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Universal Structures

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Abstract We deal with the existence of universal members in a given cardinality for several classes. First, we deal with classes of abelian groups, specifically with the existence of universal members in cardinalities which are strong limit singular of countable cofinality or $\lambda = \lambda^{\aleph_0}$. We use versions of being reduced—replacing $\mathbb{Q}$ by a subring (defined by a sequence $\bar{t}$)—and get quite accurate results for the existence of universals in a cardinal, for embeddings and for pure embeddings. Second, we deal with (variants of) the *oak property* (from a work of Džamonja and the author), a property of complete first-order theories sufficient for the nonexistence of universal models under suitable cardinal assumptions. Third, we prove that the oak property holds for the class of groups (naturally interpreted, so for quantifier-free formulas) and deals more with the existence of universals.

0 Introduction

On the existence of universal structures, see Kojman and Shelah [6] and the history therein, and a more recent survey by Džamonja [1]. Of course, a complete first-order theory T has a universal model in λ for "elementary embeddings" when $\lambda = 2^{<\lambda} > |T|$; this is true also for similar classes, that is, for abstract elementary classes (AECs) with amalgamation, the joint embedding property (JEP), and Löwenstein–Skolem–Tarski (LST) number $< \lambda$. The question we are interested in is whether there are additional cases (mainly for elementary classes and more generally for AECs as above). But here we deal with some specific classes and the notion of embeddability.

The article is organized as follows. Section 1 deals mainly with abelian groups; it continues the work of Kojman and Shelah [7] and Shelah [15], [16], [18]. Section 2 deals with the class of groups; it continues the work of Shelah and Usvyatsov [19]

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but does not rely on it. Section 3 deals with the oak property, continuing the work of Džamonja and Shelah [3], dealing with the case of singular cardinals.

Section 2 deals with the class of all groups, certainly an important one. Is this class complicated? Under several yardsticks it certainly is: its first-order theory is undecidable, and it has the quantifier-free order property (even the class of (universal) locally finite groups has this property; see Macintyre and Shelah [9]), and by [19] it has the SOP_3 (3-strong order property). But this does not exclude positive answers for other interpretations. By [19] it has the NSOP_4 (4-strong nonorder property); however, we do not know much about this family of classes (though we have hopes).

A recent relevant work is [21], where we give new sufficient conditions for "no universal," in particular, for groups.

Here we consider the oak property (a relative of the tree property, hence the name) from Džamonja and Shelah [3]. We prove that the class of groups has the oak property; hence, it follows that in some cardinals it has no universal member.

There is reasonable evidence for the class of linear orders being complicated, practically maximal for the universal spectrum problem (see [6]).

So a specific conclusion is the following.

Conclusion 0.1 *(1) The class of groups has the oak property (see Definition 2.1).*

(2) If λ satisfies, for example, $(*)$ below, then there is no universal group of cardinality λ when:

$(*)$ *(a)* $\kappa = \mathrm{cf}(\mu) < \mu$,

(b) $\lambda = \mu^{++} < \mathrm{pp}_{J^{\mathrm{bd}}_{\kappa}}(\mu)$,

(c) $\alpha < \mu \Rightarrow |\alpha|^{\kappa} < \mu$.

Proof (1) This is by Claim 2.2.

(2) This is by part (1) and [3], more exactly by Claim 3.1. □

In Section 3 we deal with the oak property per se, continuing [3], showing the nonexistence of universals in singular cardinals and dealing with a weaker relative, the weak oak property.

Concerning the first section, note that a strong limit singular cardinal λ is a case where it is easier to have a universal model, particularly when λ has cofinality $\aleph_0$.

So the canonical case seems to be $\beth_\omega$. Examples of such positive (i.e., existence) results are

(a) Shelah [10, Theorem 3.1, p. 266], where it was proved that:

if λ is a strong limit singular cardinal, then $\{G : G$ a graph with $\leq \lambda$ nodes each of valency $< \lambda\}$ has a universal member under embedding onto induced subgraphs;

(b) Grossberg and Shelah [5, Corollary 27, pp. 301, 302]:

(α) if λ is "large enough," then similar results hold for quite general classes (e.g., locally finite groups) where large enough means: λ (is a strong limit of cofinality $\aleph_0$ and) is above a compact cardinal (which is quite large); more specifically,

(β) if μ is a strong limit of cofinality $\aleph_0$ above a compact cardinal κ and, for example, if the class $\mathfrak{K}$ is the class of models of $T \subseteq \mathbb{L}_{\kappa,\aleph_0}$, $|T| < \mu$, partially ordered by $\prec_{\mathbb{L}_{\kappa,\omega}}$, then we can split $\mathfrak{K}$ into $\leq 2^{|T|+\kappa}$ classes, each having a universal model of cardinality μ under $\prec_{\mathbb{L}_{\kappa,\aleph_0}}$-embeddings.

See more in Shelah [25, Section 3] generalizing the so-called *special models*. Claim 1.16 below continues this; that is, it deals with a strong limit cardinal $\mu > \mathrm{cf}(\mu) = \aleph_0$, compared with [5] omitting the set-theoretic assumption on a compact cardinal at the expense of strengthening the model-theoretic assumption.

There are natural examples where this can be applied; for example, the class of torsion-free abelian groups G which are reduced (i.e., we cannot embed the rational into G), but the order is $G_1 \leq_{\langle n! : n<\omega \rangle} G_2$, which means that $G_1 \subseteq G_2$ but G_1 is closed inside G_2 under the $\mathbb{Z}$-adic metric, and so also G_2/G_1 is reduced. The application of Claim 1.16 to such classes is in Fact 1.14(1)(2). Earlier in Claim 1.2 we prove related positive results for the easier cases of complete members (for λ satisfying $\lambda = \lambda^{\aleph_0}$ or λ the limit of such cardinals).

We also get some negative results, that is, the nonexistence of universal members in Claim 1.7(2), 1.11. We deal more generally with $K^{\mathrm{rtf}}_{\bar{t}}$, the reduced torsion-free abelian group G such that for no $x \in G, x \neq 0$ and x is divisible by $t_{<n} = \prod_{\ell<n} t_\ell$ for every n. We sort out the existence of universal members of cardinality $\lambda = \lambda^{\aleph_0}$ for $K^{\mathrm{rtf}}_{\bar{t},\lambda}$ under embeddings and under pure embeddings, getting complete (but different) answers for $\lambda = \lambda^{\aleph_0}$.

Recall that classes of abelian groups are related to the classes of trees with $\omega + 1$ levels. The parallel of "abelian groups under pure embedding" is the case of such trees; in fact, the nonexistence of universals for abelian groups under pure embedding implies the nonexistence of such universal trees.

Notation 0.2 (1) For a set A, $|A|$ is its cardinality, but for a structure M its cardinality is $\|M\|$ while its universe is $|M|$; this applies, for example, to groups.

(2) We have that $\bar{t}$ will denote an ω-sequence of natural numbers ≥ 2.

(3) We use G, H for groups and use M, N for general models.

(4) Let $\mathfrak{k}$ denote a pair $(K_{\mathfrak{k}}, \leq_{\mathfrak{k}})$; we may say a class $\mathfrak{k}$, where:
 - (a) $K_{\mathfrak{k}}$ is a class of $\tau_{\mathfrak{k}}$-structures;
 - (b) $\leq_{\mathfrak{k}}$ is a partial order on $K_{\mathfrak{k}}$ such that $M \leq_{\mathfrak{k}} N \Rightarrow M \subseteq N$;
 - (c) both $K_{\mathfrak{k}}$ and $\leq_{\mathfrak{k}}$ are closed under isomorphisms.

(4A) We say that $f : M \to N$ is a $\leq_{\mathfrak{k}}$*-embedding* when f is an isomorphism from M onto some $M_1 \leq_{\mathfrak{k}} N$.

(5) If T is a first-order theory, then Mod_T is the pair $(\mathrm{mod}_T, \leq_T)$ where mod_T is the class of models of T and where $\leq_T$ is $\prec$ if T is complete, and $\subseteq$ if T is not complete.

(6) We may write T instead of Mod_T, for example, as in Definition 0.3 below.

Definition 0.3 (1) For a class $\mathfrak{k}$ and a cardinal λ, a set $\{M_i : i < i^*\}$ of models from $\mathfrak{k}$ is *jointly universal* when, for every $N \in K_{\mathfrak{k}}$ of size λ, there is an $i < i^*$ and an $\leq_{\mathfrak{k}}$-embedding of N into M_i.

(2) For $\mathfrak{k}$ and λ as above, let (if $\mu = \lambda$ we may omit μ)

$$\mathrm{univ}(\mathfrak{k}, \mu, \lambda) := \min\big\{|\mathcal{M}| : \mathcal{M} \text{ is a family of members of } K_{\mathfrak{k}} \text{ each of cardinality } \leq \mu \text{ which is jointly universal for models of } \mathfrak{k} \text{ of size } \lambda\big\}.$$

Remark 0.4 To help understand Definition 0.3, note that $\mathrm{univ}(T, \lambda) = 1$ if and only if there is a universal model of T of size λ. Note that some of the classes we consider are not AEC. Some have "weak failure," say, $\mathbb{Z}$-adically complete torsion-free abelian groups, where if $M_n \leq M_{n+1}$, then $\bigcup_n M_n$ is not necessarily complete. We can take a completion more seriously for some $\mathfrak{k}$, and there are contradictory completions.

Recall the following.

Definition 0.5 For an ideal J on a set A and a set B, let $\mathbf{U}_J(B) = \mathrm{Min}\{|\mathscr{P}| : \mathscr{P}$ be a family of subsets of B, each of cardinality $\leq |A|$, such that for every function f from A into B for some $u \in \mathscr{P}$ we have $\{a \in A : f(a) \in u\} \in J^+\}$. Clearly, only $|B|$ matters, so we normally write $\mathbf{U}_J(\lambda)$ (see Shelah [17]).

