Notre Dame Journal of Formal Logic
Volume 57, Number 3, 2016

Implicit Definability in Arithmetic

Stephen G. Simpson

Abstract We consider implicit definability over the natural number system $\mathbb{N}, +, \times, =$. We present a new proof of two theorems of Leo Harrington. The first theorem says that there exist implicitly definable subsets of $\mathbb{N}$ which are not explicitly definable from each other. The second theorem says that there exists a subset of $\mathbb{N}$ which is not implicitly definable but belongs to a countable, explicitly definable set of subsets of $\mathbb{N}$. Previous proofs of these theorems have used finite- or infinite-injury priority constructions. Our new proof is easier in that it uses only a nonpriority oracle construction, adapted from the standard proof of the Friedberg jump theorem.

1 Introduction

Definitions Let $\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$ = the set of all natural numbers. Let $\mathrm{Pow}(\mathbb{N})$ be the powerset of $\mathbb{N}$, that is, the set of all subsets of $\mathbb{N}$. A set $X \in \mathrm{Pow}(\mathbb{N})$ is said to be *arithmetical* if it is explicitly definable over the natural number system $\mathbb{N}, +, \times, =$. In other words,

$$X = \{n \in \mathbb{N} \mid (\mathbb{N}, +, \times, =) \models \Phi(n)\}$$

for some first-order formula $\Phi(n)$ in the language $+, \times, =$. Given two sets $X, Y \in \mathrm{Pow}(\mathbb{N})$, we say that X is *arithmetical in* Y if X is explicitly definable from Y; that is,

$$X = \{n \in \mathbb{N} \mid (\mathbb{N}, +, \times, Y, =) \models \Phi(n)\}$$

for some first-order formula $\Phi(n)$ in the language $+, \times, Y, =$. We say that X and Y are *arithmetically incomparable* if neither is arithmetical in the other. A set of sets

Received September 3, 2013; accepted January 8, 2014
First published online March 30, 2016
2010 Mathematics Subject Classification: Primary 03D55; Secondary 03D30, 03D28, 03C40, 03D80
Keywords: arithmetical hierarchy, arithmetical singletons, implicit definability, hyperarithmetical sets, Turing jump
 10.1215/00294527-3507386

$S \subseteq \text{Pow}(\mathbb{N})$ is said to be *arithmetical* if it is explicitly definable; that is,

$$S = \{X \in \text{Pow}(\mathbb{N}) \mid (\mathbb{N}, +, \times, X, =) \models \Phi\}$$

for some first-order sentence Φ in the language $+, \times, X, =$. A set $X \in \text{Pow}(\mathbb{N})$ is called an *arithmetical singleton* or *implicitly arithmetical* if the singleton set $\{X\}$ is arithmetical.

Remark 1 The purpose of this paper is to present a new proof of two theorems of Harrington [6], [7] concerning implicit definability over the natural number system $\mathbb{N}, +, \times, =$. The two theorems read as follows.

1. There exist arithmetical singletons $X, Y \in \text{Pow}(\mathbb{N})$ which are arithmetically incomparable (see Theorem 4.4 below).
2. There exists a set $Z \in \text{Pow}(\mathbb{N})$ which belongs to a countable arithmetical set of sets $S \subseteq \text{Pow}(\mathbb{N})$ but is not an arithmetical singleton (see Theorem 4.5 below).

We feel that these two theorems deserve to be better known, because they embody significant insight concerning implicit definability in arithmetic.

Remark 2 Before Harrington's work, some early theorems concerning implicit definability in arithmetic were as follows.

1. There exists $X \in \text{Pow}(\mathbb{N})$ which is implicitly arithmetical but not arithmetical. (Namely, let $X = 0^{(\omega)}$ = the Tarski truth set for $\mathbb{N}, +, \times, =$; see Rogers [11, Theorems 14-X and 15-XII]).
2. There exist $X, Y \in \text{Pow}(\mathbb{N})$ such that the pair $X \oplus Y$ is implicitly arithmetical but neither X nor Y is implicitly arithmetical. (Namely, let X and Y be Cohen generic over $\mathbb{N}, +, \times, =$ such that $X \oplus Y$ and $0^{(\omega)}$ are arithmetical in each other; see Feferman [3] or Rogers [11, Exercise 16-72]).
3. Each arithmetical singleton is arithmetical in $0^{(\alpha)}$ for some recursive ordinal α, and each such $0^{(\alpha)}$ is itself an arithmetical singleton (see, e.g., Sacks [13, Chapter II]).
4. Every nonempty countable arithmetical set of sets $S \subseteq \text{Pow}(\mathbb{N})$ contains an arithmetical singleton. (This result is due to Tanaka [15].)

Remark 3 Harrington's original proof (see [6]) of Theorem 4.4 was based on an infinite-injury priority construction. The same method has been used by Harrington [6] and others to obtain results about ω-REA arithmetical degrees (see M. F. Simpson [14, Chapters 2 and 3] and Odifreddi [10, Chapter XIII]), jump embeddings (see Hinman and Slaman [8]), nonstandard models of arithmetic (see Ash and Knight [1, Chapters 14–19, Theorem 19.19]), and generalized high/low hierarchies (see Montalbán [9]).

Remark 4 Harrington's original proof (see [7]) of Theorem 4.5 was based on a finite-injury priority construction. The same method has been extended into the transfinite by Harrington [7] and Gerdes [5] to obtain other interesting results. In particular, see Remark 12 below. For an application to effectively Borel equivalence relations, see Fokina, Friedman, and Törnquist [4].

Remark 5 Our new proof of Theorems 4.4 and 4.5 does not use a priority construction of any kind. Instead, our proof is based on a direct oracle construction, adapted from the standard proof of the Friedberg jump theorem. In this sense, our proof of Theorems 4.4 and 4.5 is much easier than the proofs in [1], [5]–[10], and [14]. On the other hand, our proof uses the recursion theorem in exactly the same way as Harrington used it. Harrington [6] has referred to this way of using the recursion theorem as "the shiny little box which was first opened by Sacks [12]."

