z|_h}{|z|_h}\Big)^{-1}|x|_h\\&=\big\|A^{-1}(y)\big\|^{-1}|x|_h\geq 2^k.\end{aligned}$$

Since $\operatorname{supp}b_k\subset B_k$, we have $b_k(A(y)x)=0$. Therefore,

$$\operatorname{supp}g_{kj}\subset B\big(0,2^k\big\|A^{-1}(y)\big\|\big)\subset B_{k+j}.$$

By the Minkowski inequality, we have

$$\|g_{kj}\|_{L^{q_2}(\mathbb{H}^n;w)}\leq\int_{2^{j-1}\leq\|A^{-1}(y)\|<2^j}\frac{|\Phi(y)|}{|y|_h^Q}\big\|b_k\big(A(y)\cdot\big)\big\|_{L^{q_2}(\mathbb{H}^n;w)}\,dy.$$

Since $q_1>q_2r_w/(r_w-1)$, there exists r satisfying $1<r<r_w$ such that $q_1=q_2r/(r-1)$. By (3.2), the reverse Hölder condition, and Proposition 2.4, we

obtain

$$\begin{aligned}
&\big\|b_k\big(A(y)\cdot\big)\big\|_{L^{q_2}(\mathbb{H}^n;w)}\\
&\quad\le \Big(\int_{A^{-1}(y)B_k}\big|b_k\big(A(y)x\big)\big|^{q_1}\,dx\Big)^{1/q_1}\Big(\int_{A^{-1}(y)B_k} w(x)^r\,dx\Big)^{1/(rq_2)}\\
&\quad\le \big|\det A^{-1}(y)\big|^{1/q_1}\Big(\int_{B_k}\big|b_k(z)\big|^{q_1}\,dz\Big)^{1/q_1}\Big(\int_{B(0,2^k\|A^{-1}(y)\|)} w(x)^r\,dx\Big)^{1/(rq_2)}\\
&\quad\le \big|\det A^{-1}(y)\big|^{1/q_1}|B_k|^{1/q_1}w\big(B\big(0,2^k\big\|A^{-1}(y)\big\|\big)\big)^{1/q_2}\frac{|B(0,2^k\|A^{-1}(y)\|)|^{1/(rq_2)}}{|B(0,2^k\|A^{-1}(y)\|)|^{1/q_2}}\\
&\qquad\times\Big(\frac{1}{w(B_k)}\int_{B_k}\big|b_k(z)\big|^{q_1}w(z)\,dz\Big)^{1/q_1}\\
&\quad\preceq \Big(\frac{|\det A^{-1}(y)|}{\|A^{-1}(y)\|^Q}\Big)^{1/q_1}\frac{w(B_{k+j})^{1/q_2}}{w(B_k)^{1/q_1}}\|b_k\|_{L^{q_1}(\mathbb{H}^n;w)}\\
&\quad\le \Big(\frac{|\det A^{-1}(y)|}{\|A^{-1}(y)\|^Q}\Big)^{1/q_1}\frac{w(B_{k+j})^{1/q_2}}{w(B_k)^{1/q_1}}w(B_k)^{-\alpha_1/Q}\\
&\quad= \Big(\frac{|\det A^{-1}(y)|}{\|A^{-1}(y)\|^Q}\Big)^{1/q_1}\Big(\frac{w(B_{k+j})}{w(B_k)}\Big)^{1/q_1+\alpha_1/Q}w(B_{k+j})^{-\alpha_2/Q},
\end{aligned}\tag{3.3}$$

where the last equality is due to $1/q_1+\alpha_1/Q=1/q_2+\alpha_2/Q$.

When $j\ge 1$, since $2^{j-1}\le\|A^{-1}(y)\|<2^j$, by Proposition 2.5, we have

$$\begin{aligned}
\big\|b_k\big(A(y)\cdot\big)\big\|_{L^{q_2}(\mathbb{H}^n;w)} &\preceq 2^{jQ(1/q_1+\alpha_1/Q)}\Big(\frac{|\det A^{-1}(y)|}{\|A^{-1}(y)\|^Q}\Big)^{1/q_1}w(B_{k+j})^{-\alpha_2/Q}\\
&= \big|\det A^{-1}(y)\big|^{1/q_1}\big\|A^{-1}(y)\big\|^{\alpha_1}w(B_{k+j})^{-\alpha_2/Q}.
\end{aligned}$$

When $j=0$, we have

$$\big\|b_k\big(A(y)\cdot\big)\big\|_{L^{q_2}(\mathbb{H}^n;w)}\preceq\Big(\frac{|\det A^{-1}(y)|}{\|A^{-1}(y)\|^Q}\Big)^{1/q_1}w(B_k)^{-\alpha_2/Q}.$$

When $j\le -1$, by (3.3) and Proposition 2.5, for $1<\delta<r_w$, we obtain

$$\begin{aligned}
&\big\|b_k\big(A(y)\cdot\big)\big\|_{L^{q_2}(\mathbb{H}^n;w)}\\
&\quad\preceq \Big(\frac{|\det A^{-1}(y)|}{\|A^{-1}(y)\|^Q}\Big)^{1/q_1}\Big(\frac{|B_{j+k}|)}{|B_k|}\Big)^{(1/q_1+\alpha_1/Q)((\delta-1)/\delta)}w(B_{k+j})^{-\alpha_2/Q}\\
&\quad= \big|\det A^{-1}(y)\big|^{1/q_1}\big\|A^{-1}(y)\big\|^{\alpha_1-(Q/q_1+\alpha_1)/\delta}w(B_{k+j})^{-\alpha_2/Q}.
\end{aligned}$$