1 More on Abelian Groups

Earlier versions of this section were originally part of [18] and [16], but as the papers were too long, it was delayed until now.

Remark 1.1 Despite all the cases dealt with in [16], there are still some "missing" cardinals (see the discussion in [18, Section 0]). Concerning λ singular satisfying $2^{\aleph_0} < \mu^+ < \lambda < \mu^{\aleph_0}$, clearly [18, 2.8], [14], and [13] indicate that, at least for most such cardinals, there is no universal: if $\chi \in (\mu^+, \lambda)$ is regular, then $\mathrm{cov}(\lambda, \chi^+, \chi^+, \chi) < \mu^{\aleph_0}$.

Let us mention positive results concerning [18, Section 0, Case 1] (see Definition 1.3 below).

Claim 1.2 *(1) If $\lambda = \lambda^{\aleph_0}$, then in the class $(K^{\mathrm{rtf}}_\lambda, \leq_{\mathrm{pr}})$, defined in Definition 1.3(5) below, there is a universal member; in fact, it is homogeneous universal.*

(2) If $\lambda = \sum_{n<\omega} \lambda_n$ and $\aleph_0 \leq \lambda_n = (\lambda_n)^{\aleph_0} < \lambda_{n+1}$, then in $(\mathfrak{K}^{\mathrm{rtf}}_\lambda, \leq_{\mathrm{pr}})$ there is a universal member (the parallel of special models for first-order theories). (See Fuchs [4] on such abelian groups.)

(3) $(K^{\mathrm{rtf}}, \leq_{\mathrm{pr}})$ has the amalgamation, and JEP is an AEC (see Shelah [20]) and is stable in λ if $\lambda = \lambda^{\aleph_0}$.

We will prove Claim 1.2 below, but first we give the following definition.

Definition 1.3 (1) K^{tf}_λ is the class of torsion-free abelian groups of cardinality λ. Let $K^{\mathrm{tf}} = \bigcup\{K^{\mathrm{tf}}_\lambda : \lambda$ a cardinal$\}$, and similarly for $K^{\mathrm{tf}}_{\leq\lambda}$.

(1A) $K^{\mathrm{rtf}}_{\bar{t},\lambda}$ is the class of $G \in K^{\mathrm{tf}}_\lambda$ such that there is no $x \in G\backslash\{0\}$ divisible by $\prod_{\ell<k} t_\ell$ for every $k < \omega$, recalling Notation 0.2(2).

(1B) Let $K^{\mathrm{rtf}}_{\bar{t}} = \bigcup\{K^{\mathrm{rtf}}_{\bar{t},\lambda} : \lambda$ a cardinal$\}$.

(1C) $G \in K^{\mathrm{rtf}}_{\bar{t}}$ is called *$\bar{t}$-complete* when every Cauchy sequence under $d_{\bar{t}}$ in G has a limit, where $d_{\bar{t}}$ is defined in Definition 1.3(3) below.

(2) We have the following.

(a) Let $\mathfrak{T} = \{\bar{t} : \bar{t} = \langle t_n : n < \omega\rangle, 2 \leq t_n \in \mathbb{N}\}$.
(b) We call $\bar{t} \in \mathfrak{T}$ *full* when

$$(\forall k \geq 2)(\exists n)\Big[k \text{ divides } \prod_{\ell<n} t_\ell\Big];$$

equivalently $(\forall n)(\exists m)[m > n \wedge n \mid \prod_{\ell=n}^m t_\ell]$, and equivalently, every prime p divides infinitely many t_n's.

(c) We call $\bar{t} \in \mathfrak{T}$ *explicitly weakly full* when, for every prime p, either p divides no t_n or it divides infinitely many t_n's.
(d) We say that G is $\bar{t}$*-divisible* when every $x \in G$ is divisible by $\prod_{\ell<n} t_\ell$ for every n.
(e) We call $\bar{t} \in \mathfrak{T}$ *weakly full* when, for some $n(*)$, the sequence $\langle t_{n(*)+n} : n < \omega\rangle$ is explicitly weakly full.

(3) For $G \in K^{\mathrm{rtf}}_{\bar{t},\lambda}$, let $G^{[\bar{t}]}$ be the $d_{\bar{t}}$-completion of G, where $d_{\bar{t}} = d_{\bar{t}}[G]$ is the metric defined by $d_{\bar{t}}(x, y) = \inf\{2^{-k} : \prod_{\ell<k} t_\ell$ divides $x - y$ in the abelian group $G\}$, justified by Observation 1.4(3); pedantically, "the d_t-completion" is determined only up to isomorphism over G.

(4) Let $K^{\mathrm{crtf}}_{\bar{t},\lambda}$ be the class of $G \in K^{\mathrm{rtf}}_{\bar{t},\lambda}$ which are $\bar{t}$-complete (i.e., $G = G^{[\bar{t}]}$).

(5) For those classes, $\leq$ means being a subgroup and $\leq_{\mathrm{pr}}$ means being a pure subgroup.

(6) We say that $\bar{t}, \bar{s} \in \mathfrak{T}$ are *equivalent* when $K^{\mathrm{rtf}}_{\bar{t}} = K^{\mathrm{rtf}}_{\bar{s}}$.

Observation 1.4 *(1) We have that $\bar{t}$ is full if and only if $\bar{t}$ is equivalent to $\langle n! : n \in \mathbb{N}\rangle$ if and only if, for every power of prime m and for some n, m divides $\prod_{\ell<n} t_\ell$.*
(2) If $\bar{t}$ is full, then every $G \in K^{\mathrm{tf}}$ can be represented (in fact, uniquely) as the direct sum $G_1 + G_2$, where G_1 is divisible, and $G_2 \in K^{\mathrm{rtf}}_{\bar{t}}$.
(3) For $G \in K^{\mathrm{rtf}}_{\bar{t}}$, $d_{\bar{t}}$ is a metric on G.
*(4) If $G \in K^{\mathrm{rtf}}_{\bar{t}}$, then there is G', called the $\bar{t}$-*completion of G*, such that*
(a) $G \leq_{\mathrm{pr}} G' \in K^{\mathrm{rtf}}_{\bar{t}}$;
(b) G' is $\bar{t}$-complete;
(c) G is dense in G' by the metric $d_{\bar{t}}$;
(d) if G'' satisfies (a), (b), (c), then G'', G' are isomorphic over G.
(5) $\bar{t}, \bar{s} \in \mathfrak{T}$ are equivalent when, for some k, ℓ, we have
- *$t_{k+n} = t_{\ell+n}$ for every n,*
- *for some m_*, for every $m \geq m_*$ there is n such that $\prod_{i=m_*}^{m} t_{k+i}$ divides $\prod_{i<n} s_{\ell+i}$ and $\prod_{\ell<m} s_{\ell+i}$ divides $\prod_{i<n} t_{\ell+i}$.*

(6) Being full and being weakly full are preserved by equivalence.

Proof The proof should be clear. □

Proof of Claim 1.2 Let $t_n = n!$, and let $\bar{t} = \langle t_n : n < \omega\rangle$.

The point is that, clearly,

(a) (α) for $G \in K^{\mathrm{rtf}}_{\bar{t}}$, $G \leq_{\mathrm{pr}} G^{[\bar{t}]} \in K^{\mathrm{rtf}}_{\bar{t}}$ and $G^{[\bar{t}]}$ has cardinality $\leq \|G\|^{\aleph_0}$ and $G^{[\bar{t}]}$ is $d_{\bar{t}}$-complete; recall that $G^{[\bar{t}]}$ is the $d_{\bar{t}}$-completion of G and that it is unique up to isomorphism over G;
(β) if $G_1 \leq_{\mathrm{pr}} G_2$, then $G_1^{[\bar{t}]} \leq_{\mathrm{pr}} G_2^{[\bar{t}]}$; more pedantically, if $G_1 \leq_{\mathrm{pr}} G_2 \leq_{\mathrm{pr}} G_3$ and G_3 is $\bar{t}$-complete, then $G_1^{[\bar{t}]}$ can be (purely) embedded into G_3 over G_1.

Recall that $K^{\mathrm{crtf}}_{\bar{t}}$ is the class of $d_{\bar{t}}$-complete $G \in K^{\mathrm{rtf}}_{\bar{t}}$.

We have easily that

(b) $(K^{\mathrm{crtf}}_{\bar{t}}, \leq_{\mathrm{pr}})$ has amalgamation, the JEP, and the LST property down to λ for any $\lambda = \lambda^{\aleph_0}$;

(c) if $G' \leq_{\mathrm{pr}} G''$ are from K^{ctrf}, then we can find a $\leq_{\mathrm{pr}}$-increasing sequence $\langle G_\alpha : \alpha \leq \alpha(*)\rangle$ of members of K^{crtf} such that
  (α) $G' = G_0, G'' = G_{\alpha(*)}$,
  (β) $x_\alpha \in G_{\alpha+1} \backslash G_\alpha$,
  (γ) $G_{\alpha+1}$ is the $\bar{t}$-completion of the pure closure of $G_\alpha \oplus \mathbb{Z}x_\alpha$ inside,
  (δ) for α limit, G_α is the $\bar{t}$-completion of $\bigcup\{G_\beta : \beta < \alpha\}$ inside G''; note that if $\mathrm{cf}(\alpha) > \aleph_0$, then the union is $\bar{t}$-complete;

