Infinitely p-Divisible Points on Abelian Varieties Defined over Function Fields of Characteristic $p > 0$

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Abstract   In this article we consider some questions raised by F. Benoist, E. Bouscaren, and A. Pillay. We prove that infinitely p-divisible points on abelian varieties defined over function fields of transcendence degree one over a finite field are necessarily torsion points. We also prove that when the endomorphism ring of the abelian variety is $\mathbb{Z}$, then there are no infinitely p-divisible points of order a power of p.

1   Introduction

Fix once and for all a prime number p.

Let K_0 be the function field of a smooth curve over a finite field $\mathbb{F}$ of characteristic p. Let B be an abelian variety over K_0.

For any abelian group G, define $G^{\#} := \bigcap_{l \geqslant 0} p^l G$. We call the elements of $G^{\#}$ the *infinitely p-divisible* points of G. Furthermore, write $\operatorname{Tor}(G)$ for the subset of G consisting of elements of finite order. If $n \in \mathbb{N}^*$, we write $\operatorname{Tor}^n(G)$ for the subset of $\operatorname{Tor}(G)$ consisting of elements of order prime to n. Similarly, we write $\operatorname{Tor}_n(G)$ for the subset of $\operatorname{Tor}(G)$ consisting of elements whose order divides a power of n.

The following conjecture is made by F. Benoist, E. Bouscaren, and A. Pillay at the very end of the article [3].

Conjecture 1.1   *The inclusion $B(K_0^{\mathrm{s}})^{\#} \subseteq \operatorname{Tor}(B(K_0^{\mathrm{s}}))$ holds.*

Here K_0^{s} is the separable closure of K_0.

In the same context, F. Benoist then asked the following question (see [2]).

Question 1.2   Suppose that there are no nonvanishing $\bar{K}_0$-homomorphisms $B_{\bar{K}_0} \to C$, where C is an abelian variety which has a model over $\bar{\mathbb{F}}$.

Do we have $B(K_0^{\mathrm{s}})^{\#} \cap \operatorname{Tor}_p(B(K_0^{\mathrm{s}})) = 0$?

We recall (in the notation of Question 1.2) that a model of C over $\bar{\mathbb{F}}$ is an abelian variety C_0 over $\bar{\mathbb{F}}$ such that $C \simeq C_0 \times_{\bar{\mathbb{F}}} \bar{K}_0$.

In response to these questions, we shall prove the following two results in this text.

Theorem 1.3   *Conjecture 1.1 holds.*

A. Pillay explained that Conjecture 1.1 can be viewed as a positive characteristic analogue of Manin's "theorem of the kernel" (see Manin [14] for the latter). He also outlined a proof of the Mordell–Lang conjecture over function fields of characteristic $p > 0$ (in the case of abelian varieties), which is based on Conjecture 1.1 and a quantifier elimination result.

Conjecture 1.1 is also linked to the Mordell–Lang conjecture in the following way. Lemma 2.4 below (which plays a key role in the proof of Conjecture 1.1) implies that the infinitely p-divisible points defined over a certain separable (but transcendental) field extension of K_0 are annulled by a fixed Weil polynomial (in particular it has no roots of unity among its roots) applied to a lifting of the Frobenius automorphism. This fact, combined with the existence of arc schemes (see, e.g., Rössler [23, before Lemma 2.3] as well as Pink and Rössler [19, Proposition 6.1]) can be used to give a quick proof of [23, Theorem 4.1]. This last theorem is the main tool in the proof of the Mordell–Lang conjecture given in [23]; more precisely, the Mordell–Lang conjecture follows quickly from it (see [23, Section 3.2] for the argument), once the existence of jet schemes (see, e.g., [23, Section 2.1] for the latter) is established.

Details about the argument outlined in the last paragraph will appear in Corpet [4], where it will also be generalized to the semiabelian situation; it would be very interesting to understand the link (if there is one) between this argument and A. Pillay's approach to the Mordell–Lang conjecture mentioned above.

Theorem 1.4   *Suppose that* $\mathrm{End}_{\bar{K}_0}(B) = \mathbb{Z}$. *Then* $B(K_0^s)^\# \cap \mathrm{Tor}_p(B(K_0^s)) = 0$.

That is, the answer to F. Benoist's question is affirmative if $\mathrm{End}_{\bar{K}_0}(B) = \mathbb{Z}$.

We actually prove a stronger result, but the stronger form is (as very often) more difficult to formulate. See the last remark of the text for this stronger form.

In particular, the answer to Question 1.2 is affirmative if B is an elliptic curve over K_0. This was also proved in [3]. In [26, after Theorem 3], F. Voloch shows that if B/K_0 is ordinary and has maximal Kodaira–Spencer rank, then $B[p](K_0^s) = 0$, and thus the answer to Question 1.2 is also affirmative in that situation.

The hypothesis $B(K_0^s)^\# \cap \mathrm{Tor}_p(B(K_0^s)) = 0$ is a crucial hypothesis in B. Poonen and J. F. Voloch's work on the Brauer–Manin obstruction over function fields (see, e.g., [20, Theorem B]).

The ideas underlying the proof of Theorem 1.3 are the following. First we notice that we may choose a (necessarily discrete) valuation on K_0 such that all the p^lth roots of a given infinitely p-divisible point lie in a corresponding maximal unramified extension of the completion of K_0 along the valuation (this is Proposition 2.2). Second, we notice that an infinitely p-divisible point on B can be recovered from the images of all its p^lth roots in the reduction of B modulo the ideal of the valuation (this is Lemma 2.4). This correspondence is Galois equivariant, and this implies that an infinitely p-divisible point must have an infinite Galois orbit, if it is not a torsion point. This is a contradiction, because we are dealing with algebraic points (see the end of the proof of Proposition 2.3).

