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# Hyperbolic Towers and Independent Generic Sets in the Theory of Free Groups

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*To Anand Pillay on the occasion of his 60th birthday*

**Abstract** We use hyperbolic towers to answer some model-theoretic questions around the generic type in the theory of free groups. We show that all the finitely generated models of this theory realize the generic type $p_0$ but that there is a finitely generated model which omits $p_0^{(2)}$. We exhibit a finitely generated model in which there are two maximal independent sets of realizations of the generic type which have different cardinalities. We also show that a free product of homogeneous groups is not necessarily homogeneous.

## 1 Introduction

This paper is motivated by the works of Pillay [9] and of Sklinos [18], which study the weight of the generic type in the free group.

Following the work of Sela [11], [12] and of Kharlampovich and Myasnikov proving that nonabelian free groups are elementarily equivalent, we denote by $T_{fg}$ their common first-order theory. Sela [15] also showed that $T_{fg}$ is stable.

Every stable theory admits a good model-theoretic notion of independence, of which we give a brief account in Section 2 for readers lacking a model theory background. (The interested reader is referred to Pillay [7].)

Poizat proved that $T_{fg}$ is connected in the sense of model theory, that is, that there is a model of $T_{fg}$ which admits no proper definable subgroup of finite index. A consequence of stability is that connectedness is equivalent to saying that $T_{fg}$ admits a unique generic type over any set of parameters. We denote by $p_0$ the generic type over the empty set. Pillay gives a characterization of elements that realize $p_0$ in nonabelian free groups. In fact, he shows more generally the following.

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**Theorem 1.1 ([9, Fact 1.10(ii) and Theorem 2.1(ii)])** *A subset of a nonabelian free group $\mathbb{F}$ is a maximal independent set of realizations of $p_0$ if and only if it is a basis of $\mathbb{F}$.*

An immediate consequence of this result is that in a nonabelian free group $\mathbb{F}$, maximal independent sets of realizations of the generic type all have the same cardinality.

The notion of weight of a type $p$ can be intuitively thought of as a generalized exchange principle (see Section 2), and when it is finite, it bounds the ratio of the cardinality of two maximal independent sets of realizations of $p$. In particular, it is straightforward to see that if a type $p$ has weight 1, then any two maximal independent sets of realizations of $p$ (in any model) have the same cardinality.

In this light, the following result might look surprising.

**Theorem 1.2 ([9], [18])** *The generic type $p_0$ of the theory of nonabelian free groups has infinite weight.*

Intuitively one can explain this behavior of the generic type by noticing that two bases of a fixed nonabelian free group have the same cardinality as a consequence of the universal property and not of some exchange principle.

It is thus natural to ask whether we can witness infinite weight in an explicit model in terms of independent sets of generic elements, or even whether we can witness that the generic type does not have weight 1.

**Question 1** Is there a model of the theory of the free group in which one can find two maximal independent sets of realizations of the generic type with different cardinalities?

Sections 2 and 3 serve as introductory material for the notions we need from model theory and geometric group theory, respectively. In Section 2 we give formal definitions of independence and weight. In Section 3 we describe in detail the geometric notion which lies at the core of this paper, namely, hyperbolic towers.

In Section 4, we examine $p_0$ with respect to the notion of isolation. We give the proof of an unpublished result of Pillay which shows that $p_0^{(2)}$ is not isolated in the theory axiomatized by $p_0$ (after adding a constant to the language of groups). We then classify the hyperbolic tower structures admitted by the fundamental group $S_4$ of the connected sum of four projective planes. We use this to deduce that the type $p_0$ is realized in every finitely generated model of $T_{fg}$, but that $S_4$ omits $p_0^{(2)}$, thus giving an explicit witness to Pillay's nonisolation result. This also enables us to see that no type in $S(T_{fg})$ (apart from the trivial one) is isolated.

In Section 5, we answer Question 1 in the affirmative by exhibiting a suitable finitely generated model of $T_{fg}$.

Finally, we use this result in Section 6 to see that homogeneity is not preserved under taking free products, thus answering a question of Jaligot.

## 2 Independence and Weight

In this section we give a quick description of the model-theoretic notions we use. The exposition is biased towards our needs and by no means complete.

We fix a stable first-order theory $T$, and we work in a "big" saturated model $\mathbb{M}$, which is usually called the *monster model* (see Marker [1, p. 218]). As mentioned in the introduction, stable theories admit a good notion of independence, the prototypical examples being linear independence in vector spaces and algebraic independence in algebraically closed fields.

In a more abstract setting Shelah gave the following definition of (forking) independence (see [16, p. 85, Definition 1.4]).

**Definition 2.1** Let $\varphi(\bar{x}, \bar{b})$ be a first-order formula in $\mathbb{M}$, and let $A \subset M$ (the underlying domain of $\mathbb{M}$). Then $\varphi(\bar{x}, \bar{b})$ forks over $A$ if there are infinitely many automorphisms $(f_i)_{i<\omega} \in \mathrm{Aut}_A(\mathbb{M})$ and some $k < \omega$, such that the set $\{\varphi(\bar{x}, f_i(\bar{b})) : i < \omega\}$ is $k$-inconsistent, that is, every subset of cardinality $k$ is inconsistent.

Recall that an $m$-type $p(\bar{x})$ over $A \subseteq M$ of the first-order theory $T$ is a consistent (with $T$) set of formulas with parameters in $A$ with at most $m$ free variables. For example, the type $\mathrm{tp}(\bar{a}/A)$ of a tuple $\bar{a} \in M$ is the set of formulas that $\bar{a}$ satisfies in $\mathbb{M}$. (In fact, saying that $\mathbb{M}$ is saturated is exactly like saying that every $m$-type $p(\bar{x})$ over a set of parameters of cardinality strictly less than $|M|$ is the type of an $m$-tuple $\bar{a} \in M$.) The type of $\bar{a}$ over $A$ can equivalently be thought of as the collection of sets which are definable over $A$ and which contain $\bar{a}$.

If $A \subseteq B$, we say that $\bar{a}$ is *independent* from $B$ over $A$ if there is no formula in $\mathrm{tp}(\bar{a}/B)$ which forks over $A$. In the opposite case we say that $\bar{a}$ *forks* with $B$ over $A$ (or $\bar{a}$ is not independent from $B$ over $A$). Heuristically, one can think of the latter case as expressing the fact that the type of $\bar{a}$ over $B$ contains much more information than the type of $\bar{a}$ over $A$ alone.

Indeed, the definition implies that there is a formula with parameters in $B$ satisfied by $\bar{a}$ which forks over $A$. Thus the set $X$ defined by this formula contains $\bar{a}$, is definable over $B$, and admits an infinite sequence of $k$-wise disjoint translates by elements of $\mathrm{Aut}_A(\mathbb{M})$. (Here $\mathrm{Aut}_A(\mathbb{M})$ denotes automorphisms of $\mathbb{M}$ fixing $A$ pointwise.)

Consider now a set $Y$ which is definable over $A$ alone and contains $\bar{a}$: any automorphism in $\mathrm{Aut}_A(\mathbb{M})$ necessarily fixes $Y$. Clearly $X$ can be assumed to be contained in $Y$, and thus so are all of its automorphic images (under $\mathrm{Aut}_A(\mathbb{M})$). Thus in some sense, $X$ is much smaller than any definable set $Y$ given by a formula in $\mathrm{tp}(\bar{a}/A)$, and the type of $\bar{a}$ over $B$ "locates" $\bar{a}$ much more precisely than its type over $A$ alone.