(d) if $\lambda = \lambda^{\aleph_0}$, then for each $G \in K^{\mathrm{crtf}}_{\bar{t},\leq\lambda}$, we can find $\langle (G_i, x_i) : i < \lambda^{\aleph_0}\rangle$ such that
  (α) $G_0 = G, G_i$ is $\leq_{\mathrm{pr}}$-increasing continuous,
  (β) $x_i \in G_{i+1} \in K^{\mathrm{crtf}}_{\bar{t},\lambda}$,
  (γ) letting G'_i be the pure closure of $G + \mathbb{Z}x_i$ inside $G_* = \bigcup\{G_j : j < \lambda^{\aleph_0}\}$, we have $G_{i+1} = G_i \oplus_G G'_i$,
  (δ) if $G \leq_{\mathrm{pr}} G', x \in G' \in K^{\mathrm{crtf}}_{\bar{t},\lambda}$ and G' is the $\bar{t}$-completion of the pure closure of $G + \mathbb{Z}x$ inside G', then we can find $i < \lambda^{\aleph_0}$ and a pure embedding h of G' into $G_{i+1}, h \restriction G =$ the identity $h(x) = x_i$ (so $h''(G_i) \leq_{\mathrm{pr}} G$); in fact, h is onto G'_i;

(e) if λ, G are as in clause (d), then we can find $G_* = \bigcup\{G_i : i < \lambda^{\aleph_0}\}$ such that
  (α) $G \leq_{\mathrm{pr}} G_* \in K^{\mathrm{rtf}}_{\lambda^{\aleph_0}}$,
  (β) if $G \leq_{\mathrm{pr}} G' \in K^{\mathrm{rtf}}_{\lambda^{\aleph_0}}$, then G' can be purely embedded into G_* over G,
  (γ) $\langle G_i : i < \lambda^{\aleph_0}\rangle$ is a $\leq_{\mathrm{pr}}$-increasing continuous sequence of members of $K^{\mathrm{rtf}}_{\lambda^{\aleph_0}}$ and $G_0 = G$;

(f) if, for $i = 1, 2, G_\ell \in K^{\mathrm{ctrf}}_{\bar{t},\lambda}$, and $\langle G^\ell_i : i < \lambda^{\aleph_0}\rangle, G^\ell_*$ are as in clause (d) or as in clause (e), and π is an isomorphism from G_1 onto G_2, then there is an isomorphism π^+ from G^1_* onto G^2_* extending π;

(g) if $\lambda = \sum\{\lambda_n : n < \omega\}, \lambda_n = \lambda_n^{\aleph_0} < \lambda_{n+1}$, and $G \in K^{\mathrm{rtf}}_{\leq\lambda}$, then we can find G', G'_n such that
  (α) $G \leq_{\mathrm{pr}} G' \in K^{\mathrm{rtf}}_\lambda$,
  (β) $G'_n \in K^{\mathrm{crtf}}_{\lambda_n}$,
  (γ) $G'_n \leq_{\mathrm{pr}} G'_{n+1}$; moreover, there is $\langle G'_{n,i}, x'_{n,i} : i < \lambda_n^{\aleph_0}\rangle$ as in (d) for G'_n such that $G'_{n+1} = \bigcup\{G'_{n,i} : i < \lambda_n^{\aleph_0}\}$,
  (δ) $G' = \bigcup\{G'_n : n < \omega\}$;

(h) with λ, λ_n as in (g), if G', G'' are as G' is in (g), then G', G'' are isomorphic;

(i) moreover, if λ, λ_n are as in clause (g) and $H \in K^{\mathrm{rtf}}_{\leq\lambda}$, then H can be purely embedded into G' (and if $H \supseteq G$, then even embedded over G).

The results now follow. □

In Claim 1.7(2) below we prove there is no universal in $\lambda = \lambda^{\aleph_0}$, using Shelah [23, Theorem 1.1]. (For the reader's convenience, we quote the special case used.)

Fact 1.5 For any λ and X, a set of cardinality $\leq \lambda$ or just $\leq \lambda^{\aleph_0}$, we can find a sequence $\bar{f} = \langle f_\eta : \eta \in {}^\omega\lambda\rangle$ such that

(a) f_η is a function from $\{\eta \restriction n : n < \omega\}$ into X;
(b) if f is a function from ${}^{\omega>}\lambda$ to X, then for some $\eta \in {}^\omega\lambda$, we have $f_\eta \subseteq f$.

Remark 1.6 (1) Concerning Fact 1.5, see Shelah [23, Fact 1.5].
(2) We use Fact 1.5 mainly for $\lambda = \lambda^{\aleph_0}$.

Claim 1.7 *Assume that $\bar{t} \in \mathfrak{T}$ is not full.*
(1) We have that $(K^{\text{rtf}}_{\bar{t}}, \leq_{\text{pr}})$ fails amalgamation.
(2) If $\lambda = \lambda^{\aleph_0}$, then in $(K^{\text{rtf}}_{\bar{t},\lambda}, \leq_{\text{pr}})$ there is no universal member, even for the $\aleph_1$-free ones.

Remark 1.8 Note that Claims 1.2 and 1.7(2) are not contradictory, as the former deals with full $\bar{t}$'s and the latter with nonfull ones.

Proof of Claim 1.7 Let p be a prime that witnesses that $\bar{t}$ is not full; that is, n_* is well defined, where $n_* = \min\{n : p$ divide no t_m with $m \geq n\}$, by Observation 1.4(5) without loss of generality $n_* = 0$.

Let $t_{<n} := \prod_{\ell<n} t_\ell$, so $t_{<0} = 1$.

We now choose a^1_n, a^0_n by induction on n such that

$(*)_1$ (a) $a^1_n, a^0_n \in \mathbb{Z}$,
(b) $a^1_n = a^0_n \mod t_{<n}$,
(c) $a^\ell_n = a^\ell_m \mod t_{<m}$ if $n = m + 1$,
(d) $a^1_n \neq a^0_n \mod p$ if $n = 0$.

[Why can we choose? For $n = 0$, clearly $t_{<0} = 1$; hence, $a^1_n = 1, a^2_n = 2$ are as required.

For $n = m + 1$, the proof is easy because p does not divide $t_{\leq n}$.]

Choose

$(*)_2$ (a) t'_n is pt_n if $n = 0$ and is t_n if $n > 0$,
(b) $t'_{<n} = \prod_{k<n} t'_k$ and $t'_{<(n+1)} = t'_{<(n+1)}$,
(c) $c^\ell_n \in \mathbb{Z}$ are chosen such that $\sum_{m\leq n}(t'_{<n}/t'_{<m})c^\ell_m = a^\ell_n$.

[Why can we choose? Just choose c^ℓ_n by induction on n.]

For every $S \subseteq {}^\omega\lambda$, we let G_S be the abelian group generated by

$$\{x_\eta : \eta \in {}^{\omega>}\lambda\} \cup \{y_{\eta,n} : \eta \in {}^\omega\lambda \text{ and } n < \omega\}$$

freely except the equations:

$(*)_3$ $t'_n y_{\eta,n+1} = y_{\eta,n} - c^\ell_n x_{\langle\rangle} \times x_{\eta\restriction n}$ $\underline{\text{if}}$ $n < \omega$ and $\eta \in S \Rightarrow \ell = 1$ and $\eta \notin S \Rightarrow \ell = 0$.

We have the following:

$(*)_4$ for $n \in {}^\omega\lambda$, let
(a) $G_\eta = \sum\{\mathbb{Z}x_{\eta\restriction n} : n < \omega\} \subseteq G_S$,
(b) $G_{S,\eta} = \sum\{\mathbb{Z}x_{\eta\restriction n} : n < \omega\} + \sum\{\mathbb{Z}y_{\eta,n} : n < \omega\} \subseteq G_S$.

We have easily that

$(*)_5$ if $S \subseteq {}^\omega\lambda$, then
(a) $G_S \in K^{\text{rtf}}_{\bar{t},\lambda^{\aleph_0}}$,
(b) $\eta \in {}^\omega\lambda \Rightarrow G_\eta \leq_{\text{pr}} G_{S,n} \leq_{\text{pr}} G_S$.

Now

$\boxplus$ if $S_0, S_1 \subseteq {}^\omega\lambda, \eta \in S_1\backslash S_0$, then G_{S_0}, G_S and even $G_{S_0,\eta}, G_{S_1,\eta}$ cannot be $\leq_{\text{pr}}$-amalgamated over G_η.

[Why? Toward a contradiction, assume that $G_\eta \leq_{\rm pr} H \in K^{\rm rtf}_{\bar t}$ and that π_ℓ is a pure embedding of G_{S_ℓ} into H over G_η, for $\ell = 0, 1$.]

Let $z_n = \pi_1(y_{\eta,n}) - \pi_0(y_{\eta,n})$ for any $n, \pi = \pi_0 \restriction G_\eta = \pi_1 \restriction G_n$.

For any n, clearly for $\ell = 1, 2$, we have

- $G_\ell \models t'_{\leq n} y'_{\eta,n+1} = y_{\eta,0} - (\sum_{m\leq n}(t'_{<n}/t'_{<m})c^\ell_m)x_{\langle\cdot\rangle} + \sum_{m\leq n}(t'_{<n}/t'_{<m})x_{\eta\restriction m}$.

So applying π_ℓ on the equation recalling $(*)_2(c)$, we have

- $H \models \pi_\ell(t'_{<n} y_{\eta,n}) = \pi_\ell(y_{\eta,0}) - a^\ell_n \pi(x_{\langle\cdot\rangle}) + \sum_{m\leq n}(t'_{<n}/t'_{<m})\pi(x_{\eta\restriction n})$.

Subtracting the equation recalling the choice of z_0, z_n, we have

- $H \models t'_{<n} z_n = z_0 - (a^1_n - a^0_n)\pi(x_{\langle\cdot\rangle})$.

But $t'_{<n}$ and $a^1_n - a^0_n$ are divisible by $t_{<n}$ in $\mathbb{Z}$ (by $(*)_2$(a), $(*)_2$(b), and $(*)_1$(c), respectively); hence,

- z_0 is divisible by $t_{<n}$ in H.

As this holds for every n and $H \in K^{\rm rtf}_{\bar t}$, we get

- $z_0 = 0$.