To sum up, we have

$$\|g_{kj}\|_{L^{q_2}(\mathbb{H}^n;w)}\preceq\mu_{kj}w(B_{k+j})^{-\alpha_2/Q},$$

where

$$\mu_{kj}=\begin{cases}\int_{2^{j-1}\le\|A^{-1}(y)\|<2^j}\frac{|\Phi(y)|}{|y|_h^Q}|\det A^{-1}(y)|^{1/q_1}\|A^{-1}(y)\|^{\alpha_1}\,dy & j>0,\\ \int_{2^{j-1}\le\|A^{-1}(y)\|<2^j}\frac{|\Phi(y)|}{|y|_h^Q}|\det A^{-1}(y)|^{1/q_1}\|A^{-1}(y)\|^{\alpha_1-(Q/q_1+\alpha_1)/\delta}\,dy & j\le 0.\end{cases}$$

Let

$$g_kj(x) = \mu_{kj} b_{kj}(x).$$

It is clear that each b_{kj} is a central $(\alpha_2, q_2; w)$-block with the support in B_{k+j}. Next we will show that $\sum_{j=-\infty}^{\infty} |\mu_{kj}|^p$ is uniformly bounded on $k \in \mathbf{Z}$. To this end, we will consider two different cases $p \geq 1$ and $0 < p < 1$.

In the case $1 \leq p < \infty$, we have that

$$\begin{aligned} \sum_{j=-\infty}^{\infty} |\mu_{kj}|^p &\leq \Big(\sum_{j=-\infty}^{\infty} |\mu_{kj}| \Big)^p \\ &\preceq \Big(\int_{\|A^{-1}(y)\| \geq 1} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{1/q_1} \big\|A^{-1}(y)\big\|^{\alpha_1} \, dy \\ &\quad + \int_{\|A^{-1}(y)\| < 1} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{1/q_1} \big\|A^{-1}(y)\big\|^{\alpha_1 - (Q/q_1 + \alpha_1)/\delta} \, dy \Big)^p. \end{aligned} \tag{3.4}$$

This proves Theorem 1.2(i).

On the other hand, when $0 < p < 1$, we have that

$$\sum_{j=-\infty}^{\infty} |\mu_{kj}|^p = \sum_{j=1}^{\infty} |\mu_{kj}|^p + \sum_{j=-\infty}^{-1} |\mu_{kj}|^p + |\mu_{k0}|^p := I_1 + I_2 + |\mu_{k0}|^p. \tag{3.5}$$

For I_1, by Hölder's inequality and the fact that $\sigma > (1-p)/p$, we have

$$\begin{aligned} I_1 &\leq \Big(\sum_{j=1}^{\infty} j^\sigma |\mu_{kj}| \Big)^p \Big(\sum_{j=1}^{\infty} j^{-\sigma p/(1-p)} \Big)^{1-p} \\ &\preceq \sum_{j=1}^{\infty} \int_{2^{j-1} \leq \|A^{-1}(y)\| < 2^j} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{1/q_1} \big\|A^{-1}(y)\big\|^{\alpha_1} \\ &\quad \times \big(1 + \log_2 \big\|A^{-1}(y)\big\|\big)^\sigma \, dy \\ &= \int_{\|A^{-1}(y)\| \geq 1} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{1/q_1} \big\|A^{-1}(y)\big\|^{\alpha_1} \big(1 + \log_2 \big\|A^{-1}(y)\big\|\big)^\sigma \, dy. \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &\leq \Big(\sum_{j=-\infty}^{-1} |j|^\sigma |\mu_{kj}| \Big)^p \Big(\sum_{j=-\infty}^{-1} |j|^{-\sigma p/(1-p)} \Big)^{1-p} \\ &\preceq \int_{\|A^{-1}(y)\| < \frac{1}{2}} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{1/q_1} \big\|A^{-1}(y)\big\|^{\alpha_1 - (Q/q_1 + \alpha_1)/\delta} \big|\log_2 \big\|A^{-1}(y)\big\|\big|^\sigma \, dy. \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_{j=-\infty}^{\infty} |\mu_{kj}|^p &\preceq \int_{\|A^{-1}(y)\| \geq 1} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{1/q_1} \big\|A^{-1}(y)\big\|^{\alpha_1} \\ &\quad \times \big(1 + \log_2 \big\|A^{-1}(y)\big\|\big)^\sigma \, dy \end{aligned}$$

$$+\int_{\|A^{-1}(y)\|<1}\frac{|\Phi(y)|}{|y|_h^Q}\left|\det A^{-1}(y)\right|^{1/q_1}\left\|A^{-1}(y)\right\|^{\alpha_1-(Q/q_1+\alpha_1)/\delta}$$
$$\times\left(1-\log_2\left\|A^{-1}(y)\right\|\right)^{\sigma}dy.$$

Theorem 1.2 is proved. □

Next we will use a different but more direct method to prove Theorems 1.3 and 1.4. To start this process, we first prove the following result.

