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Unlikely Intersections in Poincaré Biextensions over Elliptic Schemes

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Abstract This paper concerns the relations between the relative Manin–Mumford conjecture and Pink's conjecture on unlikely intersections in mixed Shimura varieties. The variety under study is the 4-dimensional Poincaré biextension attached to a universal elliptic curve. A detailed list of its special subvarieties is drawn up, providing partial verifications of Pink's conjecture in this case, and two open problems are stated in order to complete its proof.

1 Introduction

In recent work, partly in collaboration with B. Edixhoven and with D. Masser, A. Pillay, and U. Zannier (see [4], [5], [6]), semiabelian surface schemes are studied in the context of the relative Manin–Mumford conjecture. Due to the possible presence of Ribet sections (see [4], [5]), this conjecture does not hold in general, but as is shown in [6], they are the only obstruction to its validity.

More precisely, let $\mathbb{Q}^{\mathrm{alg}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and let S be an irreducible algebraic curve over $\mathbb{Q}^{\mathrm{alg}}$. For any group scheme G over S, we write G_{tor} for the union of all the torsion points of the various fibers of $G \to S$. Then, we have the following theorem.

Theorem 1 ([6, Section 1]) *Let E/S be an elliptic scheme over the curve $S/\mathbb{Q}^{\mathrm{alg}}$, and let G/S be an extension of E/S by $\mathbb{G}_{m/S}$. Further, let $s : S \to G$ be a section of G/S. Assume that the image $W = s(S)$ of s contains infinitely many points of G_{tor}. Then,*

(i) either s is a Ribet section, or
(ii) s factors through a strict subgroup scheme of G/S.

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A formal definition of Ribet sections will presently be given. But to get a grasp on Theorem 1, it suffices to know that the scheme G/S admits Ribet sections $s = s_R$ if and only if it is not isoconstant and E/S admits complex multiplications (i.e., $\mathbb{Z} \subsetneq \mathrm{End}(E/S)$). This forces E/S to be isoconstant, and so, isomorphic, after a finite base extension, to $E_0 \times S$ for some elliptic curve $E_0/\mathbb{Q}^{\mathrm{alg}}$, with complex multiplications. Under these assumptions, which imply that the extension G/S does not split, both conclusions (i) and (ii) actually occur and are mutually exclusive. The image $W_R = s_R(S)$ of a Ribet section will be called a *Ribet curve* of G.

In [4, Section 2], it was observed that although Ribet curves cannot be interpreted in terms of subgroups schemes of G/S, they *are* special curves in the moduli viewpoint provided by Pink's extension of the André–Oort and Zilber conjectures to mixed Shimura varieties (see Pink [15, Conjecture 1.2]). The relevant mixed Shimura variety is here the Poincaré biextension $\mathcal{P}_0$ attached to the CM elliptic curve E_0.

In this note, we pursue this study by looking at "unlikely intersections" for a curve W in $\mathcal{P}_0$ or, more generally, in the Poincaré biextension $\mathcal{P}$ attached to a general family of elliptic curves. The statement of Pink's conjecture for this case will be found in Section 5. Most of the present work is directly inspired by Sections 5 and 6 of Pink's preprint [15].

To conclude this introduction, here is the promised definition of a Ribet section $s_R : S \to G$. In the notations of Theorem 1, let $\hat{E}/S$ be the dual of E/S: the isomorphism class of the $\mathbb{G}_m$-torsor G over E is given by a section $q : S \to \hat{E}$. Further, let $\mathcal{P} \to E \times_S \hat{E}$ be the Poincaré biextension of E and $\hat{E}$ by $\mathbb{G}_m$, rigidified above the zero section of $E \times_S \hat{E}$. By Deligne [7, Section 10.2.13], a section $s : S \to G$ of G/S lifting a section $p : S \to E$ of E/S is entirely described by a trivialization of the $\mathbb{G}_m$-torsor $(p, q)^*\mathcal{P}$ over S. Assume now that $E = E_0 \times S$ admits complex multiplications, and let $f : \hat{E}_0 \to E_0$ be a nonzero antisymmetric isogeny (i.e., identifying E_0 and $\hat{E}_0$, a purely imaginary complex multiplication). For simplicity, suppose that f is divisible by 2. Then, $(f(q), q)^*\mathcal{P}$ is a trivial torsor in a canonical way, and the corresponding trivialization yields a well-defined section $s = s(f)$ of G/S. Any section s_R of G/S a nonzero multiple of which is of the form $s(f)$ for some antisymmetric f will be called a *Ribet section* of G/S. See [4, Section 1(i)] and its appendix for two further descriptions of these sections.

