Banach J. Math. Anal. 11 (2017), no. 2, 363–381
http://dx.doi.org/10.1215/17358787-0000008X
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

RADON–NIKODYM THEOREMS FOR OPERATOR-VALUED MEASURES AND CONTINUOUS GENERALIZED FRAMES

FENGJIE LI* and PENGTONG LI

Communicated by J. Stochel

Abstract. In this article we determine that an *operator-valued measure* (OVM) for Banach spaces is actually a weak* measure, and then we show that an OVM can be represented as an operator-valued function if and only if it has σ-finite variation. By the means of direct integrals of Hilbert spaces, we introduce and investigate *continuous generalized frames* (*continuous operator-valued frames*, or simply *CG frames*) for general Hilbert spaces. It is shown that there exists an intrinsic connection between CG frames and positive OVMs. As a byproduct, we show that a Riesz-type CG frame does not exist unless the associated measure space is purely atomic. Also, a dilation theorem for dual pairs of CG frames is given.

1. Introduction

Throughout, the scalar field $\mathbb{K}$ can be either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. Denote by (Ω, Σ) the measurable space associated with a σ-algebra Σ of a set Ω; in the present paper we also call it a *σ-algebra*. A (scalar-valued) measure space, or a *measure*, is denoted by (Ω, Σ, μ), or (Ω, μ), or just μ for short. For $1 \leq p \leq \infty$, the notation $L_p(\Omega, \mu)$—or simply, $L_p(\mu)$ or L_p—will denote the usual function spaces, and l_p the usual sequence spaces. For convenience, the term "μ-almost everywhere" is commonly abbreviated to "μ-a.e." and we will use the notation "$\bigsqcup$" to denote the union of mutually disjoint measurable sets.

Received Mar. 1, 2016; Accepted Jun. 9, 2016.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 43A15; Secondary 42C15, 46G10, 46B22, 47A20.

Keywords. Radon–Nikodym theorem, operator-valued measure, weak* measure, dilation, continuous generalized frame.

The restricted subspace of a measure space (Ω, μ) on a measurable set $E \subseteq \Omega$ will be denoted by (E, μ). If X is a Banach space, the action of x in X and φ in the Banach dual X^* will be denoted by $\varphi(x)$ or $\langle x, \varphi\rangle$ or $\langle \varphi, x\rangle$. For Banach spaces X, Y, we use $B(X, Y)$ to denote the space of all bounded linear operators from X to Y, and we write $B(X) = B(X, X)$.

A spectral measure is a self-adjoint projection-valued function on a σ-algebra. The theory of spectral measure is an important topic in the theory of operator algebras. As a generalization, positive operator-valued functions on a σ-algebra (i.e., *positive operator-valued measures*, or POVMs), play an important role in quantum theory. Recently, Han et al. in [9] developed a general dilation theory for OVMs as a generalization of the well-known Naimark's dilation theorem, which states that a POVM can be dilated to a spectral measure.

The representability of an additive function on a σ-algebra, named the *Radon–Nikodym property*, is also important in the theory of vector measures. It studies when a vector measure can be represented by a function. The terminology "Radon–Nikodym Property" comes from the following well-known Radon–Nikodym theorem (see [17, Appendix C]).

Theorem 1.1. *Let ν, μ be two finite scalar measure on a σ-algebra (Ω, Σ). If μ is positive and ν is μ-continuous (i.e., $\mu(E) = 0$ implies that $\nu(E) = 0$), then there exists a function $f \in L_1(\Omega, \Sigma, \mu)$ such that*

$$\nu(E) = \int_E f(\omega)\, d\mu(\omega)$$

for all $E \in \Sigma$.

In the present paper, we focus first on the following problem.

Problem 1.2. Let (Ω, Σ) be a σ-algebra and let X, Y be Banach spaces (or let $X = Y$ be Hilbert spaces). When can an OVM (or a POVM) $\nu : \Sigma \to B(X, Y)$ be represented as a function $F : \Omega \to B(X, Y)$ in some way?

Of course, there have been some answers to this problem; for example, see [2], [3], [12], [15]. We think that the present article provides some other answers, which will be used in the other topic we address here.

The second topic of this paper is devoted to the study of *continuous generalized frames*. The frame theory was first proposed by Duffin and Schaeffer in 1952 to study some deep problems in nonharmonic Fourier series [5]. Generally speaking, a frame is a linear representation from a abstract linear space into a concrete linear space having a special form. We list three types of frames here.

(1) A sequence $\{x_i\}_{i=1}^{\infty}$ in an abstract Hilbert space H is regarded as a frame if $x \mapsto \{\langle x, x_i\rangle\}_{i=1}^{\infty}$ is an isomorphism from H onto a closed subspace of the space l_2.
(2) Let H be a Hilbert space and let (Ω, μ) be a σ-finite, positive measure space. A weakly measurable function $F : \Omega \to H$ is called a *continuous frame* for H if $x \mapsto \langle x, f(\cdot)\rangle$ is an isomorphism from H onto a closed subspace of $L_2(\Omega, \mu)$ (see [7], [8]).

(3) Let $H, \{H_i\}_{i=1}^{\infty}$ be Hilbert spaces. A sequence $\{T_i\}_{i=1}^{\infty}$ of linear operators, where $T_i : H \to H_i$ is bounded for each i, is called a *generalized frame* (or an *operator-valued frame*) if $x \mapsto \bigoplus_{i=1}^{\infty} T_i x$ is an isomorphism from H onto a closed subspace of $\bigoplus_{i=1}^{\infty} H_i$ (see [11], [18], [19]).

By combining (2) and (3) above, one can naturally think about the concept of a "continuous generalized frame" as an "operator-valued function." Abdollahpour and Faroughi [1] introduced such a concept, but the "synthesis/concrete space" in it is unknown in general. Robinson developed this concept with some elaborations in his Ph.D. thesis [16] and figured out that the "synthesis space" is in fact a direct integral of Hilbert spaces. However, the Hilbert spaces considered in [16] are separable. This does not contain the following simple example.

Example 1.3. Let (Ω, Σ, μ) be any positive measure space satisfying $\mu(\Omega) = 1$, let H be a nonseparable Hilbert space, and let $I \in B(H)$ be the identity operator. Define $\mathcal{F} : \Omega \to B(H)$ by $\omega \mapsto I$ for every $\omega \in \Omega$. Then $\mathcal{F}$ is a continuous generalized frame for H.

Our paper is organized as follows. In Section 2, we mainly study Problem 1.2. We figure out that an OVM for Banach spaces is actually a weak* measure, and then obtain several Radon–Nikodym theorems for OVMs (or POVMs). In Section 3, by the means of Radon–Nikodym theorems and direct integrals for Hilbert spaces, we introduce and study continuous generalized frames (continuous operator-valued frames, or CG frames) for Hilbert spaces, where the separability of the Hilbert spaces is assumed. One main result shows that there is an intrinsic connection between CG frames and POVMs. As a byproduct, we show that a Riesz-type CG frame does not exist unless the associated measure space is purely atomic. Also, a dilation theorem for dual pairs of CG frames is given, which generalizes the corresponding result for continuous vector frames in [8], and essentially presents a dilation result for pairs of bounded operators.

2. Operator-valued measures and Radon–Nikodym theorems

Let (Ω, Σ, μ) be a positive measure space and let X be a Banach space. A function $G : \Omega \to X$ is called *simple* if there exist $x_1, \ldots, x_n$ in X and $E_1, \ldots, E_n$ in Σ such that $G = \sum_{i=1}^{n} x_i \chi_{E_i}$, where $\chi_{E_i}(\omega) = 1$ if $\omega \in E_i$ and $\chi_{E_i}(\omega) = 0$ if $\omega \notin E_i$. A function $G : \Omega \to X$ is called *μ-measurable*, if there exists a sequence $\{G_n\}_{n=1}^{\infty}$ of simple functions with $\lim_n \|G(\omega) - G_n(\omega)\| = 0$ on Ω μ-a.e. These concepts include the case $X = \mathbb{K}$. Let $\Gamma \subseteq X^*$. A function $G : \Omega \to X$ is called *Γ-μ-measurable* if $\langle \varphi, G \rangle : \omega \mapsto \langle \varphi, G(\omega) \rangle$ is μ-measurable for every $\varphi \in \Gamma$. We consider X as a subspace of X^{**} such that the X-μ-measurability of a function $G : \Omega \to X^*$ is well defined.