Remark 6 Beyond Theorems 4.4 and 4.5, we believe we can extend our nonpriority oracle method farther into the transfinite to obtain relatively easy proofs of at least some of the other results of Harrington [7] and Gerdes [5]. However, we reserve that extension for a future paper. In this paper we limit ourselves to providing relatively easy proofs of Theorems 4.4 and 4.5.

Remark 7 The plan of this paper is as follows. In Section 2 we review some basic recursion-theoretic notions. In Section 3 we prove a rudimentary version of Theorems 4.4 and 4.5. In Section 4 we prove Theorems 4.4 and 4.5.

2 Recursion-Theoretic Background

In this section we review some basic notions from recursion theory which are needed for our proof of Theorems 4.4 and 4.5. A good reference for this material is Rogers [11].

Natural numbers are denoted $e, i, j, k, l, m, n, \ldots$. The set of all natural numbers is denoted $\mathbb{N}$. Instead of working with $\mathrm{Pow}(\mathbb{N})$, the set of all subsets $X \subseteq \mathbb{N}$, we work with $\mathbb{N}^{\mathbb{N}}$, the set of all functions $X : \mathbb{N} \to \mathbb{N}$. The space $\mathbb{N}^{\mathbb{N}}$ with the product topology is known as the *Baire space*. Points in $\mathbb{N}^{\mathbb{N}}$ are denoted $X, Y, Z, \ldots$. Subsets of $\mathbb{N}^{\mathbb{N}}$ are denoted $P, Q, \ldots$.

Recall that a point $X \in \mathbb{N}^{\mathbb{N}}$ or a set $P \subseteq \mathbb{N}^{\mathbb{N}}$ is arithmetical if and only if it is Π^0_n for some $n \geq 1$. The hierarchy Π^0_n, where $n = 1, 2, \ldots$, is known as the *arithmetical hierarchy* (see, e.g., [11, Chapters 14–16]). (It is known (see [15]) that every arithmetical set is in arithmetical one-to-one correspondence with a Π^0_1 set. However, we will not need this result here.) A Π^0_n *singleton* is a point X such that the singleton set $\{X\}$ is Π^0_n. Thus X is an arithmetical singleton if and only if it is a Π^0_n singleton for some $n \geq 1$. A *ranked point* is a point X such that $X \in P$ for some countable Π^0_1 set P.

Points in $\mathbb{N}^{\mathbb{N}}$ may be viewed as *Turing oracles* (see, e.g., [11, Chapters 9–13]). Relativizing to a Turing oracle $A \in \mathbb{N}^{\mathbb{N}}$, a point $X \in \mathbb{N}^{\mathbb{N}}$ or a set $P \subseteq \mathbb{N}^{\mathbb{N}}$ is said to be $\Pi^{0,A}_n$ if it is Π^0_n relative to A, and *arithmetical in* A if it is $\Pi^{0,A}_n$ for some n. In particular, a set P is topologically closed if and only if it is $\Pi^{0,A}_1$ for some A. A point X such that the singleton set $\{X\}$ is $\Pi^{0,A}_n$ is called a $\Pi^{0,A}_n$ *singleton*.

For $A \in \mathbb{N}^{\mathbb{N}}$ we write $\{e\}^A(i) = j$ to mean that the eth Turing machine with oracle A and input i halts with output j. We write $\{e\}^A(i)\downarrow$ (resp., $\uparrow$) to mean that the eth Turing machine with oracle A and input i halts (resp., does not halt). Thus $\{e\}^A(i)\downarrow$ if and only if $\exists j\ (\{e\}^A(i) = j)$. For $A, B \in \mathbb{N}^{\mathbb{N}}$ we write $A \leq_{\mathrm{T}} B$ to mean that A is *Turing reducible to* B, that is, $\exists e\ \forall i\ (A(i) = \{e\}^B(i))$. We write $A \equiv_{\mathrm{T}} B$ to mean that A is *Turing equivalent to* B, that is, $A \leq_{\mathrm{T}} B$ and $B \leq_{\mathrm{T}} A$. We define $A \oplus B \in \mathbb{N}^{\mathbb{N}}$ by the equations $(A \oplus B)(2i) = A(i)$ and $(A \oplus B)(2i + 1) = B(i)$. Thus $A \oplus B \leq_{\mathrm{T}} C$ if and only if $A \leq_{\mathrm{T}} C$ and $B \leq_{\mathrm{T}} C$.

For $A \in \mathbb{N}^{\mathbb{N}}$ we write A' = the *Turing jump* of A, defined by

$$A'(e) = \begin{cases} 1 & \text{if } \{e\}^A(e)\downarrow, \\ 0 & \text{if } \{e\}^A(e)\uparrow. \end{cases}$$

We write $A^{(n)}$ = the nth Turing jump of A, defined inductively by letting $A^{(0)} = A$ and $A^{(n+1)} = (A^{(n)})'$ for all n. Recall that A is arithmetical in B if and only if $\exists n\,(A \leq_{\mathrm{T}} B^{(n)})$. For use in the proof of Theorems 3.5 and 4.5, note that for each $n \geq 1$, a set $P \subseteq \mathbb{N}^{\mathbb{N}}$ is Π^0_n if and only if $\exists e\,\forall X\,(X \in P \Leftrightarrow X^{(n)}(e) = 0)$ (see, e.g., [11, Section 14.5]).

