Functional Dependence in Strategic Games

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Abstract   The article studies properties of functional dependencies between strategies of players in Nash equilibria of multiplayer strategic games. The main focus is on the properties of functional dependencies in the context of a fixed dependency graph for payoff functions. A logical system describing properties of functional dependence for any given graph is proposed and is proven to be complete.

1 Introduction

Functional dependence. In this article we study dependency between players' strategies in Nash equilibria. For example, the coordination game described by Table 1 has two Nash equilibria: (a_1, b_1) and (a_2, b_2). Knowing the strategy of player a in a Nash equilibrium of this game, one can predict the strategy of player b. We say that player a *functionally determines* player b and denote this by $a \rhd b$.

Note that, in the case of the coordination game, we also have $b \rhd a$. However, for the game described by Table 2, the statement $a \rhd b$ is true, but $b \rhd a$ is false.

The main focus of this article is functional dependence in multiplayer games. For example, consider a "parity" game with three players a, b, c. Each of the players picks 0 or 1, and all players are rewarded if the sum of all three numbers is even. This game has four different Nash equilibria: $(0, 0, 0)$, $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$. It is easy to see that knowledge of any two players' strategies in a Nash equilibrium

Table 1 Coordination game.

	b_1	b_2
a_1	1, 1	0, 0
a_2	0, 0	1, 1

Table 2 Strategic game.

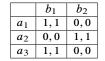|	b_1	b_2
a_1	1, 1	0, 0
a_2	0, 0	1, 1
a_3	1, 1	0, 0

Figure 1 Dependency graph Γ_1.

reveals the third. Thus, using our notation, for example, $a, b \rhd c$. At the same time, $\neg(a \rhd c)$.

As another example, consider a game between three players in which each player picks 0 or 1, and all players are rewarded if they have chosen the same strategy. This game has only two Nash equilibria: $(0, 0, 0)$ and $(1, 1, 1)$. Thus, knowledge of the strategy of player a in a Nash equilibrium reveals the strategies of the two other players. We write this as $a \rhd b, c$.

Functional dependence as a relation has been studied previously, especially in the context of database theory. Armstrong [1] presented the following sound and complete axiomatization of this relation:

  (1) *Reflexivity*: $A \rhd B$ if $B \subseteq A$,
  (2) *Augmentation*: $A \rhd B \to A, C \rhd B, C$,
  (3) *Transitivity*: $A \rhd B \to (B \rhd C \to A \rhd C)$,

where here and everywhere below A, B denotes the union of sets A and B. The above axioms are known in database literature as Armstrong's axioms (see Garcia-Molina, Ullman, and Widom [5]). Beeri, Fagin, and Howard [2] suggested a variation of Armstrong's axioms that describe properties of multivalued dependence.

Dependency graphs. As a side result, we will show that the logical system formed by the Armstrong axioms is sound and complete with respect to the strategic game semantics. Our main result, however, is a sound and complete axiomatic system for the relation $\rhd$ in games with a given dependency graph.

Dependency graphs (see Kearns, Littman, and Singh [7], [8] and Elkind, Goldberg, and Goldberg [3], [4]) put restrictions on the payoff functions that can be used in the game. For example, dependency graph Γ_1, depicted in Figure 1, specifies that the payoff function of player a can only depend on the strategy of player b in addition to the strategy of player a himself. The payoff function for player b can only depend on the strategies of players a and c in addition to the strategy of player b himself, and so on.

An example of a game over graph Γ_1 is a game between players a, b, c, and d in which these players choose real numbers as their strategies. The payoff function of players a and d is the constant 0. Player b is rewarded if his value is equal to the mean of the values of players a and c. Player c is rewarded if his value is equal to the mean of the values of players b and d. Thus, Nash equilibria of this game are all quadruples (a, b, c, d) such that $2b = a + c$ and $2c = b + d$. Hence, in this game $a, b \rhd c, d$ and $a, c \rhd b, d$, but $\neg(a \rhd b)$.

Note that although the statement $a, b \rhd c, d$ is true for the game described above, it is not true for many other games with the same dependency graph Γ_1. In this article we study properties of functional dependence that are common to all games with the same dependency graph. An example of such a statement for the graph Γ_1, as we will show in Proposition 1, is $a \rhd d \to b, c \rhd d$.

Informally, this property is true for any game over graph Γ_1 because any dependencies between players a and d must be established through players b and c. This intuitive approach, however, does not always lead to the right conclusion. For example, in graph Γ_2 depicted in Figure 2, players b and c also separate players a and d. Thus, according to the same intuition, the statement $a \rhd d \to b, c \rhd d$ must also be true for any game over graph Γ_2. This, however, is not true. Consider, for example, a game in which all four players have three strategies: *rock*, *paper*, and *scissors*. The payoff function of players a and d is the constant 0. If a and d pick the same strategy, then neither b nor c is paid. If players a and d pick different strategies, then players b and c are paid according to the rules of the standard rock-paper-scissors game. In this game, Nash equilibrium is only possible if a and d pick the same strategy. Hence, $a \rhd d$. At the same time, in any such equilibria b and c can have any possible combination of values. Thus, $\neg(b, c \rhd d)$. Therefore, the statement $a \rhd d \to b, c \rhd d$ is not true for this game.