Lemma 3.1. *Suppose that the* $(2n+1)\times(2n+1)$ *matrix* B *is invertible. Then*

$$\|B\|^{-Q}\le|\det B^{-1}|\le\|B^{-1}\|^{Q},\tag{3.6}$$

where

$$\|B\|=\sup_{x\in\mathbb{H}^n,x\neq0}\frac{|Bx|_h}{|x|_h}.$$

Proof. By definition, it is clear that $|Bx|_h\le\|B\||x|_h$ for any $x\in\mathbb{H}^n$. Therefore,

$$\|B\|^{-1}|x|_h\le|B^{-1}x|_h\le\|B^{-1}\||x|_h.$$

It yields that

$$\begin{aligned}\left|\left\{x\in\mathbb{H}^n:\|B\|^{-1}|x|_h\le1\right\}\right|&\ge\left|\left\{x\in\mathbb{H}^n:|B^{-1}x|_h\le1\right\}\right|\\&\ge\left|\left\{x\in\mathbb{H}^n:\|B^{-1}\||x|_h\le1\right\}\right|,\end{aligned}$$

which implies that

$$\Omega_Q\|B\|^{Q}\ge\Omega_Q|\det B|\ge\Omega_Q\|B^{-1}\|^{-Q}.$$

Consequently, (3.6) holds. □

3.2. **Proof of Theorem 1.3.** By definition and the Minkowski inequality,

$$\begin{aligned}&\|T_{\Phi,A}f\|^p_{\dot K^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)}\\&=\sum_{k=-\infty}^{\infty}w(B_k)^{\alpha_2p/Q}\left\|\int_{\mathbb{H}^n}\frac{\Phi(y)}{|y|_h^Q}f\left(A(y)\cdot\right)dy\right\|^p_{L^{q_2}(D_k;w)}\\&\le\sum_{k=-\infty}^{\infty}w(B_k)^{\alpha_2p/Q}\left(\int_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}\left\|f\left(A(y)\cdot\right)\right\|_{L^{q_2}(D_k;w)}dy\right)^p.\end{aligned}\tag{3.7}$$

Since $q_1>\gamma q_2r_w/(r_w-1)$, there is $1<r<r_w$ such that $q_1/\gamma=q_2r'=q_2r/(r-1)$. In view of the Hölder inequality and the reverse Hölder condition, we obtain

$$\begin{aligned}&\left\|f\left(A(y)\cdot\right)\right\|_{L^{q_2}(D_k;w)}\\&\preceq\left(\int_{D_k}\left|f\left(A(y)x\right)\right|^{q_1/\gamma}dx\right)^{\gamma/q_1}\left(\int_{D_k}w(x)^r\,dx\right)^{1/(rq_2)}\\&\preceq\left|\det A^{-1}(y)\right|^{\gamma/q_1}\left(\int_{A(y)D_k}\left|f(x)\right|^{q_1/\gamma}dx\right)^{\gamma/q_1}\left(\int_{B_k}w(x)^r\,dx\right)^{1/(rq_2)}\\&\preceq\left|\det A^{-1}(y)\right|^{\gamma/q_1}|B_k|^{-\gamma/q_1}w(B_k)^{1/q_2}\left(\int_{A(y)D_k}\left|f(x)\right|^{q_1/\gamma}dx\right)^{\gamma/q_1}.\end{aligned}$$

Proposition 2.4 and Lemma 3.1 show that

$$\begin{aligned}\Big(\int_{A(y)D_k}\big|f(x)\big|^{q_1/\gamma}\,dx\Big)^{\gamma/q_1}&\\ &\le \big|B\big(0,2^k\|A(y)\|\big)\big|^{(\gamma-1)/q_1}\Big(\int_{B(0,2^k\|A(y)\|)}\big|f(x)\big|^{q_1}\,dx\Big)^{1/q_1}\\ &\preceq \big\|A(y)\big\|^{Q\gamma/q_1}|B_k|^{\gamma/q_1}\Big(\frac{1}{w(B(0,2^k\|A(y)\|))}\int_{B(0,2^k\|A(y)\|)}\big|f(x)\big|^{q_1}w(x)\,dx\Big)^{1/q_1},\end{aligned}$$

which implies that

$$\begin{aligned}\big\|f\big(A(y)\cdot\big)\big\|_{L^{q_2}(D_k;w)}&\\ &\le \big|\det A^{-1}(y)\big|^{\gamma/q_1}\big\|A(y)\big\|^{Q\gamma/q_1}\frac{w(B_k)^{1/q_2}}{w(B(0,2^k\|A(y)\|))^{1/q_1}}\\ &\quad\times\|f\|_{L^{q_1}(B(0,2^k\|A(y)\|);w)}.\end{aligned}\tag{3.8}$$