2 Special Points

Let X be a modular curve, say, $X = Y(2)$, parameterizing isomorphism classes of elliptic curves with some level structure, let $\mathcal{E}$ be the universal elliptic scheme over X, with dual $\hat{\mathcal{E}}$, and let $\mathcal{P}$ be the Poincaré biextension of $\mathcal{E} \times_X \hat{\mathcal{E}}$ by $\mathbb{G}_m$. This is a mixed Shimura variety of dimension 4, which parameterizes points P on extensions G of elliptic curves E by $\mathbb{G}_m$. A point of $\mathcal{P}(\mathbb{C})$ can be represented by a triple (E, G, P), and is called *special* if the attached Mumford–Tate group is abelian, which is equivalent to requiring that E has complex multiplications, that G is an isotrivial extension, and that P is a torsion point on G. Denote by $\mathcal{P}_{\mathrm{sp}}$ the set of special points of $\mathcal{P}$. Following [15], we further say that an irreducible subvariety of $\mathcal{P}$ is special if it is a component of the Hecke orbit of a mixed Shimura subvariety of $\mathcal{P}$. The special subvarieties of $\mathcal{P}$ of dimension zero are the special points; those of higher dimensions are described below (for the full list, see Section 3).

Corollary 1 *Let $W/\mathbb{Q}^{\mathrm{alg}}$ be an irreducible closed algebraic curve in $\mathcal{P}$. Assume that $W \cap \mathcal{P}_{\mathrm{sp}}$ is infinite. Then, W is a special curve.*

To make this corollary more explicit—and to prove it—we distinguish the various cases provided by the projection $\varpi : \mathcal{P} \to X$ and its canonical section (rigidification) $\sigma : X \to \mathcal{P}$ above the zero section of $\mathcal{E} \times_X \hat{\mathcal{E}}$, whose image $\sigma(X)$ is made up of points of the type $(E, \mathbb{G}_m \times E, 0) \in \mathcal{P}$.

— Either the restriction of ϖ to W is dominant: the corollary then says that W lies in the Hecke orbit of the curve $\sigma(X)$. Indeed, up to Hecke transforms,

- $\sigma(X)$ is the only 1-dimensional (mixed, but actually pure) Shimura subvariety of $\mathcal{P}$ dominating X.

This case, however, is a red herring, in the sense that for $\varpi_{|W}$ dominant, the corollary follows not from Theorem 1 but from André's theorem [1, p. 12] on the special points of the mixed Shimura variety $\mathcal{E}$ (see Pila [12, Theorem 1.2] for another proof of André's theorem), combined with an easy analogue for $\mathbb{G}_{m/X}$ (which is actually covered by Pila [13, Theorem 1.1] with $n = \ell = 1, m = 0$).

— Or $\varpi(W)$ is a point x_0 of X, necessarily of CM type. In particular, W lies in the fiber $\mathcal{P}_0$ of ϖ above x_0. This fiber $\mathcal{P}_0$ is a 3-dimensional mixed Shimura subvariety of $\mathcal{P}$, which can be identified with the Poincaré biextension of $E_0 \times \hat{E}_0$ by $\mathbb{G}_m$, where E_0 denotes an elliptic curve in the isomorphism class of x_0. An analysis of the generic Mumford–Tate group of $\mathcal{P}_0$ as in Bertrand [3, p. 52] shows that, up to Hecke transforms, there are exactly four types of special curves in $\mathcal{P}_0$:

- $(\mathbb{G}_m)_{x_0}$ = the fiber above (0, 0) of the projection $\mathcal{P}_0 \to (\mathcal{E} \times_X \hat{\mathcal{E}})_{x_0} = E_0 \times \hat{E}_0$;
- (×3) the images $\psi_B(B)$ of the elliptic curves $B \subset E_0 \times \hat{E}_0$ passing through $(0,0)$ such that the $\mathbb{G}_m$-torsor $\mathcal{P}_{0|B}$ is trivial, under the corresponding (unique) trivialization $\psi_B : B \to \mathcal{P}_{0|B}$. As recalled in [4, Remark 2], there are, up to isogenies, three types of such elliptic curves B: the obvious ones $E_0 \times 0$ and $0 \times \hat{E}_0$ (whose images we will simply denote by $\psi(E_0 \times 0), \psi(0 \times \hat{E}_0)$), *and* the graphs of antisymmetric isogenies from $\hat{E}_0$ to E_0, in which case ψ_B, composed with the induced map $\hat{E}_0 \to B$, corresponds precisely to a Ribet section (of the semiabelian scheme $\mathcal{G}_0/\hat{E}_0$ to be described presently).