A consequence of stability is the existence of nonforking (independent) extensions. Let $A \subseteq B$, and let $p(\bar{x})$ be a type over $A$. Then we say that $q(\bar{x}) := \mathrm{tp}(\bar{a}/B)$ is a *nonforking extension* of $p(\bar{x})$, if $p(\bar{x}) \subseteq q(\bar{x})$ and, moreover, $\bar{a}$ does not fork with $B$ over $A$. A type over $A$ is called *stationary* if for any $B \supseteq A$ it admits a unique nonforking extension over $B$.

Let $C = \{\bar{c}_i : i \in I\}$ be a set of tuples. We say that $C$ *is an independent set over* $A$ if for every $i \in I$, $\bar{c}_i$ is independent from $A \cup C \setminus \{\bar{c}_i\}$ over $A$. If $p$ is a type over $A$ which is stationary and $(a_i)_{i<\kappa}, (b_i)_{i<\kappa}$ are both independent sets over $A$ of realizations of $p$, then $\mathrm{tp}((a_i)_{i<\kappa}/A) = \mathrm{tp}((b_i)_{i<\kappa}/A)$ (see [7, Lemma 2.28, p. 29]). This allows us to denote by $p^{(\kappa)}$ the type of $\kappa$-independent realizations of $p$.

For the purpose of assigning a dimension (with respect to forking independence) to a type, one might ask what is the cardinality of a maximal independent set of realizations of a type and whether any two such sets have the same cardinality. This naturally leads to the definition of weight.

**Definition 2.2** The preweight of a type $p(\bar{x}) := \mathrm{tp}(\bar{a}/A)$, $\mathrm{prwt}(p)$ is the supremum of the set of cardinals $\kappa$ for which there exists $\{\bar{b}_i : i < \kappa\}$ an independent set over $A$, such that $\bar{a}$ forks with $\bar{b}_i$ over $A$ for all $i$. The weight $\mathrm{wt}(p)$ of a type $p$ is the supremum of

$$\{\mathrm{prwt}(q) \mid q \text{ a nonforking extension of } p\}.$$

The special case of weight 1 can be thought of as an exchange principle: an element $a$ in the set of realizations of a weight 1 type cannot fork with more than one element from an independent set.

Thus, as in the case of the dimension theorem for vector spaces, one can easily see that any two maximal independent sets of realizations of a weight 1 type must have the same cardinality.

## 3 Hyperbolic Towers

In this section we define hyperbolic towers. Hyperbolic towers were first used by Sela [12] to describe the finitely generated models of the theory of nonabelian free groups. They also appeared in Perin [3] where the geometric structure of a group that elementarily embeds in a torsion-free hyperbolic group is characterized.

In order to define hyperbolic towers we need to give a few preliminary definitions.

**3.1 Graphs of groups and graphs of spaces** We first go briefly over the notion of graph of groups; for a more formal definition and further properties the reader is referred to Serre [17].

A *graph of groups* consists of a graph $\Gamma$, together with two collections of groups $\{G_v\}_{v \in V(\Gamma)}$ (the *vertex groups*) and $\{G_e\}_{e \in E(\Gamma)}$ (the *edge groups*), and a collection of embeddings $G_e \hookrightarrow G_v$ for each pair $(e, v)$ where $e$ is an edge and $v$ is one of its endpoints. To a graph of groups $\Gamma$ is associated a group $G$ called its *fundamental group* and denoted $\pi_1(\Gamma)$. (The use of algebraic topology terminology will be made clear below.) There is a canonical action of this group $G$ on a simplicial tree $T$ whose quotient $G \backslash T$ is isomorphic to $\Gamma$. Conversely, to any action of a group $G$ on a simplicial tree $T$ without inversions, one can associate a graph of groups $\Gamma$ whose fundamental group is isomorphic to $G$ and whose underlying graph is isomorphic to the quotient $G \backslash T$. An element or a subgroup in $G$ which fixes a point in $T$ (or, equivalently, which is contained in a conjugate in $G$ of one of the vertex groups $G_v$) is said to be *elliptic*.

A fundamental example is the special case where $\Gamma$ consists of two vertices $v$ and $w$ joined by a single edge $e$: then, the fundamental group of $\Gamma$ is the *amalgamated product* $G_v *_{G_e} G_w$. Graphs of groups can thus be thought of as a generalized version of amalgamated products.

The van Kampen lemma gives a useful perspective on graphs of groups. It states that if a topological space $X$ can be written as a union $X_1 \cup X_2$ of two of its path connected subspaces, and if $Y = X_1 \cap X_2$ is also path connected, the (usual) fundamental group $\pi_1(X)$ of the space $X$ can be written as an amalgamated product $\pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$ where the group embeddings $\pi_1(Y) \hookrightarrow \pi_1(X_i)$ are induced by the topological embeddings $Y \hookrightarrow X_i$.

Similarly, to a graph of groups $\Gamma$ we can associate a (not unique) *graph of spaces*: to each vertex $v \in V(\Gamma)$ (resp., edge $e \in E(\Gamma)$), we associate a (sufficiently nice) topological space $X_v$ (resp., $X_e$) such that $\pi_1(X_v) = G_v$ (resp., $\pi_1(X_e) = G_e$).

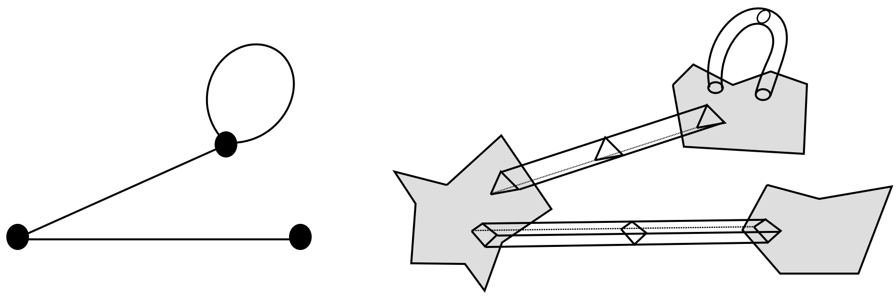

**Figure 1** A graph of groups and an associated graph of spaces.

To each pair $(e, v)$ of an edge and an endpoint is associated a topological embedding $f_{e,v} : X_e \hookrightarrow X_v$ which induces on fundamental groups the embedding $G_e \hookrightarrow G_v$. Then the fundamental group of the graph of groups $\Gamma$ is isomorphic to the fundamental group $\pi_1(X)$ of the space $X$ built by gluing the collection of spaces $\{X_v \mid v \in V(\Gamma)\}$ and $\{X_e \times [0, 1] \mid e \in E(\Gamma)\}$ using the maps $f_{e,v}$. More precisely, if $e$ is an edge joining $v$ to $w$, we identify each point $(x, 0)$ of $X_e \times \{0\}$ to the image of $x$ in $X_v$ under $f_{e,v}$ and each point $(x, 1)$ in $X_e \times \{1\}$ to the image of $x$ in $X_w$ under $f_{e,w}$. Conversely, given a graph of spaces, there is a graph of groups associated to it. Figure 1 illustrates this duality.

**Definition 3.1 (Bass–Serre presentation)** Let $G$ be a group acting on a simplicial tree $T$ without inversions; denote by $\Gamma$ the corresponding quotient graph of groups and by $p$ the quotient map $T \to \Gamma$. A Bass–Serre presentation for $\Gamma$ is a pair $(T^1, T^0)$ consisting of

- a subtree $T^1$ of $T$ which contains exactly one edge of $p^{-1}(e)$ for each edge $e$ of $\Gamma$;
- a subtree $T^0$ of $T^1$ which is mapped injectively by $p$ onto a maximal subtree of $\Gamma$.

The choice of terminology is justified by the fact that to such a pair $(T^1, T^0)$, we can associate a presentation of $G$ in terms of the subgroups $G_v$ for $v \in V(T^0)$, and elements of $G$ which send vertices of $T^1$ in $T^0$ (Bass–Serre generators).