As our final example, consider the graph Γ_3 depicted in Figure 3. We will show that $a \rhd c \to b \rhd c$ is not true for at least one game over graph Γ_3. Indeed, consider the game in which players a, b, and c use real numbers as possible strategies. Players a and c have a constant payoff of 0. The payoff of player b is equal to 0 if players a and c choose the same real number. Otherwise, it is equal to the number chosen by player b himself. Note that, in any Nash equilibrium of this game, the strategies of players a and c are equal. Therefore, $a \rhd c$, but $\neg(b \rhd c)$.

The main result of this article is a sound and complete axiomatization of all properties of functional dependence for any given dependency graph. This result is closely related to work by More and Naumov [9] on functional dependence of secrets over hypergraphs. However, the logical system presented in this article is significantly different from theirs. Another form of dependency between players in strategic games is *rationally functional* dependence introduced and completely axiomatized by Naumov and Nicholls [10]. A set of players A rationally determines a
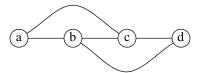

Figure 2 Dependency graph Γ_2.

Figure 3 Dependency graph Γ_3.

set of players B if, for each public commitment of all players in set A to a particular choice of strategies, players in set B have only one rational choice of strategies. As shown in [10], neither the Augmentation axiom nor the Transitivity axiom is valid for rationally functional dependence, which makes it a significantly different relation from the one studied in this article. A preliminary version of the current work (without proof of completeness) appeared as [6].

The counterexample that we have constructed for the game in Figure 3 significantly relies on the fact that player b has infinitely many strategies. However, in this article we show completeness with respect to the semantics of finite games, making the result stronger.

2   Syntax and Semantics

The graphs that we consider in this article contain no loops, multiple edges, or directed edges.

Definition 1   For any set of vertices U of a graph (V, E), the border $\mathcal{B}(U)$ is the set

$$\{v \in U \mid (v, w) \in E \text{ for some } w \in V \setminus U\}.$$

A cut (U, W) of a graph (V, E) is a partition $U \sqcup W$ of the set V. For any vertex v in a graph, by $\text{Adj}(v)$ we mean the set of all vertices adjacent to v. By $\text{Adj}^+(v)$ we mean the set $\text{Adj}(v) \cup \{v\}$.

Definition 2   For any graph $\Gamma = (V, E)$, by $\Phi(\Gamma)$ we mean the minimal set of formulas such that (i) $\bot \in \Phi(\Gamma)$, (ii) $A \rhd B \in \Phi(\Gamma)$ for each $A \subseteq V$ and $B \subseteq V$, and (iii) $\phi \to \psi \in \Phi(\Gamma)$ for each $\phi, \psi \in \Phi(\Gamma)$.

Definition 3   By a game over graph $\Gamma = (V, E)$ we mean any strategic game $G = (V, \{S_v\}_{v \in V}, \{u_v\}_{v \in V})$ such that (i) the finite set of players in the game is the set of vertices V, (ii) the finite set of strategies S_v of any player v is an arbitrary set, and (iii) the payoff function u_v of any player v only depends on the strategies of the players in $\text{Adj}^+(v)$.

By $NE(G)$ we denote the set of all Nash equilibria in the game G. The next definition is the core definition of this article. The second item in the list below gives a precise meaning of the functional dependence predicate $A \rhd B$.

Definition 4   For any game G over graph Γ and any $\phi \in \Phi(\Gamma)$, we define the binary relation $G \vDash \phi$ as follows: (i) $G \nvDash \bot$, (ii) $G \vDash A \rhd B$ if $\mathbf{s} =_A \mathbf{t}$ implies $\mathbf{s} =_B \mathbf{t}$ for each $\mathbf{s}, \mathbf{t} \in NE(G)$, and (iii) $G \vDash \psi_1 \to \psi_2$ if $G \nvDash \psi_1$ or $G \vDash \psi_2$, where here and everywhere below $\langle s_v \rangle_{v \in V} =_X \langle t_v \rangle_{v \in V}$ means that $s_x = t_x$ for each $x \in X$.

3   Axioms

The following is the set of axioms of our logical system. It consists of the original Armstrong axioms and an additional Contiguity axiom that captures properties of functional dependence specific to a given graph Γ.

(1) Reflexivity: $A \rhd B$, where $B \subseteq A$.
(2) Augmentation: $A \rhd B \to A, C \rhd B, C$.
(3) Transitivity: $A \rhd B \to (B \rhd C \to A \rhd C)$.

(4) Contiguity: $A, B \rhd C \rightarrow \mathcal{B}(U), \mathcal{B}(W), B \rhd C$, where (U, W) is a cut of the graph such that $A \subseteq U$ and $C \subseteq W$.

Note that the Contiguity axiom, unlike the Gateway axiom (see [9]), effectively requires "double layer" divider $\mathcal{B}(U), \mathcal{B}(W)$ between sets A and C. This is because, in our setting, values are assigned to the vertices and not to the edges of the graph.