Corollary 1 now follows from Theorem 1, on interpreting $\mathcal{P}_0/\hat{E}_0$ as the universal extension $\mathcal{G}_0$ of E_0 by $\mathbb{G}_m$, viewed as a group scheme over $\hat{E}_0$, so that $\mathcal{P}_{\mathrm{sp}} \cap \mathcal{P}_0 \subset (\mathcal{G}_0)_{\mathrm{tor}}$. More precisely, suppose that W dominates $\hat{E}_0$: then, it is the image of a multisection of $\mathcal{G}_0/\hat{E}_0$, and after a base extension, the theorem implies that W is a Ribet curve $\psi_B(B)$ of $\mathcal{G}_0 = \mathcal{P}_0$, or that it lies in a torsion translate of $\mathbb{G}_{m/\hat{E}_0} = \mathbb{G}_m \times \hat{E}_0$, where a new application of the theorem (or more simply, of its constant version; see Hindry [9]) shows that it must coincide with a Hecke transform of $\mathbb{G}_m = (\mathbb{G}_m)_{x_0}$ or of $\psi(0 \times \hat{E}_0)$. By biduality (i.e., reverting the roles of $\hat{\mathcal{E}}$ and $\mathcal{E}$), the same argument applies if W dominates E_0. Finally, if W projects to a point of $E_0 \times \hat{E}_0$, then, this point must be torsion, and W lies in the Hecke orbit of $(\mathbb{G}_m)_{x_0}$. Being closed, W is therefore a special curve of $\mathcal{P}$ in all cases.

3 Unlikely Intersections

Although insufficient in the presence of Ribet curves, the argument devised by Pink to relate the Manin–Mumford and the André–Oort settings often applies (see the proof of Pink [15, Theorems 5.7, 6.3] and the discussion in [5] on abelian schemes). In the present situation, one notes that given a point (E, G, P) in $\mathcal{P}(\mathbb{C})$, asking that it be special as in Corollary 1 gives 4 independent conditions, while merely asking that P be torsion on G as in Theorem 1 gives 2 conditions. Now, unlikely intersections for a curve W in $\mathcal{P}$ precisely means studying its intersection with the union of the special subvarieties of $\mathcal{P}$ of codimension ≥ 2 (i.e., of dimension ≤ 2), and according to Pink [15, Conjecture 1.2], when this intersection is infinite, W should lie in a special subvariety of dimension $\leq 1 + 2 = 3 < 4$, that is, a proper one. Similarly, if W lies in the fiber $\mathcal{P}_0$ of $\mathcal{P}$ above a CM point x_0 and meets infinitely many special curves of this 3-fold, then it should lie in a special surface of the mixed Shimura variety $\mathcal{P}_0$. In these directions, we have the following.

Corollary 2 *Let $W/\mathbb{Q}^{\mathrm{alg}}$ be an irreducible algebraic curve in $\mathcal{P}$. Assume that the intersection of W with the union of all the special surfaces of $\mathcal{P}$* dominating *X is infinite. Then, W lies in a special 3-fold of $\mathcal{P}$.*

Corollary 3 *Let $W/\mathbb{Q}^{\mathrm{alg}}$ be an irreducible algebraic curve in the fiber $\mathcal{P}_0$ of $\mathcal{P}$ above a CM point x_0 of X. Assume that the intersection of W with the union of all the special,* but not Ribet*, curves of $\mathcal{P}_0$ is infinite. Then, W lies in a special surface of $\mathcal{P}_0$.*

To see the scope of these results, we first list all the special subvarieties of $\mathcal{P}$, announcing them by 1, 2, or 3 bullets according to their dimensions, and with the occasional symbol $(\times n)$ to indicate that n types of special subvarieties are listed at that stage. By inspection, one deduces that "special surfaces" can be replaced by "special subvarieties of dimension ≤ 2" in Corollary 2; similarly, Corollary 3 can be formulated in apparently broader terms, involving special points and special curves of $\mathcal{P}$.

An analysis of the generic Mumford–Tate group of $\mathcal{P}$ as in [3, p. 59] shows that up to Hecke transforms, there are only

- • • • one type of special 3-folds of $\mathcal{P}$ dominating X, namely, its restrictions $\mathcal{P}_{|\mathcal{B}}$ to the various flat elliptic (subgroup) schemes $\mathcal{B}$ of $\mathcal{E} \times_X \hat{\mathcal{E}}$ over X;
- • • (×3) three types of special surfaces in $\mathcal{P}$ dominating X, namely, the restriction $\mathbb{G}_m \times \sigma(X)$ of $\mathcal{P}$ above the zero section of $\mathcal{E} \times_X \hat{\mathcal{E}}$, and the images $\psi_{\mathcal{B}}(\mathcal{B})$ of the elliptic subschemes $\mathcal{B}$ as above such that the $\mathbb{G}_m$-torsor $\mathcal{P}_{|\mathcal{B}}$ is trivial, under the corresponding (unique) trivialization $\psi_{\mathcal{B}} : \mathcal{B} \to \mathcal{P}_{|\mathcal{B}}$. Since $\mathcal{E}/X$ has no complex multiplications, this occurs if and only if $\mathcal{B} \subset \mathcal{E} \times_X \hat{\mathcal{E}}$ projects to the zero section of one of the factors, and in the same style as in Section 2, we will denote by $\psi(0 \times \hat{\mathcal{E}})$ and $\psi(\mathcal{E} \times 0)$ the two corresponding special surfaces of $\mathcal{P}$;
- • one special curve dominating X, namely, the already-met $\sigma(X)$.