**3.2 Surface groups** We now recall some standard facts about surfaces and surface groups. Unless otherwise mentioned, all surfaces are assumed to be compact and connected.

The classification of surfaces gives that a surface without boundary $\Sigma$ (or *closed surface*) is characterized up to homeomorphism by its orientability and its Euler characteristic $\chi(\Sigma)$. It can be easily deduced from this that a surface with (possibly empty) boundary $\Sigma$ is characterized up to homeomorphism by its orientability, its Euler characteristic $\chi(\Sigma)$, and the number of its boundary components. The orientable closed surface of characteristic 2 is the sphere, that of characteristic 0 is the torus; the nonorientable closed surface of characteristic 1 is the projective plane, and that of characteristic 0 is the Klein bottle.

The *connected sum* $\Sigma$ of two surfaces $\Sigma_1$ and $\Sigma_2$ is the surface obtained by removing an open disk from each $\Sigma_i$ and gluing the boundary components thus obtained one to the other. We then have $\chi(\Sigma) = \chi(\Sigma_1) + \chi(\Sigma_2) - 2$. One then sees, for example, that the closed nonorientable surface of characteristic $-1$ is the connected sum of three projective planes. Puncturing a surface (i.e., removing an open disk) decreases the Euler characteristic by 1.

Let $\Sigma$ be a surface with boundary. Each connected component of $\partial\Sigma$ has cyclic fundamental group, and gives rise in $\pi_1(\Sigma)$ to a conjugacy class of cyclic subgroups, which we call *maximal boundary subgroups*. A *boundary subgroup* of $\pi_1(\Sigma)$ is a nontrivial subgroup of a maximal boundary subgroup of $\pi_1(\Sigma)$.

Suppose that $\Sigma$ has $r$ boundary components, and let $\gamma_1, \ldots, \gamma_r$ be generators of nonconjugate maximal boundary subgroups. Then $\pi_1(\Sigma)$ admits a presentation of the form

$$\left\langle a_1, \ldots, a_{2m}, \gamma_1, \ldots, \gamma_r \;\middle|\; [a_1, a_2] \cdots [a_{2m-1}, a_{2m}] = \gamma_1 \cdots \gamma_r \right\rangle$$

if it is orientable and

$$\langle d_1, \ldots, d_q, \gamma_1, \ldots, \gamma_r \mid d_1^2 \cdots d_q^2 = \gamma_1 \cdots \gamma_r \rangle$$

if not. The Euler characteristic of the corresponding surface is given by $-(2m-2+r)$ in the orientable case and $-(q-2+r)$ in the nonorientable case.

Note that in particular, the fundamental group $\pi_1(\Sigma)$ of a compact surface $\Sigma$ with nonempty boundary $\partial\Sigma$ is a free group. However, we think of it as endowed with the peripheral structure given by its collection of maximal boundary subgroups.

Note also that the presentation given for the nonorientable case is equivalent to

$$\left\langle a_1, \ldots, a_{2h}, d_1, \ldots, d_p, \gamma_1, \ldots, \gamma_r \;\middle|\; [a_1, a_2] \cdots [a_{2h-1}, a_{2h}] d_1^2 \cdots d_p^2 = \gamma_1 \cdots \gamma_r \right\rangle$$

for any $h, p$ such that $2h + p = q$. This comes from the fact that the $r$-punctured connected sum of $h$ tori and $p$ projective planes (for $p > 0$) is homeomorphic to the $r$-punctured connected sum of $2h + p$ projective planes (since they are both nonorientable, have the same Euler characteristic, and have $r$ boundary components).

Let $S$ be the fundamental group of $\Sigma$ a surface with boundary, and let $\mathcal{C}$ be a set of two-sided disjoint simple closed curves on $\Sigma$. Let $\{T_c \mid c \in \mathcal{C}\}$ be a collection of disjoint open neighborhoods of the curves of $\mathcal{C}$ with homeomorphisms $c \times (-1, 1) \to T_c$ sending $c \times \{0\}$ onto $c$. Then $\Sigma$ can be seen as a graph of spaces, with edge spaces the curves in $\mathcal{C}$, and vertex spaces the connected components of $\Sigma - \bigcup_{c \in \mathcal{C}} T_c$. This gives a graph of groups decomposition for $S$, in which edge groups are infinite cyclic and boundary subgroups are elliptic. Such a decomposition is called the *decomposition of $S$ dual to $\mathcal{C}$*. The following lemma gives a useful converse; it is essentially in Morgan and Shalen [2, Theorem III.2.6].

**Lemma 3.2** *Let $S$ be the fundamental group of a surface with boundary $\Sigma$. Suppose $S$ admits a graph of groups decomposition $\Gamma$ in which edge groups are cyclic and boundary subgroups are elliptic. Then there exists a set of disjoint simple closed curves on $\Sigma$ such that $\Gamma$ is the graph of groups decomposition dual to $\mathcal{C}$.*

The idea of the proof of this lemma is to build an $S$-equivariant map $f$ between a universal cover $\tilde{\Sigma}$ of $\Sigma$ and the tree $T$ associated to $\Gamma$, and to consider the preimage by $f$ of midpoints of edges of $T$. If $f$ is suitably chosen, this preimage will be the lift of a collection of simple closed curves $\mathcal{C}$ we are looking for.

**3.3 Hyperbolic floors and towers** We will be interested in graphs of groups in which some of the vertex groups are *surface groups*, that is, fundamental groups of surfaces. (All surfaces will be compact and with possibly nonempty boundary.) Equivalently, this means that the corresponding graph of spaces will have subspaces $X_v$ which are surfaces.

**Definition 3.3** A graph of groups with surfaces is a graph of groups $\Gamma$ together with a subset $V_S$ of the set of vertices $V(\Gamma)$ of $\Gamma$, such that any vertex $v$ in $V_S$ satisfies the following.

- There exists a compact connected surface $\Sigma$ with nonempty boundary, such that the vertex group $G_v$ is the fundamental group $\pi_1(\Sigma)$ of $\Sigma$.
- For each edge $e$, and $v$ an endpoint of $e$, the injection $G_e \hookrightarrow G_v$ maps $G_e$ onto a maximal boundary subgroup of $\pi_1(\Sigma)$.
- This induces a bijection between the set of edges adjacent to $v$ and the set of conjugacy classes in $\pi_1(\Sigma)$ of maximal boundary subgroups of $\pi_1(\Sigma)$.

The vertices of $V_S$ are called *surface-type* vertices. The surfaces associated to the vertices of $V_S$ are called the *surfaces* of $\Gamma$.

Figure 2 gives an example of a graph of groups with surfaces. Each surface-type vertex $v$ of $\Gamma$ has been replaced by a picture of the corresponding surface with boundary $\Sigma_v$. Note how we represent pictorially the property that each edge group $G_e$ adjacent to a surface type vertex group $G_v$ embeds in a maximal boundary subgroup of $G_v$.

**Definition 3.4 ((Extended) hyperbolic floor)** Consider a triple $(G, G', r)$ where $G$ is a group, $G'$ is a subgroup of $G$, and $r$ is a retraction from $G$ onto $G'$ (i.e., $r$ is a morphism $G \to G'$ which restricts to the identity on $G'$).