Therefore, we infer from (3.7) and (3.8) that

$$\begin{aligned}\|T_{\Phi,A}f\|_{\dot K^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)}&\\ &\preceq\Big\{\sum_{k=-\infty}^{\infty}\Big(\int_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}\big|\det A^{-1}(y)\big|^{\gamma/q_1}\big\|A(y)\big\|^{Q\gamma/q_1}\\ &\quad\times\frac{w(B_k)^{1/q_2+\alpha_2/Q}}{w(B(0,2^k\|A(y)\|))^{1/q_1}}\|f\|_{L^{q_1}(B(0,2^k\|A(y)\|);w)}\,dy\Big)^p\Big\}^{1/p}\\ &\preceq\int_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}\big|\det A^{-1}(y)\big|^{\gamma/q_1}\big\|A(y)\big\|^{Q\gamma/q_1}\\ &\quad\times\Big\{\sum_{k=-\infty}^{\infty}\Big(\frac{w(B_k)^{1/q_2+\alpha_2/Q}}{w(B(0,2^k\|A(y)\|))^{1/q_1}}\|f\|_{L^{q_1}(B(0,2^k\|A(y)\|);w)}\Big)^p\Big\}^{1/p}dy\\ &\preceq\sum_{j=-\infty}^{\infty}\int_{2^{j-1}\le\|A(y)\|<2^j}\frac{|\Phi(y)|}{|y|_h^Q}\big|\det A^{-1}(y)\big|^{\gamma/q_1}\big\|A(y)\big\|^{Q\gamma/q_1}\\ &\quad\times\Big\{\sum_{k=-\infty}^{\infty}\Big(\frac{w(B_k)^{1/q_2+\alpha_2/Q}}{w(B_{k+j})^{1/q_1}}\|f\|_{L^{q_1}(B_{k+j};w)}\Big)^p\Big\}^{1/p}dy\\ &=\sum_{j=-\infty}^{\infty}\int_{2^{j-1}\le\|A(y)\|<2^j}\frac{|\Phi(y)|}{|y|_h^Q}\big|\det A^{-1}(y)\big|^{\gamma/q_1}\big\|A(y)\big\|^{Q\gamma/q_1}\\ &\quad\times\Big\{\sum_{k=-\infty}^{\infty}\Big[\Big(\frac{w(B_k)}{w(B_{k+j})}\Big)^{1/q_1+\alpha_1/Q}\\ &\quad\times\sum_{l=-\infty}^{j}\Big(\frac{w(B_{k+j})}{w(B_{k+l})}\Big)^{\alpha_1/Q}w(B_{k+l})^{\alpha_1/Q}\|f\|_{L^{q_1}(D_{k+l};w)}\Big]^p\Big\}^{1/p}dy,\end{aligned}\tag{3.9}$$

where the second inequality is obtained by the Minkowski inequality and the last equality holds for $1/q_1+\alpha_1/Q=1/q_2+\alpha_2/Q$.

Since $\alpha_1 < 0$ and $l \le j$, by Proposition 2.5 again, we have

$$\Big(\frac{w(B_{k+j})}{w(B_{k+l})}\Big)^{\alpha_1/Q} \preceq \Big(\frac{|B_{k+j}|}{|B_{k+l}|}\Big)^{\alpha_1(\delta-1)/(\delta Q)} = 2^{(j-l)\alpha_1(\delta-1)/\delta} \tag{3.10}$$

for any $1 < \delta < r_w$.

When $1/q_1 + \alpha_1/Q \ge 0$, Proposition 2.5 shows that, if $j \le 0$,

$$\Big(\frac{w(B_k)}{w(B_{k+j})}\Big)^{1/q_1+\alpha_1/Q} \preceq \Big(\frac{|B_k|}{|B_{k+j}|}\Big)^{\gamma(1/q_1+\alpha_1/Q)} = 2^{-jQ\gamma(1/q_1+\alpha_1/Q)}, \tag{3.11}$$

and if $j > 0$,

$$\Big(\frac{w(B_k)}{w(B_{k+j})}\Big)^{1/q_1+\alpha_1/Q} \preceq \Big(\frac{|B_k|}{|B_{k+j}|}\Big)^{(1/q_1+\alpha_1/Q)(\delta-1)/\delta} = 2^{-jQ(1/q_1+\alpha_1/Q)(\delta-1)/\delta} \tag{3.12}$$

for any $1 < \delta < r_w$.

When $1/q_1 + \alpha_1/Q < 0$, Proposition 2.5 yields that, if $j \le 0$,

$$\Big(\frac{w(B_k)}{w(B_{k+j})}\Big)^{1/q_1+\alpha_1/Q} \preceq \Big(\frac{|B_k|}{|B_{k+j}|}\Big)^{(1/q_1+\alpha_1/Q)(\delta-1)/\delta} = 2^{-jQ(1/q_1+\alpha_1/Q)(\delta-1)/\delta} \tag{3.13}$$

for any $1 < \delta < r_w$, and if $j > 0$,

$$\Big(\frac{w(B_k)}{w(B_{k+j})}\Big)^{1/q_1+\alpha_1/Q} \preceq \Big(\frac{|B_k|}{|B_{k+j}|}\Big)^{\gamma(1/q_1+\alpha_1/Q)} = 2^{-jQ\gamma(1/q_1+\alpha_1/Q)}. \tag{3.14}$$

