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Infinite Computations with Random Oracles

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Abstract We consider the following problem for various infinite-time machines. If a real is computable relative to a large set of oracles such as a set of full measure or just of positive measure, a comeager set, or a nonmeager Borel set, is it already computable? We show that the answer is independent of ZFC for ordinal Turing machines with and without ordinal parameters and give a positive answer for most other machines. For instance, we consider infinite-time Turing machines, unresetting and resetting infinite-time register machines, and α-Turing machines for countable admissible ordinals α.

1 Introduction

If a real is Turing computable relative to all oracles in a set of positive measure, then it is Turing computable by a classical theorem of Sacks (see, e.g., Downey and Hirschfeldt [5, Theorem 8.12.2]). Intuitively, this means that the use of random generators does not enrich the set of computable functions, not even when computability is weakened to computability with positive probability. This insight refutes a possible objection against the Church–Turing thesis, namely, that a computer could make randomized choices and thereby compute a function which is not computable by a purely deterministic device. The proof depends crucially on the compactness of halting Turing computations, that is, the fact that only finitely many bits of an oracle are read in the course of a halting computation.

Recently, the first author [2] considered analogues of the Church–Turing thesis for infinitary computations. This naturally leads to the question of whether a similar phenomenon can be observed concerning these machine models. The situation is quite different for ordinal-time Turing machines (OTMs; see Koepke [16]), infinite-time

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Turing machines (ITTMs; see Hamkins and Lewis [7]), weak (unresetting) infinite-time register machines (wITRMs; see Koepke [17]), (resetting) infinite-time register machines (ITRMs; see Koepke and Miller [19]), α-Turing machines (α-TMs; see Koepke and Seyfferth [20]), and ordinal-time register machines (ORMs; see Koepke and Siders [21], [22]). All of these machines can consider each bit of a real oracle in the course of a halting computation. Nevertheless, the intuitive interpretation of computing relative to oracles in a set of positive measure as a randomized computation still makes sense.

Hence, we consider the following problem for each machine model. If a real is computable relative to a large set of oracles such as a set of full measure or just of positive measure, a comeager set, or a nonmeager Borel set, is it already computable? We first show that this is independent of ZFC for OTMs with and without ordinal parameters. We then give a positive answer for most other machines. For ITTMs, writability (resp., eventual writability, accidental writability) in a nonmeager Borel set of oracles implies writability (resp., eventual writability, accidental writability). For ITRMs of both kinds, computability in a set of positive measure or a nonmeager Borel set implies computability. For all (resp., for unboundedly many) countable admissible ordinals α, computability by an α-TM from a nonmeager Borel set (resp., a set of positive measure) of oracles implies computability.

2 Ordinal Turing Machines

Ordinal Turing machines (OTMs) can roughly be thought of as Turing machines with tape length and working time the class Ord of ordinals. The machine state and tape content at limit times are obtained as the limit inferior of the earlier configurations. The definition and basic properties of OTMs can be found in Koepke [15]. We will call elements of ${}^{\omega}2$ *reals*.

Definition 1

(1) A real $x \in {}^{\omega}2$ is *OTM-computable* from a real $y \in {}^{\omega}2$ if there is an OTM P such that, on input y, P halts with output x, that is, $P^{y} = x$.

(2) A set $A \subseteq {}^{\omega}2$ is *OTM-computable* from a real y if there is an OTM P such that, for all $x \in {}^{\omega}2$, $x \in A$ if and only if P halts on input $x \oplus y$, that is, $P^{x \oplus y} \downarrow$.

It follows from the proof of Schlicht and Seyfferth [29, Corollary 2] by application of the search algorithm that, for any real x such that $\{x\}$ is OTM-computable in y, or equivalently Δ^1_2 in y, x is OTM-computable in y. Conversely, if x is OTM-computable from y, then $\{x\}$ is easily OTM-computable from y by computing x and comparing x with the input. Since these two notions do not coincide for other machine models, computable reals are called *writable* for most other machine models. We will say OTM-computable when we do not allow ordinal parameters and say OTM-computable with ordinal parameters otherwise.

Let us first collect basic facts about ordinal-time Turing machines and their halting times. Most of the results are folklore.

Definition 2 Let η^{x} denote the supremum of halting times of OTMs with oracle x.

Note that there are gaps in the OTM halting times.

Lemma 3 *Suppose that x is a real.*

(1) There are $\alpha < \beta$ such that β is an OTM halting time but α is not.
(2) All sets in L_{η^x} are countable in L_{η^x}.

Proof (1) As there are only countably many programs, there are only countably many halting times. Hence, there is some ordinal α which is not a halting time. By the absoluteness of computations, if δ is a limit ordinal, then for $\beta < \delta$, we have $L_\delta \models$ "β is a halting time" if and only if β is a halting time. Now let P be a program searching through the ordinals in their natural order for the first limit ordinal γ such that $L_\gamma \models$ "There is an ordinal which is not a halting time." As such ordinals exist, the search will eventually terminate, after at least γ many steps. Hence, if $\alpha < L_\gamma$ is such that $L_\gamma \models$ "α is not a halting time," then α and the halting time of P are as desired.

(2) We assume that $x = 0$. As L_α becomes countable in $L_{\alpha+1}$ when $L_{\alpha+1} - L_\alpha$ contains a real, it suffices to show that, for any OTM halting time α of a program P, $L_{\alpha+\omega} - L_\alpha$ contains a real. The computation of P is definable over L_α and hence in $L_{\alpha+1}$. Then $\alpha + 1$ is minimal such that P halts in $L_{\alpha+1}$. Then the hull of the empty set in $L_{\alpha+1}$ is $L_{\alpha+1}$. Hence, there is a surjection from ω onto $L_{\alpha+1}$ definable over $L_{\alpha+1}$. Hence, there is a real x coding $L_{\alpha+1}$ in $L_{\alpha+2}$, and hence, $x \in L_{\alpha+2} \setminus L_\alpha$. □

This is analogous to ITTMs (see [7, Theorem 3.4]), but different from ITRMs, where the set of halting times is downward closed (see Carl, Fischbach, Koepke, Miller, Nasfi, and Weckbecker [3, Theorem 6]).

Lemma 4 *The following conditions are equivalent for reals x, y.*

(1) x is Δ^1_2 in y.
(2) x is OTM-computable in the oracle y.
(3) $x \in L_{\eta^y}[y]$.

Proof Suppose that x is Δ^1_2 in y. Then x is OTM-computable in the oracle y by searching through the Shoenfield tree (see also the proof of [29, Corollary 2]). Since such a computation will last for fewer than η^y steps, the computation and, hence also, x are in $L_{\eta^y}[y]$.

Suppose that $x \in L_{\eta^y}[y]$. Suppose that β is the halting time of a program with oracle y such that $x \in L_\beta[y]$. Then the L-least code for $L_\beta[y]$ is Δ^1_2 in y. So x is Δ^1_2 in y. □

Thus, η^x is equal to the supremum of Δ^1_2 well-orderings in the parameter x on ω by [29, Corollary 6].

Lemma 5 *We have that η^x is an x-admissible limit of x-admissibles.*

Proof To show that η^x is x-admissible, it suffices to prove Δ_0-collection in $L_{\eta^x}[x]$. Suppose that $y \in L_{\eta^x}[x]$ and that $R \subseteq y \times L_{\eta^x}[x]$ is Δ_0-definable over $L_{\eta^x}[x]$ such that for every $u \in y$ there is some $v \in L_{\eta^y}[y]$ with $(u, v) \in R$. Let P search on input $u \in y$ for the L-least v with $(u, v) \in R$. The previous lemma implies that $\eta^y \leq \eta^x$. If we apply P successively to all $z \in y$ and halt, then the halting time is some $\gamma < \eta^x$. Hence, we can collect the witnesses in $L_\gamma[x]$.

To see that η^x is a limit of x-admissibles, let $\gamma < \eta^x$ be arbitrary, and let $\alpha > \gamma$ be the halting time of some program P^x. Then $\alpha < \eta^x$ and there is some x-admissible ordinal greater than α. Let Q^y be a program that searches through the ordinals in their natural order for the first y-admissible ordinal σ greater than the halting time of P^y. Then, as $P^x\downarrow$, Q^x will halt after at least σ many steps. By the definition of η^x, it follows that $\eta^x > \sigma > \alpha$, so there is an x-admissible ordinal between γ and η^x for every $\gamma < \eta^x$, as desired. □

Remark 6 We have that η^x is not Σ_2-x-admissible.

Proof The partial function $f\colon \omega \to \eta^x$ which maps every halting program to its halting time is Σ_1-definable over L_{η^x}, so f can be extended to a total function $\bar{f}\colon \omega \to \eta^x$ which is Δ_2-definable over L_{η^x} and whose range is cofinal in η^x. □

We will show that $\eta^x = \eta$ for Cohen reals η over L, using the following lemma. We work with ordinals α such that some Σ_1-statement is first true in L_α. Note that every such ordinal is a successor ordinal.