So as $H \models t'_0 z_1 = z_0 - (a^1_n - a^0_n)\pi(x_{\langle\cdot\rangle})$ and $n = 1$ we get $H \models t'_0 z_1 = z_0 - (a^1_0 - a^0_0)\pi(x_{\langle\cdot\rangle})$, but in $\mathbb{Z}$ we have $p \mid t'_0$ and $p \dagger (a^1_0 - a^1_1)$ and $z_0 = 0$ so p divides $x_{\langle\cdot\rangle}$ in H, which is a contradiction to purity.

This is enough for part (1); for part (2) we apply the simple black box of [23, Theorem 1.1], that is, Fact 1.5. In detail, assume that $G_* \in K^{\rm rtf}_\lambda$, and let $\bar f = \langle f_\eta : \eta \in {}^\omega\lambda\rangle$ be as in Fact 1.5 for $X = G_*$.

Define S as the set of $\eta \in {}^\omega\lambda$ such that

- there is a pure embedding g of $G_{\{\eta\},\eta}$ into G_* such that
$$n < \omega \quad \Rightarrow \quad g(x_{\eta\restriction m}) = f_\eta(\eta\restriction n),$$
and there are no $y_n \in G_*$ for $n \geq n_*$ such that
$$G_* \models \text{“}t'_n y_{n+1} = y_n - c_n f_\eta(x_{\langle\cdot\rangle}) + f_\eta(\eta\restriction n).\text{”}$$

Now $G_S \in K^{\rm rtf}_{\bar t,\lambda}$, so it is enough to prove that G_S is not purely embeddable into G_*. Toward a contradiction, assume that g is a pure embedding of G_S into G_*, and let $f : {}^{\omega>}\lambda \to X = G_*$ be $f(\eta) = g(x_\eta)$. By the choice of $\bar f$, there is $\eta \in {}^\omega\lambda$ such that $f_\eta \subseteq f$. If $\eta \in S$, then $\langle g(x_{\eta\restriction n}) : n < \omega\rangle = \langle f_\eta(x_{\eta\restriction n}) : n < \omega\rangle$ witness that $\eta \notin S$ by the definition of S.

So necessarily $\eta \in {}^\omega\lambda \backslash S$; hence, there is g_* as forbidden in the definition of S. Let $g_0 = g \restriction G_{S,\eta}$. This easily contradicts $\boxplus$. □

Remark 1.9 (1) See more in Shelah [12, Chapter II, Section 3] and [27].

(2) This holds also for $K^{{\rm rs}(p)}_\lambda$, the class of reduced separable abelian p-groups (see Definition 1.15).

We may wonder: what if we ask about $(K^{\rm rtf}_{\bar t,\lambda}, \leq)$, that is, the embedding is not necessarily pure.

Claim 1.10 *Assume that $\bar t \in \mathfrak{T}$ is weakly full, so for some n_* we have: if a prime p divides some $t_n, n \geq n_*$, then it divides infinitely many t_n's; call this* set of primes $\mathbf{P}$.

(1) If $\lambda = \lambda^{\aleph_0}$, then $(K^{\rm rtf}_{\bar t,\lambda}, \leq)$ has a universal member.

(2) If $\lambda = \sum_n \lambda_n, \lambda_n = (\lambda_n)^{\aleph_0}$ for every n, then $(K^{\mathrm{rtf}}_{\bar{t},\lambda}, \leq)$ has a universal member.

(3) Let R be the subring of $\mathbb{Q}$ generated by $\{1\} \cup \{1/p : p$ a prime $\notin \mathbf{P}\}$. Then for every $G \in K^{\mathrm{trf}}_{\bar{t},\lambda}$ there is $H \in K^{\mathrm{trf}}_{\bar{t},\lambda}$ extending G which is p-divisible for every prime $p \notin \mathbf{P}$. Hence, H can be considered to be an R-module.

(4) For the class of R-modules into which $\mathbb{Q}_R$ cannot be embedded, the results of (1) and (2) hold, replacing $\aleph_0$ by $|R| + \aleph_0$ when R is an integral domain which is not a field, $\mathbb{Q}_R$, its ring of quotients.

Proof (1), (2) This is by (4) and (3).

(3) This is easy.

(4) The proof is like the proof for full $\bar{t}$'s. $\square$

This still leaves some $\bar{t}$'s open.

Claim 1.11 *Assume that $\bar{t} \in \mathfrak{T}$ is not weakly full; hence, $\mathbf{P} := \{p : p$ a prime dividing some t_n's but only finitely many$\}$ is infinite. (This is the negation of the conditions from Claim 1.10). If $\lambda = \lambda^{\aleph_0}$, then $(K^{\mathrm{rtf}}_{\bar{t},\lambda}, \leq)$ has no universal member.*

Proof By Observation 1.4(5), *without loss of generality*,

$(*)_1$ (a) there are distinct primes p_n such that $p_k \mid t_n$ iff $k = n$,
(b) $(p_k)^{\iota(k)}$ divides t_k but $(p_k)^{\iota(k)+1}$ does not, so $\iota(k) \geq 1$.

Let $t_{<n} = \prod_{\ell<n} t_\ell$, so $t_{<0} = 1$, and let $t'_n = t_n p_n^{\ell(n)}$, $t'_{<n} = \prod_{\ell<n} t'_\ell$, $t''_n = p_n^{\ell(n)}$, and $t''_{<n} = \prod_{\ell<n} t''_\ell$. Let $(t_{\leq n}, t'_{\leq n}, t''_{\leq n}) = (t_{<(n+1)}, t'_{<(n+1)}, t''_{<(n+1)})$.

We now choose $a^1_n, a^0_n \in \mathbb{Z}$ by induction on n such that

$(*)_2$ (a) $a^1_n, a^0_n \in \mathbb{Z}$,
(b) $a^1_n = a^0_n \mod t'_{<m}$,
(c) $a^\ell_n = a^\ell_m \mod t'_{<m}$,
(d) if $k < n$, then $a^1_n \neq a^0_n \mod (p_k)^{\ell(k)+1}$.

[Why is this possible? First, for $n = 0$, let $(a^1_n, a^0_n) = (p_0, t_0, t_0)$, so $a^1_n - a^0_n$ is divisible by t_0 but not by $p_n^{\ell(n)+1}$. Second, assume $n = m + 1$ and (a^1_m, a^0_m) have been chosen. As $t_{\leq m}/t_{\leq n}$ and $k \leq m \Rightarrow p_k \pm (t_{\leq n}/t_{\leq m})$, we can find (b^1_m, b^0_m) such that $b^\ell_m = a^\ell_m \mod t^*_{\leq m}$ for $\ell = 0, 1$ and $b^1_m = b^0_m \mod t^*_{\leq n}$. Clearly requirements (a), (b), (c) hold and (d) holds for $k < m$. Let $(a^1_n, a^0_n) = (a^1_n + t^*_{\leq m} \cdot t_n, a^0_n)$; now check.]

$(*)_3$ Choose c^1_n, c^0_n by induction on n such that, for $\ell = 0, 1$, we have

$$\sum_{m \leq n} (t'_{<n}/t'_{<m}) c^\ell_n = a^\ell_n.$$

[Why is this possible? For $n = 0$ trivial for $n + 1$, note that the c^ℓ_{n+1} appear with coefficient 1.]

Next, for every $S \subseteq {}^{\omega>}\lambda$ we choose an abelian group G_S, which is generated by $\{x_\eta : \eta \in {}^{\omega>}\lambda\} \cup \{y_{\eta,n} : \eta \in {}^{\omega}\lambda$ and $n < \omega\} \cup \{x^*_n : n < \omega\}$ freely except the equations:

$(*)_4$ (a) $(t_n/p_n^{\ell(n)}) x^*_{n+1} = x^*_n$ and $x_{\langle\rangle} = x^*_0$,
(b) $t'_n y_{\eta,n+1} = y_{\eta,n} - c^\ell_n x^*_{\langle\cdot\rangle} + x_{\eta \restriction n}$ when $n < \omega, \iota < 2$, and $\ell = 1 \rightarrow \eta \in S$,

$(*)_5$ (a) for $\eta \in {}^{\omega}\lambda$, let $G_\eta = \sum_n \mathbb{Z} x_{\eta \restriction n} + \sum_n \mathbb{Z} x^*_n$,

(b) for $S \subseteq {}^{\omega}\lambda, \eta \in {}^{\omega}\lambda$, let $G_{S,\eta}$ be the following subgroup of G_S: $G_\eta + \sum \mathbb{Z} y_{\eta \restriction n}$.

We have easily that

$(*)_6$ (a) if $S \subseteq {}^{\omega}\lambda$ and $\eta \in {}^{\omega}\lambda$, then $G_\eta, G_{S,n} \in K^{\mathrm{rtf}}_{\bar{t},\lambda^{\aleph_0}}$,

(b) $G_\eta \leq_{\mathrm{pr}} G_{S,\eta} \leq_{\mathrm{pr}} G_S$.

Now,

⊞ if $S_0, S_1 \subseteq {}^{\omega}\lambda$ and $\eta \in S_1 \backslash S_0$, then G_{S_0}, G_{S_1} and even $G_{S_0,\eta}, G_{S_1,\eta}$ cannot be amalgamated over G_η in $(K^{\mathrm{rtf}}_{\bar{t}_*}, \leq)$.

We continue as in the proof of Claim 1.7, getting $\pi_1, \pi_0, \pi, \eta, z_n$ and proving that for every n

- $H \models t'_{<n} z_n = z_0 - (a^1_n - a^0_n)\pi(x_{\langle\cdot\rangle})$.