Here is the train of thought underlying the proof of Theorem 1.4. To obtain a contradiction, we suppose that we are given a sequence of nonzero points $P_i \in B(K_0^s)$ such that $p \cdot P_i = P_{i-1}$. We then consider the successive quotients of B by the groups generated by the Galois orbits of the P_i. This gives a sequence of abelian varieties, which is shown to have decreasing modular height (see (2)). Using a fundamental result of Zarhin (see Theorem 3.1), we deduce that infinitely many of the quotients are isomorphic, and thus an abelian variety isogenous to B is endowed with a nontrivial endomorphism. This contradicts one of the assumptions of Theorem 1.4.

In the following two sections, we prove the results described above in the same order. The two sections are technically and terminologically independent of each other and can be read in any order.

2 Proof of Theorem 1.3

Before describing the proof, we would like to point out the following special case (which is not needed in the proof).

Lemma 2.1 *If the p-rank of B is zero, then we have $B(K_0^s)^\# \subseteq \mathrm{Tor}(B(K_0^s))$.*

Proof of the lemma This is a consequence of the Lang–Néron theorem. The details are left to the reader. ☐

Theorem 1.3 will be shown to be a simple consequence of the following two propositions.

To formulate them, let S be the spectrum of a complete Noetherian local ring of characteristic p. Let K be the fraction field of S. Let $\mathcal{A}$ be an abelian scheme over S.

Write S^{sh} for the spectrum of the strict henselization of the ring underlying S and L for the fraction field of S^{sh} (see, e.g., Milne [15, Chapter I, Section 4, p. 38] for the definition of the strict henselization). We choose once and for all a K-embedding $L \hookrightarrow K^s$.

Proposition 2.2 *Suppose that S is a scheme of characteristic p. Suppose that the p-rank of the fibers of $\mathcal{A}$ is constant. Then we have*

$$\left(p \cdot \mathcal{A}(K^s)\right) \cap \mathcal{A}(L) = p \cdot \mathcal{A}(L).$$

The proof of Proposition 2.2 will be given below.

Proposition 2.3 *Suppose that the residue field of the closed point of S is finite. Then we have $A(L)^\# = \mathrm{Tor}(A(L)^\#)$.*

The proof of Proposition 2.3 will be given below.

Proof of Theorem 1.3 (Assuming Propositions 2.2 and 2.3) Let $x \in B(K_0^s)^\#$. We shall show that x is a torsion point.

We may assume without restriction of generality that $x \in B(K_0)$. (Otherwise, we replace K_0 by $K_0(\kappa(x))$.) After again possibly replacing K_0 by a finite extension, we may suppose that $B[p](K_0) = B[p](\bar{K}_0)$. Let U be a smooth algebraic curve over $\mathbb{F}$. We may suppose without restriction of generality that there is an abelian scheme $\mathcal{B}$ over U extending B and that the p-rank of the fibers of $\mathcal{B}$ is constant. Let now u be any closed point of U. Let S be the completion of the local ring of U at u, and let $\mathcal{A} := \mathcal{B}_S$. Let K be the fraction field of S, and let L be the fraction field of the strict henselization S^{sh} of S, as before.

Now let $x_1, x_2, \ldots \in B(K_0^s)$ be elements such that $p \cdot x_l = x_{l-1}$ for all $l \geqslant 2$ and such that $p \cdot x_1 = x$. Such a sequence exists by assumption. Applying Proposition 2.2 to x and all the x_l successively, we conclude that $x \in \mathcal{A}(L)^\#$. We conclude the proof of Theorem 1.3 by appealing to Proposition 2.3. $\qquad\square$

Proof of Proposition 2.2   Let $N := \ker[p]$ be the kernel of the multiplication by p morphism $[p] : \mathcal{A} \to \mathcal{A}$. This is a (nonreduced) finite group scheme over S. There is an exact sequence of finite, flat, S-group schemes

$$0 \to N_0 \to N \to N_{\mathrm{et}} \to 0 \quad (\%)$$

where N_0 is connected and N_{et} is étale. (See Tate [24, (1.4)] for this.) Since the unit section of N is open and closed (by the assumption on the p-rank), we see that N_0 is an infinitesimal group scheme.

Thus the decomposition ($\%$) of N leads to S-isogenies of abelian schemes

$$\mathcal{A} \xrightarrow{\varphi} \mathcal{A}_1 \xrightarrow{\mu} \mathcal{A}$$

such that μ is étale, φ is purely inseparable, and $\mu \circ \varphi = [p]$.

Let now $x \in (p \cdot A(K^s)) \cap A(L)$. Let $T \hookrightarrow \mathcal{A}_{S^{\mathrm{sh}}}$ be the closed subscheme arising from the section of $\mathcal{A}_{S^{\mathrm{sh}}}$ over S^{sh} associated to x. We have S^{sh}-morphisms $[p]^* T \to \mu^*_{S^{\mathrm{sh}}} T \to T$, where the morphism $[p]^* T \to \mu^*_{S^{\mathrm{sh}}} T$ is purely inseparable over L and the morphism $\mu^*_{S^{\mathrm{sh}}} T \to T$ is étale. Furthermore, by construction, there is an element $y \in \mathcal{A}(K^s)$ such that $y \in ([p]^* T)(K^s)$ and such that $p \cdot y = x$. Let V be the connected component of $\mu^* T$ containing the image of $\varphi_{S^{\mathrm{sh}}}(y)$. We obtain a sequence of S^{sh}-morphisms

$$\mathrm{Spec}\, K^s \to ([p]^* T)_V \to V \to T \to A$$

whose composition is x.