We write $A^{(\omega)}$ = the ωth Turing jump of A, defined by

$$A^{(\omega)}(i) = \begin{cases} A^{(n)}(e) & \text{if } i = 3^n 5^e, \\ 0 & \text{otherwise}. \end{cases}$$

Thus $A^{(\omega)} = \bigoplus_n A^{(n)}$ and $A^{(n)} \leq_{\mathrm{T}} A^{(\omega)}$ uniformly in n.

Let $0 \in \mathbb{N}^{\mathbb{N}}$ denote the constant zero function. Thus $0^{(n)}$ = the nth Turing jump of 0, and $0^{(\omega)}$ = the ωth Turing jump of 0. Note also that X is arithmetical if and only if $X \leq_{\mathrm{T}} 0^{(n)}$ for some n.

3 A Rudimentary Version of Harrington's Theorems

The purpose of this section is to prove a rudimentary version of Harrington's theorems, with "arithmetical" replaced by Π^0_n for a fixed n. Our rudimentary versions of Theorems 4.4 and 4.5 are Theorems 3.4 and 3.5, respectively.

Lemma 3.1 *Given a $\Pi^{0,A'}_1$ set P, we can find a $\Pi^{0,A}_1$ set Q and a homeomorphism $F : P \cong Q$ such that $X \oplus A \equiv_{\mathrm{T}} F(X) \oplus A$ uniformly for all $X \in P$.*

Proof Since P is a $\Pi^{0,A'}_1$ set, it follows that P is a $\Pi^{0,A}_2$ set, say, $P = \{X \mid \forall i\,\exists j\, R(X,i,j)\}$, where R is an A-recursive predicate. Define $F : P \cong Q = F(P)$ by letting $F(X) = X \oplus \widehat{X}$, where $\widehat{X}(i)$ = the least j such that $R(X,i,j)$ holds. Clearly Q is a $\Pi^{0,A}_1$ set and $X \oplus A \equiv_{\mathrm{T}} F(X) \oplus A$ uniformly for all $X \in P$. □

Lemma 3.2 *Given a $\Pi^{0,A'}_1$ set P, we can find a $\Pi^{0,A}_1$ set Q and a homeomorphism $H : P \cong Q$ such that $X \oplus A' \equiv_{\mathrm{T}} H(X) \oplus A' \equiv_{\mathrm{T}} (H(X) \oplus A)'$ uniformly for all $X \in P$.*

In order to prove Lemma 3.2, we first present some general remarks concerning strings, trees, and treemaps.

Notation (strings) Let $\mathbb{N}^* = \bigcup_{l \in \mathbb{N}} \mathbb{N}^l$ = the set of *strings*, that is, finite sequences of natural numbers. For $\sigma = \langle n_0, n_1, \ldots, n_{l-1}\rangle \in \mathbb{N}^*$ we write $\sigma(i) = n_i$ for all $i < |\sigma| = l$ = the *length* of σ. For $\sigma, \tau \in \mathbb{N}^*$ we write $\sigma^\frown\tau$ = the *concatenation*, σ followed by τ, defined by the conditions $|\sigma^\frown\tau| = |\sigma| + |\tau|$, $(\sigma^\frown\tau)(i) = \sigma(i)$ for all $i < |\sigma|$, and $(\sigma^\frown\tau)(|\sigma| + i) = \tau(i)$ for all $i < |\tau|$. We write $\sigma \subseteq \tau$ if $\sigma^\frown\rho = \tau$ for some ρ. If $|\sigma| \geq n$, we write $\sigma\restriction n = \langle\sigma(0), \sigma(1), \ldots, \sigma(n-1)\rangle$ = the unique $\rho \subseteq \sigma$ such that $|\rho| = n$. For $X \in \mathbb{N}^{\mathbb{N}}$ we write $X\restriction n = \langle X(0), X(1), \ldots, X(n-1)\rangle$ = the unique $\sigma \subset X$ such that $|\sigma| = n$. If $|\sigma| = |\tau| = n$, we define $\sigma \oplus \tau \in \mathbb{N}^*$ by the conditions $|\sigma \oplus \tau| = 2n$ and $(\sigma \oplus \tau)(2i) = \sigma(i)$ and $(\sigma \oplus \tau)(2i+1) = \tau(i)$ for all $i < n$.

Definition (trees) A *tree* is a set $T \subseteq \mathbb{N}^*$ such that

$$\forall\rho\,\forall\sigma\,\big((\rho \subseteq \sigma \text{ and } \sigma \in T) \Rightarrow \rho \in T\big).$$

For any tree T we write

$$[T] = \{\text{paths through } T\} = \big\{X \;\big|\; \forall n\,(X\restriction n \in T)\big\}.$$

Remark 8 It is well known (see, e.g., [11, Chapter 15]) that the following statements are pairwise equivalent.

1. P is a $\Pi^{0,A}_1$ set.
2. $P = [T]$ for some $\Pi^{0,A}_1$ tree T.
3. $P = [T]$ for some A-recursive tree T.
4. $P = \{X \mid X \oplus A \in [T]\}$ for some recursive tree T.

Definition (treemaps) Let T be a tree. A *treemap* is a function $F : T \to \mathbb{N}^*$ such that

$$F(\sigma^\frown\langle i\rangle) \supseteq F(\sigma)^\frown\langle i\rangle$$

for all $\sigma \in T$ and all $i \in \mathbb{N}$ such that $\sigma^\frown\langle i\rangle \in T$. We then have another tree

$$F(T) = \{\tau \mid \exists\sigma\,(\sigma \in T \text{ and } \tau \subseteq F(\sigma))\}.$$

Thus $P = [T]$ and $F(P) = [F(T)]$ are closed sets in the Baire space, and we have a homeomorphism $F : P \cong F(P)$ defined by $F(X) = \bigcup_{n\in\mathbb{N}} F(X\upharpoonright n)$ for all $X \in P$. Note also that the composition of two treemaps is a treemap. A treemap $F : T \to \mathbb{N}^*$ is said to be A-*recursive* if it is the restriction to T of a partial A-recursive function.