But $t'_{<n}$ is divisible by t_n, $(a^1_n - a^0_n)$ is divisible by $t''_{<n}$ (in H), and $x_{\langle\cdot\rangle}$ is divisible by t_n/t''_n; hence, $(a^1_n - a^0_n)x_{\langle\cdot\rangle}$ is divisible by $t_{<n}$; hence $z_0 \in t_{<n}H$ for every n. As $H \in K^{\mathrm{rtf}}_{\bar{t}}$, it follows that $z_0 = 0$.

Hence for every n

- $H \models (a^1_n - a^0_n)\pi(x_{\langle\cdot\rangle}) = -t'_{\leq n} z_n$.

Now $p_n^{\ell(n)+\ell(n)}$ divides $t'_{\leq n}$ and $p_n^{\ell(n)+1}$ does not divide $(a^1_{n+1} = a'_{n+1})$ so by $(*)_2$(d), in H, $p_n^{\ell(n)}$ divides $\pi(x_{\langle\cdot\rangle})$. As also each $t'_{\leq n}/\prod_{k\leq n} p_k^{\ell(k)}$ divides it, clearly $\pi(x_{\langle\cdot\rangle})$ contradicts $G_* \in K^{\mathrm{rtf}}_{\bar{t},\lambda}$. □

We may wonder whether the existence result of Claim 1.2 holds for a stronger embeddability notion. A natural candidate is the following.

Definition 1.12 Let $G_0 \leq_{\bar{t}} G_1$. If G_0, G_1 are abelian groups on which $\| - \|_{\bar{t}}$ is a norm, then $G_0 \leq_{\mathrm{pr}} G_1$ and G_0 is a $d_{\bar{t}}$-closed subset of G_1 (but G_ℓ is not necessarily $\bar{t}$-complete!).

Observation 1.13 *(1) We have that $(K^{\mathrm{rtf}}_{\bar{t}}, \leq_{\bar{t}})$ satisfies the axiom of being an AEC except smoothness with LST number $2^{\aleph_0}$.*

(2) If $A \subseteq G \in K^{\mathrm{rtf}}_{\bar{t}}$, then for some $G' \leq_{\bar{t}} G, A \subseteq G', |G'| = (|A| + \aleph_0)^{\aleph_0}$.

(3) If $G_1 \leq_{\bar{t}} G_2$, then $G_1 \leq_{\mathrm{pr}} G_2$.

We prove below that for μ strong limit of cofinality $\aleph_0$, the answer is positive; that is, there is a universal member for $(K^{\mathrm{rtf}}_{\bar{t},\lambda}, \leq_{\bar{t}})$, but for cardinals like $\beth^+_\omega < (\beth_\omega)^{\aleph_0}$ the question on the existence of universals remains open.

Fact 1.14 Assume that λ is strong limit and that $\aleph_0 = \mathrm{cf}(\lambda) < \lambda$.

(1) There is a universal member in $(K^{\mathrm{rtf}}_{\bar{t},\lambda}, <_{\bar{t}})$ where $\bar{t} = \langle t_\ell : \ell < \omega \rangle \in \mathfrak{T}$; hence also the case in $(K^{\mathrm{rtf}}_{\bar{t},\lambda}, \leq_{\mathrm{pr}})$.

(2) For a prime number p, similarly for $(K^{\mathrm{rs}(p)}_{\lambda}, \leq_{\langle p:\ell<\omega\rangle})$, see Definition 1.15 below.

Definition 1.15 For a prime number p and cardinal λ, we let $K^{\mathrm{rs}(p)}_{\lambda}$ be the class of abelian p-groups which are reduced and separable of cardinality λ.

Proof of Fact 1.14 Let K be the class, and let $\leq_*$ be the partial order. Let $\lambda_n < \lambda_{n+1} < \lambda = \sum_n \lambda_n$ and $2^{\lambda_n} < \lambda_{n+1}$. The idea in both cases is to analyze

$M \in K_\lambda$ as the union of increasing chains $\langle M_n : n < \omega\rangle, M_n \prec_{\mathbb{L}_{\lambda_n^+,\lambda_n^+}} M$, $\|M_n\| = 2^{\lambda_n}$.

Specifically, we will apply Claim 1.16 and Conclusion 1.18 below with

$$\mathfrak{K} = K^{\mathrm{rtf}}, \qquad \mu_n = (2^{\lambda_n})^+,$$
$$\leq_1 = \leq_0 \text{ is: } M_1 \leq_1 M_2 \qquad \underline{\text{iff}} \ (M_1, M_2 \in \mathfrak{K} \text{ and}) \ M_1 \leq_* M_2,$$

$\leq_2$ is: $M_1 \leq_2 M_2$ iff $M_1 \leq_1 M_2$ and $M_1 \prec_{\mathbb{L}_{\aleph_1,\aleph_2}} M_2$, or just:

if $G_1 \subseteq M_1, G_1 \subseteq G_2 \subseteq M_2$, and G_2 is countable,
then there is a $\leq_1$-embedding h of G_2 into M_1 over G_1.

We should check the conditions in Claim 1.16, which we postpone.
We will finish the proof after Conclusion 1.18 below. □

Claim 1.16 *Assume the following:*

(a) K is a class of models of a fixed vocabulary closed under isomorphism, and $K_\lambda \neq \emptyset$;
(b) $\lambda = \sum_{n<\omega} \mu_n, \mu_n < \mu_{n+1}, 2^{\mu_n} < \mu_{n+1}, \mu_n$ is regular, and the vocabulary of $\mathfrak{K}$ has cardinality $< \mu_0$;
(c) $\leq_1$ is a partial order on K (so $M \leq_1 M$), preserved under isomorphisms, and if $\langle M_i : i < \delta\rangle$ is $\leq_1$-increasing and continuous, then $M_\delta = \bigcup_{i<\delta} M_i \in \mathfrak{K}$ and $i < \delta \Rightarrow M_i \leq_1 M_\delta$ (so $(K, \leq_1)$ satisfies a quite weak version of an AEC (see [11], [22]));
(d) (α) $\leq_2$ is a two-place relation on K, preserved under isomorphisms;
(β) [weak LST] if $M \in K_\lambda$, then we can find $\langle M_n : n < \omega\rangle$ such that $M_n \in K_{<\mu_n}, M_n <_2 M_{n+1}$, and $M = \bigcup_{M<\omega} M_n$;
(e) [nonsymmetric amalgamation] if $M_0 \in K_{<\mu_n}$, $M_0 \leq_1 M_1 \in K_{<\mu_{n+2}}$, $N^1 \leq_2 N^2 \in K_{<\mu_{n+1}}$, and h^1 is an isomorphism from M_0 onto N^1, then we can find $M_2 \in \mathfrak{K}_{<\mu_{(n+2)}}$ such that $M_1 \leq_1 M_2$ and there is an embedding h^2 of N^2 into M extending h^1 satisfying $h(N^2) \leq_1 M_2$.

Then we can find $\langle M_n^\alpha : n \leq \omega\rangle$ for $\alpha < 2^{<\mu_0}$ such that

(α) $M_n^\alpha \in \mathfrak{K}_{<\mu_n}, M_n^\alpha \leq_1 M_{n+1}^\alpha, M_\omega^\alpha = \bigcup_{n<\omega} M_n^\alpha$;
(β) if $M \in K_\lambda$ and the sequence $\langle M_n : n < \omega\rangle$ is as in clause $(d)(\beta)$, then for some $\alpha < 2^{<\mu_0}$ we can find an embedding h of M into M_ω^α satisfying $h(M_n) \leq_1 M_{n+2}^\alpha$ (if $\mathfrak{K} = (K, \leq_1)$ is an AEC, we get that h is a $\leq_{\mathfrak{K}}$-embedding of M into M_ω^α).
(Of course, we can omit $\langle M_n^\alpha : n \leq \omega\rangle$ when $\|M_\omega^\alpha\| < \lambda$.)

Proof Let

$$\mathfrak{K}_0' = \big\{M : M \in \mathfrak{K} \text{ has universe an ordinal } < \mu_0, \text{ and there is } \langle M_n : n < \omega\rangle$$
$$\text{as in clause (d)}(\beta) \text{ with } M_0 \cong M\big\}.$$

Clearly K_0' has cardinality $\leq 2^{<\mu_0}$, and let us list it as $\langle M_0^\alpha : \alpha < \alpha^*\rangle$ with $\alpha^* \leq 2^{<\mu_0}$. We now choose, for each $\alpha < \alpha^*$, by induction on $n < \omega$, M_n^α such that

(i) $M_n^\alpha \in \mathfrak{K}$ has universe an ordinal $< \mu_n$,
(ii) $M_n^\alpha \leq_1 M_{n+1}^\alpha$,

(iii) if $N^1 \leq_2 N^2, N^1 \in K_{<\mu_n}, N^2 \in K_{<\mu_{n+1}}$ and h^1 is an embedding of N^1 into M^α_{n+1} satisfying $h^1(N^1) \leq_1 M^\alpha_{n+1}$, then we can find h^2, an embedding of N^2 into M^α_{n+2} extending h^1 such that $h^2(N^2) \leq_1 M^\alpha_{n+2}$.

For $n = 0, 1$, we do not have much to do. (If $n = 0$, use M^α_0; if $n = 1$, let $\langle M_n : n < \omega\rangle$ be as in clause (c), let $M_0 \cong M^\alpha_0$, and use M^α_1 such that $(M_1, M_0) \cong (M^\alpha_1, M^\alpha_0)$.) Assume that M^α_{n+1} has been defined, and we will define M^α_{n+2}. Let $\{(h^1_{n,\zeta}, N^1_{n,\zeta}, N^2_{n,\zeta}) : \zeta < \zeta^*_n\}$, where $\zeta^*_n \leq 2^{<\mu_{n+1}}$ lists the cases of clause (iii) that need to be taken care of, with the set of elements of $N^2_{n,\zeta}$ being an ordinal. We choose $\langle N_{n+1,\zeta} : \zeta \leq \zeta^*_n\rangle$, which is $\leq_1$-increasing continuous, satisfying $N_{n+1,\zeta} \in \mathfrak{K}_{<\mu_{n+2}}$. We choose $N_{n+1,\zeta}$ by induction on ζ. Let $N_{n+1,0} = M^\alpha_{n+1}$, for ζ limit let $N_{n+1,\zeta} = \bigcup_{\xi<\zeta} N_{n+1,\xi}$, and use clause (c) of the assumption.