Lemma 7 *Suppose that x is a real. Let us call an ordinal α Σ_1^x-*fixed *if and only if there exists a Σ_1-statement ϕ in the parameter x such that α is minimal with the property that $L_\alpha[x] \models \phi(x)$. Then η^x is the supremum of the Σ_1^x-fixed ordinals.*

Proof First, we show that there is an OTM-halting time (in the oracle x) above every Σ_1^x-fixed ordinal. To see this, let α be Σ_1^x-fixed, say, α is minimal such that $L_\alpha[x] \models \phi$, where ϕ is Σ_1^x. We will show below that there exists an OTM program P such that P^x successively writes codes for all $L_\alpha[x]$'s on the tape. Take such a program, and after each step, check whether the tape contains a code for some $L_\beta[x]$ such that $L_\beta[x] \models \phi$. Halt if this is the case. This program obviously halts after at least α many steps; hence, there is an OTM-halting time in the oracle x which is at least α. For the other direction, take an OTM program P such that P^x halts after α many steps. Hence, there exists $\beta > \alpha$ such that $L_\beta[x]$ contains the whole computation of P^x. This β is minimal such that L_β believes that P^x halts, that is, that the computation of P^x exists, which is a Σ_1^x-statement. Hence, $\beta > \alpha$ is Σ_1^x-fixed. Consequently, the suprema coincide. □

Lemma 8 *If x is Cohen generic over L, then $\eta^x = \eta$.*

Proof Suppose that x is Cohen generic over L and that P^x halts at time γ. Let $\varphi(y,\alpha)$ state that P^y halts at time α. Suppose that $\dot{x}$ is the canonical name for the Cohen real and that $p \Vdash \varphi(\dot{x},\gamma)$. Since φ is Σ_1, the existence of some α with $p \Vdash \varphi(\dot{x},\alpha)$ is Σ_1, so this holds in L_η by Lemma 7. So there is some $\alpha < \eta$ with $p \Vdash \varphi(\dot{x},\alpha)$. Then P^x halts at time $\alpha < \eta$ in $L[x]$, so $\alpha = \gamma < \eta$. □

2.1 Computations without parameters Natural numbers as oracles do not change Turing computability. Thus, there are at least two natural generalizations of Turing computability to computations of ordinal length, with and without ordinal parameters. We first consider machines without ordinal parameters.

We first show that in L there is a noncomputable real x which is computable relative to all oracles in a set of measure 1. Let us say that a set c of ordinals *codes* a transitive set x if there are some $\gamma \in \mathrm{Ord}$ and a bijection $f : \gamma \to x$ such that $c = \{p(\alpha,\beta) \mid \alpha,\beta < \gamma, f(\alpha) \in f(\beta)\}$, where $p\colon \mathrm{Ord}\times\mathrm{Ord} \to \mathrm{Ord}$ denotes Gödel pairing.

Lemma 9

1. *There is an OTM program P such that, for every $\alpha \in \mathrm{Ord}$, there is an ordinal β such that the tape content at time β is the characteristic function of a code for L_α.*
2. *There is an OTM program Q which stops with output 1 if and only if the tape content at the starting time is a code for some L_α, and Q stops with output 0 otherwise.*
3. *There is an OTM program R which, for an arbitrary real x in the oracle, stops with output 1 if and only if the tape content at the starting time is a code for some L_α with $x \in L_\alpha$.*

Proof Note that, by [15], $x \subseteq \mathrm{Ord}$ is OTM-computable from finitely many ordinal parameters if and only if $x \in L$. The program P is obtained as follows. We enumerate all tuples $(m, \alpha_0, \ldots, \alpha_n)$ with $m \in \omega$ and ordinals $\alpha_1, \ldots, \alpha_n$. Let the mth OTM program P_m run for α_0 many steps in the parameter $(\alpha_0, \ldots, \alpha_n)$. This generates codes for all elements of L, in particular, for L_α for all $\alpha \in \mathrm{Ord}$.

For the second claim, note that, by [21], bounded truth predicates can be computed by an OTM. The well-foundedness of the tape content can be tested by an exhaustive search. We can then check the sentence $\exists \alpha \in \mathrm{Ord}\ V = L_\alpha$ by evaluating the bounded truth predicate.

For the third claim, we can check whether the tape content codes some L_α by the second claim. One can check whether or not $x \in L_\alpha$ by the second claim. Whether some $\delta \in \mathrm{Ord}$ codes $n \in \omega$ can be checked as follows. If $n = 0$, then one runs through the code to check whether δ has any predecessors, for example, whether $p(\gamma, \delta)$ belongs to the code for some γ. Then, recursively, a code for $n + 1$ can be identified as having exactly the codes for $0, 1, \ldots, n$ as its predecessors. □

Theorem 10 *Suppose that $V = L$. There are a real x and a cocountable set $A \subseteq {}^\omega 2$ such that x is OTM-computable without ordinal parameters from every $y \in A$, but x is not OTM-computable without parameters.*

Proof Let $A = {}^\omega 2 \setminus L_\eta$, and suppose that x is the $<_L$-least real coding a well-ordering of order type η. We claim that x is OTM-computable without parameters relative to any $y \in A$. To see this, suppose that P is a diverging OTM program which writes L_β on the tape for all $\beta \in \mathrm{Ord}$ as in Lemma 9. We wait for the least $\beta \in \mathrm{Ord}$ with $y \in L_\beta$. Then $x \in L_\beta$, and hence $\eta < \beta$. We then write a sequence of β many 1's on the tape, succeeded by 0's. This allows us to solve the halting problem for parameter-free OTMs as follows. Whenever a program runs for β many steps, it cannot halt, since $\eta < \beta$. We compute the supremum η of the halting times and then search L_β for the L-least code x for η. However, x itself is not OTM-computable, as it would allow us to write a sequence of η many 1's on the tape succeeded by 0's, which allows a solution of the halting problem for parameter-free OTMs. □

Corollary 11 *Assume that $\mathbb{R} = \mathbb{R}^L$.*

1. *Let h be a real coding the halting problem for parameter-free OTMs. Then h is OTM-computable from every non-OTM-computable real x.*
2. *For all reals x and y, x is OTM-computable from y or y is OTM-computable from x.*

Proof The first claim follows from the previous proof. For the second claim, let α and β be minimal such that $x \in L_{\alpha+1}$ and $y \in L_{\beta+1}$. Assume without loss of generality that $\beta \geq \alpha$. Given y, we can, using the strategy from the proof of Theorem 10, compute the $<_L$-minimal real r coding an $L_{\beta+1}$. As $x \in L_{\beta+1}$, it must be coded by some fixed natural number n in r which can be given to our program in advance. It is now easy to compute x from r. Thus, x is computable from y. $\square$

The second claim shows that the analogue to the question of whether there are incomparable computably enumerable Turing degrees, also known as Post's problem, has a negative answer for OTMs in L.

It is consistent that there is no real as in Theorem 10 by the following theorem. To prove this, we will use the assumptions in the following lemma.

Lemma 12

(1) *The statement that, for every real x, the set of random reals over $L[x]$ has measure 1 is equivalent to the statement that every $\underset{\sim}{\Sigma}^1_2$-set is Lebesgue measurable. This follows from $\omega_1^{L[x]} < \omega_1$ for all reals x.*

(2) *The statement that, for every real x, the set of Cohen reals over $L[x]$ is comeager is equivalent to the statement that every $\underset{\sim}{\Sigma}^1_2$-set has the property of Baire. This follows from $\omega_1^{L[x]} < \omega_1$ for all reals x.*

Proof The claims are proved in Ikegami [9, Theorem 4.4] and Kanamori [12, Corollary 14.3]. $\square$

Theorem 13

(1) *Suppose that, for every $x \in {}^{\omega}2$, the set of random reals over $L[x]$ has measure 1. If A has positive Lebesgue measure and $x \in {}^{\omega}2$ is OTM-computable without ordinal parameters from every $y \in A$, then x is OTM-computable without ordinal parameters.*

(2) *Suppose that, for every $x \in {}^{\omega}2$, the set of Cohen reals over $L[x]$ is comeager. If A is nonmeager with the property of Baire and $x \in {}^{\omega}2$ is OTM-computable without ordinal parameters from every $y \in A$, then x is OTM-computable without ordinal parameters.*

Proof Suppose that, for every $x \in {}^{\omega}2$, the set of random reals over $L[x]$ has measure 1. This implies that every $\underset{\sim}{\Sigma}^1_2$-set of reals is Lebesgue measurable by Lemma 12. Suppose that $x \in {}^{\omega}2$ is OTM-computable without ordinal parameters from every $y \in A$. If $B \subseteq {}^{\omega}2 \times {}^{\omega}2$ is Σ^1_2 and $q \in \mathbb{Q}$, then the set $\{x \in {}^{\omega}\omega \mid \mu(B_x) > q\}$ is $\underset{\sim}{\Sigma}^1_2$; this is proved from projective determinacy in the proof of Kechris [13, Theorem 2.2.3], but the proof only uses that all $\underset{\sim}{\Sigma}^1_2$-sets are Lebesgue measurable.

Now suppose that $\mu(A) > 0$, and suppose that, for every $y \in A$, there is an OTM P such that P^y computes x. Then $\{y \mid P^y = x\}$ is provably Δ^1_2 and hence measurable by [12, Exercise 14.4]. Since there are countably many programs, $\mu(\{y \mid P^y = x\}) > 0$ for some program P. There is a basic open set U such that the relative measure of $\{y \mid P^y = x\}$ in U is greater than 0.5 by the Lebesgue density theorem.

We can assume without loss of generality that $U = {}^{\omega}2$. Then $\{x\} = \{y \mid \mu(\{z \mid y = P^z\}) > 0.5\}$, so $\{x\}$ is Σ^1_2 and thus easily Δ^1_2. Note that a set A of reals is OTM-computable if and only if it is Δ^1_2 by Seyfferth [30, Corollary 3.11]. It follows from the discussion in the beginning of this section that x is OTM-computable.