We write $\vdash_\Gamma \phi$ if $\phi \in \Phi(\Gamma)$ is provable from the combination of the axioms above and propositional tautologies in the language $\Phi(\Gamma)$ using the modus ponens inference rule. We write $X \vdash_\Gamma \phi$ if ϕ is provable using the additional set of axioms X. We often omit the parameter Γ when its value is clear from the context.

4 Examples

In this section we give examples of proofs in our formal system. The soundness and the completeness of this system will be shown in the next two sections.

Proposition 1 $\vdash_{\Gamma_1} a \rhd d \rightarrow b, c \rhd d$, where Γ_1 is the graph depicted in Figure 1.

Proof   Consider cut (U, W) of the graph Γ_1 such that $U = \{a, b\}$ and $W = \{c, d\}$. Thus, $\mathcal{B}(U) = \{b\}$ and $\mathcal{B}(W) = \{c\}$. Therefore, by the Contiguity axiom, $a \rhd d \rightarrow b, c \rhd d$. $\square$

Proposition 2 $\vdash_{\Gamma_1} a, c \rhd d \rightarrow (d, b \rhd a \rightarrow b, c \rhd a, d)$, where Γ_1 is the graph depicted in Figure 1.

Proof   Assume that $a, c \rhd d$ and $d, b \rhd a$. Consider cut (U, W) of the graph Γ_1 such that $U = \{a, b\}$ and $W = \{c, d\}$. Thus, $\mathcal{B}(U) = \{b\}$ and $\mathcal{B}(W) = \{c\}$. Therefore, by the Contiguity axiom with $A = \{a\}$, $B = \{c\}$, and $C = \{d\}$, $a, c \rhd d \rightarrow b, c \rhd d$. Thus,

$$b, c \rhd d \tag{1}$$

by the first assumption. Similarly, using the second assumption, $b, c \rhd a$. Hence, by the Augmentation axiom,

$$b, c \rhd a, b, c. \tag{2}$$

Thus, from statement (1) by the Augmentation axiom, $a, b, c \rhd a, d$. Finally, by statement (2) and the Transitivity axiom, $b, c \rhd a, d$. $\square$

Proposition 3 $\vdash_{\Gamma_4} a, c \rhd e \rightarrow b, c, d \rhd e$, where Γ_4 is the graph depicted in Figure 4.

Proof   Consider cut (U, W) of the graph Γ_4 such that $U = \{a, b, c\}$ and $W = \{d, e\}$. Thus, $\mathcal{B}(U) = \{b, c\}$ and $\mathcal{B}(W) = \{d\}$. Therefore, $a, c \rhd e \rightarrow b, c, d \rhd e$ by the Contiguity axiom with $A = \{a\}$, $B = \{c\}$, and $C = \{e\}$. $\square$

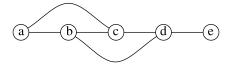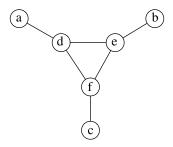**Figure 4** Dependency graph Γ_4.

Figure 5 Dependency graph Γ_5.

Proposition 4 $\vdash_{\Gamma_5} a \rhd b \to (b \rhd c \to (c \rhd a \to d, e, f \rhd a, b, c))$, *where* Γ_5 *is depicted in Figure 5.*

Proof   Assume $a \rhd b$, $b \rhd c$, and $c \rhd a$. Consider cut (U, W) of the graph Γ_5 such that $U = \{c, f\}$ and $W = \{a, b, d, e\}$. Thus, $\mathcal{B}(U) = \{f\}$ and $\mathcal{B}(W) = \{d, e\}$. Therefore, by the Contiguity axiom with $A = \{c\}$, $B = \varnothing$, and $C = \{a\}$, $c \rhd a \to d, e, f \rhd a$. Hence, $d, e, f \rhd a$ by the third assumption. Similarly, one can show that $d, e, f \rhd b$ and that $d, e, f \rhd c$. By applying the Augmentation axiom to the last three statements, $d, e, f \rhd a, d, e, f$, and $a, d, e, f \rhd a, b, d, e, f$, and $a, b, d, e, f \rhd a, b, c$. Therefore, $d, e, f \rhd a, b, c$ by the Transitivity axiom applied twice.   □

Propositions 2 and 4 are special cases of a more general principle. We will say that a subset of vertices is *sparse* if the shortest path between any two vertices in this subset contains at least three edges. The general principle states that if W is a sparse subset of vertices in the graph (V, E) and each vertex $w \in W$ is functionally determined by the set $V \setminus \{w\}$, then the subset $V \setminus W$ functionally determines the subset W:

$$\bigwedge_{w \in W} ((V \setminus \{w\})) \rhd w \to (V \setminus W) \rhd W.$$

For example, the set $\{a, d\}$ in the graph Γ_1 depicted in Figure 1 is sparse. Due to the general principle, $a, b, c \rhd d \to (d, c, b \rhd a \to b, c \rhd a, d)$. Thus, by Lemma 6, $a, c \rhd d \to (d, b \rhd a \to b, c \rhd a, d)$, which is the statement of Proposition 2. In the case of Proposition 4, the sparse set is $\{a, b, c\}$. The proof of the general principle is similar to the proof of Proposition 4.