The other special subvarieties of $\mathcal{P}$ are contained in

- • • • the fibers $\mathcal{P}_0$ of $\mathcal{P}$ above the various CM points x_0 of X, for which we recall the notations and references of the previous section; thus, up to Hecke transforms;

- •• the only type of special surfaces of a fiber $\mathcal{P}_0$ is given by its restrictions $\mathcal{P}_{0|B}$ to the various elliptic subgroups B of $E_0 \times \hat{E}_0$;
- • (×4) the already-met special curves of $\mathcal{P}_0$ are in the Hecke orbit of $(\mathbb{G}_m)_{x_0}$, $\psi(E_0 \times 0)$, $\psi(0 \times \hat{E}_0)$, or (when B is the graph of an antisymmetric isogeny) of a Ribet curve $\psi_B(B)$.

Theorem 1 (+ Masser and Zannier [10], [11]) ⇒ Corollary 2 To deduce Corollary 2 from the theorem, we follow the lines of [15] and first consider the case when W does not dominate X, that is, $\varpi(W) = x$ is a point (which we can assume to be non-CM; otherwise, the fiber $\mathcal{P}_x$ is a special 3-fold of $\mathcal{P}$ containing W). So, W lies in the nonspecial 3-fold $\mathcal{P}_x$, which we can identify with the Poincaré biextension of $(\mathcal{E} \times_X \hat{\mathcal{E}})_x = E_x \times \hat{E}_x$ by $\mathbb{G}_m$, where $E_x/\mathbb{Q}^{\mathrm{alg}}$ denotes an elliptic curve representing x. Now, we have the following.

— If W meets the Hecke orbit of $\mathbb{G}_m \times \sigma(X)$ infinitely often, then either its projection to $E_x \times \hat{E}_x$ is a torsion point, in which case W lies in the Hecke orbit of the special surface $\mathbb{G}_m \times \sigma(X)$, or this projection contains infinitely many torsion points. We then deduce from Raynaud's theorem on curves in $E_x \times \hat{E}_x$ (combined with $\mathrm{End}(E_x) = \mathbb{Z}$) that W lies in the Hecke orbit of a special 3-fold of the type $\mathcal{P}_{|\mathcal{B}}$ for some elliptic subgroup scheme $\mathcal{B}$ of $\mathcal{E} \times_X \hat{\mathcal{E}}$.

— Suppose that W projects to a point $q \in \hat{E}_x(\mathbb{Q}^{\mathrm{alg}})$, that is, lies in the extension $G_q = \mathcal{P}_{x|E_x \times q}$ of E_x by $\mathbb{G}_m$; we can assume that q is not torsion. (Otherwise, W lies in the Hecke orbit of the special 3-fold $\mathcal{P}_{|\mathcal{E}\times 0}$.) So, W does not meet the Hecke orbit of $\psi(\mathcal{E} \times 0)$ at all. And if its intersection with the Hecke orbit of $\psi(0 \times \hat{\mathcal{E}})$ is infinite, then W, viewed in the nonisotrivial extension G_q, contains infinitely many points of $(G_q)_{\mathrm{tor}}$. By Theorem 1 (or Hindry [9, Section 5, Theorem 2]), W must then lie in a torsion translate of $\mathbb{G}_m = \mathcal{P}_{x|(0\times q)}$, hence in the Hecke orbit of $\mathcal{P}_{|0\times\hat{\mathcal{E}}} = \mathbb{G}_m \times \psi(0 \times \hat{\mathcal{E}})$.

— By biduality, we can now assume that W dominates both $\hat{E}_x$ and E_x. In particular, it is the image of a multisection of the universal extension $\mathcal{G}_x := \mathcal{P}_x$ of E_x by $\mathbb{G}_m$, viewed as a group scheme over $\hat{E}_x$. If W meets the Hecke orbit of $\psi(0 \times \hat{\mathcal{E}})$ infinitely often, then W, viewed as a curve in the group scheme $\mathcal{G}_x/\hat{E}_x$, contains infinitely many points of $\mathcal{G}_{x,\mathrm{tor}}$, and the theorem implies that W lies in a torsion translate of $\mathbb{G}_{m/\hat{E}_x} \subset \mathcal{P}_{|0\times\hat{\mathcal{E}}}$. This contradicts the assumption that W dominates E_x. Inverting the roles of $\mathcal{E}$ and $\hat{\mathcal{E}}$, we similarly deduce that W cannot intersect the Hecke orbit of $\psi(\mathcal{E} \times 0)$ infinitely often.