Lemma 2.1 (Pettis measurability theorem [4, Theorem II.2])**.** *Let X be a Banach space, and let (Ω, μ) be a σ-finite, positive measure space. The followings are equivalent for a function $G : \Omega \to X$:*

(1) *G is μ-measurable,*

(2) *function* $\langle \varphi, G\rangle$ *is* μ*-measurable for every* $\varphi \in X^*$ *and* G *is* μ*-essentially separably valued (i.e., there exists a measurable set* E *with* $\mu(E) = 0$ *such that the set* $\{G(\omega) : \omega \in \Omega\backslash E\}$ *lies in a separable subspace of* X*).*

Let (Ω, Σ) be a σ-algebra and let X be a Banach space. The variation of a function $\nu : \Sigma \to X$ on $E \in \Sigma$ is defined by

$$|\nu|(E) = \sup\Big\{\sum_{i=1}^{n}\big\|\nu(E_i)\big\| : E = \bigsqcup_{i=1}^{n} E_i, \{E_i\}_{i=1}^{n} \subseteq \Sigma\Big\}.$$

Clearly, $|\nu| : \Sigma \to [0, +\infty]$ is countably additive if ν is countably additive. The notation $ca(\Omega, \Sigma)$ will denote the Banach space consisting of all scalar valued measures on (Ω, Σ) equipped with the *variation norm*

$$\|\mu\| = |\mu|(\Omega)$$

for $\mu \in ca(\Omega, \Sigma)$ (see [6]). A function $\nu : \Sigma \to X^*$ is called a *weak* measure* if $\langle x, \nu(\cdot)\rangle \in ca(\Omega, \Sigma)$ for every $x \in X$.

The following lemma describes the *weak* Radon–Nikodym Property* for conjugate Banach spaces, which is important for our purposes here. In this lemma, statement (1) comes directly from Theorem 9.1 in [13] and statement (2) can be found at Remark 3.9 in [13].

Lemma 2.2. *Let* X *be a Banach space and let* (Ω, Σ, μ) *be a* σ*-finite, positive measure space.*

(1) *Suppose that* $\nu : \Sigma \to X^*$ *is a weak* measure. Then* $|\nu|$ *is a positive measure. If* $|\nu|$ *is* σ*-finite and* μ*-continuous (i.e.,* $\mu(E) = 0$ *implies that* $|\nu|(E) = 0$*), then there exists an* X*-*μ*-measurable function* $f : \Omega \to X^*$ *such that*

$$\big\{f(\omega) : \omega \in \Omega\big\} \subseteq \overline{\operatorname{conv}}^* A_\nu(\Omega)$$

and

$$\big\langle x, \nu(E)\big\rangle = \int_E \langle x, f\rangle \, d\mu$$

for all $x \in X$ *and* $E \in \Sigma$*. Here* $\overline{\operatorname{conv}}^* A_\nu(\Omega)$ *denotes the weak* closed convex hull of the set*

$$A_\nu(\Omega) = \Big\{\frac{\nu(F)}{\mu(F)} : \mu(F) > 0, F \in \Sigma\Big\}.$$

(2) *Suppose that* $f : \Omega \to X^*$ *is an* X*-*μ*-measurable function. If*

$$\int_\Omega \langle x, f\rangle \, d\mu$$

is well-defined for each $x \in X$*, then* $\nu : \Sigma \to X^*$ *defined by*

$$\big\langle x, \nu(E)\big\rangle = \int_E \langle x, f\rangle \, d\mu$$

is a weak measure such that* $|\nu|$ *is a* μ*-continuous,* σ*-finite, positive measure.*

Definition 2.3. Let (Ω,Σ) be a σ-algebra, letX,Y be Banach spaces, and let $\nu:\Sigma\to B(X,Y)$ be a function. If $\nu_{x,\psi}:\Sigma\to\mathbb{K}$ defined by $E\mapsto\langle\nu(E)x,\psi\rangle$ is in $ca(\Omega,\Sigma)$ for every $x\in X$ and $\psi\in Y^*$, then ν is considered an OVM. In particular, if $X=Y$ is a Hilbert space and that ν takes values in the positive operators, then ν is considered a POVM.

Let X,Y be two Banach spaces with $x\in X,\psi\in Y^*$. The symbol $x\otimes\psi$ is denoted by the bounded linear functional $T\mapsto\psi(Tx)$ on $B(X,Y)$. We denote $X\hat{\otimes}_\pi Y^*$ by the closed linear span of such functionals in the Banach dual $B(X,Y)^*$. Here, the space $X\hat{\otimes}_\pi Y^*$ is clearly equivalent to the *projective tensor product* of X and Y^* defined in [17]. In the following, we write $B(X\times Y^*,Z)$ for the Banach space of all bounded bilinear mappings from the Cartesian product $X\times Y^*$ into Z, where the norm is given by

$$\|\xi\|=\sup\big\{\big\|\xi(x,\psi)\big\|:\|x\|\leq 1,\|\psi\|\leq 1\big\}$$

for $\xi\in B(X\times Y^*,Z)$.

Lemma 2.4. *Let (Ω,Σ) be a σ-algebra, let X,Y,Z be Banach spaces, and let $\nu:\Sigma\to B(X,Y)$ be an OVM. Then the following hold:*

(1) *if $\xi\in B(X\times Y^*,Z)$, then there exists a unique bounded linear operator $T_\xi:X\hat{\otimes}_\pi Y^*\to Z$ such that $T_\xi(x\otimes\psi)=\xi(x,\psi)$ holds for every $x\in X,\psi\in Y^*$, and the correspondence $\xi\leftrightarrow T_\xi$ is an isometric isomorphism between the Banach spaces $B(X\times Y^*,Z)$ and $B(X\hat{\otimes}_\pi Y^*,Z)$;*

(2) *there is an isometric isomorphism between $(X\hat{\otimes}_\pi Y^*)^*$ and $B(X,Y^{**})$, so we have an identification $B(X,Y^{**})=(X\hat{\otimes}_\pi Y^*)^*$ through which the action of an operator $T\in B(X,Y^{**})$ as a linear functional on $X\hat{\otimes}_\pi Y^*$ is given by $\langle x\otimes\psi,T\rangle=\langle\psi,Tx\rangle$;*

(3) *the bilinear mapping $\xi_\nu:X\times Y^*\to ca(\Omega,\Sigma)$ defined by $(x,\psi)\mapsto\nu_{x,\psi}$ is bounded, where $\nu_{x,\psi}(E)=\langle\nu(E)x,\psi\rangle$ for $E\in\Sigma$;*

(4) *the OVM ν is a weak* measure if we look on $B(X,Y)$ as a subspace of $B(X,Y^{**})$ and $X\hat{\otimes}Y^*$ as the predual of $B(X,Y^{**})$.*

Proof. (1) This is Theorem 2.9 in [17].

(2) Let $\xi\in B(X\times Y^*,\mathbb{K})$. Define a linear operator $S_\xi:X\to Y^{**}$ by

$$\langle\psi,S_\xi x\rangle=\xi(x,\psi)\quad\text{for } x\in X,\psi\in Y^*.$$

It is easily seen that S_ξ is bounded and that the correspondence $\xi\leftrightarrow S_\xi$ is an isometric isomorphism between $B(X\times Y^*,\mathbb{K})$ and $B(X,Y^{**})$. Then the result follows by taking $Z=\mathbb{K}$ in (1).