5   Soundness

We prove the soundness of our logical system by proving the soundness of each of our four axioms separately.

Lemma 1 (Reflexivity) $G \models A \rhd B$ *for each game G over a graph $\Gamma = (V, E)$ and each $B \subseteq A \subseteq V$.*

Proof   For any $\mathbf{s}, \mathbf{t} \in NE(G)$, if $\mathbf{s} =_A \mathbf{t}$, then $\mathbf{s} =_B \mathbf{t}$ because $A \subseteq B$.   □

Lemma 2 (Augmentation)   *If $G \models A \rhd B$, then $G \models A, C \rhd B, C$ for each game G over a graph $\Gamma = (V, E)$ and each $A, B, C \subseteq V$.*

Proof   Suppose that $G \vDash A \rhd B$, and consider any $\mathbf{s}, \mathbf{t} \in NE(G)$ such that $\mathbf{s} =_{A,C} \mathbf{t}$. We will show that $\mathbf{s} =_{B,C} \mathbf{t}$. Indeed, $\mathbf{s} =_{A,C} \mathbf{t}$ implies that $\mathbf{s} =_A \mathbf{t}$ and $\mathbf{s} =_C \mathbf{t}$. Thus, $\mathbf{s} =_B \mathbf{t}$ by the assumption $G \vDash A \rhd B$. Therefore, $\mathbf{s} =_{B,C} \mathbf{t}$. □

Lemma 3 (Transitivity)   *If $G \vDash A \rhd B$ and $G \vDash B \rhd C$, then $G \vDash A \rhd C$ for each game G over a graph $\Gamma = (V, E)$ and each $A, B, C \subseteq V$.*

Proof   Suppose that $G \vDash A \rhd B$ and $G \vDash B \rhd C$. Consider any $\mathbf{s}, \mathbf{t} \in NE(G)$ such that $\mathbf{s} =_A \mathbf{t}$. We will show that $\mathbf{s} =_C \mathbf{t}$. Indeed, $\mathbf{s} =_B \mathbf{t}$ due to the first assumption. Hence, by the second assumption, $\mathbf{s} =_C \mathbf{t}$. □

Lemma 4 (Contiguity)   *If $G \vDash A, B \rhd C$, then $G \vDash \mathcal{B}(S), \mathcal{B}(T), B \rhd C$, for each game $G = (V, E)$ over a graph Γ, each cut (U, W) of Γ, and each $A \subseteq U$, $B \subseteq V$, and $C \subseteq W$.*

Proof   Suppose that $G \vDash A, B \rhd C$. Consider any $\mathbf{s} = \langle s_v \rangle_{v \in V} \in NE(G)$ and $\mathbf{t} = \langle t_v \rangle_{v \in V} \in NE(G)$ such that $\mathbf{s} =_{\mathcal{B}(U), \mathcal{B}(W), B} \mathbf{t}$. We will prove that $\mathbf{s} =_C \mathbf{t}$. Indeed, consider the strategy profile $\mathbf{e} = \langle e_v \rangle_{v \in V}$ such that

$$e_v = \begin{cases} s_v & \text{if } v \in U, \\ t_v & \text{if } v \in W. \end{cases}$$

We will first prove that $\mathbf{e} \in NE(G)$. Assuming the opposite, let $v \in V$ be a player in the game G that can increase his payoff by changing strategy in profile $\mathbf{e}$. Without loss of generality, let $v \in U$. Then, $\mathbf{e} =_{\text{Adj}(v) \cup \{v\}} \mathbf{s}$. Thus, player v can also increase his payoff by changing strategy in profile $\mathbf{s}$, which is a contradiction with the choice of $\mathbf{s} \in NE(G)$.

Note that $\mathbf{e} =_{U,B} \mathbf{s}$ and $\mathbf{e} =_{W,B} \mathbf{t}$. Thus, $\mathbf{e} =_{A,B} \mathbf{s}$ and $\mathbf{e} =_C \mathbf{t}$. Hence, $\mathbf{e} =_C \mathbf{s}$ by the assumption $G \vDash A, B \rhd C$. Therefore, $\mathbf{s} =_C \mathbf{e} =_C \mathbf{t}$. □

6 Completeness

In this section we prove the completeness of our logical system with respect to the game semantics described above. Given a fixed dependency graph Γ and a formula $\varphi \in \Phi(\Gamma)$, not provable in our logical system for the graph Γ, the proof constructs a canonical strategic game over Γ that falsifies formula φ. The canonical game is composed of several smaller games over Γ played concurrently and independently. The composition operation is the same as the one used in the proof of completeness for rational functional dependency (see [10, p. 612]) and similar to the construction in the proof of completeness for functional dependence of secrets over hypergraphs (see [9, p. 36]). The smaller games, however, are significantly different from [10]. Each of the smaller games, in turn, consists of matching pennies games played over edges of the dependency graph Γ. We start the proof with several technical lemmas.