Since S^{sh} is strictly henselian, the morphism $V \to T$ must be an isomorphism. Let now $v \in V$ be the image of the composition $\mathrm{Spec}\, K^s \to ([p]^* T)_V \to V$. Let $w \in ([p]^* T)_V$ be the image of y. Since the morphism $([p]^* T)_V \to V$ is purely inseparable, the extension of residue fields $\kappa(w)|\kappa(v)$ is purely inseparable. Hence it must be trivial, since the field extension $\kappa(w)|\kappa(T) = L = \kappa(v)$ is separable, because it is a subextension of the separable algebraic extension $K^s|\kappa(T)$. Hence the extension $\kappa(w)|\kappa(T)$ is trivial.

This shows that we actually have $x \in p \cdot A(L)$, which completes the proof of the proposition. $\qquad\square$

Proof of Proposition 2.3   Let us temporarily use the following notation. Let $R := \Gamma(S, \mathcal{O}_S)$ be the local ring underlying S. Let $\mathfrak{m} \subseteq R$ be the maximal ideal of R, and let $k := R/\mathfrak{m}$. We write correspondingly $R^{\mathrm{sh}} := \Gamma(S^{\mathrm{sh}}, \mathcal{O}_{S^{\mathrm{sh}}})$ and $\mathfrak{m}^{\mathrm{sh}} \subseteq R^{\mathrm{sh}}$ for the maximal ideal of R^{sh}. We fix an identification of $\bar{k}$ with $R^{\mathrm{sh}}/\mathfrak{m}^{\mathrm{sh}}$. Let also $\widehat{S^{\mathrm{sh}}}$ be the spectrum of the completion $\widehat{R^{\mathrm{sh}}}$ of R^{sh} along $\mathfrak{m}^{\mathrm{sh}}$, and let $\widehat{L}$ be the fraction field of $\widehat{R^{\mathrm{sh}}}$.

Let A_0 be the special fiber of $\mathcal{A}$, that is, the fiber of $\mathcal{A}$ over the unique closed point of S.

Let $I_p(A_0(\bar{k})) := \varprojlim_{l \geqslant 0} A_0(\bar{k})$, where the transition morphisms are all given by multiplication by p. (Beware: $I_p(A_0(\bar{k}))$ is not the classical Tate module.) So an

element of $\varprojlim_{l \geqslant 0} A_0(\bar{k})$ is given by an inverse system

$$\cdots \to t_l \to t_{l-1} \to \cdots \to t_0$$

where $t_l \in A_0(\bar{k})$ and $t_{l-1} = p \cdot t_l$ for all $l \geqslant 0$. Now notice that for any $l \geqslant 1$, the kernel of the reduction map $\mathcal{A}(R^{\text{sh}}/\mathfrak{m}^{\text{sh},l}) \to A_0(\bar{k})$ is a commutative group, which is killed by multiplication by p^{l-1}. This follows from the fact that the kernel of the reduction map $\mathcal{A}(\widehat{S^{\text{sh}}}) \to A_0(\bar{k})$ can be identified with the points of a commutative formal group. See Fantechi et al. [7, Theorem 8.5.9(a), p. 213].

For a fixed $r \geqslant 1$, let $\lambda_r : I_p(A_0(\bar{k})) \to \mathcal{A}(R^{\text{sh}}/\mathfrak{m}^{\text{sh},r})$ be the map sending the inverse system

$$\cdots \to t_r \to t_{r-1} \to \cdots \to t_0$$

to the element $p^{r-1} \cdot \widetilde{t}_{r-1}$, where $\widetilde{t}_{r-1}$ is any lifting of t_{r-1} to $\mathcal{A}(R^{\text{sh}}/\mathfrak{m}^{\text{sh},r})$. Notice that the composition of λ_r with the reduction map $\mathcal{A}(R^{\text{sh}}/\mathfrak{m}^{\text{sh},r}) \to \mathcal{A}(R^{\text{sh}}/\mathfrak{m}^{\text{sh},r-1})$ is the map λ_{r-1}. Hence we obtain a homomorphism

$$\lambda : I_p\big(A_0(\bar{k})\big) \to \varprojlim_r \mathcal{A}(R^{\text{sh}}/\mathfrak{m}^{\text{sh},r}) \simeq \mathcal{A}(\widehat{S^{\text{sh}}}) \simeq \mathcal{A}(\widehat{L}).$$

Remark A variant of the map λ appears in the theory of the Serre–Tate lifting of ordinary abelian varieties over finite fields (see Katz [12]).

Lemma 2.4 *We have $\mathcal{A}(\widehat{L})^\# = \text{Im}(\lambda)$.*

Lemma 2.4 is a variant of Raynaud [22, Lemme 3.2.1]. The proofs of both lemmas are actually identical.