Remark 9 Let T be a tree, and let $F : T \to \mathbb{N}^*$ be a treemap. Given $\tau \in F(T)$, let $\sigma \in T$ be minimal such that $\tau \subseteq F(\sigma)$. Then σ is a *substring* of τ, that is, $\sigma = \langle\tau(j_0), \tau(j_1), \ldots, \tau(j_{l-1})\rangle$ for some $j_0 < j_1 < \cdots < j_{l-1} < |\tau|$. Thus, in the definition of $F(T)$, the quantifier $\exists\sigma$ may be replaced by a bounded quantifier,

$$F(T) = \{\tau \mid (\exists\sigma \text{ substring of } \tau)\,(\sigma \in T \text{ and } \tau \subseteq F(\sigma))\}.$$

This implies that, for instance, if F and T are A-recursive, then so is $F(T)$.

We are now ready to prove Lemma 3.2.

Proof of Lemma 3.2 Given A, we construct a particular A'-recursive treemap $G : \mathbb{N}^* \to \mathbb{N}^*$. We define $G(\sigma)$ by induction on $|\sigma|$ beginning with $G(\langle\rangle) = \langle\rangle$. If $G(\sigma)$ has been defined, let $e = |\sigma|$, and for each i let $G(\sigma^\frown\langle i\rangle) =$ the least $\tau \supseteq G(\sigma)^\frown\langle i\rangle$ such that $\{e\}^{\tau\oplus A}_{|\tau|}(e)\downarrow$ if such a τ exists, otherwise $G(\sigma^\frown\langle i\rangle) = G(\sigma)^\frown\langle i\rangle$. Clearly G is an A'-recursive treemap, and our construction of G implies that for all e and X, $\{e\}^{G(X)\oplus A}(e)\downarrow$ if and only if $\{e\}^{G(X\upharpoonright e+1)\oplus A}_{|G(X\upharpoonright e+1)|}(e)\downarrow$. Thus $X \oplus A' \equiv_{\mathrm{T}} G(X) \oplus A' \equiv_{\mathrm{T}} (G(X) \oplus A)'$ uniformly for all X.

Let G be the A'-recursive treemap which was constructed above. Let P be a $\Pi^{0,A'}_1$ set. By Remarks 8 and 9 we know that the restriction of G to P maps P homeomorphically onto another $\Pi^{0,A'}_1$ set $G(P)$. Applying Lemma 3.1 to $G(P)$ we obtain a $\Pi^{0,A}_1$ set Q and a homeomorphism $F : G(P) \cong Q$ such that $Y \oplus A \equiv_{\mathrm{T}} F(Y) \oplus A$ uniformly for all $Y \in G(P)$. Thus $H = F \circ G$ is a homeomorphism of P onto Q, and for all $X \in P$ we have $G(X) \oplus A \equiv_{\mathrm{T}} F(G(X)) \oplus A = H(X) \oplus A$ uniformly, and hence $X \oplus A' \equiv_{\mathrm{T}} H(X) \oplus A' \equiv_{\mathrm{T}} (H(X) \oplus A)'$ uniformly. □

Remark 10 Our proof of Lemma 3.2 via treemaps is similar to the proof of [2, Lemma 5.1]. Within our proof of Lemma 3.2, the construction of the specific treemap G is the same as the standard proof of the Friedberg jump theorem as expounded, for instance, in [11, Section 13.3].

Lemma 3.3 *Given a $\Pi^{0,0^{(n)}}_1$ set P_n, we can find a Π^0_1 set P_0 and a homeomorphism $H^n_0 : P_n \cong P_0$ such that $X_n \oplus 0^{(n)} \equiv_T X_0 \oplus 0^{(n)} \equiv_T X_0^{(n)}$ uniformly for all $X_n \in P_n$ and $X_0 = H^n_0(X_n) \in P_0$.*

Proof The proof is by induction on n. For $n = 0$ there is nothing to prove. For the inductive step, given a $\Pi^{0,0^{(n+1)}}_1$ set P_{n+1}, apply Lemma 3.2 with $A = 0^{(n)}$ to obtain a $\Pi^{0,0^{(n)}}_1$ set P_n and a homeomorphism $H_n : P_{n+1} \cong P_n$ such that $X_{n+1} \oplus 0^{(n+1)} \equiv_T H_n(X_{n+1}) \oplus 0^{(n+1)} \equiv_T (H_n(X_{n+1}) \oplus 0^{(n)})'$ uniformly for all $X_{n+1} \in P_{n+1}$. Then apply the inductive hypothesis to P_n to find a Π^0_1 set P_0 and a homeomorphism $H^n_0 : P_n \cong P_0$ such that $X_n \oplus 0^{(n)} \equiv_T X_0 \oplus 0^{(n)} \equiv_T X_0^{(n)}$ uniformly for all $X_n \in P_n$. Letting $H^{n+1}_0 = H_n \circ H^n_0$, it follows that $X_{n+1} \oplus 0^{(n+1)} \equiv_T X_0 \oplus 0^{(n+1)} \equiv_T X_0^{(n+1)}$ uniformly for all $X_{n+1} \in P_{n+1}$ and $X_0 = H^{n+1}_0(X_{n+1}) \in P_0$. □

We now use Lemma 3.3 to prove a rudimentary version of Harrington's theorems.