Lemma 5 $\mathcal{B}(X \cup Y) \subseteq \mathcal{B}(X) \cup \mathcal{B}(Y)$.

Proof   Let $v \in \mathcal{B}(X \cup Y)$. Thus, $v \in X \cup Y$ and there is $w \notin X \cup Y$ such that $(v, w) \in E$. Without loss of generality, assume that $v \in X$. Hence, $v \in X$ and $w \notin X$. Therefore, $v \in \mathcal{B}(X)$. □

Lemma 6 $\vdash A \rhd C \rightarrow A, B \rhd C$.

Proof   Assume $A \rhd C$. By the Reflexivity axiom, $A, B \rhd A$. Thus, by the Transitivity axiom, $A, B \rhd C$. □

Lemma 7 $\vdash A \rhd B, C \rightarrow A \rhd B$.

Proof   Assume $A \rhd B, C$. By the Reflexivity axiom, $B, C \rhd B$. Thus, by the Transitivity axiom, $A \rhd B$.   □

Theorem 1   *For any graph $\Gamma = (V, E)$ and any formula $\phi \in \Phi(V)$, if $\nvdash_\Gamma \phi$, then there must exist a game $(V, \{S_v\}_{v \in V}, \{u_v\}_{v \in V})$ over graph Γ such that $G \nvDash \phi$.*

Proof   Suppose that $\nvdash_\Gamma \phi$. Let M be any maximal consistent subset of formulas in $\Phi(\Gamma)$ such that $\neg\phi \in M$.

Definition 5   For any set of vertices A, let A^* be the set $\{v \in V \mid M \vdash A \rhd v\}$.

Lemma 8 $A \subseteq A^*$, *for any $A \subseteq V$.*

Proof   Let $a \in A$. By the Reflexivity axiom, $\vdash A \rhd a$. Hence, $a \in A^*$.   □

Lemma 9 $M \vdash A \rhd A^*$, *for any $A \subseteq V$.*

Proof   Let $A^* = \{a_1, \ldots, a_n\}$. By the definition of A^*, $M \vdash A \rhd a_i$, for any $i \leq n$. We will prove, by induction on k, that $M \vdash (A \rhd a_1, \ldots, a_k)$ for any $0 \leq k \leq n$.

   Base case. $M \vdash A \rhd \varnothing$ by the Reflexivity axiom.

   Induction step. Assume that $M \vdash (A \rhd a_1, \ldots, a_k)$. By the Augmentation axiom,

$$M \vdash A, a_{k+1} \rhd a_1, \ldots, a_k, a_{k+1}. \tag{3}$$

Recall that $M \vdash A \rhd a_{k+1}$. Again by the Augmentation axiom, $M \vdash (A \rhd A, a_{k+1})$. Hence, $M \vdash (A \rhd a_1, \ldots, a_k, a_{k+1})$, by (3) and the Transitivity axiom.   □

For any set of vertices A, we will now define the strategic game

$$G_A = (V, \{S_v\}_{v \in V}, \{u_v\}_{v \in V})$$

over graph Γ. For the purposes of this definition only, we assume that a direction is assigned to each edge of the graph Γ in an arbitrary way.

   Any player $v \in A^*$ may either choose strategy *pass* or opt to play "pennies" with all of his adjacent players. In the latter case, he decides on either *heads* or *tails* for each adjacent player. The player cannot choose to pass with one player and to play pennies with others. Formally, if $v \in A^*$, then

$$S_v = \{pass\} \cup \{f \mid f : \text{Adj}(v) \rightarrow \{heads, \, tails\}\}.$$

Similarly, any player $v \notin A^*$ may either choose between strategies 0 and 1 or decide to play pennies with all of his adjacent players. Thus, if $v \notin A^*$, then

$$S_v = \{0, 1\} \cup \{f \mid f : \text{Adj}(v) \rightarrow \{heads, \, tails\}\}.$$

Furthermore, it will be assumed that any isolated (one that has no adjacent vertices) vertex of the graph is prohibited from playing the pennies game. Thus such vertices either have a single strategy *pass* or a set of just two strategies: 0 and 1.

   We define the payoff function of any player v as the sum of rewards in individual pennies minigames on the edges adjacent to v or a possible penalty imposed on v for not playing the pennies.

   Penalty. If there are $u, w \in \text{Adj}^+(v) \setminus A^*$ such that u plays strategy 0 and w plays strategy 1, then a penalty in the amount of 1 is imposed on v unless v plays pennies.

For example, consider the strategy profile depicted in Figure 6. We will assume that elements of A^* are the shaded vertices. Vertices w, x, and y are subject to the penalty because player x plays 0 and player y plays 1. Players u and v have chosen to play pennies, and thus are not subject to any penalties.