Lastly, for $\zeta = \xi + 1$ use clause (e) of the assumption with $h^1_{n,\zeta}(N^1_{n,\xi})$, $N_{n+1,\xi}$, $N^1_{n,\xi}$, $N^2_{n,\xi}$, $h^1_{n,\xi}$, $N_{n+1,\xi+1}$ here standing for $M_0, M_1, N^1, N^2, h^1, h^2, M_2$ there.

Having carried out the induction on $\zeta \leq \zeta^*_n$, we let $M^\alpha_{n+2} = N_{n+1,\zeta^*_\alpha}$; so we have carried out the induction on n.

Having chosen $\langle\langle M^\alpha_n : n < \omega\rangle : \alpha < 2^{<\mu_0}\rangle$, let $M^\alpha_\omega = \bigcup\{M^\alpha_n : n < \omega\}$. Hence by clause (c) of the assumption, $M^\alpha_\omega \in K$ and $n < \omega \Rightarrow M^\alpha_n \leq_1 M^\alpha_\omega$. Clearly, clause ($\alpha$) of the desired conclusion is satisfied. For clause (β), let $M \in \mathfrak{K}_\lambda$. By clause (d) of the assumption, we can find a sequence $\langle M_n : n < \omega\rangle$ such that $M_n \in \mathfrak{K}_{<\mu_n}, M_n \leq_2 M_{n+1}$, and $M = \bigcup\{M_n : n < \omega\}$. By the choice of $\langle M^\alpha_0 : \alpha < 2^{<\mu_0}\rangle$ there is $\alpha < 2^{<\mu_0}$ such that $M_0 \cong M^\alpha_0$, and let h_0 be an isomorphism from M_0 onto M^α_0. Now by induction on $n < \omega$ we choose h_n, an embedding of M_n into M^α_{n+1} such that $h_n(M_n) \leq_1 M^\alpha_{n+1}$ and $h_n \subseteq h_{n+1}$. For $n = 0$ this has already been done as $h_0(M_0) = M^\alpha_0 \leq_1 M^\alpha_1$. For $n + 1$ we use clause (iii).

Lastly, $h = \bigcup\{h_n : n < \omega\}$ is an embedding of M into M^α_ω, as required. □

Remark 1.17 (1) We can choose $\langle M^\alpha_0 : \alpha < \alpha^*\rangle$ just to represent $\mathfrak{K}_{<\mu_0}$, and similarly later (and so ignore the "with the universe being an ordinal").

(2) Actually, the family of $\langle M_n : n < \omega\rangle$ as in clause (c) such that M_n has set of elements an ordinal, forms a tree T with ω levels with the nth level having $\leq 2^{<\mu_n}$ members, and we can use some amalgamations of it (so weakening the assumptions on $\leq_1$). This gives a variant of Claim 1.16.

(3) We can put into the axiomatization the stronger version of (d) from Claim 1.16 proved in the proof of Fact 1.14 so we can weaken (β) of Conclusion 1.18 below.

(4) That is, in (d) we can add $M_n <_* M$ and so weaken clause (β) of Claim 1.16.

Conclusion 1.18 *(1) In Claim 1.16, we can add $\bigwedge_n \bigwedge_\alpha [M^\alpha_n = M^0_n]$ provided that:*

$(f)^+$ *there is $M_* \in K_{<\lambda}$ such that every $M \in K_{<\mu_0}$ can be $\leq_1$-embeddable into M''_*.*

(2) In Claim 1.16, there is in K_λ a universal member under $\leq_1$-embedding if in addition we add to the assumptions of Claim 1.16:

$(f)^+$ *as in part (1),*

(g) *if $M_n \leq_1 M_{n+1}, M_n \leq_1 N_n, N_n \leq_2 N_{n+1}$, $M_n \in K_{<\mu_{n+2}}$ and $N_n \in K_{<\mu_{n+1}}$ for $n < \omega$, then $\bigcup_{n<\omega} M_n \leq_1 \bigcup_{n<\omega} N_n$.*

Proof The proof is easy. □

Continuation of the proof of Fact 1.14. We have to check the demands in Conclusion 1.18 and Claim 1.16.

The least trivial clause to check is (e).

Clause (e): (nonsymmetric amalgamation). Without loss of generality, $h_1 =$ the identity, $N^1 \cap M_1 = M_0 = N_0$. Just take the free amalgamation $M = N^1 *_{M_0} M_1$ (in the variety of abelian groups), and note that naturally $M_1 \leq_1 M$.

Discussion 1.19 (1) Can we in Claim 1.16 and Conclusion 1.18 replace $\mathrm{cf}(\lambda) = \aleph_0$ by $\mathrm{cf}(\lambda) = \theta > \aleph_0$? If that increasing union of chains in $K_{<\lambda}$ of length $< \theta$ behaves nicely, then yes, with no real problem. More elaborately,

(i) in Claim 1.16(c), we get $\langle M_\varepsilon : \varepsilon < \theta \rangle$ such that $M_\varepsilon \in K_{<\mu_\varepsilon}$, $\langle M_\varepsilon : \varepsilon < \theta \rangle$ is $\subseteq$-increasing continuous, $M_\varepsilon <_2 M_{\varepsilon+1}$, and $M = \bigcup\{M_\varepsilon : \varepsilon < \theta\}$;

(ii) we add: if $\langle M_i : i \leq \delta \rangle$ is $\leq_1$-increasing continuous, $M_i \in K_{<\lambda}$, and $i < \delta \Rightarrow M_i \leq_1 N$, then $M_\delta \leq_i N$.

Otherwise, we seem to be lost.

(2) Suppose $\lambda = \sum_{n<\omega} \lambda_n, \lambda_n = (\lambda_n)^{\aleph_0} < \lambda_{n+1}$, and $\mu < \lambda_0, \lambda < 2^\mu$ (i.e., [18, Section 0, Case 6b]). For $\bar{t} \in \mathfrak{T}$ which is not weakly full, is there a universal member in $(\mathfrak{K}^{\mathrm{rtf}}_{\bar{t},\lambda}, <_{\bar{t}})$?

Assume that $\mathbf{V} \models$ "$\mu = \mu^{<\mu}, \mu < \chi$" and that $\mathbb{P}$ is the forcing notion of adding χ Cohen subsets to μ (i.e., $\mathbb{P} = \{f : f$ a partial function from χ to $2, |\mathrm{Dom}(f)| < \mu\}$ ordered by inclusion). So we have that in $\mathbf{V}^{\mathbb{P}} : \lambda < \lambda^{\aleph_0}$ and $\mu < \lambda < \chi \Rightarrow$ in $(K^{\mathrm{rtf}}_{\bar{t},\lambda}, \subseteq_{\bar{t}})$ there is no universal member. The proof is easy so the answer is consistently no.

Maybe by continuing Shelah [24, Section 2] (see Shelah [26, Chapter III, Section 2]) we can get consistency of the existence.

(3) Now if $\lambda = \lambda^{\aleph_0}$, then in $(K^{\aleph_1\text{-free}}_\lambda, \subseteq)$ there is no universal member (see [23], [26, Chapter IV], [18]) because amalgamation fails badly. Putting together those results, clearly there are few cardinals which are candidates for consistency of existence. In (2), if there is a regular $\lambda' \in (\mu, \lambda)$ with $\mathrm{cov}(\lambda, \lambda^+, \lambda^+, \lambda') < 2^\mu$, then this contradicts Claim 1.2.

(4) Considering consistency of existence of universals in (2), it is natural to try to combine the independent results in [23] (see [26, Chapter IV]) and Džamonja and Shelah [2].

2 The Class of Groups

We know (see [19]) that the class of groups has NSOP_4 and SOP_3 (from [14, Section 2]). We will prove a result on the place of the class of groups in the model-theoretic classification. We know that it falls on "the complicated side" for some division: of course is unstable. Now we prove that it has the oak property (see [3]). This is formally not as well defined as the definition there was for complete first-order theories. But its meaning (and "no universal" consequences) is clear in a more general context (see below). Amenability is a condition on a theory (or class) which gives sufficient condition for the existence of somewhat universal structures, and in suitable models of set theory (see [2]), the class of groups fails it because by [21] it has no universal in λ when $\lambda = \mu^+, \mu = \mu^{<\mu}$, forcing a contradiction in the results on amenable elementary classes in [2].

Definition 2.1 (1) A theory T is said to *satisfy the oak property* as exhibited by (or just by) a formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ when for any λ, κ there are $\bar{b}_\eta$ ($\eta \in {}^{\kappa>}\lambda$), $\bar{c}_\nu$ ($\nu \in {}^\kappa\lambda$), and $\bar{a}_i$ ($i < \kappa$) in some model $\mathfrak{C}$ of T such that

(a) if $\eta \triangleleft \nu$ and $\nu \in {}^\kappa\lambda$, then $\mathfrak{C} \models \varphi[\bar{a}_{\ell g(\eta)}, \bar{b}_\eta, \bar{c}_\nu]$;

(b) if $\eta \in {}^{\kappa>}\lambda$, $\eta^\frown\langle\alpha\rangle \in \nu_1 \in {}^\kappa\lambda$, and $\eta^\frown\langle\beta\rangle \in \nu_2 \in {}^\kappa\lambda$, while $\alpha \neq \beta$ and $i > \ell g(\eta)$, then $\neg\exists\bar{y}[\varphi(\bar{a}_i, \bar{y}, \bar{c}_{\nu_1}) \wedge \varphi(\bar{a}_i, \bar{y}, \bar{c}_{\nu_2})]$;

and in addition φ satisfies

(c) $\varphi(\bar{x}, \bar{y}_1, \bar{z}) \wedge \varphi(\bar{x}, \bar{y}_2, \bar{z})$ implies $\bar{y}_1 = \bar{y}_2$ in any model of T.