Finally, assume that W dominates X. Then, after a finite base extension if necessary, the projection of W to $\hat{\mathcal{E}}$ (resp., $\mathcal{E}$) defines a section q of $\hat{\mathcal{E}}/X$, hence an extension $\mathcal{G}/X$ of $\mathcal{E}/X$ by $\mathbb{G}_m$ (resp., a section p of $\mathcal{E}/X$), and W is the image of a multisection of $\mathcal{G}/X$ lifting p. Now, we have the following.

— If W meets the Hecke orbit of $\mathbb{G}_m \times \sigma(X)$ infinitely often, its projection to $\mathcal{E} \times_X \hat{\mathcal{E}}$ either is a torsion section (and W lies in the Hecke orbit of $\mathbb{G}_m \times \sigma(X)$) or meets infinitely many torsion sections of this abelian scheme. We then deduce from the theorem of Masser and Zannier [11, Section 1] (i.e., the relative version of Raynaud's) that W lies in the Hecke orbit of a special 3-fold of the type $\mathcal{P}_{|\mathcal{B}}$.

— If W meets the Hecke orbit of $\psi(0 \times \hat{\mathcal{E}})$ infinitely often, then W, viewed as a curve in the group scheme $\mathcal{G}/X$, contains infinitely many points of $\mathcal{G}_{\mathrm{tor}}$, and the theorem implies that W lies in the Hecke orbit of the special 3-fold $\mathcal{P}_{|0\times\hat{\mathcal{E}}}$ if $\mathcal{G}/X$ is a nonisotrivial extension; otherwise, q is a torsion section of $\hat{\mathcal{E}}/X$, and W may alternatively lie in the Hecke orbit of the special surface $\psi(\mathcal{E}\times 0)\subset\mathcal{P}_{|\mathcal{E}\times 0}$.

— By biduality, the same argument applies if W intersects the Hecke orbit of $\psi(\mathcal{E}\times 0)$ infinitely often.

Theorem 1 (+ [10], [11]) ⇒ Corollary 3 Since Ribet curves are discarded, the argument goes along the same lines as the one concerning $\varpi(W)=x$, with the base X replaced by the point x_0 and $\mathcal{B}$ replaced by (the vaster choice of) an elliptic curve B in $E_0\times\hat{E}_0$.

Note that when B is the graph of a nonrational isogeny, the special surface $\mathcal{P}_{0|B}$ does not lie in any of the special 3-folds of $\mathcal{P}$ dominating X. This has no impact on Corollary 3, where these surfaces occur in the conclusion. On the other hand, they are the precise reason why we had to restrict the hypothesis of Corollary 2 to surfaces dominating X; we discuss this further in Section 5, together with the restriction to non-Ribet curves in Corollary 3.

Our last corollary of Theorem 1 concerns the remaining special 3-folds of $\mathcal{P}$, of type $\mathcal{P}_{|\mathcal{B}}$, where $\mathcal{B}$ is an elliptic (subgroup) scheme in $\mathcal{E}\times_X\hat{\mathcal{E}}$ dominating X.

Corollary 4 *Let $W/\mathbb{Q}^{\mathrm{alg}}$ be an irreducible algebraic curve in the special 3-fold $\mathcal{P}_{|\mathcal{B}}$. Assume that the intersection of W with the union of all the special curves of $\mathcal{P}_{|\mathcal{B}}$ is infinite. Then, W lies in a special surface of $\mathcal{P}_{|\mathcal{B}}$.*

As already suggested after the statement of Corollary 3, one can replace "special curves" by "special subvarieties of dimension ≤ 1," and "of $\mathcal{P}_{|\mathcal{B}}$" by "of $\mathcal{P}$," since any special point is contained in the Hecke orbit of a special curve of the type $(\mathbb{G}_m)_{x_0}$, while the special curves not contained in $\mathcal{P}_{|\mathcal{B}}$ meet this 3-fold along special points. This fact explains why Ribet curves need not be discarded in Corollary 4; $\mathcal{P}_{|\mathcal{B}}$ contains no such curve, since even above a CM point x_0, the fiber $B=\mathcal{B}_{x_0}$ of $\mathcal{B}/X$ is not the graph of a nonrational isogeny.

Theorem 1 (+ [1]) ⇒ Corollary 4 We first assume that W dominates X. Suppose that there are infinitely many points in the intersection of W with the Hecke orbit of the following special subsets.