(3) Let $\psi_0\in Y^*$. Take $\{x_n\}_{n=1}^\infty$ in X such that $x_n\to x_0$ in X and $\nu_{x_n,\psi_0}\to\mu_0$ in $ca(\Omega,\Sigma)$. For every $E\in\Sigma$, we have

$$\begin{aligned}\big|\nu_{x_n,\psi_0}(E)-\mu_0(E)\big|&=\big|(\nu_{x_n,y_0}-\mu_0)(E)\big|\\&\leq|\nu_{x_n,\psi_0}-\mu_0|(E)\\&\leq|\nu_{x_n,\psi_0}-\mu_0|(\Omega)\\&=\|\nu_{x_n,\psi_0}-\mu_0\|\\&\to 0.\end{aligned}$$

This means that $\{\nu_{x_n,\psi_0}(E)\}_{n=1}^{\infty}$ converges to $\mu_0(E)$. On the other hand,

$$\nu_{x_n,\psi_0}(E) = \big\langle \nu(E)x_n, \psi_0 \big\rangle \to \big\langle \nu(E)x_0, \psi_0 \big\rangle$$

by the continuity. So we have $\mu_0(E) = \nu_{x_0,\psi_0}(E)$. By the arbitrariness of $E \in \Sigma$, we know that $\mu_0 = \nu_{x_0,\psi_0}$. The *closed graph theorem* says that the linear mapping

$$X \to ca(\Omega, \Sigma), \qquad x \mapsto \nu_{x,\psi_0}$$

is bounded for every $\psi_0 \in Y^*$. The same argument shows that the linear mapping

$$Y^* \to ca(\Omega, \Sigma), \qquad \psi \mapsto \nu_{x_0,\psi}$$

is bounded for every $x_0 \in X$. Now we define a linear operator

$$\mathcal{T} : Y^* \to B\big(X, ca(\Omega, \Sigma)\big) \quad \text{by } (\mathcal{T}\psi)x = \nu_{x,\psi}.$$

Suppose that $\psi_n \to \psi_0$ in Y^* and that $\mathcal{T}\psi_n \to S$ in $B(X, ca(\Omega, \Sigma))$. Then, much as was shown above, we can prove that $\mathcal{T}\psi_n$ converges to $\mathcal{T}\psi_0$ in $B(X, ca(\Omega, \Sigma))$ under the strong operator topology, and so $S = \mathcal{T}\psi_0$. Applying the closed graph theorem again, we have that $\mathcal{T}$ is bounded. So

$$\big\|\xi_\nu(x, \psi)\big\| = \|\nu_{x,\psi}\| = \big\|(\mathcal{T}\psi)(x)\big\| \le \|\mathcal{T}\|\|x\|\|\psi\|$$

for every $x \in X, \psi \in Y^*$, which means that ξ_ν is bounded.

(4) The bounded bilinear mapping ξ_ν in (3) corresponds to a bounded linear operator $T_\xi : X \hat{\otimes} Y^* \to ca(\Omega, \Sigma)$ by (1). The result follows. □

Theorem 2.5. *Let X, Y be Banach spaces and let (Ω, Σ, μ) be a σ-finite, positive measure space. Then the following hold:*

(1) *if $\nu : \Sigma \to B(X, Y)$ is an OVM, then $|\nu|$ is a positive measure, and furthermore, if $|\nu|$ is σ-finite and μ-continuous, then there exits an (X, Y^*)-μ-measurable function $F : \Omega \to B(X, Y^{**})$, which means that $\langle F(\cdot)x, \psi\rangle$ is μ-measurable for every $x \in X, \psi \in Y^*$, such that*

$$\big\langle \nu(E)x, \psi \big\rangle = \int_E \big\langle F(\omega)x, \psi \big\rangle \, d\mu$$

for all $E \in \Sigma, x \in X, \psi \in Y^*$;

(2) *let $F : \Omega \to B(X, Y^{**})$ be a function such that $\langle F(\cdot)x, \psi\rangle \in L_1(\Omega, \mu)$ for $x \in X, \psi \in Y^*$, and then*

$$\big\langle \nu(E)x, \psi \big\rangle = \int_E \big\langle F(\omega)x, \psi \big\rangle \, d\mu$$

defines a function $\nu : \Sigma \to B(X, Y^{**})$ such that $\langle \nu(\cdot)x, \psi\rangle \in ca(\Omega, \Sigma)$ for $x \in X, \psi \in Y^*$, and that $|\nu|$ is a μ-continuous σ-finite measure.

Proof. (1) Lemma 2.4(4) shows that ν is a weak* measure if we consider $B(X, Y)$ as a subspace of $B(X, Y^{**})$ and if we consider $X \hat{\otimes}_\pi Y^*$ as the predual of $B(X, Y^{**})$. So by the weak* Radon–Nikodym Property for Banach dual space (i.e., Lemma 2.2(1)), we know that there exists an $X \hat{\otimes}_\pi Y^*$-μ-measurable function $F : \Omega \to B(X, Y^{**})$ such that

$$\big\langle \nu(E), u \big\rangle = \int_E \big\langle F(\omega), u \big\rangle \, d\mu$$

for every $u \in X \hat{\otimes}_\pi Y^*$, $E \in \Sigma$. By letting $u = x \otimes \psi$ for $x \in X$ and $\psi \in Y^*$, the result follows.

(2) By the closed graph theorem, it is not hard to show that

$$b_2 : X \times Y^* \to L_1(\Omega, \mu), \qquad (x, \psi) \mapsto \big\langle F(\cdot)x, \psi \big\rangle$$

is a bounded bilinear mappings. By Lemma 2.4(1), we know that b_2 can be linearized to a bounded linear operator $T \in B(X \hat{\otimes}_\pi Y^*, L_1(\Omega, \mu))$.

Under the identification $B(X, Y^{**}) = (X \hat{\otimes}_\pi Y^*)^*$ in Lemma 2.4(2), we claim that $Tu = \langle F(\cdot), u\rangle$ for every $u \in X \hat{\otimes}_\pi Y^*$. In fact, for any $u_0 \in X \hat{\otimes}_\pi Y^*$, suppose that $u_n \to u_0$ in $X \hat{\otimes}_\pi Y^*$, where $u_n = \sum_{i=1}^{k_n} x_i^{(n)} \otimes \psi_i^{(n)}$. For every $x \in X$ and $\psi \in Y^*$, we then have

$$T(x \otimes \psi) = b_2(x, \psi) = \big\langle F(\cdot)x, \psi \big\rangle = \big\langle F(\cdot), x \otimes \psi \big\rangle,$$

which implies that $Tu_n = \langle F(\cdot), u_n\rangle$ for each n. Since T is bounded, the sequence $\{\langle F(\cdot), u_n\rangle\}_{n=1}^\infty$ converges to Tu_0 in $L_1(\Omega, \mu)$. Hence there exists a subsequence $\{\langle F(\cdot), u_{n_k}\rangle\}_{k=1}^\infty$ that converges to Tu_0 μ-a.e. On the other hand, this subsequence also converges to $\langle F(\cdot), u_0\rangle$ μ-a.e. So $\langle F(\cdot), u_0\rangle = Tu_0$ in $L_1(\Omega, \mu)$.

Now Lemma 2.2(2) shows that $\nu : \Sigma \to B(X, Y^{**})$ defined by

$$\big\langle \nu(E), u \big\rangle = \int_E \big\langle F(\omega), u \big\rangle \, d\mu$$

has all we need, completing the proof. □

Recall that the Radon–Nikodym property of a Banach space X states that if (Ω, Σ, μ) is a finite, positive measure space, $\nu : \Sigma \to X$ is a vector measure which has bounded variation (i.e., $|\nu|(\Omega) < \infty$) and is μ-continuous, then there exists a μ-Bochner integrable function $f : \Omega \to X$ (i.e., $\omega \mapsto \|f(\omega)\|$ is in $L_1(\Omega, \mu)$), such that

$$\big\langle \nu(E), \varphi \big\rangle = \int_E \big\langle f(\omega), \varphi \big\rangle \, d\mu$$

for every $E \in \Sigma$, $\varphi \in X^*$. (For this property, we refer the readers to [17].) Especially, every Hilbert space has Radon–Nikodym property.