The proof of the second claim is analogous. □

It follows from Theorems 10 and 13 that ZFC does not decide whether there are a real x which is not OTM-computable and a Borel set $A \subseteq {}^{\omega}2$ which is nonmeager or has positive measure such that x is OTM-computable from every element of A.

2.2 Computations with real parameters We will see below that, for most machine concepts of transfinite computability, computability with positive probability relative to a random oracle does not exceed plain computability. Since parameter-free OTM-computation provides a natural formalization of the intuitive idea of a transfinite construction procedure, this intrinsically motivates the consideration of the statement that every real x which is OTM-computable relative to all reals y from some set A with $\mu(A) > 0$ is OTM-computable in the empty oracle. We will abbreviate this axiom by $Z(0)$. Similarly, for an arbitrary real x, we denote by $Z(x)$ the statement that every real which is OTM-computable relative to all reals $x \oplus y$ for all $y \in A$ with $\mu(A) > 0$ is OTM-computable in the oracle x. Intuitively, $\neg Z(x)$ means that x contains a way of extracting new information from randomness, so we call a real x with $\neg Z(x)$ an *extracting real*. The same intuition motivates the consideration of the statement Z that no extracting reals exist, that is, $\forall x \in {}^{\omega}2\ Z(x)$.

We easily obtain similar results as above for computability relative to real oracles.

Theorem 14 *If* ZFC *is consistent, then* Z *is independent of* ZFC.

Proof Suppose that ZFC is consistent. The failure of $Z(0)$ implies the failure of Z, so Z fails in L by Theorem 10, and thus, ZFC $+ \neg Z$ is consistent.

On the other hand, ZFC together with Martin's axiom for ω_1 is consistent, and this implies that every $\utilde{\Sigma}^1_2$-set of reals is Lebesgue measurable and, hence, that for every real x the set of random reals over $L[x]$ has measure 1 by Lemma 12. This implies Z by the proof of Theorem 13. □

As a consequence of Z, the universe V cannot be too close to L.

Theorem 15 *We have that* ZFC $+ Z$ *implies that* $V \neq L[x]$ *for all reals* x.

Proof It suffices to show that $Z(x)$ fails in $L[x]$. To see this, we relativize the proof of Theorem 10 above. □

Since $x^{\#}$ exists in $L[x^{\#}]$, the existence of $x^{\#}$ does not imply Z. However, the existence of $x^{\#}$ for all reals x implies that $\omega_1^{L[x]} < \omega_1$ for all reals x and, hence, Z by Lemma 12.

Question 16

1. Is it consistent that $Z(0)$ holds while Z fails?
2. Is it consistent that $Z(0)$ holds in $L[x]$ for some real x?
3. Does Z imply that there are random reals over L?

2.3 Computations with ordinal parameters In analogy with Turing machines, where arbitrary natural numbers are allowed as oracles, we can allow ordinals as oracles as in [20]. For this type of computation, a real x is computable from a real y if and only if there exist an OTM program P and finitely many ordinals $\alpha_0, \ldots, \alpha_n$ such that P eventually stops with x written on the tape when run in the oracle y with parameters $\alpha_0, \ldots, \alpha_n$. The computability strength corresponds to constructibility.

Lemma 17 *A real x is OTM-computable from y with ordinal parameters if and only if $x \in L[y]$.*

Proof This is a straightforward relativization of the proof from [20]. □

We aim to characterize the models of set theory where random oracles cannot add information, that is, where OTM-computability with ordinal parameters from all oracles in a set of positive measure implies OTM-computability with ordinal parameters in the empty oracle. Trivially, L has this property. Note that if ${}^{\omega}2 \not\subseteq L$ and the set of constructible reals is measurable, then it has measure 0. This follows from the fact that we can partition ω into a constructible sequence of disjoint infinite sets and translate ${}^{\omega}2 \cap L$ by some $a \in {}^{\omega}2 \setminus L$ separately on each set.

If ${}^{\omega}2 \cap L$ is not measurable, then every set of reals of positive measure contains a real in L and this real is OTM-computable with ordinal parameters. Many forcings such as random forcing and Sacks forcing preserve outer measure, so that in the generic extension the set of ground model reals is not measurable. Such extensions of L also have the required property.

We now consider the case in which ${}^{\omega}2$ has measure 0. Note that the statement that a code $c \in {}^{\omega}2$ for a Borel subset of ${}^{\omega}2$ codes a measure 1 set is absolute between transitive models of ZFC containing c by Jech [10, Lemma 26.1]. This implies that, for every generic filter g over M and every random real x over $M[g]$, x is random over M. The random reals appearing in a two-step iteration of random forcing are not mutually random generic by Bartoszyński and Judah [1, Lemma 3.2.8, Theorem 3.2.11]. However, the next lemma is sufficient for our application.

Lemma 18 *Suppose that M is a model of* ZFC. *Suppose that x is random over M and y is random over $M[x]$. Then $M[x] \cap M[y] = M$.*

Proof Let $\mathbb{P}$ denote random forcing, and let $\dot{\mathbb{P}}$ denote a $\mathbb{P}$-name for random forcing. Note that $\mathbb{P} * \dot{\mathbb{P}}$ is forcing equivalent to $\mathbb{P}$ by [1, Lemma 3.2.8]. Let $\dot{x}, \dot{y}$ be names for the random reals added by $\mathbb{P} * \dot{\mathbb{P}}$.

We claim that there is a condition $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{P}}$ with $(p, \dot{q}) \Vdash_{\mathbb{P}*\dot{\mathbb{P}}} \check{M}[\dot{x}] \cap \check{M}[\dot{y}] = \check{M}$, where $\check{M}$ is a name for the ground model M. Otherwise $1_{\mathbb{P}} \Vdash_{\mathbb{P}*\dot{\mathbb{P}}} \check{M}[\dot{x}] \cap \check{M}[\dot{y}] \neq \check{M}$. Let $\kappa = (2^{\omega})^M$. Suppose that g is generic over M for a finite support product $\mathbb{P}$ of $(\kappa^+)^M$ random forcings. Note that random forcing is σ-linked by [1, Lemma 3.1.1] and hence Knaster. Then $\mathbb{P}$ is Knaster and hence satisfies the countable chain condition by [10, Corollary 15.16].

Let $(x_\alpha)_{\alpha<(\kappa^+)^M}$ denote the sequence of random reals added by g. Suppose that y is random over $M[g]$. Then y is random over M and over $M[x_\alpha]$ for all $\alpha < (\kappa^+)^M$, so $M[x_\alpha, y]$ is a $(\mathbb{P} * \dot{\mathbb{P}})$-generic extension of M. Hence, there is some $y_\alpha \in (M[x_\alpha] \cap M[y]) \setminus M$ for each $\alpha < (\kappa^+)^M$. Then x_α, x_β are mutually generic for all $\alpha \neq \beta$. This implies $M[x_\alpha] \cap M[x_\beta] = M$ by a similar argument to that in Lemma 28 below. Then $y_\alpha \neq y_\beta$ for $\alpha \neq \beta$, and hence, $|2^\omega|^{M[G]} = |(2^\omega)^{M[y]}|^{M[G]}$. But $|2^\omega|^{M[G]} = (\kappa^+)^{M[G]} = (\kappa^+)^M$ and $|(2^\omega)^{M[y]}|^{M[G]} = |2^\omega|^{M[y]} = |2^\omega|^M = \kappa$, since random forcing and $\mathbb{P}$ are c.c.c.

Suppose that $(p, \dot{q}) \Vdash_{\mathbb{P} * \dot{\mathbb{P}}} \check{M}[\dot{x}] \cap \check{M}[\dot{y}] = \check{M}$. It follows from the isomorphism theorem for Borel measures (see Kechris [14, Theorem 17.41]) that, for every condition $r \in \mathbb{P}$, random forcing below r is forcing equivalent to $\mathbb{P}$, that is, the Boolean completions are isomorphic. Thus, for an arbitrary random real x over V not necessarily below p, there is some condition $r \in \dot{\mathbb{P}}^x$ with $r \Vdash_{\mathbb{P}}^{M[x]} \check{M}[\check{x}] \cap \check{M}[\dot{y}] = M$. Then $M[x] \cap M[y] = M$ for an arbitrary random real y over $M[x]$ by the same argument for $\dot{\mathbb{P}}^x$. □

We will use the assumptions in the following lemma.

Lemma 19

(1) *After forcing with a finite support iteration of length ω_1 of random forcings, there is a random real over $L[x]$ for every real x. The statement that, for every real x, there is a random real over $L[x]$ is equivalent to the statement that every $\utilde{\Delta}^1_2$-set is Lebesgue measurable.*

(2) *After forcing with a product of ω_1 Cohen forcings, there is a Cohen real over $L[x]$ for every real x. The statement that, for every real x, there is a Cohen real over $L[x]$ is equivalent to the statement that every $\utilde{\Delta}^1_2$-set has the property of Baire.*

Proof The first claim is proved in [9, Theorem 4.3]. The second claim is proved in [9, Theorem 4.3]. □

Theorem 20

(1) *Suppose that, for every real x, there is a random real over $L[x]$. If A has positive measure and $x \in {}^\omega 2$ is constructible from each $y \in A$, then $x \in L$.*

(2) *Suppose that, for every real x, there is a Cohen real over $L[x]$. If A is a nonmeager Borel set and $x \in {}^\omega 2$ is constructible from each $y \in A$, then $x \in L$.*

Proof Since A has a Borel subset with the same measure, we can assume that A is Borel. Suppose that a is a Borel code for A. Note that a real y is random over a model M if and only if y is in every measure 1 Borel set coded in M. Let y be random over $L[a]$ below A, and let z be random over $L[a][y]$ below A. Such reals y, z exist since random forcing below the condition A is forcing equivalent to random forcing. Then $y, z \in A$. Moreover, y is random over L, and z is random over $L[y]$ by the discussion before the previous lemma. Since x is constructible from y and from z by our assumption, we have $x \in L[y] \cap L[z] = L$ by the previous lemma.