We say that $(G, G', r)$ is an *extended hyperbolic floor* if there exists a nontrivial decomposition $\Gamma$ of $G$ as a graph of groups with surfaces, and a Bass–Serre presentation $(T^1, T^0)$ of $\Gamma$ such that

- the surfaces of $\Gamma$ which are not once punctured tori have Euler characteristic at most $-2$;
- $G'$ is the free product of the stabilizers of the non-surface-type vertices of $T^0$;
- every edge of $\Gamma$ joins a surface-type vertex to a non-surface-type vertex (bipartism);

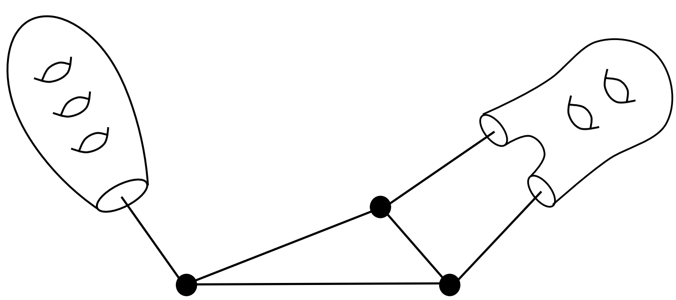

**Figure 2** A graph of groups with surfaces.

- either the retraction $r$ sends surface-type vertex groups of $\Gamma$ to nonabelian images, or $G'$ is cyclic and there exists a retraction $r' : G * \mathbb{Z} \to G' * \mathbb{Z}$ which sends surface-type vertex groups of $\Gamma$ to nonabelian images.

If the first alternative holds in this last condition, we say that $(G, G', r)$ is a *hyperbolic floor*.

**Definition 3.5 ((Extended) hyperbolic tower)** Let $G$ be a noncyclic group, let $H$ be a subgroup of $G$. We say that $G$ is an *(extended) hyperbolic tower* over $H$ if there exists a finite sequence $G = G^0 \geq G^1 \geq \cdots \geq G^m \geq H$ of subgroups of $G$ where $m \geq 0$ and

- for each $k$ in $[0, m-1]$, there exists a retraction $r_k : G^k \to G^{k+1}$ such that the triple $(G^k, G^{k+1}, r_k)$ is an (extended) hyperbolic floor, and $H$ is contained in one of the non-surface-type vertex groups of the corresponding hyperbolic floor decomposition;
- $G^m = H * F * S_1 * \cdots * S_p$ where $F$ is a (possibly trivial) free group, $p \geq 0$, and each $S_i$ is the fundamental group of a closed surface without boundary of Euler characteristic at most $-2$.

Note that all the floors $(G^k, G^{k+1}, r_k)$ are in fact (nonextended) hyperbolic floors except possibly for $(G^{m-1}, G^m, r_{m-1})$, and in this case $G^m$ is infinite cyclic, so $H$ is cyclic or trivial. In particular, extended hyperbolic towers over nonabelian subgroups are in fact hyperbolic towers.

To understand them better, let us consider hyperbolic towers in a graph of space perspective. If $G$ is an extended hyperbolic tower over $H$, it means we can build a space $X_G$ with fundamental group $G$ from a space $X_H$ with fundamental group $H$ as follows. Start with the disjoint union $X^m$ of $X_H$ with closed surfaces $\Sigma_1, \ldots, \Sigma_p$ of Euler characteristic at most $-2$, together with a graph $X_F$. When $X^k$ is built, glue surfaces with boundary to $X^k$ along their boundary components (gluing each boundary component to a nonnull homotopic curve of $X^k$) to obtain the space $X^{k-1}$.

This is represented in Figure 3: here, $X^m$ is the union of the spaces in the four small boxes, and $X^{m-1}$ the union of those in the two big square boxes. Finally, $G$ is the fundamental group of the whole space. (An edge between a surface and a box

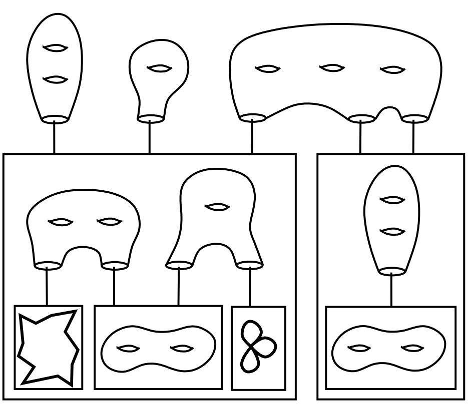

**Figure 3** A hyperbolic tower over $H$.

indicates that the corresponding boundary component is glued to a curve in the space contained in the box.) In addition, one should think that each surface retracts onto the lower floor, in a nonabelian way in the case of hyperbolic floors.

Though hyperbolic towers were introduced by Sela, their definition was slightly too restrictive, and some of the results concerning them were misstated in [12] and [3] (see Perin [5]), which is why *extended* hyperbolic floors and towers had to be introduced.

Theorem 6 of [12] characterizes finitely generated models of the free group as hyperbolic towers. The following is a corrected statement.

**Theorem 3.6** *Let $G$ be a finitely generated group. Then $G \models T_{fg}$ if and only if $G$ is an extended hyperbolic tower over the trivial subgroup.*

**Remark 3.7** One of the key steps in the proof of the "only if" direction of this result is to prove that from a map $G \to G$ preserving some of the structure of the group $G$, one can build a retraction $r : G \to G'$ to a proper subgroup which makes $(G, G', r)$ into a hyperbolic floor.

However, in a few low-complexity cases for $G$, this does not hold and the best we can get is a retraction making $(G, G', r)$ into an *extended* hyperbolic floor (see [5]).

This key step was made explicit in [3], where it is stated as Proposition 5.11 and given a more detailed proof. However, in this paper too these counterexamples were overlooked. A corrected version of the proof of this proposition can be found in Perin [4]. The "if" direction of Theorem 3.6 in these the exceptional cases is not dealt with in [12], but the proof can be extended in a straightforward way according to Sela [14].

Sela also uses the notion of hyperbolic towers in [13] to classify torsion-free hyperbolic groups up to elementary equivalence. He shows that to every torsion-free hyperbolic group $\Gamma$ can be associated a subgroup $C(\Gamma)$ which he calls its elementary core, over which $\Gamma$ admits a structure of hyperbolic tower, which is well defined up to isomorphism, and such that two torsion-free hyperbolic groups are elementarily equivalent if and only if they have isomorphic elementary cores. He also shows that if $\Gamma$ is not elementarily equivalent to the free group, then $C(\Gamma)$ is an elementary subgroup of $\Gamma$. According to Sela [14], the proof of this last result can be adapted to give, in fact, the following.

**Theorem 3.8** *Suppose that $\Gamma$ is a torsion-free hyperbolic group which admits a structure of hyperbolic tower over a nonabelian subgroup $H$. Then $H$ is an elementary subgroup of $G$.*

The converse of this result is given by [3, Theorem 1.2] so we get the following.

**Theorem 3.9** *Let $\Gamma$ be a torsion-free hyperbolic group, and let $H$ be a nonabelian subgroup of $\Gamma$. Then $H$ is an elementary subgroup of $\Gamma$ if and only if $\Gamma$ admits a structure of hyperbolic tower over $H$.*

The following result is Perin and Sklinos [6, Theorem 7.1]. Before stating it we recall that the connectedness of $T_{fg}$ implies that $p_0$ is stationary (as any nonforking extension of $p_0$ is also a generic type). Thus, following our discussion in Section 2, we denote by $p_0^{(k)}$ the type of $k$-independent realizations of $p_0$.

**Theorem 3.10** *Let $G$ be a nonabelian finitely generated group. Let $(u_1, \ldots, u_k)$ be a $k$-tuple of elements of $G$ for $k \geq 1$.*

*Then $(u_1, \ldots, u_k)$ realizes $p_0^{(k)}$ if and only if $H_u = \langle u_1, \ldots, u_k \rangle$ is free of rank $k$ and $G$ admits a structure of extended hyperbolic tower over $H_u$.*

**3.4 Hyperbolic tower structures of the connected sum of four projective planes** Admitting a structure of hyperbolic tower is quite a restrictive condition. For example, we have the following.