In the rest of this article, for Banach spaces X, Y and a measure space (Ω, Σ, μ), we say that an operator-valued function $F : \Omega \to B(X, Y)$ is *X-μ-measurable* (resp., *X-μ-Bochner integrable*) if $F(\cdot)x : \Omega \to Y$ is μ-measurable (resp., μ-Bochner integrable) for every $x \in X$.

Theorem 3.1 in [2] presents a result similar to Theorem 2.5(1), in which the associated measure space is finite and positive. We now give a σ-finite version of this result, which we will use in the next section.

Theorem 2.6. *Let X, Y be Banach spaces, let (Ω, Σ, μ) be a σ-finite, positive measure space, and let $\nu : \Sigma \to B(X, Y)$ be an OVM. Suppose that*

(1) *$|\nu|$ is σ-finite,*
(2) *$|\nu|$ is μ-continuous,*
(3) *Y has the Radon–Nikodym Property.*

Then there exits an X-μ-measurable function $F : \Omega \to B(X, Y)$ such that

$$\big\langle \nu(E)x, \psi \big\rangle = \int_E \big\langle F(\omega)x, \psi \big\rangle \, d\mu$$

for every $E \in \Sigma, x \in X, \psi \in Y^*$.

Proof. Suppose that $\Omega = \bigsqcup_{j \in \mathbb{J}} \Omega_j$ satisfying $\mu(\Omega_j) < \infty$ and $|\nu|(\Omega_j) < \infty$ for each j, where $\mathbb{J}$ is a countable index set. By Theorem 3.1 in [2], for every $j \in \mathbb{J}$ there is a function

$$F_j : \Omega_j \to B(X, Y)$$

which is X-μ-measurable and

$$\big\langle \nu(E)x, \psi \big\rangle = \int_E \big\langle F_j(\omega)x, \psi \big\rangle \, d\mu$$

for every $x \in X, \psi \in Y^*$ and $E \in \{E' : E' \in \Sigma, E' \subseteq \Omega_j\}$. Define a function

$$F : \Omega \to B(X, Y) \quad \text{by } F(\omega) = F_j(\omega), \omega \in \Omega_j.$$

Then clearly, F is X-μ-measurable. Since ν is an OVM, we have

$$\begin{aligned} \big\langle \nu(E)x, \psi \big\rangle &= \sum_{j \in \mathbb{J}} \big\langle \nu(E \cap \Omega_j)x, \psi \big\rangle \\ &= \sum_{j \in \mathbb{J}} \int_{E \cap \Omega_j} \big\langle F_j(\omega)x, \psi \big\rangle \, d\mu \\ &= \int_E \big\langle F(\omega)x, \psi \big\rangle \, d\mu \end{aligned}$$

for all $E \in \Sigma, x \in X, \psi \in Y^*$, as required. □

Theorem 2.7. *Let H be a Hilbert space, let (Ω, Σ, μ) be a σ-finite positive measure space, and let $\nu : \Sigma \to B(H)$ be a POVM. Suppose that $|\nu|$ is σ-finite and μ-continuous. Then there exists a positive operator-valued function $Q : \Omega \to B(H)$ which is weakly μ-measurable (i.e., $\langle Q(\cdot)x, y \rangle$ is μ-measurable for all $x, y \in H$) and*

$$\big\langle \nu(E)x, y \big\rangle = \int_E \big\langle Q(\omega)x, y \big\rangle \, d\mu$$

for all $E \in \Sigma$ and $x, y \in H$.

Proof. By Theorem 2.5, there exists a weakly μ-measurable operator-valued function $Q : \Omega \to B(H)$ such that the required equality in the theorem holds. On the other hand, denote $C_1(H)$ by the space of trace class operators and by $\mathrm{tr}(\cdot)$ the trace of a trace class operator. It is well known that $B(H)$ can be identified with the dual of $C_1(H)$ via the pairing $\langle T, S \rangle = \mathrm{tr}(TS)$ for $S \in B(H)$, $T \in C_1(H)$. Then in this identification, $\nu : \Sigma \to B(H)$ is a weak* measure, and it follows from Lemma 2.2 that we can assume that $\{Q(\omega) : \omega \in \Omega\} \subseteq \overline{\mathrm{conv}}^* A_\nu(\Omega)$, where

$$A_\nu(\Omega) = \Big\{ \frac{\nu(F)}{\mu(F)} : \mu(F) > 0, F \in \Sigma \Big\}$$

is a set of positive operators. Since all of the positive operators in $B(H)$ are weak* closed, we know that Q takes positive values. □

We note that in Theorem 2.7 the separability of the Hilbert space H is not assumed. In the separable case, Robinson got this result (see Theorem 3.3.2 in [16]). However, without the help of Lemma 2.2, we can give a simple proof which is different from that in [16]. In fact, let $Q : \Omega \to B(H)$ be the function found in the preceding proof. Take a countable dense subset $\{x_i\}_{i=1}^{\infty}$ in H, and let

$$E_i = \big\{\omega \in \Omega : \big\langle Q(\omega)x_i, x_i\big\rangle \ngeq 0\big\}$$

for each i. Denoting by $E_0 = \bigcup_{i=1}^{\infty} E_i$, then $\mu(E_0) = 0$. It is easy to derive that $\langle Q(\omega)x, x\rangle \geq 0$ for $x \in H$ and $\omega \in \Omega - E_0$. This means that Q is of positive operator values μ-a.e.

3. Continuous generalized frames

In this section, we detailedly discuss continuous generalized frames. The direct integrals of Hilbert spaces will play a key role.

Let (Ω, Σ, μ) be a positive measure space, let X be a Banach space, and let $1 \leq p < \infty$. Denote by $L_p(\Omega, \mu, X)$ the Banach space of all (equivalence classes of) X-valued Bochner integrable functions F defined on Ω with $\int_\Omega \|F(\omega)\|^p \, d\mu < \infty$ (see [4]). If $p = 2$ and if X is a Hilbert space, then $L_2(\Omega, \mu, X)$ is a Hilbert space under the inner product

$$\langle F, G\rangle = \int_\Omega \big\langle F(\omega), G(\omega)\big\rangle \, d\mu, \quad F, G \in L_2(\Omega, \mu, X).$$

The concept of direct integrals of separable Hilbert spaces was first introduced in 1949 by von Neumann in one paper in his "On Rings of Operators" series. The nonseparable case was first presented by Wils in 1970 (see [21]). The terminologies of direct integrals in this paper can be mostly found in [14]. We note that in the definitions below, the measure space need not be σ-finite.

For a Hilbert space H, the notation $\dim H$ is the cardinal number of one of its orthonormal basis. For each cardinal number γ, we can fix a set C_γ such that C_γ has cardinal number γ. We write $l_2(\gamma)$ for the Hilbert space of absolutely square summable (scalar valued) families indexed by C_γ, where "sum" means "unordered sum".

Let (Ω, Σ, μ) be a positive measure space. A *field of Hilbert spaces $\mathcal{H}$ on Ω* is a Hilbert space-valued function on Ω (i.e., a rule which assigns each point in Ω to a Hilbert space). Elements in $\prod_{\omega\in\Omega} \mathcal{H}(\omega)$ are called *vector fields over $\mathcal{H}$*. A *coherence* α for $\mathcal{H}$ is a choice, for each point $\omega \in \Omega$, of a linear isometry $\alpha(\omega)$ of $\mathcal{H}(\omega)$ onto $l_2(\dim \mathcal{H}(\omega))$. Denote

$$\Gamma_0 = \big\{\dim \mathcal{H}(\omega) : \omega \in \Omega\big\}$$

and for $\gamma \in \Gamma_0$, let

$$\Omega_\gamma = \big\{\omega \in \Omega : \dim \mathcal{H}(\omega) = \gamma\big\}.$$

Note that Γ_0 is a set of cardinal numbers and, moreover, let

$$\Gamma = \{\gamma \in \Gamma_0 : \mu(\Omega_\gamma) > 0\},$$
$$\Omega_0 = \{\omega \in \Omega : \dim \mathcal{H}(\omega) \in \Gamma_0 \setminus \Gamma\}.$$

We say that $\mathcal{H}$ is a *μ-measurable field of Hilbert spaces* if $\Omega_0 \in \Sigma$, $\mu(\Omega_0) = 0$ and $\Omega_\gamma \in \Sigma$ for every $\gamma \in \Gamma$. Furthermore, an *(α, μ)-measurable vector field over $\mathcal{H}$* is a vector field v over $\mathcal{H}$ such that, for each $\gamma \in \Gamma$, the map $\omega \mapsto \alpha(\omega)v(\omega)$ from Ω_γ to $l_2(\gamma)$ is μ-measurable. The set $\tilde{L}_2(\Omega, \mu; \mathcal{H}, \alpha)$, consisting of all (α, μ)-measurable vector fields v over $\mathcal{H}$ for which $\|v(\cdot)\|$ belongs to $L_2(\mu)$, is a semi-inner product space with respect to the pointwise linear operations and the semi-inner product

$$\langle v, w\rangle = \int_\Omega \langle v(\omega), w(\omega)\rangle \, d\mu.$$

The symbol $L_2(\Omega, \mu; \mathcal{H}, \alpha)$ or $\int_\Omega^\oplus \mathcal{H}\, d\mu$ (if α is clear) will be used to denote the associated inner-product space.