Theorem 3.4 *Given n, we can find Π^0_1 singletons X, Y such that $X \not\leq_T Y^{(n)}$ and $Y \not\leq_T X^{(n)}$.*

Proof It is well known (see [11, Section 13.3]) that there exist incomparable Turing degrees between 0 and $0'$. Relativizing to $0^{(n)}$, let X_n, Y_n be such that $0^{(n)} \leq_T X_n \leq_T 0^{(n+1)}$ and $0^{(n)} \leq_T Y_n \leq_T 0^{(n+1)}$ and such that $X_n \not\leq_T Y_n$ and $Y_n \not\leq_T X_n$. Note that X_n and Y_n are $\Delta^{0,0^{(n)}}_2$; hence X_n and Y_n are $\Pi^{0,0^{(n)}}_2$ singletons. Therefore, by the proof of Lemma 3.1, we may safely assume that X_n and Y_n are $\Pi^{0,0^{(n)}}_1$ singletons. Apply Lemma 3.3 to $P_n = \{X_n, Y_n\}$ to get $X_0 = H^n_0(X_n)$ and $Y_0 = H^n_0(Y_n)$. Note that $P_0 = \{X_0, Y_0\}$ is a Π^0_1 set; hence X_0 and Y_0 are Π^0_1 singletons. Since $X_n \not\leq_T Y_n \oplus 0^{(n)} \equiv_T Y_0^{(n)}$ and $X_n \oplus 0^{(n)} \equiv_T X_0 \oplus 0^{(n)}$, we have $X_0 \not\leq_T Y_0^{(n)}$, and similarly $Y_0 \not\leq_T X_0^{(n)}$. Letting $X = X_0$ and $Y = Y_0$, we obtain our theorem. □

Theorem 3.5 *Given n, we can find a countable Π^0_1 set P such that some $Z \in P$ is not a Π^0_n singleton.*

Proof Let P_n be a countable Π^0_1 set such that some $Z_n \in P_n$ is not isolated in P_n. (For instance, let $P_n = \{X \mid \forall i\, \forall j\, (X(i) \neq 0 \neq X(j) \Rightarrow i = j)\}$, and let $Z_n = 0$.) Treating P_n as a $\Pi^{0,0^{(n)}}_1$ set, apply Lemma 3.3. Then P_0 is a countable Π^0_1 set and, because $H^n_0 : P_n \cong P_0$ is a homeomorphism, $Z_0 = H^n_0(Z_n)$ is not isolated in P_0. We claim that Z_0 is not a Π^0_n singleton. Otherwise, let e be such that $\{Z_0\} = \{X \mid X^{(n)}(e) = 0\}$. Since $Z_0^{(n)}(e) = 0$ and $Z_0 \in P_0$ and $X_0^{(n)} \equiv_T X_n \oplus 0^{(n)}$ uniformly for all $X_n \in P_n$ and $X_0 = H^n_0(X_n) \in P_0$, there exists j such that $X_0^{(n)}(e) = 0$ for all $X_n \in P_n$ such that $X_n \restriction j = Z_n \restriction j$. But Z_n is not isolated in P_n, so there exists $X_n \in P_n$ such that $X_n \restriction j = Z_n \restriction j$ and $X_n \neq Z_n$. Thus $X_0^{(n)}(e) = 0$ and $X_0 \neq Z_0$, which is a contradiction. Letting $P = P_0$ and $Z = Z_0$, we obtain our theorem. □

4 Proof of Harrington's Theorems

In order to prove the full version of Harrington's theorems, we need to show that Lemma 3.3 holds with n replaced by ω. To this end, we first draw out some effective uniformities which are implicit in the proofs of Lemmas 3.1 and 3.2.

Notation Let W_e^A for $e = 0, 1, 2, \ldots$ be a standard enumeration of all A-recursively enumerable subsets of $\mathbb{N}^*$. Then

$$T_e^A = \{\sigma \in \mathbb{N}^* \mid (\forall n \leq |\sigma|)\,(\sigma \restriction n \notin W_e^A)\}$$

for $e = 0, 1, 2, \ldots$ is a standard enumeration of all $\Pi_1^{0,A}$ trees. Hence $P_e^A = [T_e^A]$ for $e = 0, 1, 2, \ldots$ is a standard enumeration of all $\Pi_1^{0,A}$ sets.

Remark 11 If F is an A-recursive treemap and T is a $\Pi_1^{0,A}$ tree, then $F(T)$ is again a $\Pi_1^{0,A}$ tree. Moreover, this holds uniformly in the sense that there is a primitive recursive function f such that $T_{f(e)}^A = F(T_e^A)$ and $P_{f(e)}^A = F(P_e^A)$ for all e, and we can compute a primitive recursive index of f knowing only an A-recursive index of F.

The next two lemmas are refinements of Lemmas 3.1 and 3.2, respectively.

Lemma 4.1 (refining Lemma 3.1) *There is a primitive recursive function f with the following property. Given e, we can effectively find an A-recursive treemap $F : T_e^{A'} \to T_{f(e)}^A$ which induces a homeomorphism $F : P_e^{A'} \cong P_{f(e)}^A$. It follows that $X \oplus A \equiv_{\mathrm{T}} F(X) \oplus A$ uniformly for all $X \in P_e^{A'}$.*