— The special curve $\sigma(X)$: As explained above (and after a base extension), we can view W as a curve in a semiabelian scheme $\mathcal{G}/X$ and view these intersections as points in $\mathcal{G}_{\mathrm{tor}}$. Theorem 1 implies that W lies in the Hecke orbit of $\mathbb{G}_m\times\sigma(X)$, $\mathcal{P}_{|0\times\hat{\mathcal{E}}}$, or $\mathcal{P}_{|\mathcal{E}\times 0}$, and all intersect $\mathcal{P}_{|\mathcal{B}}$ along its special surface $\mathbb{G}_m\times\sigma(X)$ if $\mathcal{B}$ is not one of the factors of $\mathcal{E}\times_X\hat{\mathcal{E}}$. Otherwise, we can assume by biduality that $\mathcal{B}=\mathcal{E}\times 0$, in which case W lies in the Hecke orbit of $\mathbb{G}_m\times\sigma(X)$ or of the special surface $\psi(\mathcal{E}\times 0)$ of $\mathcal{P}_{|\mathcal{E}\times 0}$.

— The union of the special curves contained in the fibers $\mathcal{P}_0$ above the various CM points x_0 of X (i.e., those of type $(\mathbb{G}_m)_{x_0}$, $\psi(E_0\times 0)$, $\psi(0\times\hat{E}_0)$, and the Ribet curves): By André's theorem [1], applied to the special points of the mixed Shimura variety $\mathcal{B}/X$, the projection of W to $\mathcal{B}$ must be a torsion section, and W lies in the Hecke orbit of $\mathbb{G}_m\times\sigma(X)$.

The case when $\varpi(W)$ is a non-CM point x is proved along similar lines. Finally, assume that $\varpi(W)$ is a CM point x_0, and let $B\subset E_0\times\hat{E}_0$ be the fiber above x_0 of the elliptic scheme $\mathcal{B}/X$. Then, W lies in the special surface $\mathcal{P}_{0|B}=\mathcal{P}_{|\mathcal{B}}\cap\mathcal{P}_0$.

4 Back to Group Schemes

First of all, since the four "corollaries" above have requested the help of [1, p. 12] and of [10] and [11], it is fairer to gather them under a new heading, as follows. We recall from [15] that the special closure of an irreducible curve W in the mixed Shimura variety $\mathcal{P}$ is the intersection of all the special subvarieties of $\mathcal{P}$ containing W.

Theorem 2 *Let $W/\mathbb{Q}^{\mathrm{alg}}$ be an irreducible curve in the mixed Shimura 4-fold $\mathcal{P}$, and let δ_W be the dimension of the special closure of W.*

- *(i) Suppose that $\delta_W = 4$; then, the intersection of W with the union of all the special surfaces of $\mathcal{P}$ dominating X is finite.*
- *(ii) Suppose that $\delta_W = 3$; then, the intersection of W with the union of all the special non-Ribet curves of $\mathcal{P}$ is finite.*
- *(iii) Suppose that $\delta_W = 2$; then, the intersection of W with the union of all the special points of $\mathcal{P}$ is finite.*

This statement is equivalent to the union of the four corollaries; we leave it to the reader to check that (ii) is equivalent to Corollaries 3 + 4 and that (i) + (ii) + (iii) implies Corollary 1. (The Hecke orbit of any special point meets a special curve of type $(\mathbb{G}_m)_{x_0}$, which is contained in the special surface $\mathbb{G}_m \times \sigma(X)$.)

In the next section, we will discuss how far Theorem 2 stands from Pink's general conjecture for curves. In the reverse direction, following [15, Section 6] (and Pink [14, Remark 2.13]), we now prove the following.

Theorem 2 ⇒ Theorem 1 In fact, the following weaker version of Theorem 2 will suffice:

- (i′) in (i), constrain the conclusion to the Hecke orbit of the special surface $\psi(0 \times \hat{\mathcal{E}})$;
- (ii′) in (ii), constrain the conclusion to the Hecke orbits of the special curves $\sigma(X)$ and $\psi(0 \times \hat{E}_0)$, where E_0 runs through the CM fibers of $\mathcal{E} \to X$.

So, let $G \to S$ and $W = s(S)$ satisfy the hypotheses of Theorem 1. The universal property of $\mathcal{P}$ provides canonical morphisms $\varphi : S \to \hat{\mathcal{E}}$ (above the "modular" map λ attached to the elliptic scheme E/S) and $\Phi : G \to \mathcal{P}$ above φ such that the following diagram commutes:

$$\begin{array}{ccccccc} W & \hookrightarrow & G & \xrightarrow{\Phi} & \mathcal{P} & & \\ & \nwarrow^{s} & \downarrow & & \downarrow & \searrow & \\ & & S & \xrightarrow{\varphi} & \hat{\mathcal{E}} & & \varpi \\ & & & \searrow^{\lambda} & \downarrow & \swarrow & \\ & & & & X & & \end{array}$$

Furthermore, Φ induces a morphism of group schemes from G/S to $\mathcal{P}/\hat{\mathcal{E}}$, where we view the latter as the canonical extension of the elliptic scheme $\mathcal{E}_{\hat{\mathcal{E}}}$ by $\mathbb{G}_m$.