**Lemma 3.11** *If $\mathbb{F}$ is a free group, it does not admit any structure of extended hyperbolic floor over a subgroup.*

**Proof** Lemma 5.19 in [3] states that free groups do not admit structures of hyperbolic floors. The argument given for the proof does not use the fact that surface-type vertex groups have nonabelian images by the retraction $r$, and thus it can be applied to *extended* hyperbolic floors as well. □

Let $S_4$ denote the fundamental group of the surface $\Sigma_4$ which is the connected sum of four projective planes (i.e., the nonorientable closed surface of characteristic $-2$). It has a trivial structure of hyperbolic tower over $\{1\}$.

But this is not the only extended hyperbolic tower structure it admits. The following lemma gives some structures of extended hyperbolic floor for $S_4$.

**Lemma 3.12** *Suppose that $H$ is a nontrivial subgroup of $S_4$, over which $S_4$ admits a structure of extended hyperbolic floor. Then $H$ is cyclic, and $S_4$ admits one of the following presentations:*

- $\langle h, a, b, c \mid h^2 = a^2b^2c^2 \rangle$,
- $\langle h, a, b, t \mid htht^{-1} = a^2b^2 \rangle$,

*where $h$ generates $H$. Conversely, given such a presentation, $S_4$ admits a structure of extended hyperbolic floor over the subgroup $H$ generated by $h$.*

These two structures are illustrated by Figure 4. In both pictures, the fundamental group of the space inside the box is $H = \langle h \rangle$. The fundamental group of the upper surface in the picture on the left is the subgroup generated by $a, b, c$ in $S_4$. In the

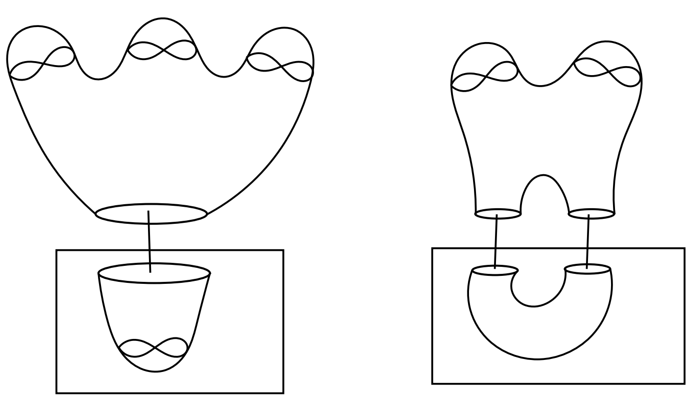

**Figure 4** Hyperbolic floor structures of the connected sum of four projective planes.

picture on the right, the fundamental group of the upper surface is the subgroup of $S_4$ generated by $a$ and $b$.

**Proof** Suppose that $S_4$ admits a structure of hyperbolic floor over a proper subgroup $H$, and denote by $\Gamma$ the associated graph of groups decomposition. By Lemma 3.2, $\Gamma$ is dual to a set of simple closed curves on $\Sigma_4$. In particular, the surfaces of $\Gamma$ correspond to $\pi_1$-injected subsurfaces of $\Sigma_4$. Denote by $\Sigma$ the (possibly disconnected) subsurface of $\Sigma_4$ formed by all these subsurfaces, and denote by $\Sigma'$ the closure of its complement in $\Sigma_4$. We have $\chi(\Sigma) + \chi(\Sigma') = \chi(\Sigma_4) = -2$. Since the connected components of $\Sigma$ are surfaces of a hyperbolic floor decomposition, they are punctured tori or have characteristic at most $-2$. This implies in particular that $\Sigma$ can have at most two connected components. If it has exactly two, they must be punctured tori, and $\Sigma'$ has characteristic 0 with two boundary components. Thus $\Sigma'$ must be a cylinder, but this contradicts the nonorientability of $\Sigma_4$. So in fact, $\Sigma$ is connected, and there are only two possibilities: either $\chi(\Sigma) = -1$ and $\Sigma$ is a punctured torus, or $\chi(\Sigma) = -2$.

In the first case, $\Sigma'$ has one boundary component, characteristic $-1$, and is nonorientable: it must be a punctured Klein bottle. However, there cannot be a retraction of $S_4$ on the fundamental group of $S'$ of this punctured Klein bottle. Indeed, $S_4$ then admits a presentation of the form $\langle a, b, d_1, d_2 \mid [a, b] = d_1^2 d_2^2 \rangle$ where $S'$ is the free subgroup of rank 2 of $S_4$ generated by $d_1$ and $d_2$. If there exists a retraction $r : S_4 \to S'$, it fixes $d_1$ and $d_2$, and thus $d_1^2 d_2^2 = r([a, b])$ is a commutator; this is a contradiction.

Thus the second alternative holds. This implies that $\chi(\Sigma') = 0$. In particular, each connected component of $\Sigma'$ has characteristic 0 and hence must be a cylinder or a Möbius band. Since $H$ is contained in a subgroup of $S_4$ corresponding to one of these connected components, it must be cyclic, say, $H = \langle h \rangle$. If $H$ is contained in a connected component of $\Sigma'$ which is a Möbius band, its complement is a punctured connected sum of three projective planes, so $S_4$ admits a presentation as $\langle h, a, b, c \mid h^2 = a^2 b^2 c^2 \rangle$. If $H$ is contained in a connected component of $\Sigma'$ which is a cylinder, its complement is a twice-punctured Klein bottle, so $S_4$ admits a presentation as $\langle h, a, b, t \mid hth t^{-1} = a^2 b^2 \rangle$.

Let us now prove the converse. Let $\Gamma$ be the graph of groups decomposition for $S_4$ which consists of a single non–surface-type vertex with corresponding vertex group $\langle h \rangle$, let a single surface-type vertex with corresponding group be the subgroup generated by $\{a, b, c\}$ (resp., $\{a, b, h, tht^{-1}\}$), and let the corresponding surface be a punctured connected sum of three projective planes (resp., a twice-punctured Klein bottle). Consider the retraction $r' : S_4 * \langle x \rangle \to \langle h \rangle * \langle x \rangle$ defined by $r'(a) = h$ and $r'(b) = r'(c^{-1}) = x$ (resp., $r'(a) = r'(b^{-1}) = x$ and $r'(t) = 1$), and $r'(x) = x$. The conditions of Definition 3.4 are satisfied by $\Gamma$ and $r'$, which proves the result. $\square$

Noting that $S_4$ is freely indecomposable, one gets the following.

**Corollary 3.13** *If $S_4$ admits a structure of hyperbolic tower over a subgroup $H$, then either $H$ is trivial or $H = \langle h \rangle$ is cyclic and $S_4$ admits one of the two presentations in terms of $h$ given in Lemma 3.12.*

## 4 Omitting, Realizing, and Isolating Types in the Theory of Free Groups

We observe that Theorem 1.1 implies that for $n \geq 2$, the free group $\mathbb{F}_n$ on $n$ generators realizes $p_0^{(n)}$ but omits $p_0^{(n+1)}$. Thus, it is natural to ask whether this holds also for $n = 1$: is there a group $G$ which realizes $p_0$ yet omits $p_0^{(2)}$?

Pillay answered the above question in the affirmative in a nonconstructive way, using purely model-theoretic methods. He then naturally asked whether an explicit model realizing $p_0$, but omitting $p_0^{(2)}$, exists. Such a group is exhibited in Proposition 4.6; however, we first give Pillay's elegant, but nonconstructive, argument.

His proof is based on the notion of semi-isolation, which we recall below, together with the following result (see Sklinos [18, Theorem 2.1]).