Therefore, if $1 \le p < \infty$ and $1/q_1 + \alpha_1/Q \ge 0$, we infer from (3.9)–(3.12) that, for any $1 < \delta < r_w$,

$$\begin{aligned}
&\|T_{\Phi,A}f\|_{\dot K^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)} \\
&\preceq \sum_{j=-\infty}^{0} \int_{2^{j-1}\le\|A(y)\|<2^j} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{\gamma/q_1} \big\|A(y)\big\|^{-\gamma\alpha_1} \\
&\quad\times \Big\{\sum_{k=-\infty}^{\infty}\Big[\sum_{l=-\infty}^{j} 2^{(j-l)\alpha_1(\delta-1)/\delta} w(B_{k+l})^{\alpha_1/Q}\|f\|_{L^{q_1}(D_{k+l};w)}\Big]^p\Big\}^{1/p} dy \\
&\quad+ \sum_{j=1}^{\infty} \int_{2^{j-1}\le\|A(y)\|<2^j} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{\gamma/q_1} \big\|A(y)\big\|^{-Q(1/q_1+\alpha_1/Q)(\delta-1)/\delta+Q\gamma/q_1} \\
&\quad\times \Big\{\sum_{k=-\infty}^{\infty}\Big[\sum_{l=-\infty}^{j} 2^{(j-l)\alpha_1(\delta-1)/\delta} w(B_{k+l})^{\alpha_1/Q}\|f\|_{L^{q_1}(D_{k+l};w)}\Big]^p\Big\}^{1/p} dy \\
&\preceq \sum_{j=-\infty}^{0} \int_{2^{j-1}\le\|A(y)\|<2^j} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{\gamma/q_1} \big\|A(y)\big\|^{-\gamma\alpha_1} \\
&\quad\times \Big\{\sum_{l=-\infty}^{j} 2^{(j-l)\alpha_1(\delta-1)/\delta}\Big(\sum_{k=-\infty}^{\infty} w(B_{k+l})^{\alpha_1 p/Q}\|f\|^p_{L^{q_1}(D_{k+l};w)}\Big)^{1/p}\Big\} dy \\
&\quad+ \sum_{j=1}^{\infty} \int_{2^{j-1}\le\|A(y)\|<2^j} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{\gamma/q_1} \big\|A(y)\big\|^{-Q(1/q_1+\alpha_1/Q)(\delta-1)/\delta+Q\gamma/q_1}
\end{aligned}$$

$$\times \Big\{ \sum_{l=-\infty}^{j} 2^{(j-l)\alpha_1(\delta-1)/\delta} \Big(\sum_{k=-\infty}^{\infty} w(B_{k+l})^{\alpha_1 p/Q} \|f\|^p_{L^{q_1}(D_{k+l};w)} \Big)^{1/p} \Big\} \, dy$$

$$\preceq \|f\|_{\dot K^{\alpha_1,p}_{q_1}(\mathbb{H}^n;w)} \Big(\int_{\|A(y)\|<1} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{\gamma/q_1} \big\|A(y)\big\|^{-\gamma\alpha_1} \, dy$$

$$+ \int_{\|A(y)\|\geq 1} \frac{|\Phi(y)|}{|y|_h^Q} \big|\det A^{-1}(y)\big|^{\gamma/q_1} \big\|A(y)\big\|^{-Q(1/q_1+\alpha_1/Q)(\delta-1)/\delta + Q\gamma/q_1} \, dy \Big).$$

This proves Theorem 1.3(i).

When $1 \leq p < \infty$ and $1/q_1 + \alpha_1/Q < 0$, by virtue of (3.9), (3.13), (3.14), and a similar argument as above, we complete the proof of Theorem 1.3. □

From this point forward, for the sake of convenience, we will sometimes use $w(\cdot)$ for $|\cdot|_h^\beta$.

3.3. **Proof of Theorem 1.4.** Similar to the proof of the preceding theorem, noting that for any $k \in \mathbf{Z}$,

$$w(B_k) = \int_{|x|\leq 2^k} |x|_h^\beta \, dx \simeq 2^{k(Q+\beta)}, \tag{3.15}$$

we have

$$\|T_{\Phi,A} f\|_{\dot K^{\alpha,p}_q(\mathbb{H}^n;|\cdot|_h^\beta)} \preceq \Big\{ \sum_{k=-\infty}^{\infty} \Big(\int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} 2^{k\alpha(Q+\beta)/Q} \big\| f\big(A(y)\cdot\big)\big\|_{L^q(D_k;|\cdot|_h^\beta)} \, dy \Big)^p \Big\}^{1/p}. \tag{3.16}$$

On the other hand, it follows from Lemma 3.1 that

$$\begin{aligned} &\big\| f\big(A(y)\cdot\big)\big\|_{L^q(D_k;|\cdot|_h^\beta)} \\ &= \big|\det A^{-1}(y)\big|^{1/q} \Big(\int_{A(y)D_k} \big|f(x)\big|^q \big|A^{-1}(y)x\big|_h^\beta \, dx \Big)^{1/q} \\ &\preceq \begin{cases} (|\det A^{-1}(y)| \|A^{-1}(y)\|^\beta)^{1/q} \|f\|_{L^q(A(y)D_k;|\cdot|_h^\beta)} & \beta > 0, \\ (|\det A^{-1}(y)| \|A(y)\|^{-\beta})^{1/q} \|f\|_{L^q(A(y)D_k;|\cdot|_h^\beta)} & \beta \leq 0. \end{cases} \end{aligned} \tag{3.17}$$

Next we estimate $\|f\|_{L^q(A(y)D_k;|\cdot|_h^\beta)}$. By the definition of D_k and Lemma 3.1, it is clear that

$$A(y)D_k \subset \big\{x : \big\|A^{-1}(y)\big\|^{-1} 2^{k-1} \leq |x|_h < \big\|A(y)\big\| 2^k \big\}.$$

