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An Inner Model Proof of the Strong Partition Property for δ^2_1

Grigor Sargsyan

Abstract Assuming $V = L(\mathbb{R})+AD$, using methods from inner model theory, we give a new proof of the strong partition property for $\underset{\sim}{\delta}^2_1$. The result was originally proved by Kechris et al.

The main theorem of this note is the following special case of [3, Theorem 1.1] originally due to Kechris, Kleinberg, Moschovakis, and Woodin.

Theorem 0.1 *Assume $V = L(\mathbb{R}) + \mathrm{AD}$. Then $\underset{\sim}{\delta}^2_1$ has the strong partition property, that is, $\underset{\sim}{\delta}^2_1 \to (\underset{\sim}{\delta}^2_1)^{\underset{\sim}{\delta}^2_1}$ holds.*

Our proof uses techniques from inner model theory and resembles Martin's proof of strong partition property for ω_1 (see Jackson [2]). We expect that it will have other applications and, in particular, can be used to show that under AD^+, if Γ is any Π^1_1-like (i.e., closed under $\forall^{\mathbb{R}}$ and non-self-dual) scaled point class and $\delta = \delta(\Gamma)$, then δ has the strong partition property. Our motivation to find a new proof of Theorem 0.1 comes from a desire to prove Kechris–Martin-like results for Π^1_1-like scaled point classes which will settle Schimmerling [8, Question 19] and most likely, several other questions in the same neighborhood. We are optimistic that inner model-theoretic techniques will settle this question, and our optimism comes from the fact that the literature is already full of descriptive set-theoretic results that have been proved using methods from inner model theory (see, e.g., Hjorth [1], Sargsyan [5], and Steel [11]). More importantly for us, recently, Neeman, in [4], found a proof of the Kechris–Martin theorem for Π^1_3 using techniques from inner model theory. Finally, we believe that our proof can be used to prove the strong partition property for many cardinals $\delta = \delta(\Gamma)$ where Γ has strong closure properties. In fact, we expect that it can be used to prove [3, Theorem 1.1], but we certainly have not done so.

We now start proving Theorem 0.1.

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Proof of Theorem 0.1 Let $\kappa = \underset{\sim}{\delta}^2_1$. By Martin's theorem (see [2, Theorem 2.31, Definition 2.30]), it is enough to show that κ is κ-reasonable, that is, there is a non-self-dual point class $\underset{\sim}{\Gamma}$ closed under $\exists^{\mathbb{R}}$ and a map φ with domain $\mathbb{R}$ satisfying

1. $\forall x(\varphi(x) \subseteq \kappa \times \kappa)$.
2. $\forall F : \kappa \to \kappa, \exists x \in \mathbb{R}(\varphi(x) = F)$.
3. $\forall \beta < \kappa, \forall \gamma < \kappa, R_{\beta,\gamma} \in \underset{\sim}{\Delta}$ where
$$x \in R_{\beta,\gamma} \leftrightarrow \varphi(x)(\beta,\gamma) \wedge \forall \gamma' < \kappa\big(\varphi(x)(\beta,\gamma') \to \gamma' = \gamma\big).$$
4. Suppose $\beta < \kappa$, $A \in \exists^{\mathbb{R}}\underset{\sim}{\Delta}$, and $A \subseteq R_\beta = \{x : \exists\gamma < \kappa R_{\beta,\gamma}(x)\}$. Then $\exists\gamma_0 < \kappa$ such that $\forall x \in A\exists\gamma < \gamma_0 R_{\beta,\gamma}(x)$.

Let $\Gamma = \Sigma^2_1$. We claim that $\underset{\sim}{\Gamma}$ is as desired and spend the rest of the proof to argue for it. In what follows, we will freely use the terminology developed for analyzing HOD of models of AD^+. This terminology has been exposited in many places including Sargsyan [5], [6], [7], Schindler and Steel [9], Steel [11], and more recently in Steel and Woodin [12]. In particular, recall the definitions of suitable premouse, short tree, maximal tree, and short tree iterable. Given a suitable premouse $\mathcal{P}$, we let $\delta_{\mathcal{P}}$ be its Woodin cardinal and let $\lambda_{\mathcal{P}}$ be the least cardinal which is $< \delta_{\mathcal{P}}$-strong in $\mathcal{P}$.