Proof of the lemma Let $x_0, x_1, \ldots \in A(\widehat{L})$ be such that $p \cdot x_i = x_{i-1}$ for all $i \geqslant 1$. Let $\rho : A(\widehat{L}) \to A_0(\bar{k})$ be the reduction map. Consider the inverse system $q \in I_p(A_0(\bar{k}))$ given by

$$\cdots \to \rho(x_i) \to \rho(x_{i-1}) \to \cdots \to \rho(x_0).$$

We claim that $\lambda(q) = x_0$. For this, it is sufficient to prove that

$$\lambda_r(q) = x_0 \ (\text{mod } \mathfrak{m}^{\text{sh},r})$$

for all $r \geqslant 1$. We compute

$$\lambda_r(q) = p^{r-1} \cdot \big(x_{r-1} \ (\text{mod } \mathfrak{m}^{\text{sh},r})\big) = (p^{r-1} \cdot x_{r-1}) \ (\text{mod } \mathfrak{m}^{\text{sh},r})$$
$$= x_0 \ (\text{mod } \mathfrak{m}^{\text{sh},r}).$$

This shows that $\bigcap_{l \geqslant 0} p^l A(\widehat{L}) \subseteq \text{Im}(\lambda)$.

For the opposite inclusion (which is actually not needed in the text), let

$$\cdots \to t_1 \to t_0$$

be an element of $I_p(A_0(\bar{k}))$. This element is the image under multiplication by p of the element

$$\cdots \to t_2 \to t_1$$

(in other words, $I_p(A_0(\bar{k}))^\# = I_p(A_0(\bar{k}))$). Hence any image by λ of an element of $I_p(A_0(\bar{k}))$ is infinitely p-divisible. $\qquad \square$

Remark Until now in the proof, we only used the fact that k is a field of characteristic p (not that it is a finite field). In particular, the last lemma is true in that generality.

Now notice that the map λ is compatible with the natural action of $\mathrm{Gal}(\bar{k}|k)$ on $I_p(A_0(\bar{k}))$ and $\mathcal{A}(\widehat{L})$. Let $\sigma \in \mathrm{Gal}(\bar{k}|k) \simeq \widehat{\mathbb{Z}}$ be the Frobenius automorphism. Let $x \in \mathcal{A}(L)^\# \subseteq \mathcal{A}(\widehat{L})^\# = \mathrm{Im}(\lambda)$. Since the residue field of x lies in an algebraic extension of K, there exists $r \geqslant 1$ such that $\sigma^r(x) = x$. On the other hand, by the Weil conjectures, there is a polynomial Q with integer coefficients and no roots of unity among its roots, such that $Q(\sigma^r) = 0$ on $I_p(A_0(\bar{k}))$. Hence $Q(\sigma^r(x)) = Q(1) \cdot x = 0$. Since $Q(1) \neq 0$, this implies that x is a torsion point. This proves Proposition 2.3.   □

Remark   Let L_1 be the algebraic closure of L in $\widehat{L}$. (Notice that if S is a discrete valuation ring, then $L_1 = L$; see, e.g., Liu [13, Example 8.3.34, p. 360].) The above proof of Proposition 2.3 actually shows that

$$\mathcal{A}(\widehat{L})^\# \cap \mathcal{A}(L_1) = \mathrm{Tor}\big(\mathcal{A}(L_1)\big),$$

which is stronger.

3 Proof of Theorem 1.4

This section can be read independently of the previous one.

To prove Theorem 1.4, the following theorem is crucial. Let U be a smooth, proper curve over $\mathbb{F}$ whose function field is K_0. A semiabelian scheme $\mathcal{A}/U$ is a commutative group scheme over U whose geometric fibers are semiabelian varieties. We shall say that a semiabelian scheme over U is *generically abelian* if its geometric generic fiber is an abelian variety. If $\mathcal{A}/U$ is a semiabelian scheme with zero-section ϵ, we shall write

$$\omega^0_{\mathcal{A}/U} := \epsilon^*\big(\det(\Omega_{\mathcal{A}/U})\big)$$

for the determinant of the sheaf of differentials of $\mathcal{A}$ over U, restricted to the zero section.

Theorem 3.1 (Zarhin)   Let $\beta, g, n \geqslant 0$. Suppose that $(n, p) = 1$ and that $n \geqslant 3$. Up to isomorphism, there is only a finite number of generically abelian semiabelian schemes $\mathcal{A}/U$ of relative dimension g over U such that

- $\deg_U(\omega^0_{\mathcal{A}/U}) = \beta$;
- there exists an isomorphism of group schemes $\mathcal{A}_{K_0}[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}_{K_0}$.

Theorem 3.1 is well known to specialists, but I could find no formal proof of it in the literature.

Proof of Theorem 3.1   We shall first prove the following statement. Let $\beta \in \mathbb{N}$.

Up to isomorphism, there is only a finite number of generically abelian semiabelian schemes $\mathcal{A}/U$ of relative dimension g over U, such that $(*)$

- there exists a principal polarization on $\mathcal{A}_{K_0}$;
- there exists a symplectic isomorphism $(\mathbb{Z}/n\mathbb{Z})^{2g}_{K_0} \simeq \mathcal{A}_{K_0}[n]$;
- $\deg_U(\omega^0_{\mathcal{A}/U}) = \beta$.

To prove this, we shall use the following deep results of D. Mumford, A. Grothendieck, L. Moret-Bailly, C.-L. Chai, and G. Faltings.