The argument for Cohen forcing is similar. □

There is a forcing extension of L such that there is a nonconstructible real x which is constructible from all elements of a measure 1 set (see Judah and Shelah [11, Section 3]).

Theorem 21 (Judah–Shelah [11, Section 3]) *There is a forcing $\mathbb{P}$ in L such that, in any $\mathbb{P}$-generic extension of L, there is a measure 1 set A such that every $x \in A$ can be constructed from every $y \in A$, but A contains no constructible real.*

Proof Blass–Shelah forcing has this property (see [11, Section 3]). We include a much shorter proof via a simplification of a forcing of Martin Goldstern, whom we thank for allowing us to include this. We define a forcing $\mathbb{P}$ with the property that every new real constructs the generic real, that is, the forcing is minimal for reals, and the set of ground model reals has measure 0. Suppose that $(a_n)_{n\in\omega}$ is a strictly increasing sequence of natural numbers with $a_{n+1} - a_n \geq n$. Let $I_n = [a_n, a_{n+1})$. The forcing $\mathbb{P}$ consists of trees t whose nodes are of the form $(C_0, \ldots, C_n)$ with $C_i = 2^{I_i} \setminus \{t_i\}$ for some $t_i \in 2^{I_i}$. Then $\mu(C_i) \geq 1 - \frac{1}{2^i}$. Every splitting node $(C_0, \ldots, C_n)$ splits into $(C_0, \ldots, C_{n+1})$ for all such C_{n+1}. The trees have no end nodes and cofinally many splitting nodes. The conditions are ordered by reverse inclusion.

Suppose that $(C_n)_{n\in\omega}$ is $\mathbb{P}$-generic over V. Then $\mu(\{x \mid \forall^\infty n\ x \upharpoonright I_n \in C_n\}) = 1$. Let $X = \{x \mid \exists^\infty n\ x \upharpoonright I_n \notin C_n\}$. Then $\mu(X) = 0$. Suppose that $x \in {}^\omega 2 \cap V$. Then for any $t \in \mathbb{P}$ with the stem $(D_0, \ldots, D_n)$, we can find some $s \leq t$ by choosing D_{n+1} with $x \upharpoonright I_{n+1} \notin D_{n+1}$; hence, $(D_0, \ldots, D_{n+1})$ forces that $x \upharpoonright I_{n+1} \notin \dot{C}_{n+1}$, where $\dot{C}_{n+1}$ is a name for C_{n+1}. This implies that $x \in X$. Thus, $\mu({}^\omega 2 \cap V) = 0$.

We claim that $\mathbb{P}$ has the pure decision property, that is, given any $s \in \mathbb{P}$ and any sentence φ, there is some $t \leq s$ with the same stem as s which decides φ. As for Sacks forcing, we enumerate the direct successors of the stem t_0 of t as $u_0, \ldots, u_n$ and choose trees $t^i \leq t/u_i = \{r \in t \mid u \subseteq u_i \text{ or } u_i \subseteq r\}$ deciding φ. Then $s = \bigcup_{i\leq n} t^i$ has the stem t_0 and decides φ.

If t forces that $\dot{x}$ is a name for a new real, we can build a subtree $s \leq t$ by using the pure decision property such that, at every splitting node p in s, the parts of $\dot{x}$ decided by s/q for direct successors of p are incompatible. This can easily be done by considering all pairs of direct successors, since the trees are finitely splitting. Then the generic real y is the unique branch in s which is compatible with $\dot{x}^y$ and hence is constructible from $\dot{x}^y$. □

It is independent of ZFC whether there are a real x and a set $X \subseteq {}^\omega 2$ of positive measure such that x is OTM-computable with parameters from each element of X, by Theorems 20 and 21. The same statement, but with sets of positive measure replaced by nonmeager Borel sets, is independent of ZFC by Theorem 20 and the following property of Laver forcing.

Theorem 22 (Gray [6]) *Laver forcing adds a minimal real such that the set of ground model reals is meager.*

Proof Laver forcing is minimal (see [6]). Since a Laver real dominates the ground model reals by [1, Theorem 7.3.28], the set of ground model reals is meager in the generic extension. □

Remark 23 The results in this section hold verbatim for ordinal register machines (ORMs) (introduced in [21]), which are identical to OTMs in computational strength with and without ordinal parameters. This is shown in [22] in the case with parameters. We leave out the proof for the case without parameters, which is not hard to obtain, but technical and not very informative.

Note that in the situation of Theorem 21, for any new real x, we can search through all $\mathbb{P}$-names $\dot{x}$ in the ground model M and thin out trees as in the proof of Theorem 21. For each such tree t, we compute the unique branch y with $\dot{x}^y = x$ if it exists, and then we check whether it is $\mathbb{P}$-generic over L. Thus, we have an OTM program which computes a $\mathbb{P}$-generic real over L from each new real.

Question 24 Is it consistent that there are a nonconstructible real x and a Borel set A of measure 1 such that x is OTM-computable without parameters from every $y \in A$?

More generally, we ask which combinations of the following statements are consistent (with $\mu({}^{\omega}2 \cap L) = 0$). If A is a Borel set of positive measure (resp., measure 1) and x is OTM-computable (resp., OTM-computable with ordinal parameters) from each $y \in A$, then x is OTM-computable (resp., with ordinal parameters).

3 Infinite-Time Turing Machines

Historically, infinite-time Turing machines (ITTMs) were the first machine model of transfinite computations. Roughly speaking, an ITTM is a classical Turing machine with transfinite ordinal running time: whenever the time reaches a limit ordinal, the tape content at each cell is the limit inferior of the earlier contents and the machine assumes a special limit state. The definitions of ITTMs, writability, eventual writability, and accidental writability can be found in [7].

In this section, we will show that every real x which is writable (resp., eventually writable, accidentally writable) from every real in a nonmeager Borel set is already writable (resp., eventually writable, accidentally writable). The proofs use Cohen forcing over L_α. A similar argument in which a ranked forcing language is used can be found in Welch [31, Theorem 3.1]. In ongoing work, we are attempting to use a similar strategy for random forcing instead of Cohen forcing, which would lead to the analogous result for positive Lebesgue measure. The difficulty is that random forcing in L_α is a proper class.

Definition 25 Suppose that y is a real. Let λ^y denote the supremum of the ordinals writable in the oracle y, let ζ^y be the supremum of the ordinals eventually writable in the oracle y, and let Σ^y be the supremum of the ordinals accidentally writable in the oracle y. Let $\lambda = \lambda^0$, $\zeta = \zeta^0$, and $\Sigma = \Sigma^0$.

Welch [32] characterized the writable, eventually writable, and accidentally writable reals.

Theorem 26 (Welch) *For every real x, we have the following.*

(1) The reals writable in the oracle x are exactly those in $L_{\lambda^x}[x]$.
(2) The reals eventually writable in the oracle x are exactly those in $L_{\zeta^x}[x]$.
(3) The reals accidentally writable in the oracle x are exactly those in $L_{\Sigma^x}[x]$.

Note that ζ is Σ_2-admissible and Σ is a limit of Σ_2-admissibles by Welch [33, Lemma 7, p. 19] (that ζ is an admissible limit of admissibles due to [7, Theorem 8.3]), but Σ is not admissible by [33, Fact 2]. Moreover, λ is an admissible limit of admissibles by [33, Fact 2.2, p. 11]. Since adding an oracle can only increase the supremum of the writable, eventually writable, and accidentally writable ordinals, we have $\lambda \leq \lambda^x$, $\zeta \leq \zeta^x$, and $\Sigma \leq \Sigma^x$ for all reals x.

Our goal is to show that $\lambda^x = \lambda$, $\zeta^x = \zeta$, and $\Sigma^x = \Sigma$ for Cohen generic reals x over $L_{\Sigma+1}$, using the following characterization. The proof of the unrelativized version can be found in [32, Theorems 2.1, 2.3]. The relativized version is discussed in the proof of [32, Lemma 2.4].

Theorem 27 (Welch) *Suppose that x is a real. Then (ζ^x, Σ^x) is the lexically minimal pair of ordinals such that $L_{\zeta^x}[x] \prec_{\Sigma_2} L_{\Sigma^x}[x]$. Moreover, λ^x is minimal with the property that $L_{\lambda^x}[x] \prec_{\Sigma_1} L_{\zeta^x}[x]$.*

Although we only need to force over L_α where α is admissible or a limit of admissibles, let us phrase the results in a stronger form. Mathias [24] developed set forcing over models of a weak fragment PROV of ZFC such that the transitive models of PROV, the *provident sets*, are the transitive sets closed under functions defined by recursion along rudimentary functions and containing ω. The definitions and basic facts about rudimentary functions and provident sets can be found in Mathias [24] and [25]. For example, L_α is provident if and only if α is an infinite indecomposable ordinal. We would like to thank Adrian Mathias for discussions on this topic.