**Theorem 4.1** *The generic type $p_0$ is not isolated in $T_{fg}$.*

**Definition 4.2** Let $\mathbb{M}$ be a big saturated model of a stable theory $T$, and let $\bar{a}, \bar{b}$ be tuples in $\mathbb{M}$. The type $\mathrm{tp}(\bar{a}/\bar{b})$ is semi-isolated if there is a formula $\varphi(\bar{x}, \bar{y})$ (over the empty set) such that

(i) $\mathbb{M} \models \varphi(\bar{a}, \bar{b})$;
(ii) $\mathbb{M} \models \varphi(\bar{x}, \bar{b}) \to \mathrm{tp}(\bar{a})$.

The following lemma connecting the notions of semi-isolation and forking will be useful (see Pillay [8, Lemma 9.53(ii)]).

**Lemma 4.3** *Suppose that* $\mathrm{tp}(\bar{a}/\bar{b})$ *is semi-isolated and* $\mathrm{tp}(\bar{a})$ *is not isolated. Then* $\mathrm{tp}(\bar{a}/\bar{b})$ *forks over* $\emptyset$.

We are now ready to give Pillay's proof.

**Theorem 4.4 (Pillay [10])** *There exists a group $G$ such that $G \models p_0$ and $G \not\models p_0^{(2)}$.*

**Proof** By the omitting types theorem (see [1, Theorem 4.2.3, p. 125]), it is enough to prove that $p_0^{(2)}$ is not isolated in $p_0(c)$, that is, the complete theory in $\mathcal{L} = \{\cdot, ^{-1}, 1, c\}$ axiomatized by $p_0(c)$. Note that if $\mathbb{F}_2 = \langle e_1, e_2 \rangle$, then $(\mathbb{F}_2, e_1)$ is a model of $p_0(c)$.

Suppose, for the sake of contradiction, that $\varphi(x, y, c)$ isolates $p_0^{(2)}$ in $p_0(c)$. Let $(a, b)$ be a realization of $\varphi(x, y, e_1)$ in $\mathbb{F}_2$. As $\mathbb{F}_2 \models \varphi(x, y, e_1) \to p_0^{(2)}$ we have by Theorem 1.1 that $a, b$ form a basis of $\mathbb{F}_2$. In particular, there is a word $w(x, y)$, such that $w(a, b) = e_2$. Now it is easy to see that the formula $\psi(z, u) := \exists x, y(\varphi(x, y, u) \wedge z = w(x, y))$ semi-isolates $\mathrm{tp}(e_2/e_1)$. But as $\mathrm{tp}(e_2)$ is not isolated, Lemma 4.3 gives that $\mathrm{tp}(e_2/e_1)$ forks over $\emptyset$, contradicting Theorem 1.1. □

We note that the above proof does not give much information about the group $G$, apart from the fact that it is countable.

Theorem 4.1 implies that there exists a model of $T_{fg}$ omitting the generic type. Using the results above, we can show that this model cannot be finitely generated.

**Proposition 4.5** *Suppose that $G$ is a finitely generated model of the theory $T_{fg}$ of nonabelian free groups. Then $G$ realizes $p_0$.*

**Proof** By Theorem 3.6, $G$ admits a structure of extended hyperbolic tower over $\{1\}$. The ground floor $G^m$ of this tower is a nontrivial free product of a (possibly trivial) free group $\mathbb{F}$ with fundamental groups $S_1, \dots, S_q$ of closed hyperbolic surfaces. By Theorem 3.10, it is enough to show that $G$ has a structure of extended hyperbolic tower over a cyclic group $Z$.

We may assume that $\mathbb{F}$ is trivial, since otherwise any cyclic free factor of $\mathbb{F}$ will do.

If $q$ is nonzero, $S_1$ admits a presentation as

$$\big\langle a_1, \dots, a_g, b_1, \dots, b_g \;\big|\; [a_1, b_1] \cdots [a_g, b_g] = 1 \big\rangle$$

if it is orientable, and

$$\langle d_1, \dots, d_p \mid d_1^2 \cdots d_p^2 = 1 \rangle$$

if not. Let $H$ be the subgroup generated by $a_1, b_1$ in the first case and by $d_1, d_2$ in the second.

The map $r$ fixing $a_1, b_1$, sending $a_2$ to $b_1$, $b_2$ to $a_1$, and $a_j, b_j$ to 1 for $j > 2$ (resp., fixing $d_1, d_2$, sending $d_3$ to $d_2^{-1}$, $d_4$ to $d_1^{-1}$, and $d_j$ to 1 for $j > 4$) is a retraction of $S_1$ onto the subgroup $H \simeq \mathbb{F}_2$, which we can extend into a retraction of $G^m$ onto $H * S_2 * \cdots * S_p$. However, this retraction makes $(G^m, r(G^m), r)$ an extended hyperbolic floor only if the surface corresponding to $a_2, b_2, \dots, a_g, b_g$ (resp., $d_3, \dots, d_p$) is a punctured torus or has characteristic at most $-2$. This fails to be true only in the nonorientable case and if $p = 4$, that is, if $S_1$ is the connected sum of four projective planes. In all the other cases, we can take $Z$ to be any cyclic free factor of $H$, and the result is proved.

If $S_1$ is the fundamental group of the connected sum of four projective planes, choose a presentation of $S_1$ as $\langle h, a, b, c \mid h^2 = a^2 b^2 c^2 \rangle$. If $q \geq 2$, we define a retraction $r : S_1 * \cdots * S_q \to \langle h \rangle * S_2 * \cdots * S_q$ by $r(a) = h$, $r(b) = r(c^{-1}) = s$ for some nontrivial element $s$ of $S_2$. Then $(G^m, r(G^m), r)$ is a hyperbolic floor. If $q = 1$, we have seen in Lemma 3.12 that $G^m$ admits a structure of extended hyperbolic floor over $\langle h \rangle$, so $G$ is an extended hyperbolic tower over $\langle h \rangle$. □

On the other hand, the following proposition gives an alternative proof of Theorem 4.4.

**Proposition 4.6** *Let $S_4$ be the fundamental group of the connected sum of four projective planes. Then $S_4$ omits $p_0^{(2)}$.*

**Proof** Suppose, for the sake of contradiction, that $S_4 \models p_0^{(2)}(u, v)$. Then by Theorem 3.10, $u, v$ generate a free group $H$ of rank 2 over which $S_4$ admits a structure of hyperbolic tower. By Corollary 3.13 we know that no such structure exists. □

We conclude this section by giving another application of Corollary 3.13.

The following result is easily deduced from [6, Propositions 5.9, 6.2].

**Proposition 4.7** *Let $G$ and $G'$ be torsion-free hyperbolic groups. Let $\bar{u}$ and $\bar{v}$ be nontrivial tuples of elements of $G$ and $G'$, respectively, and let $U$ be a finitely presented subgroup of $G$ which contains $\bar{u}$ and is freely indecomposable with respect to it.*

*If $\mathrm{tp}^G(\bar{u}) = \mathrm{tp}^{G'}(\bar{v})$, then either there exists an embedding $U \hookrightarrow G'$ which sends $\bar{u}$ to $\bar{v}$, or $U$ admits the structure of hyperbolic floor over $\langle \bar{u} \rangle$.*

We can now prove the following.