It can be proved that the operator

$$\Phi : L_2(\Omega, \mu; \mathcal{H}, \alpha) \to \bigoplus_{\gamma\in\Gamma} L_2(\Omega_\gamma, \mu, l_2(\gamma)),$$
$$v \mapsto \bigoplus_{\gamma\in\Gamma} v_\gamma$$

is an isometric isomorphism between $L_2(\Omega, \mu; \mathcal{H}, \alpha)$ and $\bigoplus_{\gamma\in\Gamma} L_2(\Omega_\gamma, \mu, l_2(\gamma))$. Here v_γ is defined by

$$v_\gamma : \Omega_\gamma \to l_2(\gamma),$$
$$\omega \mapsto \alpha(\omega)v(\omega).$$

Hence $L_2(\Omega, \mu; \mathcal{H}, \alpha)$ is a Hilbert space.

The Hilbert space $L_2(\Omega, \mu; \mathcal{H}, \alpha)$ (interchangeably, $\int_\Omega^\oplus \mathcal{H}\, d\mu$) will be called the *direct integral Hilbert space* of $\mathcal{H}$ with respect to μ and α, or more generally, *direct integral of Hilbert spaces*. Note that in the separable case,this concept is usually defined by a sequence of vector fields but not a coherence (see [10], [16], [20]).

Let $L_2(\Omega, \mu; \mathcal{H}, \alpha)$ be a direct integral Hilbert space and let H be a fixed Hilbert space. Elements in $\prod_{\omega\in\Omega} B(H, H_\omega)$ (interchangeably, $\{B(H, H_\omega)\}_{\omega\in\Omega}$) are called *operator fields*, where H_ω means $\mathcal{H}(\omega)$. An operator field $\mathcal{F}$ is called *(α, μ)-measurable* if $\omega \mapsto \alpha(\omega)\mathcal{F}(\omega)$ is μ-measurable on each Ω_γ, and *H-(α, μ)-measurable* if the vector field

$$\mathcal{F}x \triangleq \{\mathcal{F}(\omega)x\}_{\omega\in\Omega}$$

is (α, μ)-measurable for each $x \in H$.

Lemma 3.1. *Let $L_2(\Omega, \mu; \mathcal{H}, \alpha)$ be a direct integral Hilbert space. Suppose that H is another Hilbert space and that $\mathcal{F} \in \prod_{\omega\in\Omega} B(H, H_\omega)$ is an operator field. If the vector field $\mathcal{F}x$ is in $L_2(\Omega, \mu; \mathcal{H}, \alpha)$ for every $x \in H$, then*

$$T : H \to L_2(\Omega, \mu; \mathcal{H}, \alpha), \qquad x \mapsto \mathcal{F}x$$

is a bounded linear operator.

Proof. Obviously T is linear. We will use the closed graph theorem to prove that T is bounded. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in H that converges to $x_0 \in H$ and $Tx_n \to f_0$ in $L_2(\Omega, \mu; \mathcal{H}, \alpha)$. Then we have

$$\int_{\Omega} \left\| \mathcal{F}(\omega)x_n - f_0(\omega) \right\|^2 d\mu \to 0.$$

By the property of L_2-spaces, we know that there is a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that $\{\mathcal{F}(\cdot)x_{n_j}\}_{j=1}^{\infty}$ converges to f_0 on Ω μ-a.e. On the other hand, $\{\mathcal{F}(\cdot)x_{n_j}\}_{j=1}^{\infty}$ converges to $\mathcal{F}(\cdot)x_0$ pointwise on Ω. So we have $f_0 = \mathcal{F}x$ in $L_2(\Omega, \mu; \mathcal{H}, \alpha)$, which means that T is bounded. □

Definition 3.2. Let $L_2(\Omega, \mu; \mathcal{H}, \alpha)$ be a direct integral Hilbert space. Suppose that H is another Hilbert space and that $\mathcal{F} \in \prod_{\omega \in \Omega} B(H, H_\omega)$ is an operator field. We say that $\mathcal{F}$ is an $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-*frame* (continuous operator-valued frame, continuous generalized frame, or simply, a cg-frame) for H, if

(1) the vector field $\mathcal{F}x \in L_2(\Omega, \mu; \mathcal{H}, \alpha)$ for every $x \in H$,
(2) there exist constants $A, B > 0$, such that

$$A\|x\| \le \|\mathcal{F}x\| \le B\|x\|$$

holds for every $x \in H$.

If the inequality holds only in the right side, $\mathcal{F}$ is called an $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-*Bessel operator field for* H.

For the case of Bessel operator fields, the condition (2) in the definition is redundant since Lemma 3.1 guarantees the existence of the upper bound. By definition, if $\mathcal{F}$ is an $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-Bessel operator field for H, we then can define a bounded linear operator $T_{\mathcal{F}} : H \to L_2(\Omega, \mu; \mathcal{H}, \alpha)$ by

$$(T_{\mathcal{F}}x)(\omega) = \mathcal{F}(\omega)x, \quad \omega \in \Omega.$$

This operator is called the *analysis operator* for $\mathcal{F}$. An $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-frame is said to be of *Riesz-type* if its analysis operator is surjective. Following the *open mapping theorem*, an $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-Bessel operator field $\mathcal{F}$ is an $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-frame if and only if its analysis operator has a bounded inverse on its range or, equivalently, if its analysis operator is injective and has closed range.

Let $\mathcal{F}, \mathcal{G}$ be two $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-frames. Then $\mathcal{G}$ is called a *dual frame of* $\mathcal{F}$ if

$$\langle x, y \rangle = \int_{\Omega} \left\langle \mathcal{F}(\omega)x, \mathcal{G}(\omega)y \right\rangle d\mu$$

holds for every $x, y \in H$. Let T be the analysis operator for $\mathcal{F}$. Obviously T^*T is invertible. The $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-frame

$$\mathcal{F}(T^*T)^{-1} \triangleq \mathcal{F}(\cdot)(T^*T)^{-1}$$

is called the *canonical dual frame of* $\mathcal{F}$. The analysis operator of this cg-frame is clearly $T(T^*T)^{-1}$.

Given a Hilbert space H, denote by $\bar{H}$ the associated conjugate Hilbert space (see [10]). Note that there is obviously a linear or conjugate-linear, isometric bijection between every pair of H, $\bar{H}$, and the Banach dual H^*. In particular, we have identifications $H^* = \bar{H}$ and $H^{**} = H$.