(2) A theory T has the Δ-*oak property* if it is exhibited by some $\varphi(\bar{x}, \bar{y}, \bar{z}) \in \Delta$.

Claim 2.2 *The class of groups has the oak property by some quantifier-free formula.*

Remark 2.3 The original proof goes as follows.

Let $w(x, y)$ be a complicated enough word, say, of length $k^* = 100$ (see demands below). For cardinals κ, λ, let $G = G_{\lambda,\kappa}$ be defined as follows.

Let G be the group generated by $\{x_i : i < \kappa\} \cup \{y_\eta : \eta \in {}^{\kappa>}\lambda\} \cup \{z_\nu : \nu \in {}^\kappa\lambda\}$ freely except the set of equations

$$\Gamma = \left\{y_{\nu \restriction i} = w(z_\nu, x_i) : \nu \in {}^\kappa\lambda, i < \kappa\right\}.$$

Clearly, it suffices to show that

$(*)_1$ if $\nu \in {}^\kappa\lambda, i < \kappa$, and $\rho \in {}^i\lambda \backslash \{\nu \restriction i\}$, then $G \models$ "$y_\rho \neq w(z_\nu, x_i)$."

Now,

$(*)_2$ each word $y^{-1}_{\nu \restriction i} w(z_\nu, x_i)$ is so-called *cyclically reduced*, that is, both $w_1 = y^{-1}_{\nu \restriction i} w(z_\nu, x_i)$ and $w_2 = w(z_\nu, x_i) y^{-1}_{\nu \restriction i}$ are reduced, that is, we do not have a generator and its inverse in adjacent places;

$(**)$ for any two such words or cyclical permutations of them which are not equal, any common segment has length $< k^*/6$.

For an explanation and why this is enough, see Lyndon and Schupp [8]. There is nothing to elaborate on as this is not used.

But we prefer to use the more ad hoc but accessible proof.

Proof of Claim 2.2 Let $G = G_0$ be the group generated by

$$Y = \{x_i : i < \kappa\} \cup \{z_\nu : \nu \in {}^\kappa\mu\}$$

freely except (recalling $[xy] = xyx^{-1}y^{-1}$, the commutator) the set of equations $\Gamma_2 = \{[z_\nu, x_i] = [z_\eta, x_i] : i < \kappa, \nu \in {}^\kappa\lambda, \eta \in {}^\kappa\lambda$ satisfy $\nu \restriction i = \eta \restriction i\}$. So for $i < \kappa, \rho \in {}^i\lambda$, we can choose $y_\rho \in G$ such that $\eta \in {}^\kappa\lambda, \eta \restriction i = \rho \Rightarrow y_\rho = [z_\eta, x_i]$. Let G_1 be the group generated by set Y freely, and let h be the homomorphism from G_1 onto G mapping the members of Y to themselves (using abelian groups where no two members of Y are identified in G_1). Let $N = \mathrm{Kernel}(h)$.

Clearly, it suffices to prove that

$(*)_1$ in $G = G_1/N$, if $\nu, \eta \in {}^\kappa\lambda$ and $i < \kappa$, then $[z_\nu, x_i] = [z_\eta, x_i] \Leftrightarrow \nu \restriction i = \eta \restriction i$.

The implication $\Leftarrow$ holds trivially. For the other direction, let $j < \kappa$ and $\eta, \nu \in {}^\kappa\lambda$ be such that $\eta \restriction j \neq \nu \restriction j$, and we will prove that $G \models$ "$y_{\eta \restriction j} \neq y_{\nu \restriction j}$."

Let N_1 be the normal subgroup of G_1 generated by

$$(*)_2 \quad X_* = \{x_i : i < \kappa \text{ and } i \neq j\} \cup \{z_\rho : \rho \in {}^\kappa\lambda \text{ and } \rho \restriction j \notin \{\eta \restriction j, \nu \restriction j\}\}$$
$$\cup \{z_\rho z_\eta^{-1} : \rho \in {}^\kappa\lambda \text{ and } \rho \restriction j = \eta \restriction j\}$$
$$\cup \{z_\rho z_\nu^{-1} : \rho \in {}^\kappa\lambda \text{ and } \rho \restriction j = \nu \restriction j\}.$$

Clearly, by inspection N_1 includes N. Let $N_0 = h(N_1)$. Clearly, N_1 is a normal subgroup of G_1 and h induces a homomorphism $\hat{h}$ from G_1/N_1 onto G_0/N_0. Now by looking at the members of X_*, G_1/N_1 is generated by $\{x_i\} \cup \{z_\eta, z_\nu\}$. Checking the equations in Γ_2, we see clearly that G_1/N_1 is generated by $\{x_i\} \cup \{z_\eta, z_\nu\}$ freely. Hence $G_1/N_1 \models$ "$[z_\eta, x_i] \neq [z_\nu, x_i]$," which means $[z_\eta, x_i]^{-1}[z_\nu, x_i] \notin N_1$ and hence $\notin N$. So recalling the choice of G in $(*)_1$ we have $G \models$ "$y_{\eta\restriction j} \neq y_{\nu\restriction j}$," as required. □

3 More on the Oak Property

Through the "no universal" results in [3], we can also deal with the case of singular cardinals. We also note that the so-called weak oak property suffices.

Claim 3.1 *We have* $\mathrm{univ}(\lambda_1, T) \geq \lambda_2$ *when:*

(a) T *is a complete first-order theory with the oak property,* $\mathfrak{K} = (\mathrm{Mod}_T, \prec)$*;*

(b) *(i)* $\kappa = \mathrm{cf}(\mu) \leq \sigma < \mu < \lambda = \mathrm{cf}(\lambda) < \lambda_1 \leq \lambda_2$,

(ii) $\kappa \leq \sigma \leq \lambda_1, |T| \leq \lambda_2$,

(iii) $\mu^\kappa \geq \lambda_2$*;*

(c) *(i)* $S \subseteq \lambda$ *is stationary,*

(ii) $\bar{C} = \langle C_\delta : \delta \in S\rangle, C_\delta \subseteq \delta, \mathrm{otp}(C_\delta) = \mu, S \subseteq \lambda$,

(iii) $J =: \{A \subseteq \lambda :$ *for some club* E *of* $\lambda, \delta \in S \cap A \Rightarrow C_\delta \nsubseteq E\}$,

(iv) $\lambda \notin J$ *and* $\alpha < \lambda \Rightarrow \lambda > |\{C_\delta \cap \alpha : \alpha \in \mathrm{nacc}(C_\delta), \delta \in S\}|$,

(v) *let* $\langle \alpha_{\delta,\zeta} = \alpha(\delta, \zeta) : \zeta < \mu\rangle$ *list* C_δ *in increasing order;*

(d) $\mathbf{U}_J(\lambda_1) < \lambda_2$*;*

(e) *for some* $\mathcal{P}_1, \mathcal{P}_2$*, we have*

(i) $\mathcal{P}_1 \subseteq [\lambda_1]^\kappa, \mathcal{P}_2 \subseteq [\sigma]^\kappa$,

(ii) *if* $g : \sigma \to \lambda_1$ *is one to one, then for some* $X \in \mathcal{P}_2$*, we have* $\{g(i) : i \in X\} \in \mathcal{P}_1$,

(iii) $|\mathcal{P}_1| < \lambda_2$,

(iv) $|\mathcal{P}_2| \leq \lambda_1$.

Remark 3.2 (1) We can in Claim 3.1 replace clause (a) by

(a)′ $\mathfrak{k}$ is an AEC which has the φ-oak property (see Definition 2.1 and $\mathrm{LST}(\mathfrak{k}) \leq \lambda_2$).

(2) The proof also gives $\mathrm{univ}(\lambda, \lambda_1, T) \geq \lambda_2$.

Recall the following.

Definition 3.3 Assume that $T, \lambda, \mu, S, \bar{C}$ are as in Claim 3.1 (see (a), (c)).

(1) For $\bar{N} = \langle N_\gamma : \gamma < \lambda\rangle$ an elementary-increasing continuous sequence of models of T of size $< \lambda$ and for $a, c \in N_\lambda = \bigcup_{\gamma<\lambda} N_\gamma$ and $\delta \in S$, we let $\mathrm{inv}_{\varphi,\bar{N}}(c, C_\delta, a) = \{\zeta < \mu$: there is $b \in N_{\alpha(\delta,\zeta+2)} \setminus N_{\alpha(\delta,\zeta+1)}$ such that $N_\lambda \models \varphi[a, b, c]\}$.

(2) For $\delta, \bar{N}$ as above and a set $A \subseteq N_\lambda$, let $\mathrm{inv}^A_{\varphi,\bar{N}}(c, C_\delta) = \bigcup\{\mathrm{inv}_{\varphi,\bar{N}}(c, C_\delta, a) : a \in A\}$.

Proof of Claim 3.1 *Step* A: Assume toward a contradiction that $\theta =: \mathrm{univ}(\lambda_1, T) < \lambda_2$, so let $\langle N^*_j : j < \theta\rangle$ exemplify this, and let $\theta_1 = \theta + |\mathscr{P}_1| + |\mathscr{P}_2| + |T| + \mathbf{U}_J(\lambda_1)$, hence $\theta_1 < \lambda_2$.

Without loss of generality, the universe of N^*_j is λ_1.

Step B: By the definition of $\mathbf{U}_J(\lambda_1)$, there is $\mathscr{A}$ such that

(a) $\mathscr{A} \subseteq [\lambda_1]^\lambda$,
(b) $|\mathscr{A}| \le \mathbf{U}_J(\lambda_1)$,
(c) if $f : \lambda \to \lambda_1$, then for some $A \in \mathscr{A}$ we have $\{\delta \in S : f(\delta) \in A\} \neq \emptyset \bmod J$.