**Theorem 4.8** *Let $\bar{v}$ be a nontrivial tuple of elements in a nonabelian free group $\mathbb{F}$. Then $\mathrm{tp}^{\mathbb{F}}(\bar{v})$ is not isolated.*

**Proof** Suppose that $\mathrm{tp}^{\mathbb{F}}(\bar{v})$ is isolated; then there exists a tuple $\bar{u}$ in $S_4$ such that $\mathrm{tp}^{S_4}(\bar{u}) = \mathrm{tp}^{\mathbb{F}}(\bar{v})$. As $S_4$ does not embed in $\mathbb{F}$, Proposition 4.7 applied to $U = S_4$ gives that $S_4$ is a hyperbolic floor over a subgroup $H$ containing $\langle \bar{u} \rangle$. Lemma 3.12 implies that $H$ is a cyclic group, whose generator $h$ realizes $p_0$ by Theorem 3.10. The tuple $\bar{u}$ is thus of the form $(h^{k_1}, \ldots, h^{k_n})$, and since its type is isolated, the type of $h^{k_1}$ is isolated: by a formula $\theta(x)$, say. Let $\psi(x)$ be a formula in $p_0$. By uniqueness of roots in $S_4$, the only $k_1$th root of $h^{k_1}$ is $h$, so the formula $F(u) : \forall z (z^{k_1} = u \to \psi(z))$ is in the type of $h^{k_1}$. In particular,

$$S_4 \models \forall z \big( \theta(z^{k_1}) \to \psi(z) \big).$$

Thus $\theta(z^{k_1})$ isolates $p_0(z)$, which contradicts Theorem 4.1. □

## 5 Maximal Independent Sequences

The following result gives an example witnessing that $p_0$ has weight greater than 1.

**Proposition 5.1** *Let $S$ be the fundamental group of the orientable closed surface of characteristic $-2$, and let $G$ be the free product $\mathbb{Z} * S$. Then $G$ admits maximal independent sets of realizations of $p_0$ of cardinality 2 and 3.*

**Proof** We choose the following presentation for $G$:

$$\big\langle a, a', b, b', z \;\big|\; [a, b][a', b'] = 1 \big\rangle.$$

The group $G$ admits at least three distinct hyperbolic tower structures (see Figure 5).

1. **The trivial structure:** We have $m = 0$, and $G = G^0 = \langle z \rangle * S$ is a free product of a free group and a closed surface group.
2. **The structure over the subgroup** $H_1 = \langle a, b, z \mid \rangle \simeq \mathbb{F}_3$**.** There is a hyperbolic floor $(G, H_1, r)$ described as follows.

   The hyperbolic floor decomposition $\Lambda_1$ consists of
   - one vertex with vertex group $H_1$,
   - one surface vertex with vertex group generated by $a', b'$ (the corresponding surface being a punctured torus).

   The edge group is generated by $[a, b]$. The retraction $r : G \to H_1$ is given by $r(a) = a$, $r(b) = b$, $r(z) = z$, and $r(a') = b$, $r(b') = a$.
3. **The structure over the subgroup** $H_2 = \langle a, za'z^{-1} \rangle \simeq \mathbb{F}_2$**.** There is a hyperbolic floor $(G, H_2, r)$ described as follows.

   The hyperbolic floor decomposition $\Lambda_2$ consists of
   - one vertex with vertex group $H_2$,
   - one surface vertex corresponding to a four times punctured sphere, whose maximal boundary subgroups are generated by $a$, $ba^{-1}b^{-1}$, $a'$, and $b'a'^{-1}b'^{-1}$, respectively.

   The embeddings of the corresponding edge groups into $H_2$ send them on the subgroups generated by $a$, $a^{-1}$, $za'z^{-1}$, and $za'^{-1}z^{-1}$, respectively (so the Bass–Serre elements are $b$, $z^{-1}$, and $b'z^{-1}$).

   The retraction $r : G \to H_2$ is given by $r(a) = a$, $r(b) = 1$, $r(a') = za'z^{-1}$, $r(b') = 1$, and $r(z) = 1$.

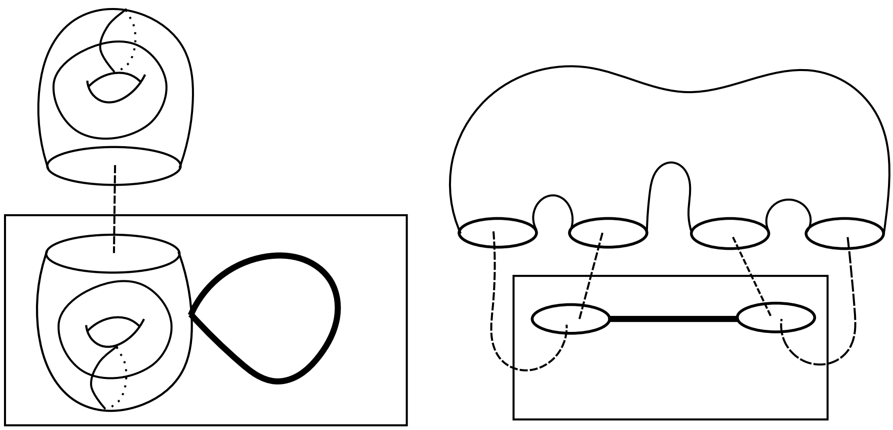

**Figure 5** The tower structures of $G$ over $H_1$ and $H_2$.

We claim that $G$ does not admit a tower structure over any rank 4 subgroup $K_1$ in which $H_1$ is a free factor, nor over any rank 3 subgroup $K_2$ in which $H_2$ is a free factor.

Indeed, suppose that such a subgroup $K_i$ exists for $i = 1$ or $i = 2$. In the associated graph of group decomposition $\Gamma_i$, the subgroup $H_i$ is elliptic, so the boundary subgroups of the surface group $S$ of $\Lambda_i$ are all elliptic. By Lemma 3.2, the induced graph of group decomposition for $S$ is dual to a set of nonparallel simple closed curves on the punctured torus if $i = 1$, or on the four times punctured sphere if $i = 2$. In other words, the surfaces of $\Gamma_i$ are proper subsurfaces of the punctured torus or the four-punctured sphere. But the only proper $\pi_1$-embedded subsurfaces that these surfaces admit are thrice-punctured spheres or cylinders, and these are not permitted in tower structures.

Thus any basis for $H_1$ realizes $p_0^{(3)}$ in $G$, but cannot be extended to a realization of $p_0^{(4)}$; hence it is maximal. Similarly, any basis for $H_2$ is a maximal realization of $p_0^{(2)}$. □

**Remark 5.2** A similar proof would show that $S * \mathbb{F}_n$ admits maximal independent sets of realizations of $p_0$ of size both $n + 1$ and $n + 2$.

In this low complexity case, it is easy to compute all the different possible tower structures that a given group may admit. In a more general setting, this becomes trickier. In particular, it would be interesting to find examples of finitely generated models of $T_{fg}$ (i.e., extended hyperbolic towers over the trivial subgroup) which admit maximal independent sets of realizations of $p_0$ of sizes whose ratio is arbitrarily large, thus witnessing directly infinite weight.

## 6 Homogeneity and Free Products

We mention one last application of the notion of hyperbolic towers: the free product of two homogeneous groups is not necessarily homogeneous.

For this we show the following.

**Lemma 6.1** *Let $\Sigma$ denote the closed orientable surface of characteristic $-2$. The fundamental group $S$ of $\Sigma$ is homogeneous.*

Note that [6, Corollary 8.5] states that the fundamental group of a surface of characteristic at most $-3$ is not homogeneous.

**Proof** Suppose that $\bar{u}$ and $\bar{v}$ are tuples which have the same type in $S$.

Suppose first that there exist embeddings $i : S \hookrightarrow S$ and $j : S \hookrightarrow S$ such that $i(\bar{u}) = \bar{v}$ and $j(\bar{v}) = \bar{u}$. Then since $S$ is freely indecomposable, the relative co-Hopf property of torsion-free hyperbolic groups (see [3, Corollary 4.19]) implies that $j \circ i$ is an isomorphism; hence so are $i$ and $j$.