As usual, if $\mathbb{P} \subseteq L_\alpha$ is a partial order and $G \subseteq \mathbb{P}$ is a filter, let $L_\alpha[G] = \{\sigma^G \mid \sigma \in L_\alpha\}$ denote the *generic extension* of L_α by G. Let L_α^x denote L_α built relative to the language $\{\in, x\}$, where x is a real. If L_α is provident and x is Cohen generic over L_α, then $L_\alpha[x] = L_\alpha^x$ by [24, Section 9].

Lemma 28 *Suppose that L_α is provident, $\mathbb{P}, \mathbb{Q} \in L_\alpha$ are forcings, and $G \times H$ is $\mathbb{P} \times \mathbb{Q}$-generic over L_α. Then $L_\alpha[G] \cap L_\alpha[H] = L_\alpha$.*

Proof The forcing relation for atomic formulas is definable by a rudimentary recursion over provident sets by [24, Section 2], and the forcing relation for Δ_0-formulas is rudimentary in the forcing relation for atomic formulas (see [24, Section 3]). Hence, $\{(p,q) \in \mathbb{P} \times \mathbb{Q} \mid p \Vdash \check{q} \in \sigma\} \in L_\alpha$ for any $\mathbb{P}$-name $\sigma \in L_\alpha$. Thus, a filter $F \subseteq \mathbb{P} \times \mathbb{P}$ is $(\mathbb{P} \times \mathbb{P})$-generic over L_α if and only if there are a $\mathbb{P}$-generic filter G over L_α and a $\mathbb{P}$-generic filter H over $L_\alpha[G]$ with $F = G \times H$. (This is proved in [10, Lemma 15.9] for transitive models of ZFC, and the same proof works for provident sets.)

Let $\dot{G}$, $\dot{H}$ denote the canonical names for G, H. Suppose that x is of minimal rank with $x \in L_\alpha[G] \cap L_\alpha[H]$ and $x \notin L_\alpha$. Suppose that $\sigma \in M^{\mathbb{P}}$ and $\tau \in M^{\mathbb{Q}}$ with $\sigma^G = x$ and $\tau^H = x$. Then there are conditions $p \in \mathbb{P}$ and $q \in \mathbb{Q}$ with $(p,q) \Vdash_{\mathbb{P}\times\mathbb{Q}} \sigma^{\dot{G}} = \tau^{\dot{H}}$. Suppose that $x \notin M$. Then for some $y \in M$, p does not decide if $y \in \sigma^{\dot{G}}$, and hence, q does not decide if $y \in \tau^{\dot{H}}$. Suppose that $p' \leq p$ and $q' \leq q$ with $p' \Vdash_{\mathbb{P}} y \in \sigma^{\dot{G}}$ and $q' \Vdash_{\mathbb{Q}} y \notin \tau^{\dot{H}}$. Then $(p', q') \Vdash_{\mathbb{P}\times\mathbb{Q}} \sigma^{\dot{G}} \neq \tau^{\dot{H}}$, contradicting the assumption that $(p,q) \Vdash_{\mathbb{P}\times\mathbb{Q}} \sigma^{\dot{G}} = \tau^{\dot{H}}$. $\square$

Lemma 29 *Suppose that $\alpha \in \omega_1$ and $a \subseteq \omega$. Then the set C_α of Cohen generic reals over $L_\alpha[a]$ is comeager.*

Proof The set C_α is the intersection of all dense subsets of Cohen forcing contained in $L_\alpha[a]$. As $L_\alpha[a]$ is countable, C_α is hence an intersection of countably many dense sets and thus comeager. $\square$

Lemma 30 *Suppose that $A \subseteq {}^\omega 2$ is a nonmeager Borel set and $\alpha < \omega_1$. There are reals $x, y \in A$ such that x is Cohen generic over L_α and y is Cohen generic over $L_\alpha[x]$.*

Proof Let C_α denote the set of Cohen reals over L_α. Then C_α is comeager, and hence, $A \cap C_\alpha$ is comeager. Suppose that $x \in A \cap C_\alpha$, and let C denote the set of Cohen reals over $L_\alpha[x]$. Since C is comeager, suppose that $y \in A \cap C$. Then y is

Cohen generic over $L_\alpha[x]$. Hence, $x, y \in A$ are mutually Cohen generic over L_α. □

Lemma 31 *Let $\mathbb{P}$ denote Cohen forcing. Suppose that L_α is provident, $p \in \mathbb{P}$, $\vec{\sigma} \in L_\alpha$, and φ is a formula.*

(1) If φ is a Δ_0-formula, then $p \Vdash_{\mathbb{P}}^{L_\alpha} \varphi$ is Δ_1 over L_α.
(2) If φ is a Σ_n-formula, then $p \Vdash_{\mathbb{P}}^{L_\alpha} \varphi$ is Σ_n over L_α.
(3) If φ is a Π_n-formula, then $p \Vdash_{\mathbb{P}}^{L_\alpha} \varphi$ is Π_n over L_α.

Proof This is proved for Δ_0-formulas in [24, Section 3]. The rest follows inductively from the definition of the forcing relation. □

Lemma 32 *Let $\mathbb{P}$ denote Cohen forcing. Suppose that L_α is provident, $p \in \mathbb{P}$, φ is a formula, and $\vec{\sigma} \in L_\alpha$.*

(1) $p \Vdash \varphi(\vec{\sigma})$ if and only if $L_\alpha[G] \vDash \varphi(\vec{\sigma}^G)$ for all Cohen generic filters G over $L_{\alpha+1}$.
(2) Suppose that G is Cohen generic over $L_{\alpha+1}$. Then $L_\alpha[G] \vDash \varphi(\vec{\sigma})$ if and only if $p \Vdash_{\mathbb{P}} \varphi(\vec{\sigma})$ for some $p \in G$.

Proof This follows from the proof of the forcing theorem (see, e.g., Kunen [23, Theorems 3.5, 3.6]). □

The following lemma is implicit in [31, Lemma 3.3] and Coskey and Hamkins [4, Theorem 4.8].

Lemma 33 *Suppose that x is Cohen generic over $L_{\Sigma+1}$.*

(1) $L_\lambda[x] \prec_{\Sigma_1} L_\zeta[x] \prec_{\Sigma_2} L_\Sigma[x]$.
(2) $\lambda^x = \lambda$, $\zeta^x = \zeta$, and $\Sigma^x = \Sigma$.

Proof The previous lemma shows that $L_\alpha[x] \prec_{\Sigma_n} L_\beta[x]$ for all $n \geq 1$ and provident sets $L_\alpha \subseteq L_\beta$. This immediately implies the claims. □

Theorem 34 *Suppose that x is a real and that A is a comeager set of reals.*

- *If x is writable in every oracle $y \in A$, then x is writable.*
- *If x is eventually writable in every oracle $y \in A$, then x is eventually writable.*
- *If x is accidentally writable in every oracle $y \in A$, then x is accidentally writable.*

Proof The set C of Cohen generic reals over $L_{\Sigma+1}$ is comeager by Lemma 29, so $A \cap C$ is comeager. We may assume without loss of generality that $A \subseteq C$. The reals writable in every $y \in A$ are those in $\bigcap_{y\in A} L_\lambda[y]$, the reals eventually writable in every $y \in A$ are those in $\bigcap_{y\in A} L_\zeta[y]$, and the reals accidentally writable in every $y \in A$ are those in $\bigcap_{y\in A} L_\Sigma[y]$, by Lemma 33 and Theorem 26.

Since A is comeager, A contains two mutually Cohen generic reals u and v by Theorem 30. Since λ, ζ, and Σ are limits of admissibles, it is readily seen that L_λ, L_ζ, and L_Σ are provident. Then

$$L_\lambda \subseteq \bigcap_{y\in A} L_\lambda[y] \subseteq L_\lambda[u] \cap L_\lambda[v] = L_\lambda,$$

$$L_\zeta \subseteq \bigcap_{y\in A} L_\zeta[y] \subseteq L_\zeta[u] \cap L_\zeta[v] = L_\zeta,$$

$$L_\Sigma \subseteq \bigcap_{y \in A} L_\Sigma[y] \subseteq L_\Sigma[u] \cap L_\Sigma[v] = L_\Sigma,$$

by Theorem 28. Hence, we have equalities in each case, and the claim follows from Theorem 26. □

Theorem 35 *Suppose that x is a real and that A is a nonmeager Borel set of reals.*

- *If x is writable in every oracle $y \in A$, then x is writable.*
- *If x is eventually writable in every oracle $y \in A$, then x is eventually writable.*
- *If x is accidentally writable in every oracle $y \in A$, then x is accidentally writable.*

Proof Since A has the Baire property, there is some finite t such that, for the corresponding basic open set $N_t := \{x \mid t \subseteq x\}$, $(A \cap N_t) \triangle N_t$ is meager. Consequently, $A \cap N_t$ is comeager in N_t. We define a translation function $t : [0, 1] \to N_t$, where $t(x)$ is obtained from x by replacing the sequence of the first $|t|$ many bits of x with t. Then range$(f) = N_t$, and $X := f^{-1}[A \cap N_t]$ is comeager in $[0, 1]$. Furthermore, t is clearly ITTM-computable. Now, if some y is writable in every $a \in A$, then it is writable in every $t(x)$ with $x \in X$. So we can compute y from every element of X by first applying f and then applying the reduction from N_t to y. Hence, y is writable in all elements of a comeager set, so y is writable by Theorem 34. The same argument shows the analogous statement for eventual and accidental writability. □

4 Infinite-Time Register Machines

Before we consider infinite-time register machines (ITRMs), let us briefly mention the unresetting version of these machines. Unresetting ITRMs (see [17]), also called weak ITRMs (wITRMs), work like classical register machines. In particular, they use finitely many registers, each of which can store a single natural number, but with transfinite ordinal running time. At limit times, the program line is the limit inferior of the earlier program lines, and there is a similar limit rule for the register contents. If the limit inferior is infinite, then the computation is undefined. A real x is wITRM-computable if and only if $x \in L_{\omega_1^{CK}}$ by [17], and the proof relativizes.