Proposition 3.3. *Let (Ω, Σ, μ) be a σ-finite, positive measure space, and let H_1, H_2 be Hilbert spaces. Denote by P_E the orthogonal projection from $L_2(\Omega, \mu, H_2)$ onto the closed subspace $L_2(E, \mu, H_2)$ for $E \in \Sigma$.*

(1) *Suppose that $T : H_1 \to L_2(\Omega, \mu, H_2)$ is a bounded linear operator and that $\nu : \Sigma \to B(H_1)$ defined by $E \mapsto T^* P_E T$ is a POVM satisfying that $|\nu|$ is σ-finite. Then there exists an H_1-μ-measurable function $F : \Omega \to B(H_1, H_2)$ such that, for every $x \in H_1$, we have $(Tx)(\cdot) = F(\cdot)x$ on Ω μ-a.e.*

(2) *Conversely, let $F : \Omega \to B(H_1, H_2)$ be a H_1-μ-measurable function such that the operator $T : H_1 \to L_2(\Omega, \mu, H_2)$ given by $x \mapsto F(\cdot)x$ is well defined. Then $\nu : \Sigma \to B(H_1)$ defined by $E \mapsto T^* P_E T$ is a POVM satisfying that $|\nu|$ is σ-finite.*

Proof. (1) Fix $g \in L_2(\Omega, \mu)$. Then by the Hölder's inequality, the bilinear mapping

$$\xi_g : H_1 \times \bar{H}_2 \to L_1(\Omega, \mu) \quad \text{by } (x, y) \mapsto \big\langle (Tx)(\cdot) g(\cdot), y \big\rangle$$

is clearly bounded. So by Lemma 2.4(1), there is a corresponding bounded operator $S_g : H_1 \hat{\otimes} \bar{H}_2 \to L_1(\Omega, \mu)$ such that $S_g(x \otimes y) = \xi_g(x, y)$ for $x \in H_1, y \in \bar{H}_2$. Keeping Lemma 2.4(2) in mind, we have the identification $(H_1 \hat{\otimes} \bar{H}_2)^* = B(H_1, H_2)$, in which the action of an operator $R \in B(H_1, H_2)$ as a linear functional on $H_1 \hat{\otimes} \bar{H}_2$ is given by $\langle x \otimes y, R \rangle = \langle Rx, y \rangle$. Consider the bounded adjoint operator $S_g^* : L_\infty(\Omega, \mu) \to B(H_1, H_2)$ and define

$$\lambda_g : \Sigma \to B(H_1, H_2) \quad \text{by } E \mapsto S_g^* \chi_E.$$

Then clearly, λ_g is a weak* measure and we have

$$\begin{aligned}
\big\|\lambda_g(E)\big\| &= \sup_{\|x\| \le 1, \|y\| \le 1} \big|\big\langle (S_g^* \chi_E) x, y \big\rangle\big| \\
&= \sup_{\|x\| \le 1, \|y\| \le 1} \big|\big\langle \xi_g(x, y), \chi_E \big\rangle\big| \\
&= \sup_{\|x\| \le 1, \|y\| \le 1} \Big|\int_E \big\langle (Tx)(\omega) g(\omega), y \big\rangle \, d\mu\Big| \\
&\le \sup_{\|x\| \le 1} \int_E \big\|(Tx)(\omega)\big\| \cdot \big|g(\omega)\big| \, d\mu \\
&\le \frac{1}{2} \sup_{\|x\| \le 1} \int_E \big(\big\|(Tx)(\omega)\big\|^2 + \big|g(\omega)\big|^2\big) \, d\mu \\
&= \frac{1}{2} \big(\|P_E T\|^2 + \|g \chi_E\|^2\big) \\
&= \frac{1}{2} \big(\big\|\nu(E)\big\| + \|g \chi_E\|^2\big)
\end{aligned}$$

for every $E \in \Sigma$, where $x \in H_1, y \in \bar{H}_2$. So the σ-finiteness of $|\nu|$ provides the σ-finiteness of $|\lambda_g|$, and we can easily see that $|\lambda_g|$ is μ-continuous from the above inequalities. Hence by Theorem 2.6, there exists an H_1-μ-measurable function $F_g : \Omega \to B(H_1, H_2)$ such that

$$\int_E \langle F_g(\omega)x, y\rangle \, d\mu = \langle \lambda_g(E)x, y\rangle = \int_E \langle (Tx)(\omega)g(\omega), y\rangle \, d\mu \tag{3.1}$$

hold for $E \in \Sigma, x \in H_1, y \in H_2$.

By the σ-finiteness of μ, we can suppose that the partition $\Omega = \bigsqcup_{j=1}^{\infty} \Omega_j$ with $\mu(\Omega_j) < \infty$ for each j. Let $g_0 \in L_2(\Omega, \mu)$ given by

$$g_0(\omega) = \begin{cases} \frac{1}{j\sqrt{\mu(\Omega_j)}}, & \mu(\Omega_j) \neq 0, \\ 1, & \mu(\Omega_j) = 0 \end{cases} \quad \text{when } \omega \in \Omega_j.$$

Then define a function $F : \Omega \to B(H_1, H_2)$ by

$$F(\omega) = \frac{F_{g_0}(\omega)}{g_0(\omega)}.$$

Clearly, F is also H_1-μ-measurable. Replacing g with g_0 in (3.1), and then per the arbitrariness of $E \in \Sigma$, we infer that for $x \in H_1, y \in H_2$, $\langle F(\cdot)x, y\rangle = \langle (Tx)(\cdot), y\rangle$ on Ω μ-a.e. For every $x \in H_1$, since $(Tx)(\cdot)$ and $F(\cdot)x$ are μ-measurable, it follows from the Pettis measurability theorem (Lemma 2.1) that $(Tx)(\cdot)$ and $F(\cdot)x$ are μ-essentially separably valued and so, $(Tx)(\cdot) = F(\cdot)x$ on Ω μ-a.e.

(2) Define $Q : \Omega \to B(H_1)$ by $Q(\omega) = F(\omega)^* F(\omega)$. Then we have

$$\begin{aligned} \langle \nu(E)x_1, x_2\rangle &= \langle T^* P_E T x_1, x_2\rangle \\ &= \langle P_E T x_1, P_E T x_2\rangle \\ &= \int_E \langle F(\omega)x_1, F(\omega)x_2\rangle \, d\mu \\ &= \int_E \langle F(\omega)^* F(\omega)x_1, x_2\rangle \, d\mu \\ &= \int_E \langle Q(\omega)x_1, x_2\rangle \, d\mu \end{aligned}$$

for $E \in \Sigma, x_1, x_2 \in H_1$. Therefore, by Theorem 2.5(2), it is easy to show that ν is a POVM and that $|\nu|$ is σ-finite. The proof is complete. □

We are now ready to prove one main result which can show that there is an intrinsic connection between CG frames (or Bessel operator fields) and POVMs.

Theorem 3.4. *Let H be a Hilbert space, let (Ω, Σ, μ) be a σ-finite, positive measure space, and let $L_2(\Omega, \mu; \mathcal{H}, \alpha)$ be a direct integral Hilbert space. Denote by P_E the orthogonal projection from $L_2(\Omega, \mu; \mathcal{H}, \alpha)$ onto the closed subspace $L_2(E, \mu; \mathcal{H}, \alpha)$ for $E \in \Sigma$. Then following statements are true.*

(1) *Let $T : H \to L_2(\Omega, \mu; \mathcal{H}, \alpha)$ be a bounded linear operator and also let $\nu : \Sigma \to B(H)$ defined by $E \mapsto T^* P_E T$ be a POVM satisfying that $|\nu|$ is σ-finite. Then there exists an operator field $\mathcal{F} \in \{B(H, H_\omega)\}_{\omega \in \Omega}$ such that*

for each $x \in H$, $(Tx)(\cdot) = \mathcal{F}(\cdot)x$ on Ω μ-a.e. This means that $\mathcal{F}$ is an $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-Bessel operator field for H with analysis operator T.