Let $\mathbf{A}_{g,n}$ be the functor from the category of locally Noetherian $\mathbb{F}_p$-schemes to the category of sets such that

$$\mathbf{A}_{g,n}(S) = \{\text{isomorphism classes of the following objects:}$$
$$\text{principally polarized abelian schemes over } S \text{ endowed}$$
$$\text{with a symplectic isomorphism } (\mathbb{Z}/n\mathbb{Z})_S^{2g} \simeq \mathcal{A}[n]\}.$$

Then D. Mumford proves (see [18]) that the functor $\mathbf{A}_{g,n}$ is representable by a scheme, which is separated and of finite type over $\mathbb{F}_p$. We shall also denote this scheme by $\mathbf{A}_{g,n}$.

Furthermore, in [6, Chapter V, Section 2, Theorem. 2.5], G. Faltings and C. Chai prove that there exists

- a scheme $\mathbf{A}_{g,n}^*$, which is proper over $\mathbb{F}_p$;
- an open immersion $\mathbf{A}_{g,n} \hookrightarrow \mathbf{A}_{g,n}^*$;
- a semiabelian scheme $\mathcal{B}$ over $\mathbf{A}_{g,n}^*$ such that $\mathcal{B}_{\mathbf{A}_{g,n}}$ is isomorphic to the abelian scheme underlying the universal object over $\mathbf{A}_{g,n}$.

Also, they show that $\omega^0_{\mathcal{B}/\mathbf{A}_{g,n}^*}$ is an ample line bundle.

Now write $Z := U \times_{\mathbb{F}_p} \mathbf{A}_{g,n}^*$. Recall that the Hilbert scheme $\mathrm{Hilb}(Z/\mathbb{F}_p)$ is a scheme over $\mathbb{F}_p$, which is locally of finite type and such that

$$\mathrm{Hilb}(Z/\mathbb{F}_p)(T) = \{\text{closed subschemes of } Z_T, \text{ which are proper and flat over } T\}$$

for any locally Noetherian scheme T over $\mathbb{F}_p$ (see Grothendieck [8]).

Furthermore, fix $\Phi \in \mathbb{Q}[\lambda]$, a polynomial with rational coefficients, and fix L/Z an ample line bundle. The $\mathbb{F}_p$-scheme $\mathrm{Hilb}_\Phi(Z/\mathbb{F}_p)$ is then characterized by the property that

$$\mathrm{Hilb}_\Phi(Z/\mathbb{F}_p)(T)$$
$$:= \{\text{closed subschemes } W \text{ of } Z_T, \text{ which are proper and flat over } T$$
$$\text{and such that } \chi(W_t, L_{W_t}^{\otimes\lambda}) = \Phi(\lambda) \text{ for all } \lambda \in \mathbb{N} \text{ and all } t \in T\}.$$

Here W_t is the fiber at $t \in T$ of the morphism $W \to T$ and L_{W_t} is the pullback of L to W_t by the natural morphism $W_t \to Z$. The symbol $\chi(\cdot)$ refers to the Euler characteristic; by definition,

$$\chi(W_t, L_{W_t}^{\otimes\lambda}) = \sum_{r \geqslant 0} (-1)^r \dim_{\kappa(t)} H^r(W_t, L_{W_t}^{\otimes\lambda}).$$

It is shown in [8] that $\mathrm{Hilb}_\Phi(Z/\mathbb{F}_p)$ is projective over $\mathbb{F}_p$ (as a consequence of the projectivity of Z). Notice that by construction, we have a decomposition

$$\mathrm{Hilb}(Z/\mathbb{F}_p) = \coprod_{\Phi \in \mathbb{Q}[\lambda]} \mathrm{Hilb}_\Phi(Z/\mathbb{F}_p).$$

Finally, it is shown in [7, part II, 5.23] that the functor $\mathrm{Mor}_{\mathbb{F}_p}(U, \mathbf{A}_{g,n}^*)$ from locally Noetherian $\mathbb{F}_p$-schemes T to the category of sets, such that

$$\mathrm{Mor}_{\mathbb{F}_p}(U, \mathbf{A}_{g,n}^*)(T) = \{T\text{-morphisms from } U_T \text{ to } \mathbf{A}_{g,n,T}^*\},$$

is representable by an open subscheme of $\mathrm{Hilb}(Z/\mathbb{F}_p)$. The open immersion $\mathrm{Mor}_{\mathbb{F}_p}(U, \mathbf{A}_{g,n}^*) \hookrightarrow \mathrm{Hilb}(U \times_{\mathbb{F}_p} \mathbf{A}_{g,n}^*/\mathbb{F}_p)$ is described by the natural transformation of functors

$$T\text{-morphism } f \text{ from } U_T \text{ to } \mathbf{A}_{g,n,T}^* \mapsto \text{graph of } f.$$

Now fix an ample line bundle M on U. Let L be the line bundle $M \boxtimes \omega^0_{\mathcal{B}/\mathbf{A}^*_{g,n}}$ on $Z = U \times_{\mathbb{F}_p} \mathbf{A}^*_{g,n}$.