We can now assume without loss of generality that there does not exist any embedding $i : S \hookrightarrow S$ such that $i(\bar{u}) = \bar{v}$. By Proposition 4.7 applied to $U = S$, this implies that $S$ admits a structure of extended hyperbolic floor over a proper subgroup $U$ containing $\bar{u}$.

By Lemma 3.2, the hyperbolic floor decomposition $\Gamma$ is dual to a set $\mathcal{C}$ of disjoint simple closed curves on $\Sigma$. This decomposes $\Sigma$ into two (possibly disconnected) subsurfaces $\Sigma_0$ and $\Sigma_1$, corresponding, respectively, to non-surface-type vertices and surface-type vertices. Now $\Sigma_0$ and $\Sigma_1$ satisfy $\chi(\Sigma_0) + \chi(\Sigma_1) = -2$, and either $\Sigma_1$ is a punctured torus, or it has characteristic at most $-2$.

In the first case, $\Sigma_0$ is also a punctured torus and $U = \pi_1(\Sigma_0) \simeq \mathbb{F}_2$. In the second case, we get that $\chi(\Sigma_0) = 0$, so $\Sigma_0$ must be a cylinder and $\bar{u}$ lies in $\pi_1(\Sigma_0)$. Let $\Sigma_0'$ be a punctured torus containing $\Sigma_0$. Since $S$ admits a structure of hyperbolic floor over $\pi_1(\Sigma_0')$ which contains $\bar{u}$, we may assume that we are also in the first case. Thus $S$ admits the following presentation:

$$\left\langle \alpha_0, \alpha_1, \beta_0, \beta_1 \;\middle|\; [\alpha_0, \beta_0] = [\alpha_1, \beta_1] \right\rangle$$

with $U = \langle \alpha_0, \beta_0 \rangle$ and $\pi_1(\Sigma_1) = \langle \alpha_1, \beta_1 \rangle$.

*Case 1.* If there does not exist any embedding $j : S \hookrightarrow S$ such that $j(\bar{v}) = \bar{u}$, we can deduce similarly that $S$ has a structure of hyperbolic floor over a subgroup $V \simeq \mathbb{F}_2$ which contains $\bar{v}$, and that $S$ admits a presentation as $\langle \alpha_0', \alpha_1', \beta_0', \beta_1' \mid [\alpha_0', \beta_0'] = [\alpha_1', \beta_1'] \rangle$ with $V = \langle \alpha_0', \beta_0' \rangle$.

Note that if there exists an isomorphism $f : U \to V$ sending $\bar{u}$ to $\bar{v}$, we must have $f([\alpha_0, \beta_0]) = g[\alpha_0', \beta_0']g^{-1}$ for some $g$ in $V$. (In $\mathbb{F}_2$, all the commutators of two elements forming a basis are conjugate.) Thus $f$ can be extended to an automorphism of $S$ by letting $f(\alpha_1) = g\alpha_1' g^{-1}$ and $f(\beta_1) = g\beta_1' g^{-1}$. We will now show that such an isomorphism $U \to V$ always exists.

If $U$ is freely indecomposable with respect to $\bar{u}$, by Lemma 3.11 and by Proposition 4.7, there is an embedding $f : U \hookrightarrow V$ sending $\bar{u}$ to $\bar{v}$. The smallest free factor of $V$ containing $\bar{v}$ contains $f(U)$; thus it cannot be cyclic. In particular, we have that $V$ is freely indecomposable with respect to $\bar{v}$. This implies in a similar way that there is an embedding $h : V \hookrightarrow U$ sending $\bar{v}$ to $\bar{u}$. Considering $h \circ f$ and using the relative co-Hopf property for torsion-free hyperbolic groups shows $f$ is in fact an isomorphism, again proving the claim.

If $\bar{u}$ is contained in a cyclic free factor $\langle u_0 \rangle$ of $U$, then $\bar{v}$ is contained in a cyclic free factor $\langle v_0 \rangle$ of $V$. Then $\bar{u} = (u_0^{k_1}, \dots, u_0^{k_l})$, but since $\bar{u}$ and $\bar{v}$ have the same type, we have $\bar{v} = (v_0^{k_1}, \dots, v_0^{k_l})$. Thus we can easily find an isomorphism $f$ as required.

*Case 2.* Suppose now that there exists an embedding $j : S \hookrightarrow S$ such that $j(\bar{v}) = \bar{u}$. The hyperbolic floor decomposition $\Gamma$ of $S$ over $U$ (namely, the amalgamated product $U *_{\langle c\rangle} S_1$) induces via $j$ a splitting of $S$ as a graph of groups with cyclic edge groups. By Lemma 3.2, this splitting is dual to a set $\mathcal{C}$ of simple closed curves on $\Sigma$. Since $\bar{u}$ is elliptic in the splitting $U *_{\langle c\rangle} S_1$, the tuple $\bar{v}$ is elliptic in this induced splitting. Thus $\bar{v}$ is contained in the fundamental group $S'_0$ of one of the connected components $\Sigma'_0$ of the complement in $\Sigma$ of $\mathcal{C}$, and $j(S'_0)$ is contained in $U$.

We claim that $\Sigma'_0$ is a punctured torus and that $j$ sends $S'_0$ isomorphically onto $U$ (as a surface group with boundary). This is enough to finish the proof, since we can then easily extend $j|_{S'_0}$ to an isomorphism $S \to S$.

Let us thus prove the claim. The morphism $j$ is injective and sends elements corresponding to curves of $\mathcal{C}$ (in particular, boundary subgroups of $S'_0$) to edge groups of $\Gamma$, that is, to conjugates of $\langle[\alpha_0, \beta_0]\rangle$. By [3, Lemmas 3.10, 3.12] we deduce that the complexity of $\Sigma'_0$ is at least that of $\Sigma_0$ and that if we have equality, then $j|_{S'_0}$ is an isomorphism of surface groups. In particular, if $\chi(\Sigma'_0) = -1$, $\Sigma'_0$ must have exactly one boundary component; hence it is a punctured torus, and the claim is proved. If $\chi(\Sigma'_0) = -2$, the surface $\Sigma'_0$ is a twice-punctured torus. This implies that $S$ is generated by $S'_0$ together with an element $t$ which conjugates two maximal boundary subgroups $\langle d_1\rangle$ and $\langle d_2\rangle$ of $S'_0$ which are not conjugate in $S'_0$. Now $j(d_1)$ and $j(d_2)$ are conjugate in $S$, and both contained in $U$: they must be conjugate by an element $t'$ of $U$ since $U$ is a retract of $S$. Now $j(t)^{-1}t'$ commutes to $j(d_1)$, so $j(t)^{-1}t'$, and thus $j(t)$, is contained in $U$. Finally, $j(S) = \langle j(S'_0), j(t)\rangle \leq U$, but this is a contradiction since $U$ is free and $j$ is injective. □

On the other hand, the following result is an immediate consequence of Proposition 5.1.

**Lemma 6.2** *Let $G = \mathbb{Z} * S$. Then $G$ is not homogeneous.*

Since $\mathbb{Z}$ is homogeneous, this gives an example of a free product of two homogeneous groups which fails to be homogeneous.

**Proof** By Proposition 5.1, there exist maximal realizations $(u_1, u_2)$ of $p_0^{(2)}$ and $(v_1, v_2, v_3)$ of $p_0^{(3)}$ in $G$. If $G$ were homogeneous, there would be an automorphism $\theta$ of $G$ sending $(v_1, v_2)$ to $(u_1, u_2)$ since they both realize $p_0^{(2)}$. But then $(u_1, u_2, \theta(v_3))$ would realize $p_0^{(3)}$, contradicting maximality of $(u_1, u_2)$. □

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