Assume first that $\Phi(W)$ is a point of $\mathcal{P}$. Then, G/S must be an isoconstant scheme, of which s is an isoconstant section. Since W meets G_{tor}, s is a torsion section, and W does lie in a strict subgroup scheme of G/S. So, we can now assume that $W' = \Phi(W)$ is a curve in $\mathcal{P}$.

By the universal property of $\mathcal{P}$, the image of G_{tor} under Φ lies in the Hecke orbit of the special surface $\psi(0 \times \hat{\mathcal{E}})$. Theorem 2(i′) then implies that $\delta_{W'} < 4$, so W' lies in the Hecke orbit of a special 3-fold. Performing a Hecke transform, we henceforth assume that W' lies in $\mathcal{P}_{|\mathcal{B}}$ for some $\mathcal{B} \subset \mathcal{E} \times \hat{\mathcal{E}}$, or in the fiber $\mathcal{P}_0$ of $\mathcal{P}$ above some CM point x_0, represented by a CM curve E_0.

Let us first assume that W' lies in $\mathcal{P}_{|0 \times \hat{\mathcal{E}}}$. Then, up to a torsion translate, W lies in the fiber of $G \to E$ above the zero section, that is, in the strict subgroup scheme $\mathbb{G}_{m/S}$ of G/S.

Let us now assume that W' lies in $\mathcal{P}_{|\mathcal{B}}$, where $\mathcal{B}$ is the graph of a homomorphism $\mathcal{E} \to \hat{\mathcal{E}}$. Then, the intersection of $\mathcal{P}_{|\mathcal{B}}$ with $\psi(0 \times \hat{\mathcal{E}})$ is the special curve $\sigma(X)$, and since $\Phi(G_{\mathrm{tor}})$ lies in the Hecke orbit of $\psi(0 \times \hat{\mathcal{E}})$, the curve $W' \subset \mathcal{P}_{|\mathcal{B}}$ meets the Hecke orbit of $\sigma(X)$ infinitely often. By Theorem 2(ii′), W' must lie in the Hecke orbit of a special surface. Postponing to the next step the case when this surface is above a CM point, we conclude that up to a Hecke transform, W' lies in $\mathbb{G}_m \times \sigma(X)$, in which case a torsion translate of W lies in the subgroup scheme $\mathbb{G}_{m/S}$ of G/S, or that $\mathcal{B} = \mathcal{E} \times 0$, with W' lying in $\psi(\mathcal{E} \times 0)$, in which case G/S must be an isotrivial extension and, after an isogeny, W lies in a torsion translate of its subgroup scheme E/S.

Let us finally assume that W' lies in a CM fiber $\mathcal{P}_0$. Then, up to an isogeny, G is an extension of $E_0 \times S$ by $\mathbb{G}_m$, and $\Phi(G_{\mathrm{tor}})$ lies in the Hecke orbit of the special curve $\psi(0 \times \hat{E}_0)$ of $\mathcal{P}_0$. By Theorem 2(ii′), W' must then lie in the Hecke orbit of a special surface of the type $\mathcal{P}_{0|B}$, where B is an elliptic (subgroup) curve in $E_0 \times \hat{E}_0$. We can assume that $B \neq 0 \times \hat{E}_0$; otherwise, W lies in a torsion translate of the fiber $\mathbb{G}_{m/S}$ of $G \to E_0 \times S$ above the zero section.

We are at last reduced to the case when B is the graph of a homomorphism $E_0 \to \hat{E}_0$ and W' lies in the special surface $\mathcal{P}_{0|B}$. But again, $\Phi(G_{\mathrm{tor}})$ lies in the Hecke orbit of the special curve $\psi(0 \times \hat{E}_0)$, which meets such a surface $\mathcal{P}_{0|B}$ transversally, and therefore along special points of $\mathcal{P}$. So, W' contains infinitely many special points and by Theorem 2(iii) must be a special curve of $\mathcal{P}_{0|B}$, necessarily of type $(\mathbb{G}_m)_{x_0}$, or $\psi(E_0 \times 0)$ if $B = E_0 \times 0$, or a Ribet curve of $\mathcal{P}_0$ otherwise. In the first case, W lies in a torsion translate of $\mathbb{G}_{m/S}$; in the second one, G/S is an isotrivial extension and W lies in a torsion translate of $E_0 \times S$; and in the last one, G/S is not isoconstant, and W is a Ribet curve of G/S.

5 Pink's Conjecture for Curves in $\mathcal{P}$

We close this note by discussing how far we now stand from Pink's conjecture for a curve W in the mixed Shimura variety $\mathcal{P}$. As is timely to recall, this asserts that, *whatever the dimension δ_W of the special closure of W is*, the following holds.