Now we are finally ready to tackle our proof of finiteness. Suppose that we are given $\mathcal{A}/U$ as in $(*)$. Restricting to the generic point of U, we get a morphism $\operatorname{Spec}\kappa(U) \to \mathbf{A}^*_{g,n}$ (whose image is in $\mathbf{A}_{g,n}$), and since $\mathbf{A}^*_{g,n}$ is proper over $\mathbb{F}_p$ and U is a Dedekind scheme, this extends to a morphism $\varphi : U \to \mathbf{A}^*_{g,n}$. We thus get a point $\varphi \in \operatorname{Mor}_{\mathbb{F}_p}(U, \mathbf{A}^*_{g,n})(\mathbb{F}_p)$. Let $\Gamma_\varphi \hookrightarrow U \times_{\mathbb{F}_p} \mathbf{A}^*_{g,n}$ be the graph of φ. We compute its Hilbert polynomial

$$
\begin{aligned}
\chi(\Gamma_\varphi, L_\Gamma^{\otimes\lambda}) &= \chi\big(U, \big(M \otimes \varphi^*(\omega^0_{\mathcal{B}/\mathbf{A}^*_{g,n}})\big)^{\otimes\lambda}\big) \\
&= \deg_U\big(\big(M \otimes \varphi^*(\omega^0_{\mathcal{B}/\mathbf{A}^*_{g,n}})\big)^{\otimes\lambda}\big) + 1 - g_U \\
&= \lambda \cdot \deg_U\big(M \otimes \varphi^*(\omega^0_{\mathcal{B}/\mathbf{A}^*_{g,n}})\big) + 1 - g_U \\
&= \lambda \cdot \deg_U(M \otimes \omega_{\mathcal{A}/U}) + 1 - g_U \\
&= \lambda \cdot \deg_U(M) + \lambda \cdot \deg_U(\omega_{\mathcal{A}/U}) + 1 - g_U \\
&= \lambda \cdot \deg_U(M) + \lambda \cdot \beta + 1 - g_U := Q(\lambda). \quad (1)
\end{aligned}
$$

Here g_U is the genus of U. The second equality is justified by the Riemann–Roch theorem on U. From (1) we see that the Hilbert polynomial of $\varphi \in \operatorname{Mor}_{\mathbb{F}_p}(U, \mathbf{A}^*_{g,n})(\mathbb{F}_p)$ only depends on β (once M is given), and thus $\varphi \in \operatorname{Hilb}_{Q(\lambda)}(U \times_{\mathbb{F}_p} \mathbf{A}^*_{g,n}/\mathbb{F}_p)(\mathbb{F}_p)$.

Now to prove that there are only a finite number of generically abelian semiabelian schemes $\mathcal{A}/U$ satisfying $(*)$, just notice that the set $\operatorname{Hilb}_{Q(\lambda)}(U \times_{\mathbb{F}_p} \mathbf{A}^*_{g,n}/\mathbb{F}_p)(\mathbb{F}_p)$ is finite, since $\operatorname{Hilb}_{Q(\lambda)}(U \times_{\mathbb{F}_p} \mathbf{A}^*_{g,n}/\mathbb{F}_p)$ is projective and hence of finite type over $\mathbb{F}_p$.

To conclude, recall the following facts.

First, a generically abelian semiabelian scheme $\mathcal{A}/U$ is completely determined by its generic fiber $\mathcal{A}_{K_0}$ (see Raynaud [21, Chapitre IX, Corollaire 1.4, p. 130]). Second, by Zarhin's "trick" (see Moret-Bailly [17, Chapitre IX, 1]), for any abelian variety C/K_0, the abelian variety $(C \times_{K_0} C^\vee)^4$ can be principally polarized. Third, a given abelian variety C/K_0 has only a finite number of direct factors (see [16, Theorem 18.7]). Fourthly, if C/K_0 is an abelian variety which extends to a semi-abelian scheme $\mathcal{C}$ over U, then the dual abelian variety $C^\vee$ has the same property (by Grothendieck's criterion in Grothendieck, Raynaud, and Rim [9, Chapitre IX, Proposition 5.13(c)]), and furthermore $\deg(\omega^0_{\mathcal{C}/U}) = \deg(\omega^0_{\mathcal{C}^\vee/U})$. (Here we wrote somewhat sloppily $\mathcal{C}^\vee$ for the semiabelian extension of $C^\vee$; see [6, Chapter V, Section 3, Lemma 3.4] for the latter equality.)

Putting all these facts together, the theorem readily follows from the just proven fact that there are only a finite number of generically abelian semiabelian schemes $\mathcal{A}/U$ satisfying $(*)$. $\qquad\square$

Proof of Theorem 1.4   The proof is by contradiction. So suppose that there exists a point $P \in B(K_0^s)^\#$ such that $P \neq 0$ and $p^r \cdot P = 0$ for some $r \geq 1$. Then there exists a sequence of points $(P_n \in B(K_0^s))_{n \in \mathbb{N}^*}$ such that $P = P_1$ and $p \cdot P_{i+1} = P_i$ for all $i \in \mathbb{N}^*$. Let G_i be the subgroup of $B(K_0^s)^\#$ generated by the elements $\gamma(P_i)$, where γ runs through $\operatorname{Gal}(K_0^s|K_0)$. By Galois descent, the groups G_i are naturally defined over K_0 and we have natural inclusion morphisms $G_1 \hookrightarrow G_2 \hookrightarrow \cdots$ over

K_0. Furthermore, the order d_i of the group G_i is strictly increasing as a function of $i \in \mathbb{N}^*$.

Now we may replace K_0 by a finite extension and U by the corresponding projective curve over k, without restricting generality. Hence we may suppose that B extends to a generically abelian semiabelian scheme $\mathcal{B}$ over U (by Grothendieck's semiabelian reduction theorem; see [9, IX]). Furthermore, we may suppose that there is an isomorphism $\rho_B : (\mathbb{Z}/l\mathbb{Z})^{2g}_{K_0} \simeq \mathcal{B}_{K_0}[l]$, where $l \geqslant 3$ is a natural number prime to p. We fix such an l for the rest of the proof.