Suppose $a \in HC$. We say that an a-premouse $\mathcal{Q}$ is *good* if

1. $\mathcal{Q}$ is (ω, ω_1)-iterable,
2. $\mathcal{Q} \vDash$ ZFC − Powerset + "there are no Woodin cardinals" + "there is a largest cardinal,"
3. $\mathcal{Q}$ is full, that is, for every cutpoint ξ of $\mathcal{Q}$, $Lp(\mathcal{Q}|\xi) \trianglelefteq \mathcal{Q}$.

If $\mathcal{Q}$ is good, then it has a unique (ω, ω_1)-iteration strategy with the Dodd–Jensen property. We let $\Sigma_{\mathcal{Q}}$ be this strategy. Also, let $\eta_{\mathcal{Q}}$ be the largest cardinal of $\mathcal{Q}$. Given an iteration tree $\mathcal{T}$ on $\mathcal{Q}$ according to $\Sigma_{\mathcal{Q}}$ with last model $\mathcal{R}$ such that $\pi^{\mathcal{T}}$ exists, we let $\pi_{\mathcal{Q},\mathcal{R}} : \mathcal{Q} \to \mathcal{R}$ be the iteration embedding. Notice that because $\Sigma_{\mathcal{Q}}$ has the Dodd–Jensen property, $\pi^{\mathcal{T}}$ is independent of $\mathcal{T}$. We say that $\mathcal{Q}$ is *excellent* if whenever $\mathcal{R}$ is a $\Sigma_{\mathcal{Q}}$-iterate of $\mathcal{Q}$ such that $\pi_{\mathcal{Q},\mathcal{R}}$ is defined $\mathcal{R}$ is good. In this case, we also say that $\Sigma_{\mathcal{Q}}$ is fullness preserving.

Suppose now that $\alpha < \kappa$ is such that it ends a weak gap (see Steel [10]). We then let
$$\mathcal{F}(\alpha, a) = \big\{\mathcal{Q} : J_\alpha(\mathbb{R}) \vDash \text{“}\mathcal{Q} \text{ is an excellent } a\text{-premouse”}\big\}.$$
Given an a-premouse $\mathcal{P}$ such that $J_\alpha(\mathbb{R}) \vDash$ "$\mathcal{P}$ is suitable and short tree iterable," we let $\mathcal{F}(\alpha, a, \mathcal{P})$ be the set of $\mathcal{Q}$ such that in $J_\alpha(\mathbb{R})$, there is a correctly guided short tree $\mathcal{T}$ on $\mathcal{P}$ with last suitable model $\mathcal{P}^*$ such that for some $\mathcal{P}^*$-cardinal $\eta \leq \lambda_{\mathcal{P}^*}$, $\mathcal{Q} = \mathcal{P}^*|(\eta^+)^{\mathcal{P}^*}$.

Lemma 0.2 *Suppose that $\alpha < \kappa$ ends a weak gap, $a \in HC$, and $\mathcal{P}$ is an a-premouse such that $J_\alpha(\mathbb{R}) \vDash$"$\mathcal{P}$ is suitable and short tree iterable." Then $\mathcal{F}(\alpha, a, \mathcal{P}) \subseteq \mathcal{F}(\alpha, a)$.*

Proof Fix $\mathcal{Q} \in \mathcal{F}(\alpha, a, \mathcal{P})$. Work in $J_\alpha(\mathbb{R})$. Let $\mathcal{T}$ be a correctly guided short tree on $\mathcal{P}$ with last suitable model $\mathcal{P}^*$ such that for some $\mathcal{P}^*$-cardinal $\eta \leq \lambda_{\mathcal{P}^*}$, $\mathcal{Q} = \mathcal{P}^*|(\eta^+)^{\mathcal{P}^*}$. Because $\mathcal{P}$ is short tree iterable, we have that $\mathcal{Q}$ is (ω, ω_1)-iterable via a unique iteration strategy Σ. As the iterations of $\mathcal{Q}$ can also be viewed as iterations of $\mathcal{P}^*$, we have that Σ is fullness preserving, implying that $\mathcal{Q}$ is excellent. □

Notice that if $\beta > \alpha$ is such that β ends a weak gap and $J_\beta(\mathbb{R}) \models$ "$\mathcal{P}$ is a suitable and short tree iterable a-premouse," then there could be $\mathcal{Q} \in \mathcal{F}(\beta, a, \mathcal{P})$ which is not in $\mathcal{F}(\alpha, a, \mathcal{P})$. However, we always have the following easy lemma.