For any $y \in \operatorname{supp}(\Phi)$, there is $j_0 \in \mathbf{Z}$ such that

$$2^{j_0} \leq \big\|A^{-1}(y)\big\|^{-1} < 2^{j_0+1}. \tag{3.18}$$

Since $\|A^{-1}(y)\|^{-1} \leq \|A(y)\|$, there must exist a nonnegative integer m_0 satisfying

$$2^{j_0+m_0} \leq \big\|A(y)\big\| < 2^{j_0+m_0+1}. \tag{3.19}$$

The inequalities (3.18) and (3.19) imply that

$$\log_2\big(\big\|A^{-1}(y)\big\|\big\|A(y)\big\|/2\big) < m_0 < \log_2\big(2\big\|A^{-1}(y)\big\|\big\|A(y)\big\|\big)$$

and

$$A(y)D_k \subset \left\{x : 2^{j_0+k-1} \le |x|_h < 2^{j_0+m_0+k+1}\right\}.$$

Therefore,

$$\|f\|_{L^q(A(y)D_k;|\cdot|_h^\beta)} \preceq \sum_{l=j_0}^{j_0+m_0+1} \|f\|_{L^q(D_{k+l};|\cdot|_h^\beta)}. \tag{3.20}$$

When $\beta > 0$, it follows from (3.16), (3.17), and (3.20) that

$$\begin{aligned}
&\|T_{\Phi,A}f\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;|\cdot|_h^\beta)} \\
&\quad\preceq \Big\{\sum_{k=-\infty}^{\infty}\Big(\int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q}\big(|\det A^{-1}(y)|\|A^{-1}(y)\|^\beta\big)^{1/q} \\
&\qquad \times 2^{k\alpha(Q+\beta)/Q} \sum_{l=j_0}^{j_0+m_0+1} \|f\|_{L^q(D_{k+l};|\cdot|_h^\beta)}\, dy\Big)^p\Big\}^{1/p} \\
&\quad\preceq \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q}\big(|\det A^{-1}(y)|\|A^{-1}(y)\|^\beta\big)^{1/q} \\
&\qquad \times \Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{l=j_0}^{j_0+m_0+1} 2^{k\alpha(Q+\beta)/Q}\|f\|_{L^q(D_{k+l};|\cdot|_h^\beta)}\Big)^p\Big\}^{1/p} dy \\
&\quad\preceq \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q}\big(|\det A^{-1}(y)|\|A^{-1}(y)\|^\beta\big)^{1/q} \\
&\qquad \times \sum_{l=j_0}^{j_0+m_0+1} 2^{-l\alpha(Q+\beta)/Q}\Big(\sum_{k=-\infty}^{\infty} 2^{(k+l)\alpha(Q+\beta)p/Q}\|f\|^p_{L^q(D_{k+l};|\cdot|_h^\beta)}\Big)^{1/p} dy \\
&\quad\preceq \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;|\cdot|_h^\beta)} \\
&\qquad \times \Big(\int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q}\big(|\det A^{-1}(y)|\|A^{-1}(y)\|^\beta\big)^{1/q} \sum_{l=j_0}^{j_0+m_0+1} 2^{-l\alpha(Q+\beta)/Q}\, dy\Big).
\end{aligned} \tag{3.21}$$

Noting that, for $\alpha = 0$,

$$\sum_{l=j_0}^{j_0+m_0+1} 2^{-l\alpha(Q+\beta)/Q} = m_0 + 2 \preceq 1 + \log_2\big(\|A^{-1}(y)\|\|A(y)\|\big), \tag{3.22}$$

and for $\alpha \ne 0$,

$$\begin{aligned}
\sum_{l=j_0}^{j_0+m_0+1} 2^{-l\alpha(Q+\beta)/Q} &= 2^{-j_0\alpha(Q+\beta)/Q}\frac{1-2^{-\alpha(Q+\beta)(m_0+2)/Q}}{1-2^{-\alpha(Q+\beta)/Q}} \\
&\preceq \begin{cases} \|A^{-1}(y)\|^{\alpha(Q+\beta)/Q} & \alpha > 0, \\ \|A(y)\|^{-\alpha(Q+\beta)/Q} & \alpha < 0, \end{cases}
\end{aligned} \tag{3.23}$$

we complete the proof of part (i) of the theorem from (3.21)–(3.23). By a similar argument as above, we also conclude the proof of part (ii).

For part (iii), similar to the proof of Theorem 1.2, it suffices to show that for every central $(\alpha, q; |\cdot|_h^\beta)$-block b_k with the support in B_k, we have

$$\widetilde{H}(b_k) = \sum_{j=-\infty}^{\infty} \mu_{kj} b_{kj},$$

where each b_{kj} is a central $(\alpha, q; w)$-block with the support in B_{k+j}, and

$$\Big(\sum_{j=-\infty}^{\infty} |\mu_{kj}|^p\Big)^{1/p} \preceq C_7$$

uniformly for $k \in \mathbf{Z}$.