Proof Let $T = T_e^{A'}$, and let $P = P_e^{A'}$. Since $T_e^{A'}$ is uniformly $\Pi_1^{0,A'}$, it is uniformly $\Pi_2^{0,A}$, say, $T = T_e^{A'} = \{\sigma \mid \forall i\,\exists j\, R(\sigma, e, i, A \restriction j)\}$, where $R \subseteq \mathbb{N}^* \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}^*$ is a fixed primitive recursive predicate. Let $(-,-)$ be a fixed primitive recursive one-to-one mapping of $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$ such that $m \leq (m, n)$ and $n \leq (m, n)$ for all m and n. Define $Q = [\widehat{T}]$, where $\widehat{T} = \{\sigma \oplus \tau \mid |\sigma| = |\tau|$ and $(\forall (n,i) < |\tau|)\,(\tau((n,i)) =$ the least j such that $R(\sigma \restriction n, e, i, A \restriction j))\}$. Thus $Q = \{X \oplus \widehat{X} \mid X \in P\}$, where $\widehat{X}((n,i)) =$ the least j such that $R(X \restriction n, e, i, A \restriction j)$. Moreover, we have an A-recursive treemap $F : T \to \widehat{T}$ given by $F(\sigma) = \sigma \oplus \widehat{\sigma}$ for all $\sigma \in T$, where $|\sigma| = |\widehat{\sigma}|$ and $(\forall (n,i) < |\sigma|)\,(\widehat{\sigma}((n,i)) =$ the least j such that $R(\sigma \restriction n, e, i, A \restriction j))$. Although we cannot expect to have $F(T) = \widehat{T}$, we nevertheless have $F : [T] \cong [\widehat{T}]$; that is, $F : P \cong F(P) = Q$, and $F(X) = X \oplus \widehat{X}$ and $X \oplus A \equiv_{\mathrm{T}} F(X) \oplus A$ uniformly for all $X \in P$. The definition of $\widehat{T}$ shows that $\widehat{T}$ is uniformly A-recursive, and hence uniformly $\Pi_1^{0,A}$, so we can find a fixed primitive recursive function f such that $T_{f(e)}^A = \widehat{T_e^{A'}}$ for all e and A. □

Lemma 4.2 (refining Lemma 3.2) *There is a primitive recursive function h with the following property. Given e, we can effectively find an A'-recursive treemap $H : T_e^{A'} \to T_{h(e)}^A$ which induces a homeomorphism $H : P_e^{A'} \cong P_{h(e)}^A$ such that $X \oplus A' \equiv_{\mathrm{T}} H(X) \oplus A' \equiv_{\mathrm{T}} (H(X) \oplus A)'$ uniformly for all $X \in P_e^{A'}$.*

Proof Let G be the specific A'-recursive treemap which was constructed in the proof of Lemma 3.2. By Remark 11 we can find a primitive recursive function g such that for all e we have $G(T_e^{A'}) = T_{g(e)}^{A'}$, and the restriction of G to $T_e^{A'}$ is a treemap from $T_e^{A'}$ to $T_{g(e)}^{A'}$ which induces a homeomorphism $G : P_e^{A'} \cong P_{g(e)}^{A'}$. By the construction of G we have $X \oplus A' \equiv_T G(X) \oplus A' \equiv_T (G(X) \oplus A)'$ uniformly for all $X \in P_e^{A'}$. Now applying Lemma 4.1 we obtain an A-recursive treemap $F : T_{g(e)}^{A'} \to T_{f(g(e))}^{A}$ which induces a homeomorphism $F : P_{g(e)}^{A'} \cong P_{f(g(e))}^{A}$ such that $Y \oplus A \equiv_T F(Y) \oplus A$ uniformly for all $Y \in P_{g(e)}^{A}$. Thus the treemap $H = F \circ G : T_e^{A'} \to T_{f(g(e))}^{A}$ induces a homeomorphism $F \circ G = H : P_e^{A'} \cong P_{f(g(e))}^{A}$ such that $X \oplus A' \equiv_T H(X) \oplus A' \equiv_T (H(X) \oplus A)'$ uniformly for all $X \in P_e^{A'}$. Our lemma follows upon defining $h(e) = f(g(e))$. □

We now show that Lemma 3.3 holds with n replaced by ω.

Lemma 4.3 *Given a $\Pi_1^{0,0^{(\omega)}}$ set P_ω, we can effectively find a Π_1^0 set P_0 and a homeomorphism $H_0^\omega : P_\omega \cong P_0$ such that $X_\omega \oplus 0^{(\omega)} \equiv_T X_0 \oplus 0^{(\omega)} \equiv_T X_0^{(\omega)}$ uniformly for all $X_\omega \in P_\omega$ and $X_0 = H_0^\omega(X_\omega) \in P_0$.*

Proof Since P_ω is a $\Pi_1^{0,0^{(\omega)}}$ set, Remark 8 gives a recursive tree T such that $P_\omega = \{X \mid X \oplus 0^{(\omega)} \in [T]\}$. Moreover, from the definition of $0^{(\omega)}$ we know that $0^{(\omega)} \upharpoonright n$ is computable from $0^{(n)}$ uniformly for all n. Thus, letting $T_\omega = \{\sigma \mid \sigma \oplus 0^{(\omega)} \upharpoonright |\sigma| \in T\}$, we see that $P_\omega = [T_\omega]$ and $\{\sigma \mid |\sigma| \leq n, \sigma \in T_\omega\} \leq_T 0^{(n)}$ uniformly for all n. Define

$$T_{e,n} = \big\{\sigma \;\big|\; |\sigma| \leq n\big\} \cup \big\{\sigma \;\big|\; |\sigma| > n, \sigma \upharpoonright n \in T_\omega, \sigma \in T_e^{\langle n\rangle^\frown 0^{(n)}}\big\}.$$

Thus $T_{e,n}$ is a $\Pi_1^{0,0^{(n)}}$ tree, and hence $P_{e,n} = [T_{e,n}]$ is $\Pi_1^{0,0^{(n)}}$ uniformly for all n.

In the vein of Lemma 4.2, we claim that there is a primitive recursive function h^* with the following property. Given e and n we can effectively find a $0^{(n+1)}$-recursive treemap

$$H_{e,n} : T_{e,n+1} \to T_{h^*(e),n}$$

which induces a homeomorphism $H_{e,n} : P_{e,n+1} \cong P_{h^*(e),n}$ such that $X \oplus 0^{(n+1)} \equiv_T H_{e,n}(X) \oplus 0^{(n+1)} \equiv_T (H_{e,n}(X) \oplus 0^{(n)})'$ uniformly for all $X \in P_{e,n+1}$, and in addition $H_{e,n}(\sigma) = \sigma$ for all σ such that $|\sigma| \leq n$.