Conjecture 1.2 of [15] for curves in $\mathcal{P}$ *The intersection of a curve W of $\mathcal{P}$ with the union of the special subvarieties of $\mathcal{P}$ of dimension $\leq \delta_W - 2$ is finite.*

We again point out that, in the case of our mixed Shimura variety $\mathcal{P}$, the sign $\leq$ can equivalently be replaced by $=$ in this statement. So, to prove the conjecture, it suffices to lift the restrictions "dominating X" and "non-Ribet" in the conclusions of Theorem 2. We now state the two corresponding problems in concrete terms and mention possible approaches.

Going back to the list of special subvarieties of $\mathcal{P}$ established above, we see that the only special surfaces left out by the restriction "dominating X" are of the type

$\mathcal{P}_{0|B}$ for $B \subset E_0 \times \hat{E}_0$, where $E_0 = \mathcal{E}_{x_0}$ is the fiber of $\mathcal{E}/X$ above a CM point x_0. Therefore, lifting this restriction amounts to a positive answer to the following.

Question 1 Let E be a nonisoconstant elliptic scheme over a curve $S/\mathbb{Q}^{\text{alg}}$, and let p, q be two sections of E/S defined over $\mathbb{Q}^{\text{alg}}$. Assume that there are infinitely many points $\overline{\lambda} \in S(\mathbb{Q}^{\text{alg}})$ such that the fiber $E_{\overline{\lambda}}$ of E/S above $\overline{\lambda}$ admits complex multiplications and such that the points $p(\overline{\lambda})$ and $q(\overline{\lambda})$ are linearly dependent over $\text{End}(E_{\overline{\lambda}})$. Must the sections p and q then be linearly dependent over $\mathbb{Z}$?

This problem can alternatively be viewed as a special case of Pink's conjecture for curves in the nth fibered power of $\mathcal{E}$ over X, with $n = 2$. As already mentioned, this special case is established by André [1] for $n = 1$. The theorem of Masser and Zannier [11] used above provides a partial answer for $n = 2$. See Habegger [8] for further results in higher dimensions.

Although it addresses constant elliptic schemes, the work of Buium and Poonen on Heegner-type points in a given group of finite rank may provide an approach to Question 1. See Pila's proof in [12] of André's theorem for further suggestions.

As for lifting the restriction "non-Ribet," here is a way to state the problem. We fix a CM elliptic curve $E_0/\mathbb{Q}^{\text{alg}}$ and a curve $S/\mathbb{Q}^{\text{alg}}$. Given an extension G/S of $E_0 \times S$ by $\mathbb{G}_m$ which admits Ribet sections, we define a Ribet point as any point (not necessarily torsion) of $G(\mathbb{Q}^{\text{alg}})$ lying on a Ribet curve of G. We denote by $q \in E_0(S)$ the image under the standard polarization $\hat{E}_0 \simeq E_0$ of the section of $\hat{E}_0 \times S$ representing the isomorphism class of the extension G/S. Given a section s of G/S, we denote by $p = \pi \circ s \in E_0(S)$ its composition with the projection $\pi : G \to E_0 \times S$.

Question 2 Let G/S be a nonisoconstant extension of $E_0 \times S$ by $\mathbb{G}_m$, and let s be a section of G/S defined over $\mathbb{Q}^{\text{alg}}$. Assume that there exist infinitely many points $\overline{\lambda} \in S(\mathbb{Q}^{\text{alg}})$ such that $s(\overline{\lambda})$ is a Ribet point of G. Must the sections p and q then be linearly dependent over $\text{End}(E_0)$?

Notice that at each $\overline{\lambda}$ such that $s(\overline{\lambda})$ is a Ribet point, the points $p(\overline{\lambda})$ and $q(\overline{\lambda})$ of $E_0(\mathbb{Q}^{\text{alg}})$ are linearly dependent over $\text{End}(E_0)$. That the lift $s(\overline{\lambda})$ of $p(\overline{\lambda})$ is a Ribet point gives a second condition, and both constraints are unlikely to hold infinitely often.

As a possible approach to Question 2, we mention the existence of a canonical *relative* height on semiabelian varieties, which has the property that the relative height of any Ribet point on $G(\mathbb{Q}^{\text{alg}})$ vanishes (see Bertrand [2, Proposition 4, Theorem 4]).

6 Conclusion

Apart from clearing the way towards Pink's conjecture, presenting the viewpoint of mixed Shimura varieties was motivated by two aims:

— To get a more uniform statement of Theorem 1; obviously, Theorem 2 is not a satisfactory answer; Corollary 1 is better in this respect, but is too weak.

— To put some order into the array of cases which the proof of [6, Theorem 1] leads to, particularly when E/S is isoconstant.

This second aim is only partially fulfilled; the list of cases to be distinguished during this proof does not always parallel the list of cases encountered in the present note. The basic reason is that differential Galois groups are not fully controlled by

Mumford–Tate groups. So, a unified proof of Theorem 1 should probably put more emphasis on the study of the generic Mumford–Tate groups of the special subvarieties coming into play.

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