More formally, for any profile $\mathbf{s} = \langle s_v \rangle_{v \in V}$, penalty of the player $v \in V$ is defined as follows:

$$penalty_v(\mathbf{s}) = \begin{cases} 1 & \text{if there are } u, w \in \text{Adj}^+(v) \setminus A^* \text{ such that} \\ & s_u = 0, s_w = 1, \text{ and } s_v \in \{0, 1, pass\}, \\ 0 & \text{otherwise.} \end{cases}$$

Rewards. In an individual minigame along any edge of the graph Γ, rewards are only given if both players are playing pennies. The rewards are given according to the rules of a variation of the standard matching pennies game: the player who is, say, at the beginning of the directed edge is rewarded 1 for matching his opponent's strategy and the player at the opposite end of the edge is rewarded 1 for not matching his opponent's strategy.

For example, in the strategy profile depicted in Figure 6, player u gets the reward 1 for matching player v. Player v is not rewarded. Players x, y, and w also do not receive any rewards since they are not playing pennies.

More formally, for any profile $\mathbf{s} = \langle s_v \rangle_{v \in V}$, reward of the player $v \in V$ is defined as follows:

$$reward_v(\mathbf{s}) = |u \in \{(v, u) \in E \mid s_u \neq s_v\}| + |u \in \{(u, v) \in E \mid s_u = s_v\}|.$$

Finally, the payoff function of player v is $u_v(\mathbf{s}) = reward_v(\mathbf{s}) - penalty_v(\mathbf{s})$. This concludes the definition of the game G_A.

Lemma 10   *In any Nash equilibrium of the game G_A, no player is playing the pennies game.*

Proof   Assume that a vertex v is playing pennies games in a strategy profile $\mathbf{s}$. Due to the definition of the set of strategies in the game G_A, vertex v cannot be an isolated vertex in the graph Γ. Let u be any vertex adjacent to v. If player u is not playing pennies in $\mathbf{s}$, then he can increase his payoff by playing pennies. If player u is playing pennies in $\mathbf{s}$, then either player u or player v would want to switch his strategy in the minigame along edge (u, v) since the two-player matching pennies game has no Nash equilibria. Therefore, strategy profile $\mathbf{s}$ is not a Nash equilibrium.   □
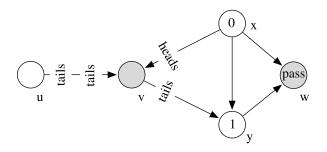

Figure 6 Strategy profile.

Lemma 11 $s_{w_1} = s_{w_2}$ *for each* $w_1, w_2 \in \mathrm{Adj}^+(v) \setminus A^*$, *each* $v \in V$, *and each*
$\mathbf{s} = \langle s_w \rangle_{w \in V} \in NE(G_A)$.

Proof   By Lemma 10, no player is playing pennies in profile $\mathbf{s}$. Suppose that
$s_{w_1} \neq s_{w_2}$ for some $w_1, w_2 \in \mathrm{Adj}^+(v) \setminus A^*$. Thus, player v is subject to penalty
in the strategy profile $\mathbf{s}$. Then, he can increase his payoff by starting to play pennies
and avoiding the penalty. Therefore, $\mathbf{s} \notin NE(G_A)$.   □

Definition 6   For any $u, v \in V$, let $u \sim v$ if there is a path from u to v in graph Γ
such that no two consecutive vertices of the path belong to the set A^*.

Lemma 12   *Relation* $\sim$ *is an equivalence relation on the set* V.

By $[v]$ we will denote the equivalence class of vertex v with respect to this relation.

Lemma 13   *If* $u \sim v$, *then* $s_u = s_v$, *for each* $u, v \notin A^*$ *and each* $\langle s_w \rangle_{w \in V} \in$
$NE(G_A)$.

Proof   Let $\pi = (w_0, w_1, \dots, w_k)$ be a path connecting vertices u and v ($w_0 = u$
and $w_k = v$) such that no two consecutive vertices in π belong to A^*. We prove the
statement by induction on k. If $k = 0$, then $u = v$. Thus, $s_u = s_v$. Suppose now
that $k > 0$.
     Case I: $w_1 \notin A^*$. Thus, by Lemma 11, $s_u = s_{w_1}$. By the induction hypothesis,
$s_{w_1} = s_v$. Therefore, $s_u = s_v$.
     Case II: $w_1 \in A^*$. Thus, $w_1 \neq v$ and, since no two consecutive vertices in π
belong to A^*, we have $w_2 \notin A^*$. Hence, $s_u = s_{w_2}$ by Lemma 11 and $s_{w_2} = s_v$ by
the induction hypothesis. Therefore, $s_u = s_v$.   □

Lemma 14 $\mathcal{B}([v]) \subseteq A^*$ *for each* $v \in V$.

Proof   Let $w \in \mathcal{B}([v])$, but $w \notin A^*$. By Definition 1, there is $u \notin [v]$ such
that $(w, u) \in E$. Consider the two-vertex path $\pi = (w, u)$. Since $w \notin A^*$, no
two consecutive vertices in π belong to A^*. Hence, $w \sim u$. Note that $v \sim w$ by
Definition 1. Thus, $v \sim u$, which is a contradiction.   □

Lemma 15   *If* $C \rhd D \in M$, *then* $G_A \vDash C \rhd D$.