Now look at the sequence of K_0-isogenies

$$B \to B/G_1 \to B/(G_1 \cdot G_2) \to \cdots B/(G_1 \cdot G_2 \cdot G_3) \to \cdots . \tag{2}$$

Let $B_i := B/(G_1 \cdots G_i)$ and $B_0 := B$. Let $\pi_i : \mathcal{B}_i \to U$ be the connected component of the Néron model of B_i over U. Since all the B_i are isogenous, the criterion [9, IX, Proposition 5.13(c)] shows that all the $\mathcal{B}_i$ are generically abelian semiabelian schemes over U. Furthermore, by the universal property of Néron models, the morphisms in (2) extend to U-morphisms

$$\mathcal{B} \xrightarrow{\varphi_0} \mathcal{B}_1 \xrightarrow{\varphi_1} \mathcal{B}_2 \to \cdots . \tag{3}$$

Finally, since the isogenies φ_i are of degree prime to l, we obtain isomorphisms $\varphi_i \circ \varphi_{i-1} \circ \cdots \circ \varphi_0 \circ \rho_B : (\mathbb{Z}/l\mathbb{Z})^{2g}_{K_0} \simeq \mathcal{B}_{i,K_0}[l]$ for all i.

By construction, the morphisms φ_i are generically étale. Looking at differentials, we see that there is an exact sequence of coherent sheaves

$$\varphi_i^* \Omega_{\mathcal{B}_{i+1}/U} \to \Omega_{\mathcal{B}_i/U} \to \Omega_{\varphi_i} \to 0.$$

Since $\Omega_{\mathcal{B}_{i+1}/U}$ is locally free and since the morphism of sheaves $\varphi_i^* \Omega_{\mathcal{B}_{i+1}/U} \to \Omega_{\mathcal{B}_i/U}$ is injective at the generic point of $\mathcal{B}_i$, we see that the morphism $\varphi_i^* \Omega_{\mathcal{B}_{i+1}/U} \to \Omega_{\mathcal{B}_i/U}$ is actually injective, so that we have an exact sequence

$$0 \to \varphi_i^* \Omega_{\mathcal{B}_{i+1}/U} \to \Omega_{\mathcal{B}_i/U} \to \Omega_{\varphi_i} \to 0. \tag{4}$$

Restricting the sequence (4) to the zero section $\epsilon_i : U \to \mathcal{B}_i$, we get

$$0 \to \epsilon_i^*(\Omega_{\mathcal{B}_{i+1}/U}) \to \epsilon_i^*(\Omega_{\mathcal{B}_i/U}) \to \epsilon_i^*(\Omega_{\varphi_i}) \to 0. \tag{5}$$

Here the exactness on the left is again justified by the fact that the sequence (5) is generically exact on U and the fact that $\epsilon_{i+1}^*(\Omega_{\mathcal{B}_{i+1}/U})$ is locally free. Applying $\deg_U(\cdot)$ to the objects in the last sequence, we conclude that

$$\deg_U(\omega^0_{\mathcal{B}_i/U}) - \deg_U(\epsilon_i^*(\Omega_{\varphi_i})) = \deg_U(\omega^0_{\mathcal{B}_{i+1}/U}).$$

Since $\epsilon_i^*(\Omega_{\varphi_i})$ is a torsion sheaf, we have $\deg_U(\epsilon_i^*(\Omega_{\varphi_i})) \geqslant 0$ (see Hartshorne [10, Chapter 7, Example 6.12, p. 149]).

So we conclude that $\deg_U(\omega^0_{\mathcal{B}_i/U}) \leqslant \deg_U(\omega^0_{\mathcal{B}/U})$ for all $i \in \mathbb{N}$. From Theorem 3.1, we conclude that there exists $j, l \in \mathbb{N}$ such that $j < l$ and such that $\mathcal{B}_j \simeq \mathcal{B}_l$. Since there is also an étale K_0-isogeny $B_j \to B_l$ of degree divisible by p^{l-j}, this means that there is an étale K_0-isogeny $B_j \to B_j$ of degree divisible by p^{l-j}. Since this last isogeny cannot coincide with the isogeny given by a power of p (because the isogeny given by multiplication by p always has an inseparable part), this implies that $\text{End}_{\bar{K}_0}(B_j)_\mathbb{Q} \neq \mathbb{Q}$. Thus $\text{End}_{\bar{K}_0}(B)_\mathbb{Q} \neq \mathbb{Q}$. This concludes the proof. $\qquad\square$

Remark   The Proof of Theorem 1.4 gives a somewhat stronger result. In fact, it shows that the conclusion of Theorem 1.4 holds if the following weaker assumption holds: there is no $\bar{K}_0$-isogeny $\varphi : B_{\bar{K}_0} \to B'$ such that

- φ is étale;
- $\deg(\varphi)$ is a power of p;
- B' carries an étale and finite $\bar{K}_0$-endomorphism, whose degree is a power of p.

References

[1] Abramovich, D., and J. F. Voloch, "Toward a proof of the Mordell-Lang conjecture in characteristic p," *International Mathematics Research Notices*, vol. 1992, no. 5, pp. 103–15. MR 1162230.