(2) *Conversely, let $\mathcal{F} \in \{B(H, H_\omega)\}_{\omega\in\Omega}$ be an H-(α, μ)-measurable $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-Bessel operator field for H and let T be its analysis operator. Then we have that $\nu : \Sigma \to B(H)$ defined by $E \mapsto T^*P_ET$ is a POVM satisfying that $|\nu|$ is σ-finite.*

Proof. (1) By the σ-finiteness of μ, we have the identification

$$L_2(\Omega, \mu; \mathcal{H}, \alpha) = \bigoplus_{j\in\mathbb{J}} L_2(\Omega_j, \mu, H_j)$$

for some countable index set $\mathbb{J}$ and Hilbert spaces $\{H_j\}_{j\in\mathbb{J}}$, in which the isometric isomorphism is as follows:

$$\Phi : L_2(\Omega, \mu; \mathcal{H}, \alpha) \to \bigoplus_{j\in\mathbb{J}} L_2(\Omega_j, \mu, H_j),$$
$$v \mapsto \bigoplus_{j\in\mathbb{J}} v_j,$$

where $v_j : \Omega_j \to H_j$ is defined by $\omega \mapsto \alpha(\omega)v(\omega)$ and where $\alpha(\omega)$ is an isometry of H_ω onto H_j. (In fact, we can choose $\mathbb{J} = \Gamma$ and $H_j = l_2(j)$ for $j \in \Gamma$.) Denote by $\hat{P}_j$ the orthogonal projection from $\bigoplus_{j\in\mathbb{J}} L_2(\Omega_j, \mu, H_j)$ onto $L_2(\Omega_j, \mu, H_j)$ for $j \in \mathbb{J}$. Without loss of generality, we can further assume that $\Omega = \bigsqcup_{j\in\mathbb{J}} \Omega_j$.

Fix $j \in \mathbb{J}$. We already know that the operator $\hat{P}_j\Phi T : H \to L_2(\Omega_j, \mu, H_j)$ is bounded. Denote by (Ω_j, Σ_j) the restriction of the σ-algebra (Ω, Σ) on Ω_j, and denote by $P_E^{(j)}$ the orthogonal projection from $L_2(\Omega_j, \mu, H_j)$ onto the closed subspace $L_2(E, \mu, H_j)$ for $E \in \Sigma_j$. Then clearly, $\nu_j : \Sigma_j \to B(H)$ defined by

$$\nu_j(E) = (\hat{P}_j\Phi T)^* P_E^{(j)} (\hat{P}_j\Phi T)$$

is a POVM on Σ_j. The σ-finiteness of $|\nu|$ provides the σ-finiteness of $|\nu_j|$. Moreover, it follows from Proposition 3.3(1) that there exists an H-μ-measurable function $F_j : \Omega_j \to B(H, H_j)$ such that for every $x \in H$, $(\hat{P}_j\Phi Tx)(\cdot) = F_j(\cdot)x$ in $L_2(\Omega_j, \mu, H_j)$.

We now define an operator field $\mathcal{F} \in \{B(H, H_\omega)\}_{\omega\in\Omega}$ by

$$\mathcal{F}(\omega) = \alpha(\omega)^{-1}F_j(\omega) \quad \text{for } \omega \in \Omega_j, j \in \mathbb{J}.$$

Since $(\hat{P}_j\Phi Tx)(\cdot) = F_j(\cdot)x$ in $L_2(\Omega_j, \mu, H_j)$ for $j \in \mathbb{J}, x \in H$, we get $(Tx)(\cdot) = \mathcal{F}(\cdot)x$ in $L_2(\Omega, \mu; \mathcal{H}, \alpha)$ for $x \in H$. And by Definition 3.2, $\mathcal{F}$ is clearly a $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-Bessel operator field for H with analysis operator T.

(2) Consider the function $Q : \Omega \to B(H)$ given by $Q(\omega) = \mathcal{F}(\omega)^*\mathcal{F}(\omega)$. Then similar to the proof of Proposition 3.3(2), for all $E \in \Sigma, x, y \in H$ we have

$$\begin{aligned}\big\langle \nu(E)x, y\big\rangle &= \langle P_E T x, P_E T y\rangle \\ &= \int_E \big\langle \mathcal{F}(\omega)x, \mathcal{F}(\omega)y\big\rangle \, d\mu \\ &= \int_E \big\langle \mathcal{F}(\omega)^* \mathcal{F}(\omega)x, y\big\rangle \, d\mu \\ &= \int_E \big\langle Q(\omega)x, y\big\rangle \, d\mu.\end{aligned}$$

By Theorem 2.5(2), it is easily seen that ν is a POVM and that $|\nu|$ is σ-finite. We are done. □

Let (Ω, Σ, μ) be a σ-finite, positive measure space. Recall that $A \in \Sigma$ is an atom if $0 < \mu(A) < \infty$ and for each measurable set $B \subset A$, either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. The space (Ω, Σ, μ) is called *purely atomic*, if the set $\Omega - \bigcup\{A \in \Sigma : A \text{ is an atom}\}$ has measure zero. Since atoms are essentially disjoint and μ is countable additive and σ-finite, Ω only contain at most countably many atoms.

Corollary 3.5. *Let (Ω, μ) be a σ-finite, positive measure space, and let $L_2(\Omega, \mu; \mathcal{H}, \alpha)$ be a direct integral Hilbert space. Then there exists a Riesz-type $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-frame if and only if (Ω, μ) is purely atomic.*

Proof. If (Ω, μ) is purely atomic, then clearly a Riesz-type $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-frame exists by the discrete case (see [11], [18]). Conversely, suppose that $\mathcal{F}$ is a Riesz-type $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-frame and that T is its analysis operator. By Theorem 3.4 we know $\nu : E \mapsto T^* P_E T$ is a POVM satisfying that $|\nu|$ is σ-finite. On the other hand, since $\mathcal{F}$ is Riesz-type we have that T is invertible. For arbitrary $E_0 \in \Sigma$ and $\mu(E_0) > 0$, the relations

$$1 = \|P_{E_0}\| = \|P_{E_0} T T^{-1}\| \le \|P_{E_0} T\| \|T^{-1}\|$$

show that $\|P_{E_0} T\| \ge \frac{1}{\|T^{-1}\|}$, which implies that $|\nu|(E_0) \ge \|\nu(E_0)\| \ge \frac{1}{\|T^{-1}\|^2}$. So the σ-finiteness of $|\nu|$ guarantees that (Ω, μ) is purely atomic. □

Remark 3.6. Corollary 3.5 is a generalization of [9, Theorem 3.20]. Suppose that the space $L_2(\Omega, \mu; \mathcal{H}, \alpha)$ degenerates into $L_2(\Omega, \mu)$. From the proof of Corollary 3.5, we can see that the POVM ν satisfies that

$$\inf\big\{\big\|\nu(E)\big\| : \big\|\nu(E)\big\| \ne 0, E \in \Sigma\big\} \ge \frac{1}{\|T^{-1}\|^2} > 0.$$

It follows directly from [9, Theorem 3.20] that (Ω, μ) is purely atomic. It should be mentioned that the proof of Corollary 3.5 use the σ-finiteness of $|\nu|$ explicitly, while Theorem 3.20 in [9] use this property implicitly.

Remark 3.7. Let H be a Hilbert space, (Ω, Σ) be a σ-algebra and let $\nu : \Sigma \to B(H)$ be a POVM. The classical Naimark's dilation theorem (see [9, Theorem 3.4]) says that there are a Hilbert space K, a bounded linear operator $T : H \to K$, and a spectral measure (i.e., orthogonal projection valued POVM) $\nu_2 : \Sigma \to B(K)$ such that

$$\nu(E) = T^* \nu_2(E) T$$

for every $E \in \Sigma$. If K is a direct integral Hilbert space $L_2(\Omega, \mu; \mathcal{H}, \alpha)$, T is the analysis operator for some $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-Bessel operator field, and $\nu_2(E)$ is the orthogonal projection from $L_2(\Omega, \mu; \mathcal{H}, \alpha)$ onto $L_2(E, \mu; \mathcal{H}, \alpha)$ for $E \in \Sigma$, then we will call (K, T, ν_2) a *frame dilation*. If μ is σ-finite further, then we will call (K, T, ν_2) a *σ-frame dilation*.

If a POVM ν has a σ-frame dilation, then clearly ν has σ-finite variation (i.e., $|\nu|$ is σ-finite) by Theorem 3.4. Conversely, an alternative result of [16] shows that if H is separable and $\nu : \Sigma \to B(H)$ has σ-finite variation, then ν has a σ-frame dilation. For general case, we give the following conjecture.