Lemma 36 *A real x is wITRM-computable in the oracle y if and only if $x \in L_{\omega_1^{CK,y}}[y]$.*

Hence, the question is whether there are a set A of positive measure and a real $x \notin L_{\omega_1^{CK}}$ such that $x \in L_{\omega_1^{CK,y}}[y]$. We will use the following result (see Nies [26, Theorem 9.1.13]), where $\leq_h$ denotes hyperarithmetic reducibility.

Theorem 37 (Sacks) *Suppose that x is a real. Then $x \notin \Delta^1_1$ if and only if $x \notin L_{\omega_1^{CK}}$ if and only if $\mu(\{a \mid x \leq_h a\}) = 0$.*

Theorem 38 *Suppose that x is a real and A is a set of reals with $\mu(A) > 0$ such that x is wITRM-computable from every $y \in A$. Then x is wITRM-computable.*

Proof By Sacks [27, Chapter IV, Corollary 1.6], we have $\mu(\{y \mid \omega_1^{CK,y} = \omega_1^{CK}\}) = 1$. Hence, we may assume that $\omega_1^{CK,y} = \omega_1^{CK}$, and thus, $L_{\omega_1^{CK,y}}[y] =$

$L_{\omega_1^{CK}}[y]$ for all $y \in A$. If y is not wITRM-computable, then y is not hyperarithmetic (see [17]). Then $\mu(\{x \mid y \leq_h x\}) = 0$ by Theorem 37, contradicting the assumption $\mu(A) > 0$. □

For the rest of this section, we consider (resetting) ITRMs. They differ from weak ITRMs only in their behavior when the limit inferior is infinite. In this case, the register in question is assigned the value 0 and the computation continues. This leads to a huge increase in terms of computability strength. An introduction to ITRMs can be found in [19].

A real x is ITRM-computable if and only if $x \in L_{\omega_\omega^{CK}}$ by Koepke [18], and the proof relativizes.

Lemma 39 *A real x is ITRM-computable in a real y if and only if $x \in L_{\omega_\omega^{CK,y}}[y]$.*

The question now is whether there are a real $x \notin L_{\omega_\omega^{CK}}$ and a set A of positive measure such that $x \in L_{\omega_\omega^{CK,y}}[y]$ for every $y \in A$. To show that there is no such real, we first relativize Theorem 37.

Lemma 40 *Suppose that x, y are reals. Then $x \notin L_{\omega_1^{CK,y}}[y]$ if and only if $\mu(\{a \mid x \leq_h a \oplus y\}) = 0$.*

Proof We follow the proof of [13, Theorem 3.1.1]. Suppose that $x \notin L_{\omega_1^{CK,y}}[y]$. The set $\{a \mid x \leq_h a \oplus y\}$ is Π^1_1 in y. Let us assume that it has positive measure. Since there are only countably many hyperarithmetic reductions, there is some hyperarithmetic reduction P such that, for a positive measure set of a, P reduces x to $a \oplus y$. Then there is a rational interval I in which this set has relative measure greater than 0.5 by the Lebesgue density theorem. The set $\{b \in I \mid P^{a\oplus y} = P^{b\oplus y}\}$ is Π^1_1 in a and hence measurable. We define Y as the set of a with $\mu_I(\{b \in I \mid P^{a\oplus y} = P^{b\oplus y}\}) > 0.5$, where $\mu_I(A) = \frac{\mu(A\cap I)}{\mu(I)}$ denotes the relative measure. Then Y is Π^1_1 in y by [13, Theorem 2.2.3]. The set $Z := \{z \in {}^{\omega}2 \mid \omega_1^{CK,z} = \omega_1^{CK}\}$ has measure 1 by [26, Corollary 9.1.15]. Since $\mu(Y) > 0.5$, there is some $z \in Y$ with $\omega_1^{CK,z} = \omega_1^{CK}$. Since Y is Π^1_1 in y, there is a tree T computable in y such that $z \in Y$ if and only if T_z is well founded, for all $z \in {}^{\omega}2$. Since Z has measure 1, there is some $z \in Y \cap Z$. Then $\alpha := \mathrm{rank}(T_z) < \omega_1^{CK,z} = \omega_1^{CK}$ by Hjorth [8, Theorem 4.4]. Since α is computable, the set $X := \{z \in {}^{\omega}2 \mid \mathrm{rank}(T_z) \leq \alpha\}$ is a nonempty subset of Y which is Δ^1_1 in y. Then $P^{a\oplus y} = x$ for all $a \in Y$. Then $z = x$ if and only if $P^{u\oplus y} = z$ for some (for all) $u \in X$. Since $P^{u\oplus y}$ halts for all $u \in X$, $P^{u\oplus y} = z$ can be equivalently replaced by the statement that for every halting run of P on input $u \oplus y$ the output is z. Thus, x is hyperarithmetic in y. □

Lemma 41 *Suppose that x is a real.*

(1) $\mu(\{y \in {}^{\omega}2 \mid \omega_1^{CK,x\oplus y} = \omega_1^{CK,x}\}) = 1$.
(2) $\mu(\{y \in {}^{\omega}2 \mid \forall i \in \omega\ \omega_i^{CK,y} = \omega_i^{CK}\}) = 1$.
(3) $\mu(\{y \in {}^{\omega}2 \mid \omega_\omega^{CK,y} = \omega_\omega^{CK}\}) = 1$.

Proof (1) Let $c(x)$ denote the $<_{L[x]}$-least real r which codes a well-ordering of length $\omega_1^{CK,x}$. Now suppose that y is such that $\omega_1^{CK,x\oplus y} > \omega_1^{CK,x}$. Then $x \in L_{\omega_1^{CK,x}}[x] \in L_{\omega_1^{CK,x\oplus y}}[x \oplus y]$ and $L_{\omega_1^{CK,x\oplus y}}[x] \subseteq L_{\omega_1^{CK,x\oplus y}}[x \oplus y]$. Let

H denote the hull of x in $L_{\omega_1^{CK,x}}[x]$ for the canonical Skolem functions. Then $H = L_{\omega_1^{CK,x}}[x]$ by condensation (see Schindler and Zeman [28, Theorem 1.16]) and since $L_{\omega_1^{CK,x}}[x]$ is the least model of KP containing x. Then there are a bijection between ω and $\omega_1^{CK,x}$ and, hence, a code c for $\omega_1^{CK,x}$ in $L_{\omega_1^{CK,x}+\omega}[x]$. As $c(x) \leq_{L[x]} c$ by the minimality of $c(x)$, we have $c(x) \in L_{\omega_1^{CK,x\oplus y}}[x \oplus y]$ and $c(x) \leq_h x \oplus y$. Moreover, $c(x) \not\leq_h x$ implies that $\mu(\{y \mid c(x) \leq_h x \oplus y\}) = 0$ by Theorem 40.

(2) Let $c(i)$ denote the $<_L$-least code for ω_i^{CK}. We have $\mu(\{x \in {}^{\omega}2 \mid \omega_1^{CK,x\oplus y} = \omega_1^{CK,y}\}) = 1$ for all $y \in {}^{\omega}2$ by Lemma 41. Then

$$X_i := \{x \in {}^{\omega}2 \mid \omega_1^{CK,x\oplus c(i)} = \omega_1^{CK,c(i)}\}$$

has measure 1 for all $i \in \omega$, and hence, $X = \bigcap_{i\in\omega} X_i$ has measure 1. We claim that $\omega_i^{CK,x} = \omega_i^{CK}$ for all $x \in X$. To see this, let us denote by $c(i, y)$ the $<_L$-least code for $\omega_i^{CK,y}$ for $y \in {}^{\omega}2$. Then $c(0, y)$ is a code for ω and $\omega_1^{CK,y\oplus c(i,y)} = \omega_{i+1}^{y}$ for all reals y. Now suppose that $x \in X$. Since $x \in X_1$, we have $\omega_1^{CK,x} = \omega_1^{CK,x\oplus c(0,x)} = \omega_1^{CK,c(0)} = \omega_1^{CK}$. If $\omega_i^{CK,x} = \omega_i^{CK}$, then $c(i, x) = c(i)$. Since $x \in X_{i+1}$, we have $\omega_{i+1}^{CK,x} = \omega_1^{CK,c(i,x)\oplus x} = \omega_1^{CK,c(i)\oplus x} = \omega_1^{CK,x\oplus c(i)} = \omega_1^{CK,c(i)} = \omega_{i+1}^{CK}$. Hence, $\omega_i^{CK,x} = \omega_i^{CK}$ for all $i \in \omega$.