Lemma 0.3 *Suppose that $a \in HC$, $\mathcal{P}$ is an a-premouse and $\alpha < \beta < \kappa$. Suppose that α and β end weak gaps such that both $J_\alpha(\mathbb{R})$ and $J_\beta(\mathbb{R})$ satisfy that $\mathcal{P}$ is suitable and short tree iterable. Then $\mathcal{F}(\alpha, a, \mathcal{P}) \subseteq \mathcal{F}(\beta, a, \mathcal{P})$.*

Proof The lemma follows because any iteration tree on $\mathcal{P}$ which is correctly guided and short in the sense of $J_\alpha(\mathbb{R})$ is also correctly guided and short in the sense of $J_\beta(\mathbb{R})$. □

Next we define $\leq_{\alpha,a}$ on $\mathcal{F}(\alpha, a)$ by setting $\mathcal{Q} \leq_{\alpha,a} \mathcal{R}$ if and only if there is an iteration tree $\mathcal{T}$ on $\mathcal{Q}$ according to $\Sigma_\mathcal{Q}$ with last model $\mathcal{S}$ such that $\pi^\mathcal{T}$ exists, $\mathcal{S} \trianglelefteq \mathcal{R}$, and $\mathcal{S} = \mathcal{R}|(\eta^+_\mathcal{S})^\mathcal{R}$. Also, let $\leq_{\alpha,a,\mathcal{P}} = \leq_{\alpha,a} \restriction \mathcal{F}(\alpha, a, \mathcal{P})$. As usual, we have the following.

Lemma 0.4 *We have that $\leq_{\alpha,a}$ and $\leq_{\alpha,a,\mathcal{P}}$ are directed, and $\leq_{\alpha,a,\mathcal{P}}$ is dense in $\leq_{\alpha,a}$.*

Let then $\mathcal{M}_\infty(\alpha, a)$ be the direct limit of $(\mathcal{F}(\alpha, a), \leq_{\alpha,a})$ under the iteration embeddings $\pi_{\mathcal{Q},\mathcal{R}}$. Also, let $\mathcal{M}_\infty(\alpha, a, \mathcal{P})$ be the direct limit of $(\mathcal{F}(\alpha, a, \mathcal{P}), \leq_{\alpha,a,\mathcal{P}})$ under the iteration embeddings $\pi_{\mathcal{Q},\mathcal{R}}$. The next lemma follows from Lemma 0.4.

Lemma 0.5 *We have $\mathcal{M}_\infty(\alpha, a) = \mathcal{M}_\infty(\alpha, a, \mathcal{P})$.*

We let $\pi_{\mathcal{Q},\infty} : \mathcal{Q} \to \mathcal{Q}^* \trianglelefteq \mathcal{M}_\infty(\alpha, a, \mathcal{P})$ be the direct-limit embedding.[1]

We can now define φ. First let S be the set of those reals x which code a pair $(y_x, \mathcal{P}_x)$ such that

1. $y_x \in \mathbb{R}$,
2. for some $\alpha < \kappa$ ending a weak gap, $J_\alpha(\mathbb{R}) \models$ "$\mathcal{P}_x$ is a suitable and short tree iterable y_x-premouse."

Clearly S is Σ^2_1. We let $f : \kappa^2 \to \kappa$ be such that $f(\beta, \gamma)$ is the least α such that $J_\alpha(\mathbb{R}) \models \max(\beta, \gamma) < \underset{\sim}{\delta}^2_1$. We also let $g : S \times \kappa^2 \to \kappa$ be the function defined as follows: for all $(\beta, \gamma) \in \kappa^2$ and $x \in S$, if there is an ordinal $\alpha > f(\beta, \gamma)$ such that $J_\alpha(\mathbb{R}) \models$ "$\mathcal{P}_x$ is suitable and short tree iterable y_x-premouse," then $g(\beta, \gamma)$ is the least such α, and otherwise $g(x, \beta, \gamma) = 0$. Notice that g is Σ^2_1 in codes. We define φ as follows.