Let g_{kj} be as in (3.1). By (3.2), $\operatorname{supp} g_{kj} \subset B_{k+j}$, and

$$\|b_{kj}\|_{L^q(\mathbb{H}^n;|\cdot|_h^\beta)} \preceq \int_{2^{j-1}\le\|A^{-1}(y)\|<2^j} \frac{|\Phi(y)|}{|y|_h^Q} \big\|a_k\big(A(y)\cdot\big)\big\|_{L^q(\mathbb{H}^n;|\cdot|_h^\beta)}\,dy.$$

Since $2^{j-1} \le \|A^{-1}(y)\| < 2^j$, by (3.15) we have

$$\begin{aligned}
&\big\|a_k\big(A(y)\cdot\big)\big\|_{L^q(\mathbb{H}^n;|\cdot|_h^\beta)}\\
&\quad\preceq \big(\big\|A(y)\big\|^{-\beta}\big|\det A^{-1}(y)\big|\big)^{1/q}\big\|a_k(\cdot)\big\|_{L^q(\mathbb{H}^n;|\cdot|_h^\beta)}\\
&\quad\preceq \big(\big\|A(y)\big\|^{-\beta}\big|\det A^{-1}(y)\big|\big)^{1/q} 2^{-k\alpha(Q+\beta)/Q}\\
&\quad\preceq \big(\big\|A(y)\big\|^{-\beta}\big|\det A^{-1}(y)\big|\big)^{1/q}\big\|A^{-1}(y)\big\|^{\alpha(1+\beta/Q)}\Big(\int_{B_{k+j}} |x|_h^\beta\,dx\Big)^{-\alpha/Q}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|g_{kj}\|_{L^q(\mathbb{H}^n;|\cdot|_h^\beta)}\\
&\quad\preceq \Big(\int_{2^{j-1}\le\|A^{-1}(y)\|<2^j} \frac{|\Phi(y)|}{|y|_h^Q}\big(\big\|A(y)\big\|^{-\beta}\big|\det A^{-1}(y)\big|\big)^{1/q}\big\|A^{-1}(y)\big\|^{\alpha(1+\beta/Q)}\,dy\Big)\\
&\qquad\times\Big(\int_{B_{k+j}} |x|_h^\beta\,dx\Big)^{-\alpha/Q}\\
&\quad:= \mu_{kj}\Big(\int_{B_{k+j}} |x|_h^\beta\,dx\Big)^{-\alpha/Q}.
\end{aligned}$$

Let

$$g_{kj} = \mu_{kj} b_{kj}.$$

It is easy to check that each b_{kj} is a central $(\alpha, q; |\cdot|_h^\beta)$-block with the support in B_{k+j}. By a similar discussion as in (3.4) and (3.5), respectively, we have

$$\sum_{j=-\infty}^{\infty} |\mu_{kj}|^p \preceq \Big(\int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q}\big(\big\|A(y)\big\|^{-\beta}\big|\det A^{-1}(y)\big|\big)^{1/q}\big\|A^{-1}(y)\big\|^{\alpha(1+\beta/Q)}\,dy\Big)^p,$$

if $1 \le p < \infty$ and

$$\begin{aligned}\sum_{j=-\infty}^{\infty} |\mu_{kj}|^p &\\ \preceq \Big(\int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} &\big(\big\|A(y)\big\|^{-\beta}\big|\det A^{-1}(y)\big|\big)^{1/q}\big\|A^{-1}(y)\big\|^{\alpha(1+\beta/Q)} \\ &\times \big(1+\big|\log_2\big\|A^{-1}(y)\big\|\big|\big)^{\sigma}\, dy\Big)^p, \end{aligned} \tag{3.24}$$

if $0 < p < 1$ and $(1-p)/p < \sigma$. This completes the proof. □

3.4. **Proof of Theorem 1.5.** If $\|A^{-1}(y)\| \preceq \|A(y)\|^{-1}$, then Lemma 3.1 yields that

$$\big\|A(y)\big\|^{-Q} \simeq \big|\det A^{-1}(y)\big| \simeq \big\|A^{-1}(y)\big\|^{Q}. \tag{3.25}$$

The "if" part of Theorem 1.5 is easily obtained from Theorem 1.4. Next we will show the "only if" part.

When $\alpha \ge 0$, for any $\epsilon > 0$, let

$$f_\epsilon(x) = |x|_h^{-(Q+\beta)(\alpha/Q+1/q)-\epsilon/q}\chi_{\{|x|_h>1\}}.$$

A simple calculation shows that, for any $k \ge 1$,

$$\|f_\epsilon\|_{L^q(D_k;|\cdot|_h^\beta)} \simeq \Big(\frac{2^{(Q+\beta)\alpha q/Q+\epsilon}-1}{(Q+\beta)\alpha q/Q+\epsilon}2^{-k((Q+\beta)\alpha q/Q+\epsilon)}\Big)^{1/q}, \tag{3.26}$$

which gives that

$$\begin{aligned}\|f_\epsilon\|_{\dot K_q^{\alpha,p}(\mathbb{H}^n;|\cdot|_h^\beta)} &\simeq \Big(\sum_{k=1}^{\infty} 2^{k(Q+\beta)\alpha p/Q}\|f_\epsilon\|^p_{L^q(D_k;|\cdot|_h^\alpha)}\Big)^{1/p} \\ &\simeq \Big(\frac{2^{(Q+\beta)\alpha q/Q+\epsilon}-1}{(Q+\beta)\alpha q/Q+\epsilon}\Big)^{1/q}\Big(\sum_{k=1}^{\infty} 2^{-kp\epsilon/q}\Big)^{1/p} \\ &= \Big(\frac{2^{(Q+\beta)\alpha q/Q+\epsilon}-1}{(Q+\beta)\alpha q/Q+\epsilon}\Big)^{1/q}\frac{1}{(2^{\epsilon p/q}-1)^{1/p}}. \end{aligned} \tag{3.27}$$