To prove our claim, let r be a 3-place primitive recursive function such that $T_{r(e,n,\sigma)}^{0^{(n)}} = \{\tau \mid \sigma^\frown\tau \in T_{e,n}\}$ for all e, n, σ. We can then write

$$T_{e,n+1} = \big\{\sigma \;\big|\; |\sigma| \leq n\big\} \cup \big\{\sigma^\frown\tau \;\big|\; |\sigma| = n, \tau \in T_{r(e,n+1,\sigma)}^{0^{(n+1)}}\big\}.$$

Since n is uniformly computable from $\langle n\rangle^\frown 0^{(n)}$, let h^* be a primitive recursive function such that

$$T_{h^*(e),n} = \big\{\sigma \;\big|\; |\sigma| \leq n\big\} \cup \big\{\sigma^\frown\tau \;\big|\; |\sigma| = n, \tau \in T_{h(r(e,n+1,\sigma))}^{0^{(n)}}\big\},$$

where h is as in Lemma 4.2. For all σ and τ such that $|\sigma| = n$ and $\tau \in T_{r(e,n+1,\sigma)}^{0^{(n+1)}}$ let $H_{e,n}(\sigma^\frown\tau) = \sigma^\frown H(\tau)$, where $H : T_{r(e,n+1,\sigma)}^{0^{(n+1)}} \to T_{h(r(e,n+1,\sigma))}^{0^{(n)}}$ is as in Lemma 4.2. Clearly $h^*(e)$ and $H_{e,n}$ have the required properties, so our claim is proved.

Let h^* and $H_{e,n}$ be as in the above claim. By the recursion theorem (see [11, Chapter 11]), let e^* be a *fixed point* of h^*, so that $T_{h^*(e^*)}^A = T_{e^*}^A$ for all A, and hence $T_{h^*(e^*),n} = T_{e^*,n}$ for all n. Let $H_n = H_{e^*,n}$ and $T_n = T_{e^*,n}$ and $P_n = P_{e^*,n} = [T_n]$ for all n. As in the proof of Lemma 3.3 we have uniformly for each $s > n$ a $0^{(s)}$-recursive treemap $H^s_n = H_n \circ \cdots \circ H_{s-1} : T_s \to T_n$ which induces a homeomorphism $H^s_n : P_s \cong P_n$ such that $X \oplus 0^{(s)} \equiv_T H^s_n(X) \oplus 0^{(s)} \equiv_T (H^s_n(X))^{(s-n)}$ uniformly for all $X \in P_s$, and in addition $H^s_n(\sigma) = \sigma$ for all σ such that $|\sigma| \leq n$. We also have for each n a $0^{(\omega)}$-recursive treemap $H^\omega_n : T_\omega \to T_n$ which induces a homeomorphism $H^\omega_n : P_\omega \cong P_n$; namely, $H^\omega_n(\sigma) = H^{|\sigma|}_n(\sigma)$ if $|\sigma| > n$ and $H^\omega_n(\sigma) = \sigma$ if $|\sigma| \leq n$. Note also that for all $n < s < t < \omega$ we have $H^t_n = H^s_n \circ H^t_s$ and $H^\omega_n = H^s_n \circ H^\omega_s$. Finally, given $X_\omega \in P_\omega$, let $X_n = H^\omega_n(X_\omega)$ for all n. Then $X_\omega \restriction n = X_n \restriction n$ and $X_n \oplus 0^{(n)} \equiv_T X_0 \oplus 0^{(n)} \equiv_T X_0^{(n)}$ uniformly for all n and all $X_\omega \in P_\omega$, and hence $X_\omega \oplus 0^{(\omega)} \equiv_T X_0 \oplus 0^{(\omega)} \equiv_T X_0^{(\omega)}$ uniformly for all $X_\omega \in P_\omega$. This completes the proof. □

We now present Harrington's construction of arithmetically incomparable arithmetical singletons.

Theorem 4.4 *There is a pair of arithmetically incomparable Π^0_1 singletons.*

Proof As in the proof of Theorem 3.4, let X_ω, Y_ω be such that $0^{(\omega)} \leq_T X_\omega \leq_T 0^{(\omega+1)}$ and $0^{(\omega)} \leq_T Y_\omega \leq_T 0^{(\omega+1)}$ and such that $X_\omega \not\leq_T Y_\omega$ and $Y_\omega \not\leq_T X_\omega$. Note that X_ω and Y_ω are $\Delta^{0,0^{(\omega)}}_2$ and hence $\Pi^{0,0^{(\omega)}}_2$ singletons. Therefore, by the proof of Lemma 3.1, we may safely assume that X_ω and Y_ω are $\Pi^{0,0^{(\omega)}}_1$ singletons. Apply Lemma 4.3 to $P_\omega = \{X_\omega, Y_\omega\}$ to get a Π^0_1 set P_0 and a homeomorphism $H^\omega_0 : P_\omega \cong P_0$. Let $X_0 = H^\omega_0(X_\omega)$, and let $Y_0 = H^\omega_0(Y_\omega)$. Since $P_0 = \{X_0, Y_0\}$, it follows that X_0 and Y_0 are Π^0_1 singletons. Since $X_\omega \not\leq_T Y_\omega \oplus 0^{(\omega)} \equiv_T Y_0^{(\omega)}$ and $X_\omega \oplus 0^{(\omega)} \equiv_T X_0 \oplus 0^{(\omega)}$, we have $X_0 \not\leq_T Y_0^{(\omega)}$, and similarly $Y_0 \not\leq_T X_0^{(\omega)}$. In particular, X_0 and Y_0 are arithmetically incomparable. □

Finally, we present Harrington's construction of a ranked point which is not an arithmetical singleton. This refutes a conjecture which had been known as McLaughlin's conjecture and which was suggested by the result of Tanaka [15] mentioned in Remark 2 above.