Definition 0.6 If $x \notin S \cap \mathbb{R}$, then let $\varphi(x) = \emptyset$. Suppose now $x \in S$. Let $(y_x, \mathcal{P}_x)$ be the pair coded by x. Given $\beta, \gamma < \kappa$, we let $(\beta, \gamma) \in \varphi(x)$ if and only if letting $\mathcal{P} = \mathcal{P}_x$ and $g(x, \beta, \gamma) = \alpha$, then $\alpha > 0$ and for some $a \in \mathcal{P}$ the following holds in $J_\alpha(\mathbb{R})$:

1. $\mathcal{P}$ is suitable and short tree iterable;
2. a is the collapse of $x(0)$;
3. $a \subseteq \lambda_\mathcal{P} \times \lambda_\mathcal{P}$;
4. there is a correctly guided short tree $\mathcal{T}$ on $\mathcal{P}$ with last model $\mathcal{S}$ such that $\pi_{\mathcal{P},\mathcal{S}}$ exists and an $\mathcal{S}$-cardinal η such that
   (a) $(\eta^+)^\mathcal{S} < \lambda^\mathcal{S}$,
   (b) if $\mathcal{Q} = \mathcal{S}|(\eta^+)^\mathcal{S}$ and $a^\mathcal{Q} = \pi_{\mathcal{P},\mathcal{S}}(a) \cap (\eta \times \eta)$, then $(\beta, \gamma) \in \pi_{\mathcal{Q},\infty}(a^\mathcal{Q}) \cap \mathrm{rng}(\pi_{\mathcal{Q},\infty})$.

Given $\alpha < \Theta$, we let S_α, f_α, g_α, and φ_α be what the above definitions give over $J_\alpha(\mathbb{R})$. The following lemmas establish that φ is as desired. We start with the following easy lemma.

Lemma 0.7 *For each $x \in \mathbb{R}$, $\varphi(x) = \bigcup_{\alpha<\kappa} \varphi_\alpha(x)$.*

Proof Suppose $(\beta, \gamma) \in \varphi(x)$. Let $\alpha > g(x, \beta, \gamma)$ be such that it ends a weak gap. Then $(\beta, \gamma) \in \varphi_\alpha(x)$. The other direction is similar. □

Lemma 0.8 *For every $x \in \mathbb{R}$, $\varphi(x) \subseteq \kappa \times \kappa$.*

Proof The claim follows from the fact that for every α and a, $\mathcal{M}_\infty(\alpha, a) \subseteq J_\alpha(\mathbb{R})$. □

Lemma 0.9 *Suppose $F : \kappa \to \kappa$. Then there is $x \in \mathrm{dom}(\varphi)$ such that $\varphi(x) = F$.*

Proof Fix y such that $F \in \mathrm{HOD}_y$. There is then a suitable $\mathcal{P}$ over y such that $F \in \mathrm{rng}(\pi_{\mathcal{P},\emptyset,\infty})$.[2] Notice that $\pi_{\mathcal{P},\emptyset,\infty}(\lambda_\mathcal{P}) = \kappa$ (see [11, Chapter 8]). Let then $a \subseteq \lambda_\mathcal{P} \times \lambda_\mathcal{P}$ be such that $\pi_{\mathcal{P},\emptyset,\infty}(a) = F$, and let x code the pair $(y, \mathcal{P})$ such that $x(0) = a$. It is then easy to see that $\varphi(x) = F$ (use Lemma 0.7).[3] □

Lemma 0.10 *Suppose $\beta, \gamma < \kappa$. Let*

$$x \in R_{\beta,\gamma} \leftrightarrow \varphi(x)(\beta, \gamma) \wedge \forall \gamma' < \kappa\big(\varphi(x)(\beta, \gamma') \to \gamma' = \gamma\big).$$

Then $R_{\beta,\gamma}$ is $\underset{\sim}{\Delta}^2_1$.

Proof We have that the following are equivalent.

1. We have $x \in R_{\beta,\gamma}$.
2. There is α such that $J_\alpha(\mathbb{R}) \vDash$ "$x \in \mathrm{dom}(\varphi_\alpha)$ and γ is the unique ordinal such that $(\beta, \gamma) \in \varphi_\alpha(x)$."
3. For all $\alpha > f(\beta, \gamma)$ such that $J_\alpha(\mathbb{R}) \vDash$ "$x \in \mathrm{dom}(\varphi_\alpha)$," γ is the unique ordinal such that $(\beta, \gamma) \in \varphi_\alpha(x)$.