Proof   Let $\mathbf{s} = \langle s_v \rangle_{v \in V}, \mathbf{s}' = \langle s'_v \rangle_{v \in V} \in NE(G_A)$ such that $\mathbf{s} =_C \mathbf{s}'$. It will be
sufficient to show that $s_d = s'_d$ for each $d \in D$. Consider any $d \in D$. If $d \in A^*$,
then player d has only two options in the game G_A: to play pennies or to decide to
pass. By Lemma 10, player d chooses the strategy *pass* under strategy profiles $\mathbf{s}$ and
$\mathbf{s}'$. Therefore, $s_d = s'_d$. We will now assume that $d \notin A^*$.
     If $d \sim c_0$ for some $c_0 \in C \setminus A^*$, then, by Lemma 13, $s_d = s_{c_0} = s'_{c_0} = s'_d$.
We will now assume that $d \not\sim c$ for each $c \in C \setminus A^*$. Thus, $C \setminus A^* \subseteq \bigcup_{v \notin [d]} [v]$.
Consider cut $(\bigcup_{v \notin [d]} [v], [d])$. By the Contiguity axiom,

$$C \setminus A^*, A^* \rhd d \rightarrow \mathcal{B}\Big(\bigcup_{v \notin [d]} [v] \Big), \mathcal{B}([d]), A^* \rhd d.$$

Due to the assumption $M \vdash C \rhd D$ and Lemmas 6 and 7,

$$M \vdash C \setminus A^*, A^* \rhd d.$$

Thus,

$$M \vdash \mathcal{B}\Big(\bigcup_{v \notin [d]} [v] \Big), \mathcal{B}([d]), A^* \rhd d.$$

By Lemma 5, $\mathcal{B}(\bigcup_{v \notin [d]}[v]) \subseteq \bigcup_{v \notin [d]} \mathcal{B}([v])$. Hence, by Lemma 6,

$$M \vdash \bigcup_{v \notin [d]} \mathcal{B}([v]), \mathcal{B}([d]), A^* \rhd d.$$

Then, by Lemma 14, $M \vdash A^* \rhd d$. Thus, by Lemma 9 and the Transitivity axiom, $M \vdash A \rhd d$. Therefore, $d \in A^*$, which is a contradiction. $\qquad \square$

Definition 7   For any $A \subseteq V$ and any $k \in \{0, 1\}$, let strategy profile $\mathbf{s}^{k,A}$ be defined as

$$s_v^{k,A} = \begin{cases} pass & \text{if } v \in A^*, \\ k & \text{otherwise.} \end{cases}$$

Lemma 16 $\mathbf{s}^{k,A} \in NE(G_A)$ for each $k \in \{0, 1\}$.

Proof   By the definition of the game G_A, no player is paying a penalty in the strategy profile $\mathbf{s}^{k,A}$. At the same time, no player can get a reward by unilaterally switching to playing pennies. $\qquad \square$

Lemma 17   If $G_A \vDash A \rhd b$, then $b \in A^*$.

Proof   Assume that $b \notin A^*$. By Lemma 8, $\mathbf{s}^{0,A} =_A \mathbf{s}^{1,A}$. At the same time, $s_b^{0,A} = 0 \neq 1 = s_b^{1,A}$ since $b \notin A^*$. Therefore, $G_A \nvDash A \rhd b$. $\qquad \square$

The product construction below defines a way to combine several games played over the same graph into a single game. The payoff for a given player in the combined game is the sum of his payoffs in the individual games.

Definition 8   Let $G^i = (V, \{S_v^i\}_{v \in V}, \{u_v^i\}_{v \in V})$ for $i \in I$ be any family of games over the same graph $\Gamma = (V, E)$. By $\prod_{i \in I} G^i$ we mean game $(V, \{S_v\}_{v \in V}, \{u_v\}_{v \in V})$ such that

(1) S_v is the Cartesian product $\prod_{i \in I} S_v^i$,
(2) $u_v = \sum_{i \in I} u_p^i$.

Lemma 18   If $\langle\langle s_v^i \rangle_{i \in I}\rangle_{v \in V} \in NE(\prod_{i \in I} G^i)$, then $\langle s_v^{i_0}\rangle_{v \in V} \in NE(G^{i_0})$ for each $i_0 \in I$.

Lemma 19   If $\langle s_v^i\rangle_{v \in V} \in NE(G^i)$ for each $i \in I$, then

$$\langle\langle s_v^i\rangle_{i \in I}\rangle_{v \in V} \in NE\Big(\prod_{i \in I} G^i\Big).$$

Lemma 20   If $\{G^i\}_{i \in I}$ is a family of games over a graph Γ such that each of these games has a nonempty set of Nash equilibria, then $\prod_{i \in I} G^i \vDash C \rhd D$ if and only if, for each $i \in I$, $G^i \vDash C \rhd D$.