On the other hand, changing variables and Lemma 3.1 yield that

$$\begin{aligned}&T_{\Phi,A}f_\epsilon(x) \\ &\quad= \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q}\big|A(y)x\big|_h^{-(Q+\beta)(\alpha/Q+1/q)-\epsilon/q}\chi_{\{|A(y)x|_h>1\}}\, dy \\ &\quad\succeq \Big(\int_{\|A(y)\|\succeq 1/|x|_h} \frac{\Phi(y)}{|y|_h^Q}\big\|A(y)\big\|^{-(Q+\beta)(\alpha/Q+1/q)-\epsilon/q}\, dy\Big)|x|_h^{-(Q+\beta)(\alpha/Q+1/q)-\epsilon/q},\end{aligned}$$

which implies that

$$\begin{aligned}&\|T_{\Phi,A}f_\epsilon\|_{\dot K_q^{\alpha,p}(\mathbb{H}^n;|\cdot|_h^\beta)} \\ &\quad\simeq \Big(\sum_{k=-\infty}^{\infty} 2^{k(Q+\beta)\alpha p/Q}\|T_{\Phi,A}f_\epsilon\|^p_{L^q(D_k;|\cdot|_h^\beta)}\Big)^{1/p}\end{aligned}$$

$$\succeq \Big\{ \sum_{k=-\infty}^{\infty} 2^{k(Q+\beta)\alpha p/Q} \Big(\int_{2^{k-1}\le |x|_h < 2^k} |x|_h^{-(Q+\beta)(\alpha/Q+1/q)q-\epsilon+\beta} \times \Big| \int_{\|A(y)\|\succeq 2^{-k}} \frac{\Phi(y)}{|y|_h^Q} \big\|A(y)\big\|^{-(Q+\beta)(\alpha/Q+1/q)-\epsilon/q} \, dy \Big|^q \, dx \Big)^{p/q} \Big\}^{1/p}. \quad (3.28)$$

For any $\epsilon > 0$, there is an integer k_0 such that $2^{-k_0} \le \epsilon < 2^{-k_0+1}$. Then (3.28) means that

$$\begin{aligned} &\|T_{\Phi,A} f_\epsilon\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;|\cdot|_h^\beta)} \\ &\succeq \int_{\|A(y)\|\succeq \epsilon} \frac{\Phi(y)}{|y|_h^Q} \big\|A(y)\big\|^{-(Q+\beta)(\alpha/Q+1/q)-\epsilon/q} \, dy \\ &\quad \times \Big\{ \sum_{k=k_0}^{\infty} 2^{k(Q+\beta)\alpha p/Q} \Big(\int_{2^{k-1}\le |x|_h < 2^k} |x|_h^{-(Q+\beta)(\alpha/Q+1/q)q-\epsilon+\beta} \, dx \Big)^{p/q} \Big\}^{1/p}. \end{aligned} \quad (3.29)$$

By (3.26),

$$\begin{aligned} &\Big\{ \sum_{k=k_0}^{\infty} 2^{k(Q+\beta)\alpha p/Q} \Big(\int_{2^{k-1}\le |x|_h < 2^k} |x|_h^{-(Q+\beta)(\alpha/Q+1/q)q-\epsilon+\beta} \, dx \Big)^{p/q} \Big\}^{1/p} \\ &\simeq \Big(\frac{2^{(Q+\beta)\alpha q/Q+\epsilon} - 1}{(Q+\beta)\alpha q/Q+\epsilon} \Big)^{1/q} \Big(\sum_{k=k_0}^{\infty} 2^{-kp\epsilon/q} \Big)^{1/p} \\ &\simeq \Big(\frac{2^{(Q+\beta)\alpha q/Q+\epsilon} - 1}{(Q+\beta)\alpha q/Q+\epsilon} \Big)^{1/q} \frac{\epsilon^{\epsilon/q} 2^{\epsilon/q}}{(2^{\epsilon p/q} - 1)^{1/p}}. \end{aligned} \quad (3.30)$$

Therefore, (3.27), (3.29), and (3.30) tell us that the inequality

$$\epsilon^{\epsilon/q} 2^{\epsilon/q} \Big(\int_{\|A(y)\|\succeq \epsilon} \frac{\Phi(y)}{|y|_h^Q} \big\|A(y)\big\|^{-(Q+\beta)(\alpha/Q+1/q)-\epsilon/q} \, dy \Big) \preceq 1$$

holds uniformly on $\epsilon > 0$. Letting $\epsilon \to 0^+$, we obtain the desired conclusion.

When $\alpha < 0$, for any $\epsilon > 0$, let

$$f_\epsilon(x) = |x|_h^{-(Q+\beta)(\alpha/Q+1/q)+\epsilon/q} \chi_{\{|x|_h \le 1\}}.$$

We finish the proof of Theorem 1.5 by a similar argument as above. □

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