Theorem 4.5 *There is a countable Π^0_1 set P such that some $Z \in P$ is not an arithmetical singleton.*

Proof As in the proof of Theorem 3.5, let P_ω be a countable Π^0_1 set such that some $Z_\omega \in P_\omega$ is not isolated in P_ω. Apply Lemma 4.3, and note that P_0 is a countable Π^0_1 set and that $Z_0 = H^\omega_0(Z_\omega) \in P_0$ is not isolated in P_0. We claim that Z_0 is not an arithmetical singleton. Otherwise, let i be such that $\{Z_0\} = \{X \mid X^{(\omega)}(i) = 0\}$. Since $Z_0^{(\omega)}(i) = 0$ and $Z_0 \in P_0$ and $X_0^{(\omega)} \equiv_T X_\omega \oplus 0^{(\omega)}$ uniformly for all $X_\omega \in P_\omega$ and $X_0 = H^\omega_0(X_\omega) \in P_0$, there exists j such that $X_0^{(\omega)}(i) = 0$ for all $X_\omega \in P_\omega$ such that $Z_\omega \restriction j \subset X_\omega$. But Z_ω is not isolated in P_ω, so there exists $X_\omega \in P_\omega$ such that $Z_\omega \restriction j \subset X_\omega$ and $X_\omega \neq Z_\omega$. Thus $X_0^{(\omega)}(i) = 0$ and $X_0 \neq Z_0$, which is a contradiction. Letting $P = P_0$ and $Z = Z_0$, we obtain our theorem. □

Remark 12 Modifying the proof of Lemma 4.3, it is easy to replace ω by a small recursive ordinal such as $\omega + \omega$ or $\omega \cdot \omega$ or ω^ω. Harrington [7] and Gerdes [5] have shown that Lemma 4.3 and consequently Theorems 4.4 and 4.5 hold generally with ω replaced by any recursive ordinal.

References

[1] Ash, C. J., and J. F. Knight, *Computable Structures and the Hyperarithmetical Hierarchy*, vol. 144 of *Studies in Logic and the Foundations of Mathematics*, North-Holland, Amsterdam, 2000. MR 1767842. 330, 331

[2] Cole, J. A., and S. G. Simpson, "Mass problems and hyperarithmeticity," *Journal of Mathematical Logic*, vol. 7 (2007), pp. 125–43. Zbl 1150.03013. MR 2423947. DOI 10.1142/S0219061307000652. 333

[3] Feferman, S., "Some applications of the notions of forcing and generic sets," *Fundamenta Mathematicae*, vol. 56 (1964/1965), pp. 325–45. Zbl 0129.26401. MR 0176925. 330

[4] Fokina, E. B., S.-D. Friedman, and A. Törnquist, "The effective theory of Borel equivalence relations," *Annals of Pure and Applied Logic*, vol. 161 (2010), pp. 837–50. Zbl 1223.03031. MR 2601014. DOI 10.1016/j.apal.2009.10.002. 330

[5] Gerdes, P. M., "Harrington's solution to McLaughlin's conjecture and non-uniform self-moduli," preprint, arXiv:1012.3427v2 [math.LO]. 330, 331, 337

[6] Harrington, L. A., "Arithmetically incomparable arithmetic singletons," handwritten, 28 pp., April 1975. 330, 331

[7] Harrington, L. A., "McLaughlin's conjecture," handwritten, 11 pp., September 1976. 330, 331, 337

[8] Hinman, P. G., and T. A. Slaman, "Jump embeddings in the Turing degrees," *Journal of Symbolic Logic*, vol. 56 (1991), pp. 563–91. Zbl 0745.03036. MR 1133085. DOI 10.2307/2274700. 330

[9] Montalbán, A., "There is no ordering on the classes in the generalized high/low hierarchies," *Archive for Mathematical Logic*, vol. 45 (2006), pp. 215–31. Zbl 1099.03030. MR 2209744. DOI 10.1007/s00153-005-0304-0. 330

[10] Odifreddi, P., *Classical Recursion Theory, Vol. II*, vol. 143 of *Studies in Logic and the Foundations of Mathematics*, North-Holland, Amsterdam, 1999. MR 1718169. 330, 331

[11] Rogers, H., Jr., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, New York, 1967. MR 0224462. 330, 331, 332, 333, 334, 336

[12] Sacks, G. E., "On a theorem of Lachlan and Martin," *Proceedings of the American Mathematical Society*, vol. 18 (1967), pp. 140–41. Zbl 0154.00603. MR 0207558. 331

[13] Sacks, G. E., *Higher Recursion Theory*, vol. 2 of *Perspectives in Mathematical Logic*, Springer, Berlin, 1990. MR 1080970. DOI 10.1007/BFb0086109. 330

[14] Simpson, M. F., "Arithmetic degrees: Initial segments, ω-REA operators and the ω-jump (recursion theory)," Ph.D. dissertation, Cornell University, Ithaca, N.Y., 1985. MR 2634405. 330, 331

[15] Tanaka, H., "A property of arithmetic sets," *Proceedings of the American Mathematical Society*, vol. 31 (1972), pp. 521–24. Zbl 0251.02044. MR 0286661. 330, 331, 337

Acknowledgments

This paper is based on a two-hour tutorial of the same title given February 22–23, 2013 at the Sendai Logic School, Tohoku University, Sendai, Japan. The author is grateful to Professor Kazuyuki Tanaka for organizing the school and inviting him to Sendai. The author's research is supported by the Eberly College of Science at the Pennsylvania State University, and by Simons Foundation Collaboration Grant 276282.

Department of Mathematics
Vanderbilt University
Nashville, Tennessee 37240
USA
and
Department of Mathematics
Pennsylvania State University
University Park, Pennsylvania 16802
USA
simpson@math.psu.edu
http://www.math.psu.edu/simpson