Clearly (1) implies (2) and (3). Also, that (3) implies (1) is straightforward. We show that (2) implies (1). Fix then α such that $J_\alpha(\mathbb{R}) \vDash$ "$x \in \mathrm{dom}(\varphi_\alpha)$ and γ is the unique ordinal such that $(\beta, \gamma) \in \varphi_\alpha(x)$." It follows from the definition of φ_α that $\alpha > g(x, \beta, \gamma)$. Let $(y, \mathcal{P})$ be the pair coded by x, and let $a \in \mathcal{P}$ be the transitive collapse of $x(0)$. Working in $J_\alpha(\mathbb{R})$, let $\mathcal{T}$ be a correctly guided short tree on $\mathcal{P}$ with last model $\mathcal{S}$ such that $\pi_{\mathcal{P},\mathcal{S}}$ exists and an $\mathcal{S}$-cardinal η such that

1. $(\eta^+)^\mathcal{S} < \lambda_\mathcal{S}$;
2. if $\mathcal{Q} = \mathcal{S}|(\eta^+)^\mathcal{S}$ and $a^\mathcal{Q} = \pi_{\mathcal{P},\mathcal{S}}(a) \restriction \eta$, then $(\beta, \gamma) \in \pi_{\mathcal{Q},\infty}(a^\mathcal{Q}) \cap \mathrm{rng}(\pi_{\mathcal{Q},\infty})$.

Suppose now there is some ξ such that for some γ', $(\beta, \gamma') \in \varphi_\xi(x)$. Working in $J_\xi(\mathbb{R})$, let $\mathcal{T}^*$ be a correctly guided short tree on $\mathcal{P}$ with last model $\mathcal{S}^*$ such that $\pi_{\mathcal{P},\mathcal{S}^*}$ exists and an $\mathcal{S}^*$-cardinal ν such that

1. $(\nu^+)^{\mathcal{S}^*} < \lambda_{\mathcal{S}^*}$;
2. if $\mathcal{R} = \mathcal{S}^*|(\nu^+)^{\mathcal{S}^*}$ and $a^\mathcal{R} = \pi_{\mathcal{P},\mathcal{S}^*}(a) \restriction \nu$, then $(\beta, \gamma') \in \pi_{\mathcal{R},\infty}(a^\mathcal{R}) \cap \mathrm{rng}(\pi_{\mathcal{R},\infty})$.

Let $\nu = \max(\xi, \alpha)$. The following is an easy claim.

Claim $J_\nu(\mathbb{R}) \vDash$"*$\mathcal{S}$ and $\mathcal{S}^*$ are suitable and short tree iterable.*"

Proof We have that $\mathcal{P}$ is suitable and short tree iterable in both $J_\alpha(\mathbb{R})$ and $J_\xi(\mathbb{R})$. We also have $J_\alpha(\mathbb{R}) \vDash$ "$\mathcal{T}$ is short" and $J_\xi(\mathbb{R}) \vDash$ "$\mathcal{T}^*$ is short." It then follows that $J_\nu(\mathbb{R}) \vDash$ "$\mathcal{T}$ and $\mathcal{T}^*$ are short trees on $\mathcal{P}$." It then follows that $J_\nu(\mathbb{R}) \vDash$ "$\mathcal{S}$ and $\mathcal{S}^*$ are suitable and short tree iterable." □

We work now in $J_\nu(\mathbb{R})$. Using the claim we can find $\mathcal{S}^{**}$ which is a suitable correct iterate of both $\mathcal{S}$ and $\mathcal{S}^*$. Notice that since $\mathcal{S}^{**}$ is suitable, the iteration embeddings $i : \mathcal{S}|(\lambda_{\mathcal{S}}^+)^{\mathcal{S}} \to \mathcal{S}^{**}|(\lambda_{\mathcal{S}^{**}}^+)^{\mathcal{S}^{**}}$ and $j : \mathcal{S}^*|(\lambda_{\mathcal{S}^*}^+)^{\mathcal{S}^*} \to \mathcal{S}^{**}|(\lambda_{\mathcal{S}^{**}}^+)^{\mathcal{S}^{**}}$ exist.