For each $X \in \mathscr{P}_1$, $j < \theta$, and $A \in \mathscr{A}$, let $M_{j,X,A}$ be an elementary submodel of N^*_j of cardinality λ which includes $X \cup A \subseteq \lambda_1$, and let $\bar{M}_{j,X,A} = \langle M_{j,X,A,\varepsilon} : \varepsilon < \lambda\rangle$ be a filtration of $M_{j,X,A}$.

Lastly, consider

$$\mathscr{B} = \big\{\mathrm{inv}^X_{\bar{M}_{j,X,A}}(a, C_\delta) : j < \theta, X \in \mathscr{P}_1, A \in \mathscr{A}, \delta \in S, \text{ and } a \in M_{j,X,A}\big\}.$$

Step C: Easily we have $|\mathscr{B}| \le \theta_1 < \lambda_2$ and $\mathscr{B} \subseteq [\mu]^\kappa$; hence there is $B^* \in [\mu]^\kappa \setminus \mathscr{B}$. Without loss of generality, $\mathrm{otp}(B) = \kappa$, where each $\alpha \in B$ is a successor ordinal.

[Why? Let $h : \mu \to \mu$ be such that $(\forall \alpha < \mu)(\exists^\mu \beta < \mu)(h(\beta) = \alpha + 1)$, and let $\mathscr{B}' = \{\{h(\beta) : \beta \in B\} : B \in \mathscr{B}\}$, so $|\mathscr{B}'| \le |\mathscr{B}|$. Hence we can choose $B' \in [\mu]^\kappa \setminus \mathscr{B}'$. Let $\langle \beta_i : i < \kappa\rangle$ list B', and by induction on $i < \kappa$ choose $\alpha_i < \mu$ which is $> \bigcup_{j<i} \alpha_j$ and satisfies $h(\alpha_i + 1) = \beta_i$. So $\{\alpha_i + 1 : i < \kappa\}$ is as required.]

Let $\langle \alpha^*_i : i < \kappa\rangle$ list B in increasing order. For $\delta \in S$, let $\alpha_{\delta,i}$ be the α^*_ith member of C_δ. Now for $\delta \in S$ and $j < \delta$, let $\nu_{\delta,j} = \langle \alpha_{\delta,i} : i < j\rangle$.

Now let M^* be a λ^+-saturated model of T in which a_i, b_η (for $\eta \in {}^{\kappa>}(\lambda_2)$), c_ν (for $\nu \in {}^\kappa(\lambda_2)$), and φ are as in the definition of the oak property, and for each $Y \in \mathscr{P}_2$, choose $\langle N_{Y,\varepsilon} : \varepsilon < \lambda\rangle$, $\langle c_{Y,\varepsilon,\delta} : \delta \in S\rangle$ such that

(a) $N_{Y,\varepsilon}$ is increasing continuous with ε,
(b) $N_{Y,\varepsilon}$ has cardinality $< \lambda$ for $\varepsilon < \lambda$,
(c) $a_i \in N_{Y,0}$ for $i < \kappa$,
(d) $b_{\nu_\delta \restriction (i+1)} \in N_{Y,\nu_\delta(i)+1}$ for $\delta \in S, i < \kappa$,
(e) $c_{\nu_\delta} \in N_{Y,\delta+1}$ for $\delta \in S$.

As $|\mathscr{P}_2| \le \lambda_1$, we can choose $N \prec M^*$, $\|N\| = \lambda_1$ such that

$$\{a_i : i < \sigma\} \cup \bigcup\{N_{Y,\varepsilon} : Y \in \mathscr{P}_2, \varepsilon < \lambda\} \subseteq N.$$

Step D: By our choice of $\langle N^*_j : j < \theta\rangle$, there is $j(*) < \theta$ and elementary embedding $f : N \to N^*_j$. By an assumption, there is $Y \in \mathscr{P}_2$ such that $X := \{f(a_i) : i \in Y\} \in \mathscr{P}_1$. Also by the choice of $\mathscr{A}$, there is $A \in \mathscr{A}$ such that $\{\delta \in S : f(c_{Y,\delta}) \in A\} \neq \emptyset \mod J$.

Now we can finish. (Note that we use here again the last clause in the definition of the oak property.) □

Definition 3.4 (1) The formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ has the weak oak property in T (a first-order complete theory) when it is as in Definition 2.1 omitting clause (c) (i.e., in [3, Definition 1.8]).

(2) A complete first-order theory T has the weak oak property when some $\varphi(\bar{x}, \bar{y}, \bar{z})$ has it in T.

(3) For a noncomplete first-order property T (or class $\mathfrak{k} = (K_{\mathfrak{k}}, \leq_{\mathfrak{k}})$), we mean that φ is quantifier-free.

Claim 3.5 *Assume that*

(a) T has the weak oak property, $|T| \leq \lambda = \mathrm{cf}(\lambda)$,

(b) $\bar{C} = \langle C_\delta : \delta \in S\rangle$, J are as in clause (c) of Claim 3.1,

(c) $\kappa = \mathrm{cf}(\mu) < \sigma < \mu < \lambda = \mathrm{cf}(\lambda)$, and $\mathcal{P} \subseteq \{u \subseteq \sigma : \mathrm{otp}(u) = \kappa\}$ has cardinality $\leq \lambda$.

Then for each $B^* \subseteq \mu$ of order type κ, T has a model N^* of cardinality λ and sequence $\langle a_i : i < \sigma\rangle$ of members of N^* satisfying the following:

⊛ *if N is a model of T of cardinality λ with filtration $\bar{N} = \langle N_\alpha : \alpha < \lambda\rangle$ and f is an elementary embedding of N^* into N, then for every increasing sequence $\bar{\varepsilon} = \langle \varepsilon(i) : i < \kappa\rangle$ enumerating in increasing order some $u \in \mathcal{P}$, we have*

$$\big\{\delta \in S : \text{for some } a \in N^* \text{ we have } B^* = \mathrm{inv}^{\{f(a_{\varepsilon(i)}):i<\kappa\}}_{\varphi,\bar{N}}(C_\delta, a)\big\} = S \mod J.$$

Proof Without loss of generality, some $\varphi = \varphi(x, y, z)$ witnesses that T has the weak oak property (as we can replace T by such T' with $\mathrm{univ}(\lambda, T) = \mathrm{univ}(\lambda, T')$).

As usual, there is $N^* \models T$ with filtration $\bar{N}^* = \langle N^*_i : i < \lambda\rangle$ and $I \subseteq {}^{\kappa>}\lambda$ of cardinality λ, $\langle a_i : i < \kappa\rangle$, $\langle b_\eta : \eta \in \mathcal{T}\rangle$, and $\nu_\delta \in {}^\kappa(C_\delta) \cap \lim_\kappa(T)$ for $\delta \in S$ and $\langle c_{\nu_\delta} : \delta \in S\rangle$ such that

(a) $\langle a_i : i < \kappa\rangle$, $\langle b_\eta : \eta \in \mathcal{T}\rangle$, $\langle c_{\nu_\delta} : \delta \in S\rangle$ are as in Definition 3.4,

(b) $\mathrm{otp}(\nu_\delta(i) \cap C_\delta) = (\text{the } i\text{th member of } B^*) + 1$.

So let N, $\langle N_\varepsilon : \varepsilon < \lambda\rangle$, f be as in the assumption of ⊛ of the claim. Without loss of generality, the universes of N^* and of N are λ.

Let

$$E_* = \big\{\delta < \lambda : \delta \text{ limit}, f''(\delta) = \delta, |N_\delta| = \delta = |N^*_\delta| \\ \text{and } (N_\delta, N^*_\delta, f) \prec (N, N^*, f)\big\}.$$

It is a club of λ. For each $i < \sigma$, let

$$W_i = \big\{\alpha : \text{for some } \delta \in S, \alpha \in C_\delta \subseteq E, \nu_\delta(i) > \alpha, \\ \text{but } \varphi\big(f(a_i), y, f(c_{\nu_\delta})\big) \text{ is satisfied (in } N) \text{ by some } b \in N_\alpha\big\}.$$

Now,

⊛ W_i is not stationary.

[Why? Otherwise, let $\mathfrak{B} \prec (\mathcal{H}(\lambda^+), \in, <^*)$ be such that $\bar{N}, \bar{N}^*, a_i$ (and even $\langle a_j : j < \sigma\rangle$ and $\mathcal{P}$ but not used) and $\langle b_\eta : \eta \in \mathcal{T}\rangle$, $\langle c_{\nu_\delta} : \delta \in S\rangle$ belong to $\mathfrak{B}$ and $\mathfrak{B} \cap \lambda = \alpha \in W_i$, and assume $b \in \mathfrak{B} \cap \alpha$, $N \models \varphi\,[f(a_i), b, f(c_{\nu_\delta})]$. So there is $\delta(*) \in S \cap \delta$ such that $N \models \varphi[f(a_1), b, f(c_{\nu_{\delta(*)}})]$. But $\nu_\delta(i) \geq \alpha > \nu_{\delta(*)}(i)$, hence $\varphi(a_i, y, c_{\nu_\delta}), \varphi(a_i, y, c_{\nu_{\delta'}})$ are incompatible (in N^*), hence their images by f are incompatible in N by b satisfying both, which is a contradiction, so W_i is not stationary.]

So there is a club E^* of λ included in E_* and disjoint to W_i for each $i < \sigma$. So there is $\delta \in S$ such that $C_\delta \subseteq E^*$, and we get a contradiction as earlier. □

Question 3.6 Can we combine Claims 3.1 and 3.5?

(For many singular λ_1's, the answer is certainly yes).

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