Proof   ($\Rightarrow$): Assume that $\mathbf{s}^{i_0} = \langle s_v^{i_0}\rangle_{v \in V} \in NE(G^{i_0})$ and $\mathbf{t}^{i_0} = \langle t_v^{i_0}\rangle_{v \in V} \in NE(G^{i_0})$ are such that $\mathbf{s}^{i_0} =_C \mathbf{t}^{i_0}$. We will show that $\mathbf{s}^{i_0} =_D \mathbf{t}^{i_0}$.

By the assumption of the lemma, for each $i \in I$, the game G^i has at least one Nash equilibrium. We denote an arbitrary one of them by $\mathbf{e}^i = \langle e_v^i\rangle_{v \in V}$. Consider

strategy profiles $\mathbf{S} = \langle\langle s_v^i \rangle_{i \in I} \rangle_{v \in V}$ and $\mathbf{T} = \langle\langle t_v^i \rangle_{i \in I} \rangle_{v \in V}$ for the game $\prod_{i \in I} G^i$ such that

$$s_v^i = \begin{cases} s_v^{i_0} & \text{if } i = i_0, \\ e_v^i & \text{otherwise,} \end{cases}$$

$$t_v^i = \begin{cases} t_v^{i_0} & \text{if } i = i_0, \\ e_v^i & \text{otherwise.} \end{cases}$$

By Lemma 19, we have $\mathbf{S}, \mathbf{T} \in NE(\prod_{i \in I} G^i)$. Note that $\mathbf{S} =_C \mathbf{T}$ due to the assumption $\mathbf{s}^{i_0} =_C \mathbf{t}^{i_0}$. Hence, $\mathbf{S} =_D \mathbf{T}$ by the assumption of the lemma. Therefore, $\mathbf{s}^{i_0} =_D \mathbf{t}^{i_0}$.

($\Leftarrow$): Consider Nash equilibria $\mathbf{S} = \langle\langle s_v^i \rangle_{i \in I} \rangle_{v \in V}$ and $\mathbf{T} = \langle\langle t_v^i \rangle_{i \in I} \rangle_{v \in V}$ of the game $\prod_{i \in I} G^i$ such that $\mathbf{S} =_C \mathbf{T}$. We will show that $\mathbf{S} =_D \mathbf{T}$. It will be sufficient to show that $\langle s_v^i \rangle_{v \in V} =_D \langle t_v^i \rangle_{v \in V}$ for each $i \in I$. Indeed, $\langle s_v^i \rangle_{v \in V} =_C \langle t_v^i \rangle_{v \in V}$ due to the assumption $\mathbf{S} =_C \mathbf{T}$. By Lemma 18, $\langle s_v^i \rangle_{v \in V}, \langle t_v^i \rangle_{v \in V} \in NE(G^i)$. Therefore, $\langle s_v^i \rangle_{v \in V} =_D \langle t_v^i \rangle_{v \in V}$ by the assumption of the lemma. $\qquad\square$

Lemma 21   *For any $\psi \in \Phi(\Gamma)$, $\psi \in M$ if and only if $\prod_{A \subseteq V} G_A \vDash \psi$.*

Proof   The proof is by induction on the structural complexity of formula ψ. The case $\psi \equiv \bot$ follows from the assumption of consistency of the set M and Definition 4. The case $\psi \equiv \psi_1 \to \psi_2$ follows from the maximality and consistency of the set M and Definition 4 in the standard way. Assume now that $\psi \equiv E \rhd F$.

If $E \rhd F \in M$, then, by Lemma 15, $G_A \vDash E \rhd F$ for each $A \subseteq V$. By Lemma 16, each of the games G_A has at least one Nash equilibrium: $\mathbf{s}^{0,A}$. Therefore, $\prod_{A \subseteq V} G_A \vDash E \rhd F$ by Lemma 20.

If $\prod_{A \subseteq V} G_A \vDash E \rhd F$, then, by Lemma 20, $G_E \vDash E \rhd F$. Hence, by Definition 4, $G_E \vDash E \rhd f$ for each $f \in F$. Hence, by Lemma 17, $f \in E^*$ for each $f \in F$. Thus, $F \subseteq E^*$. Note that $M \vdash E \rhd E^*$ by Lemma 9. Hence, $M \vdash E \rhd F$ by Lemma 7. Therefore, $E \rhd F \in M$ due to the maximality of M. $\qquad\square$

To finish the proof of the theorem, recall that $\neg\phi \in M$. Thus, $\phi \notin M$ due to the consistency of M. Therefore, by Lemma 21, $\prod_{A \subseteq V} G_A \nvDash \phi$. $\qquad\square$

7   Conclusion

In this article we have described a sound and complete logical system for functional dependence in strategic games over a fixed dependency graph. The dependency graph puts restrictions on the type of payoff functions that can be used in the game. If no such restrictions are imposed, then the logical system for functional dependence in strategic games is just the set of original Armstrong axioms. This statement follows from our results since the absence of restrictions corresponds to the case of a complete (in the graph theory sense) dependency graph. In the case of a complete graph, the Contiguity axiom follows from the Armstrong axioms because, for any cut (U, W), the set $\mathcal{B}(U) \cup \mathcal{B}(W)$ is the set of all vertices in the graph.

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