(3) This follows from the previous claim, since $\omega_\omega^{CK,x} = \sup_{i\in\omega} \omega_i^{CK,x}$. □

We can now show that ITRM-computability relative to oracles in a set of positive measure implies ITRM-computability.

Theorem 42 *Suppose that x is a real and A is a set of positive measure such that x is ITRM-computable from all $y \in A$. Then x is ITRM-computable.*

Proof It is sufficient to show that $\bigcap_{y\in A} L_{\omega_\omega^{CK,y}}[y] = L_{\omega_\omega^{CK}}$. Suppose that $x \in \bigcap_{y\in A} L_{\omega_\omega^{CK,y}}[y] \setminus L_{\omega_\omega^{CK}}$. Then for each $y \in A$, there is a least $i(y) \geq 1$ with $x \in L_{\omega_i^{CK,y}}[y]$. Let $A_j := \{y \in A \mid i(y) = j\}$ for $j \in \omega$. Then $A = \bigcup_{j\in\omega} A_j$, and since the sets A_j are provably Δ_2^1, they are measurable by [12, Exercise 14.4]. Hence, $\mu(A_k) > 0$ for some $k \geq 1$. If $k = 1$, then $x \in L_{\omega_1^{CK,y}}[y]$ for all $y \in A_1$ and $\mu(A_1) > 0$, so x is ITRM-computable. Suppose that $k = j + 1$. Let c denote the $<_L$-least code for a well-ordering of length ω_j^{CK}. Then there is a partial surjection of ω onto ω_j^{CK} which is Σ_1 over $L_{\omega_j^{CK,y}}[y]$ for all $y \in A_k$ and hence $c \in L_{\omega_j^{CK,y}+1}[y]$. Then $x \in L_{\omega_k^{CK,y}}[y] = L_{\omega_1^{CK,c\oplus y}}[c \oplus y]$ and hence $x \leq_h c \oplus y$ for $y \in A_k$. Then $x \in L_{\omega_1^{CK,c}}[c] = L_{\omega_k^{CK}} \subseteq L_{\omega_\omega^{CK}}$ by Theorem 40, since $\mu(A_k) > 0$. □

Let us call a real x *ITRM-extracting* if and only if there is a real y which is not ITRM-computable from x, but the set of reals z such that y is ITRM-computable from $x \oplus z$ has positive measure. A slight generalization of the above idea shows that there are also no extracting reals for ITRMs, in contrast to the case of OTMs, where this is independent of ZFC.

Lemma 43 *Suppose that x, y are reals and $\omega_j^{CK} = \omega_j^{CK,y}$ for all $j \in \omega$. Suppose that $i \in \omega$ and $c(i)$ is the $<_L$-least code for ω_i^{CK}. Then $x \notin L_{\omega_{i+1}^{CK,y}}[y]$ if and only if $\mu(\{z \mid x \leq_h z \oplus y \oplus c(i)\}) = 0$.*

Proof Since $\omega_j^{CK} = \omega_j^{CK,y}$ for all $j \in \omega$, we have $\omega_{i+1}^{CK,y} = \omega_1^{CK,c(i)\oplus y}$. Then $x \in L_{\omega_{i+1}^{CK,y}}[y] \subseteq L_{\omega_1^{CK,y\oplus c(i)}}[y \oplus c(i)]$ implies that $x \leq_h y \oplus c(i)$. For the other direction, suppose that $x \notin L_{\omega_{i+1}^{CK,y}}[y] = L_{\omega_1^{CK,y\oplus c(i)}}[y \oplus c(i)]$. Then $\{z \mid x \leq_h z \oplus y \oplus c(i)\} = \{z \mid x \leq_h z \oplus (y \oplus c(i))\}$ has measure 0 by Theorem 40 applied to $y \oplus c(i)$. □

Corollary 44 *There is no ITRM-extracting real.*

Proof Assume for a contradiction that x is ITRM-extracting, witnessed by a real y. Then $y \notin L_{\omega_\omega^{CK,x}}[x]$ and $y \in L_{\omega_\omega^{CK,x\oplus z}}[x \oplus z]$ for a set of reals z of positive measure. We have $y \not\leq_h c(i) \oplus x$ if and only if $\mu(\{z \mid y \leq_h z \oplus x \oplus c(i)\}) = 0$ for all $i \in \omega$ by Lemma 43. Hence,

$$
\begin{aligned}
y \notin L_{\omega_\omega^{CK,x}}[x] &\Leftrightarrow \forall i \in \omega\ y \notin L_{\omega_i^{CK,x}}[x] \\
&\Leftrightarrow \forall i \in \omega\ y \not\leq_h c(i) \oplus x \\
&\Leftrightarrow \forall i \in \omega\ \mu\big(\{z \mid y \leq_h c(i) \oplus z \oplus x\}\big) = 0 \\
&\Leftrightarrow \forall i \in \omega\ \mu\big(\{z \mid y \in L_{\omega_i^{CK,z\oplus x}}[z \oplus x]\}\big) = 0 \\
&\Leftrightarrow \mu\big(\{z \mid y \in L_{\omega_\omega^{z\oplus x}}[z \oplus x]\}\big) = 0,
\end{aligned}
$$

contradicting the assumption on y. □

Remark 45 A similar strategy works for the other machine types considered in this paper besides OTMs and ORMs, and the arguments relativize in a straightforward manner.

We now prove an analogous result for nonmeager Borel sets of oracles.

Lemma 46

(1) *If g is Cohen generic over $L_{\omega_\omega^{CK}}$, then $\omega_\omega^{CK,g} = \omega_\omega^{CK}$.*
(2) *If g is Cohen generic over $L_{\omega_i^{CK}+1}$, then $\omega_i^{CK,g} = \omega_i^{CK}$.*

Proof (1) If α is admissible and h is a Cohen generic filter over $L_{\alpha+1}$, then $L_\alpha[h]$ is admissible by [24, Theorem 10.1]. Note that g is Cohen generic over $L_{\omega_i^{CK}+1}$ for all $i \in \omega$. Then $\omega_i^{CK,g} = \omega_i^{CK}$ for all $i \in \omega$. Hence, $\omega_\omega^{CK,g} = \bigcup_{i\in\omega} \omega_i^{CK,g} = \bigcup_{i\in\omega} \omega_i^{CK} = \omega_\omega^{CK}$.

(2) As in the proof of the previous claim, ω_j^{CK} is g-admissible for all $j \leq i$, so that $\omega_j^{CK,g} = \omega_j^{CK}$ for all $j \leq i$. □

Theorem 47 *Suppose that x is a real and A is a nonmeager Borel set such that x is ITRM-computable from all $y \in A$. Then x is ITRM-computable.*

Proof We can assume that there is some ITRM program P which computes x from all $y \in A$. The set C of Cohen reals over $L_{\omega_\omega^{CK}}$ is comeager, so we can assume that

$A \subseteq C$. There are mutually Cohen generic reals $u, v \in A$ over $L_{\omega_\omega^{CK}}$ by Lemma 30. Then $L_{\omega_\omega^{CK,u}}[u] \cap L_{\omega_\omega^{CK,v}}[v] = L_{\omega_\omega^{CK}}[u] \cap L_{\omega_\omega^{CK}}[v] = L_{\omega_\omega^{CK}}$ by Lemma 28. Then

$$L_{\omega_\omega^{CK}} \subseteq \bigcap_{y \in A} L_{\omega_\omega^{CK}}[y] \subseteq L_{\omega_\omega^{CK}}[u] \cap L_{\omega_\omega^{CK}}[v] = L_{\omega_\omega^{CK}},$$

and hence, x is ITRM-computable. □

Remark 48 Following the same line of reasoning, if x is wITRM-computable from all oracles in a nonmeager Borel set A of oracles, then x is wITRM-computable.

5 α-Turing Machines

Suppose that $\alpha > \omega$ is a countable admissible ordinal. In this section, we consider computability relative to a set of oracles of positive measure for parameter-free α-Turing machines (α-TMs) as defined in [20]. These machines are similar to ITTMs, but have tape length α. We crucially use the following characterization of the computability strength of α-TMs. This is a minor modification of [20, Lemma 3].

Lemma 49 *Suppose that $\alpha > \omega$ is exponentially closed. A real x is computable by an α-TM in an oracle y if and only if x is Δ_1-definable in the parameter y over $L_\alpha[y]$.*

In particular, for reals x, y with $\omega_i^{CK,y} = \omega_i^{CK}$, x is ω_i^{CK}-computable from y if and only if $x \in L_{\omega_i^{CK}}[y]$.

If α is an ordinal, let α^+ denote the least admissible ordinal $\gamma > \alpha$. Let $\bar{\alpha} = \omega_{\bar{\iota}}$ denote the least admissible ordinal γ such that L_{γ^+} does not contain a real coding γ. Then for every admissible $\alpha < \bar{\alpha}$, the $<_L$-least real c_α coding α is in L_{α^+}. We will extend the preceding results to all admissible ordinals $\alpha < \bar{\alpha}$.