Suppose now that $\gamma \neq \gamma'$. Let $(\bar{\beta}, \bar{\gamma}, \bar{\gamma}') \in \mathcal{S}^{**}$ be such that letting $\zeta = \max(i(\eta_{\mathcal{Q}}), j(\eta_{\mathcal{R}}))$ and $\mathcal{W} = \mathcal{S}^{**}|(\zeta^+)^{\mathcal{S}^{**}}$, $\pi_{\mathcal{W},\infty}(\bar{\beta}, \bar{\gamma}, \bar{\gamma}') = (\beta, \gamma, \gamma')$. It then follows that $(\bar{\beta}, \bar{\gamma}) \in i(\pi^{\mathcal{T}}_{\mathcal{P},\mathcal{S}}(a))$ and $(\bar{\beta}, \bar{\gamma}') \in j(\pi^{\mathcal{T}^*}_{\mathcal{P},\mathcal{S}^*}(a))$. However, $i \circ \pi^{\mathcal{T}}_{\mathcal{P},\mathcal{S}} = j \circ \pi^{\mathcal{T}^*}_{\mathcal{P},\mathcal{S}^*}$, implying that $i(\pi^{\mathcal{T}}_{\mathcal{P},\mathcal{S}}(a)) = j(\pi^{\mathcal{T}^*}_{\mathcal{P},\mathcal{S}^*}(a))$ and that

$$S^{**} \vDash (\bar{b}, \bar{\gamma}) \in i\left(\pi^{\mathcal{T}}_{\mathcal{P},\mathcal{S}}(a)\right) \wedge (\bar{b}, \bar{\gamma}') \in i\left(\pi^{\mathcal{T}}_{\mathcal{P},\mathcal{S}}(a)\right).$$

Let now $(\tau, \tau^*) \in \mathcal{Q}$ be such that $\pi_{\mathcal{Q},\infty}(\tau, \tau^*) = (\beta, \gamma)$. By elementarity of i, we then get that $\mathcal{S} \vDash$ "there is $\tau^{**} \neq \tau^*$ such that $(\tau, \tau^{**}) \in \pi_{\mathcal{P},\mathcal{S}}(a)$." Fix such a τ^{**}, and let $\varsigma \in (\tau^{**}, \lambda_{\mathcal{S}})$ be an $\mathcal{S}$-cardinal. Then letting $\mathcal{Q}^* = \mathcal{S}|(\varsigma^+)^{\mathcal{S}}$ we have that $(\beta, \pi_{\mathcal{Q}^*,\infty}(\tau^{**})) \in \varphi_\alpha(x)$ and $\pi_{\mathcal{Q}^*,\infty}(\tau^{**}) \neq \gamma$, a contradiction. □

The next lemma finishes the proof.

Lemma 0.11 *Suppose $\beta < \lambda$, $A \in \underset{\sim}{\Delta}_1^2$ and $A \subseteq R_\beta = \{x : \exists\gamma < \kappa R_{\beta,\gamma}(x)\}$. Then $\exists\gamma_0 < \kappa$ such that $\forall x \in A \exists\gamma < \gamma_0 R_{\beta,\gamma}(x)$.*

Proof Let $h : A \to \kappa$ be defined by $h(x) = \nu$ if ν is the least such that ν ends a weak gap and $J_\nu(\mathbb{R}) \vDash x \in R_\beta$. Then f is $\underset{\sim}{\Sigma}_1$ over $J_\kappa(\mathbb{R})$, and hence, as κ is $\mathbb{R}$-admissible, f is bounded. □

This completes the proof of Theorem 0.1. □

Notes

1. Notice that because $\mathcal{Q}$ has a unique iteration strategy, $\pi_{\mathcal{Q},\infty}$ is independent of α and a. Because of this we dropped them from our notation.

2. Recall the direct limit construction that converges to HOD$|\Theta$. Here $\pi_{\mathcal{P},\emptyset,\infty}$ is the direct limit embedding given by $\emptyset$-iterability embeddings. For more details see either of the aforementioned papers.

3. Notice that by reflection there is α such that α ends a weak gap and $J_\alpha(\mathbb{R}) \vDash$ "$\mathcal{P}$ is suitable and short tree iterable." It is then the case that for any $\beta \in (\alpha, \kappa)$ which ends a weak gap, $J_\beta(\mathbb{R}) \vDash$ "$\mathcal{P}$ is suitable and short tree iterable."

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Department of Mathematics
Rutgers University
Hill Center for the Mathematical Sciences
110 Frelinghuysen Rd.
Piscataway, New Jersey 08854
USA
grigor@math.rutgers.edu