[2] Benoist, F., "A theorem of the kernel in characteristic p," preprint, 2011. 579

[3] Benoist, F., E. Bouscaren, and A. Pillay, "Semiabelian varieties over separably closed fields, maximal divisible subgroups, and exact sequences," preprint, arXiv:0904.2083v1 [math.LO]. 579, 580

[4] Corpet, C., "Around the Mordell–Lang and Manin–Mumford conjectures in positive characteristic," preprint, arXiv:1212.5193v1 [math.AG]. 580

[5] Demazure, A., and A. Grothendieck, *Schémas en groupes, II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*, Séminaire de Géométrie Algébrique du Bois Marie 1962–1964 (SGA 3), vol. 152 of *Lecture Notes in Mathematics*, Springer, Berlin, 1970. MR 0274459.

[6] Faltings, G., and C.-L. Chai, *Degeneration of Abelian Varieties*, with an appendix by David Mumford, vol. 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*, Springer, Berlin, 1990. MR 1083353. 585, 586

[7] Fantechi, B., L. Göttsche, L. Illusie, S. L. Kleiman, N. Nitsure, and A. Vistoli, *Fundamental Algebraic Geometry: Grothendieck's FGA Explained*, vol. 123 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, 2005. MR 2222646. 583, 585

[8] Grothendieck, A., "Techniques de construction et théorèmes d'existence en géométrie algébrique, IV: Les schémas de Hilbert, pp. 249–76 in *Séminaire Bourbaki, Vol. 6*, (1960/1961) no. 221, Société Mathématique de France, Montrouge, 1995. MR 1611822. 585

[9] Grothendieck, A., M. Raynaud, and D. S. Rim, *Groupes de monodromie en géométrie algébrique, I*, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I), vol. 288 of *Lecture Notes in Mathematics*, Springer, Berlin, 1972. MR 0354656. 586, 587

[10] Hartshorne, R., *Algebraic Geometry*, vol. 52 of *Graduate Texts in Mathematics*, Springer, New York, 1977. MR 0463157. 587

[11] Hrushovski, E., "The Mordell-Lang conjecture for function fields," *Journal of the American Mathematical Society*, vol. 9 (1996), pp. 667–90. MR 1333294.

[12] Katz, N., "Serre–Tate local moduli," pp. 138–202 in *Algebraic Surfaces (Orsay, 1976)*, vol. 868 of *Lecture Notes in Mathematics*, Springer, Berlin, 1981. MR 0638600. 583

[13] Liu, Q., *Algebraic Geometry and Arithmetic Curves*, vol. 6 of *Oxford Graduate Texts in Mathematics*, Oxford University Press, Oxford, 2002. MR 1917232. 584

[14] Manin, Y. I., "Letter to the editors: 'Rational points on algebraic curves over function fields'" (in Russian), *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, vol. 53 no. 2 (1989), pp. 447–48; English translation in *Mathematics of the USSR-Izvestiya*, vol. 34 (1990), pp. 465–66. MR 0998307. 580

[15] Milne, J. S., *Étale Cohomology*, vol. 33 of *Princeton Mathematical Series*, Princeton University Press, Princeton, 1980. MR 0559531. 581

[16] Milne, J. S., "Abelian varieties," pp. 103–50 in *Arithmetic Geometry (Storrs, Conn., 1984)*, Springer, New York, 1986. MR 0861974. 586

[17] Moret-Bailly, L., "Pinceaux de variétés abéliennes," *Astérisque*, vol. 129 (1985), 266 pp. MR 0797982. 586

[18] Mumford, D., J. Fogarty, and F. Kirwan, *Geometric Invariant Theory*, 3rd ed., vol. 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2)*, Springer, Berlin, 1994. MR 1304906. 585

[19] Pink, R., and D. Rössler, "On ψ-invariant subvarieties of semiabelian varieties and the Manin–Mumford conjecture," *Journal of Algebraic Geometry*, vol. 13 (2004), pp. 771–98. MR 2073195. 580

[20] Poonen, B., and J. F. Voloch, "The Brauer–Manin obstruction for subvarieties of abelian varieties over function fields," *Annals of Mathematics (2)*, vol. 171 (2010), pp. 511–32. MR 2630046. DOI 10.4007/annals.2010.171.511. 580

[21] Raynaud M., *Faisceaux amples sur les schémas en groupes et les espaces homogènes*, vol. 119 of *Lecture Notes in Mathematics*, Springer, Berlin, 1970. MR 0260758. 586

[22] Raynaud M., "Sous-variétés d'une variété abélienne et points de torsion, pp. 327–52 in *Arithmetic and Geometry, Vol. I*, vol. 35 of *Progress in Mathematics*, Birkhäuser, Boston, 1983. MR 0717600. 583

[23] Rössler, D., "On the Manin-Mumford and Mordell-Lang conjectures in positive characteristic," preprint, arXiv:1104.4311v3 [math.AG]. 580

[24] Tate, J., "*p*-divisible groups," pp. 158–83 in *Proceedings of the Conference on Local Fields (Driebergen, 1966)*, Springer, Berlin, 1967. MR 0231827. 582

[25] Tate, J., and J. F. Voloch, "Linear forms in *p*-adic roots of unity," *International Mathematics Research Notices*, vol. 1996, no. 12, pp. 589–601. MR 1405976.

[26] Voloch, J. F., "Diophantine approximation on abelian varieties in characteristic *p*," *American Journal of Mathematics*, vol. 117 (1995), pp. 1089–95. MR 1342843. DOI 10.2307/2374961. 580

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