Lemma 50 *If $\iota < \bar{\iota}$, then $\mu(\{x \in {}^\omega 2 \mid \omega_\iota^{CK,x} = \omega_\iota^{CK}\}) = 1$.*

Proof The proof is similar to Lemma 41, where the case $\iota < \omega$ was proved. Suppose that $\iota < \bar{\alpha}$ and that the claim is known for all $\gamma < \iota$. Let $M_\gamma := \{y \mid \omega_\gamma^{CK,y} = \omega_\gamma^{CK}\}$ for $\gamma < \iota$ and $M := \bigcap_{\delta < \iota} M_\delta$. Then $\mu(M) = 1$. If $\iota = \gamma + 1$, then $\mu(\{z \mid \omega_1^{CK,z \oplus c_\gamma} = \omega_1^{CK,c_\gamma}\}) = 1$ by Lemma 41. Since $\omega_1^{CK,c_\gamma} = \omega_{\gamma+1}^{CK} = \omega_\iota^{CK}$, this implies $\mu(\{y \in M \mid \omega_1^{CK,y \oplus c_\gamma} = \omega_\iota^{CK}\}) = 1$. For all $y \in M$, we have $\omega_\gamma^{CK,y} = \omega_\gamma^{CK}$ and $\omega_1^{CK,y \oplus c_\gamma} = \omega_{\gamma+1}^{CK,y} = \omega_\iota^{CK}$, so $\omega_\iota^{CK,y} = \omega_\iota^{CK}$. If ι is a limit ordinal, then $\omega_\gamma^{CK,y} = \omega_\gamma^{CK}$ for all $\gamma < \iota$ and $y \in M$. Then $\omega_\iota^{CK,y} = \bigcup_{\gamma < \iota} \omega_\gamma^{CK,y} = \bigcup_{\gamma < \iota} \omega_\gamma^{CK} = \omega_\iota^{CK}$ for all $y \in M$ and $\mu(\{x \in {}^\omega 2 \mid \omega_\iota^{CK,x} = \omega_\iota^{CK}\}) = 1$. □

Consequently, $L_{\omega_\iota^{CK,x}}[x] = L_{\omega_\iota^{CK}}[x]$ for almost all x and all $\iota < \bar{\iota}$.

Theorem 51 *Suppose that $\alpha = \omega_\iota^{CK} < \bar{\alpha}$ is admissible, x is a real, A is a set of positive measure, and P is an α-Turing program such that $P^y = x$ for all $y \in A$. Then x is α-computable.*

Proof Suppose that $\iota < \bar{\iota}$ and that the claim holds for all $\gamma < \iota$. We have $x \in L_{\omega_\iota^{CK,y}}[y]$ for all $y \in A$, so we can assume that $\omega_\gamma^{CK,y} = \omega_\gamma^{CK}$ for all $\gamma \leq \iota$ by Lemma 50. Then $x \in L_{\omega_\iota^{CK}}[y]$ for all $y \in A$.

If $\iota = 1$, then we can assume that $\omega_1^{CK,y} = \omega_1^{CK}$ for all $y \in A$, since $\mu(\{y \subseteq \omega \mid \omega_1^{CK,y} = \omega_1^{CK}\}) = 1$ by Lemma 41. Then $x \in \bigcap_{y \in A} L_{\omega_1^{CK}}[y]$ by Lemma 49, so $x \leq_h y$ for all $y \in A$. Then $x \in L_{\omega_1^{CK}}$ by Theorem 37, and hence, x is ω_1^{CK}-computable by Lemma 49.

If $\iota = \gamma + 1 > 1$, then $x \in L_{\omega_1^{CK, c_\gamma \oplus y}}[c_\gamma \oplus y] = L_{\omega_\iota^{CK}}[y]$ and, hence, $x \leq_h c_\gamma \oplus y$ for all $y \in A$. If $x \leq_h c_\gamma$, then $x \in L_{\omega_1^{CK, c_\gamma}}[c_\gamma] = L_{\omega_\iota^{CK}} = L_\alpha$ and x is α-computable, as desired. If $x \not\leq_h c_\gamma$, then $\mu(\{z \mid x \leq_h z \oplus c_\gamma\}) = 0$ by Lemma 40. Since $x \leq c_\gamma \oplus a$ for all $a \in A$, this implies $\mu(A) = 0$, contradicting the assumption on A.

If ι is a limit ordinal, then $x \in \bigcup_{\gamma < \iota} L_{\omega_\gamma^{CK}}[y]$ for all $y \in A$. There are $\gamma_y < \iota$ with $x \in L_{\omega_{\gamma_y}^{CK}}[y]$ for all $y \in A$. Let $A_\gamma := \{y \in A \mid \gamma_y = \gamma\}$ for $\gamma < \iota$. Since A_γ is provably Δ^1_2, it is measurable (see [12, Exercise 14.4]). Then $\mu(A_\gamma) > 0$ for some $\gamma < \iota$. Hence, $x \in L_{\omega_\gamma^{CK}}[y]$ for all $y \in A_\gamma$ and $x \in L_{\omega_\gamma^{CK}} \subseteq L_\alpha$. □

This can be extended to unboundedly many countable admissibles.

Theorem 52 *There are unboundedly many countable admissible ordinals α such that every real x which is α-computable from all elements of a set A of positive measure is α-computable.*

Proof Suppose that T is a finite fragment of ZFC which is sufficient for the proof of Lemma 18. Then $L_\alpha \vDash T$ for unboundedly many countable admissible ordinals. Suppose that $L_\alpha \vDash T$. Since $\mu(A) > 0$, there are $y, z \in A$ such that y is random generic over L_α and z is random generic over $L_\alpha[y]$. Then $L_\alpha[y] \cap L_\alpha[z] = L_\alpha$ by Lemma 18 and hence $x \in L_\alpha$. □

Let us calculate bounds on $\bar{\alpha}$. Let α_0 denote the least β such that L_α is elementarily equivalent to L_β for some $\alpha < \beta$. Recall that η denotes the supremum of the halting times of OTMs.

Lemma 53 *We have $\alpha_0 \leq \bar{\alpha} < \eta$.*

Proof Suppose that $\gamma < \alpha_0$ is admissible. To see that L_{γ^+} contains a real coding L_γ, let S denote the set of all sentences which hold in $(L_\gamma, \in)$. Since $\gamma < \alpha_0$, L_γ is minimal such that $L_\gamma \models S$. Let H denote the elementary hull of the empty set in L_γ with respect to the canonical Skolem functions, and let $L_{\bar{\gamma}}$ denote the transitive collapse of H. Then $H = L_{\bar{\gamma}} = L_\gamma$ by the minimality of γ. Then there are a surjection from ω onto H and a real coding L_γ in L_{γ^+}.

To see that $\bar{\alpha} < \eta$, recall that η is the supremum of the Σ_1-fixed ordinals by Lemma 7. The existence of admissibles $\alpha < \beta$ such that there is a real $x \in L_\beta \setminus L_\alpha$ is expressed by a Σ_1-formula which first becomes true in some L_γ with $\gamma > \bar{\alpha}$. This implies $\eta > \bar{\alpha}$. □

A generalization of the argument for ITRMs shows an analogous result for α-TMs for admissible ordinals α and nonmeager Borel sets of oracles.

Theorem 54 *Suppose that x is a real, α is a countable admissible ordinal, A is a nonmeager Borel set of reals, and P is an α-Turing program such that $P^y = x$ for all $y \in A$. Then x is α-computable.*

Proof Suppose that $\alpha = \omega_\iota^{CK}$. If x is Cohen generic over $L_{\alpha+1}$, then $\omega_\iota^{CK,x} = \omega_\iota^{CK}$. This follows from the fact that, for admissible $\beta < \alpha$, $L_\beta[x]$ is admissible by [24, Theorem 10.1]. The set C of Cohen generic reals over $L_{\alpha+1}$ is comeager by Lemma 29, so we can assume that $A \subseteq C$. Then $\omega_\iota^{CK,y} = \omega_\iota^{CK}$ for all $y \in A$. There are mutual Cohen generics $u, v \in A$ over $L_{\alpha+1}$ by Lemma 30. Then

$$L_{\omega_\iota^{CK,u}}[u] \cap L_{\omega_\iota^{CK,v}}[v] = L_{\omega_\iota^{CK}}[u] \cap L_{\omega_\iota^{CK}}[v] = L_{\omega_\iota^{CK}} = L_\alpha$$

by Lemma 28. Hence, $x \in L_\alpha$ is α-computable. □

6 Conclusion

We considered the question of whether computability from all oracles in a set of positive measure implies computability for various machine models. This is the case for most models while, for OTMs it holds under the additional assumption that, for all $x \in {}^{\omega}2$, the set of random reals over $L[x]$ has measure 1. Thus, these machine models share the intuitive property of Turing machines that no information can be extracted from random information.

Question 55 Suppose that $\alpha \leq \beta < \omega_1$ are admissible. Are there analogous results for (α, β)-TMs with tape length α and running time bounded by β?

Analogous results fail for other natural notions of largeness even in the computable setting, for example, for Sacks measurability. For any $x \in {}^{\omega}2$, there is a perfect tree $T \subseteq {}^{<\omega}2$ such that x is computable from every branch $y \in [T]$. Moreover, $[T]$ is Sacks measurable and not Sacks null (see [9, Definition 2.6]).

Question 56 Suppose that A is Borel and ${}^{\omega}2 \setminus A$ is Sacks null. If $x \in {}^{\omega}2$ is computable from every $y \in A$, is x computable?

Various machine types correspond in a natural manner to variants of Martin-Löf randomness. A fascinating subject is how far the analogy goes in each case. In particular, for which machine types is it true that if x is computable from two mutually Martin-Löf random reals y and z, then x must be computable? We are pursuing this in ongoing work.

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