Conjecture 3.8. *A POVM has a σ-frame dilation if and only it has σ-finite variation.*

Next, we will give a dilation theorem for dual pairs of CG frames, which generalizes Theorem 1.1 in [8]. For an operator T, denote $\mathcal{R}(T)$ by the range of T, and $R_1 \ominus R_2$ by the space $R_1 \cap R_2^{\perp}$ if R_1, R_2 are closed subspace in a Hilbert space with $R_2 \subseteq R_1$.

Theorem 3.9. *Let $L_2(\Omega, \mu; \mathcal{H}, \alpha)$ be a direct integral of Hilbert spaces and let H_1, H_2 be other Hilbert spaces. Suppose that*

(1) *$\mathcal{F}$ is an $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-frame for H_1,*
(2) *$\mathcal{G}$ is a dual of $\mathcal{F}$,*
(3) *$\mathcal{F}_0$ is an $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-frame for H_2,*
(4) *T_1, T_2 and S are analysis operators of $\mathcal{F}, \mathcal{G}$, and $\mathcal{F}_0$, respectively,*
(5) *$\mathcal{R}(T_1) \cup \mathcal{R}(T_2) \subseteq \mathcal{R}(S)$.*

Then there are a Hilbert space K_0 and an $L_2(\Omega, \mu; \mathcal{H}, \alpha)$-frame $\mathcal{F}_1$ for $H_1 \oplus K_0$, such that $\mathcal{F}(\omega)x = \mathcal{F}_1(\omega)(x \oplus 0), \mathcal{G}(\omega)x = \mathcal{G}_1(\omega)(x \oplus 0)$ μ-a.e. on Ω and such that the ranges of the analysis operators of $\mathcal{F}_1, \mathcal{F}_0$ are the same. Here $\mathcal{G}_1$ is the canonical dual of $\mathcal{F}_1$.

Proof. Denote $H_o = L_2(\Omega, \mu; \mathcal{H}, \alpha)$, $R_1 = \mathcal{R}(T_1)$, $R_2 = \mathcal{R}(T_2)$, and $R_3 = \mathcal{R}(S)$. Since $\mathcal{G}$ is a dual of $\mathcal{F}$, we clearly have that $T_2^* T_1 = I$ and that $T_1 T_2^*$ is an oblique projection onto R_1. Symmetrically, $T_1^* T_2 = I$ and $T_2 T_1^*$ is an oblique projection onto R_2.

We claim that there exists an isomorphism $\psi : R_3 \ominus R_1 \to R_3 \ominus R_2$. In fact, $P_0 : R_3 \to R_3$ defined by $P_0 = (T_1 T_2^*)|_{R_3}$ is obviously an oblique projection with the range R_1. Applying the *first isomorphism theorem* for Banach spaces to the operator $I_3 - P_0$, where I_3 is the identity operator on R_3, we have

$$R_3 \ominus R_2 = \mathcal{R}(I_3 - P_0) \cong R_3 / \ker(I_3 - P_0) = R_3 / R_1 \cong R_3 \ominus R_1,$$

as required.

Denote K_0 by the space $R_3 \ominus R_1$ and define

$$\phi : H_1 \oplus K_0 \to H_2 \quad \text{by } x \oplus y \mapsto (S^* S)^{-1} S^* (T_1 x + \psi y).$$

We will verify that K_0, ϕ satisfy

$$T_1 x = S \phi(x \oplus 0) \tag{3.2}$$

and

$$T_2x = S\phi\big((S\phi)^*S\phi\big)^{-1}(x\oplus 0). \tag{3.3}$$

for $x \in H_1$.

In fact, ϕ is clearly a linear bijection between $H_1 \oplus K_0$ and H_2. It is easy to check that $S\phi(x\oplus 0) = S(S^*S)^{-1}S^*T_1x = T_1x$. So (3.2) holds. We next verify (3.3). For $z \in H_2, x\oplus y \in H_1 \oplus K_0$, we have

$$\begin{aligned}\langle \phi^*z, x\oplus y\rangle &= \big\langle z, (S^*S)^{-1}S^*(T_1x + \psi y)\big\rangle \\ &= \big\langle S(S^*S)^{-1}z, T_1T_2^*T_1x + \psi y\big\rangle \\ &= \big\langle T_1^*S(S^*S)^{-1}z, x\big\rangle + \big\langle S(S^*S)^{-1}z, \psi y\big\rangle.\end{aligned}$$

Noting that $S^*|_{R_3}$ is an isomorphism from $\mathcal{R}_3$ onto H_2, we derive that

$$\begin{aligned}&\big\langle \phi^*(S^*|_{R_3})T_2x, \bar{x}\oplus\bar{y}\big\rangle \\ &= \big\langle T_1^*S(S^*S)^{-1}(S^*|_{R_3})T_2x, \bar{x}\big\rangle + \big\langle S(S^*S)^{-1}(S^*|_{R_3})T_2x, \psi\bar{y}\big\rangle \\ &= \langle T_1^*T_2x, \bar{x}\rangle + \langle T_2x, \psi\bar{y}\rangle \\ &= \langle x, \bar{x}\rangle = \langle x\oplus 0, \bar{x}\oplus\bar{y}\rangle\end{aligned}$$

for every $x, \bar{x} \in H_1$ and $\bar{y} \in K_0$. So for $x \in H_1$, we have

$$\begin{aligned}&\phi^*(S^*|_{R_3})T_2x = x\oplus 0 \\ \Longrightarrow \quad & (S^*|_{R_3})^{-1}(\phi^*)^{-1}(x\oplus 0) = T_2x \\ \Longrightarrow \quad & S(S^*S)^{-1}S^*(S^*|_{R_3})^{-1}(\phi^*)^{-1}(x\oplus 0) = S(S^*S)^{-1}S^*T_2x \\ \Longrightarrow \quad & S(S^*S)^{-1}(\phi^*)^{-1}(x\oplus 0) = T_2x \\ \Longrightarrow \quad & (S\phi)\big((S\phi)^*(S\phi)\big)^{-1}(x\oplus 0) = T_2x.\end{aligned}$$

So (3.3) holds.

Now consider the vector field $\mathcal{F}_1 \in \{B(H_1\oplus K_0, H_\omega)\}_{\omega\in\Omega}$ defined by $\mathcal{F}_1(\omega) = \mathcal{F}_0(\omega)\phi$ for $\omega \in \Omega$. Clearly, the analysis operator of $\mathcal{F}_0$ is $S\phi$, which gives that the ranges of the analysis operators of $\mathcal{F}_1, \mathcal{F}_0$ are the same. Equations (3.2) and (3.3) show that $\mathcal{F}(\omega)x = \mathcal{F}_1(\omega)(x\oplus 0)$ and $\mathcal{G}(\omega)x = \mathcal{G}_1(\omega)(x\oplus 0)$ μ-a.e. on Ω. This completes the proof. □

From the proof of Theorem 3.9, we can show the following dilation result for dual pairs of operators.

Proposition 3.10. *Let H_1, H_2, H_o be Hilbert spaces, $T_1, T_2 \in B(H_1, H_o)$, $S \in B(H_2, H_o)$. Suppose that T_1, T_2, S have bounded inverses on their ranges and that $\mathcal{R}(T_1)\cup\mathcal{R}(T_2) \subseteq \mathcal{R}(S)$. If $T_2^*T_1 = I$, then there exit a Hilbert space K_0 and an isomorphism $\phi : H_1\oplus K_0 \to H_2$ such that*

$$T_1x = S\phi(x\oplus 0)$$

and

$$T_2x = S\phi\big((S\phi)^*S\phi\big)^{-1}(x\oplus 0).$$

for $x \in H_1$.

Acknowledgments. The authors would like to thank the referees for their very thorough reading and many helpful comments.

P. Li's work was partially supported by National Natural Science Foundation of China (NSFC) grants 11171151 and 11571247. F. Li's work was partially supported by Jiangsu Province of China graduate research and innovation project KYZZ_0086.

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Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, People's Republic of China.
E-mail address: lfj_2@126.com; pengtongli@nuaa.edu.cn