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Lattices in Locally Definable Subgroups of $\langle R^n, +\rangle$

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Abstract Let $\mathcal{M}$ be an o-minimal expansion of a real closed field R. We define the notion of a lattice in a locally definable group and then prove that every connected, definably generated subgroup of $\langle R^n, +\rangle$ contains a definable generic set and therefore admits a lattice.

The goal of this note is to reformulate some problems which appeared in Eleftheriou and Peterzil [4], introduce the notion of a lattice in a locally definable group (a notion which also appeared in that paper, but not under this name), and establish connections between various related concepts. Finally, we return to the main conjecture from [4].

Every locally definable, connected abelian group which is generated by a definable set contains a definable generic set.

We prove the conjecture for subgroups of $\langle R^n, +\rangle$, in the context of an o-minimal expansion $\mathcal{M}$ of a real closed field R.

1 Locally Definable Groups and Lattices

We first recall some definitions. Let $\mathcal{M}$ be an arbitrary κ-saturated o-minimal structure (for κ sufficiently large). By a *locally definable* group we mean a group $\langle \mathcal{U}, \cdot\rangle$, whose universe $\mathcal{U} = \bigcup_{n\in\mathbb{N}} X_n$ is a countable union of definable subsets of M^k, for some fixed k, and whose group operation is definable when restricted to each $X_m \times X_n$ (equivalently, to each definable subset of $\mathcal{U} \times \mathcal{U}$). We say that a function $f : \mathcal{U} \to M^n$ is *locally definable* if its restriction to each X_i (equivalently, to each definable subset of $\mathcal{U}$) is definable. We let $\dim \mathcal{U}$ be the maximum of $\dim X_n$, $n \in \mathbb{N}$. While some notions treated here make sense under the more general "$\bigvee$-definable group" (no restriction on the number of X_i's), we mostly work in the context of a group which is generated, as a group, by a definable subset and hence is locally definable. Note that another related concept, that of an *ind-definable group* (see

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Hrushovski, Peterzil, and Pillay [6]), is identical to our definition when one further assumes that the group is a subset of a fixed M^k.

As was shown in Peterzil and Starchenko [7], every locally definable group admits a group topology. This topology agrees with the M^k-topology in neighborhoods of generic points, namely, points $g \in \mathcal{U}$ such that $\dim(g/A) = \dim(\mathcal{U})$. (We assume here that all the X_i's above are defined over A.) We therefore obtain a definable family of neighborhoods $\{U_t : t \in T\}$ of the identity element, such that $\{gU_t : t \in T, g \in \mathcal{U}\}$ is a basis for the group topology on $\mathcal{U}$. In Baro and Otero [2] it was further shown that the topology can be realized by countably many definable open charts, each definably homeomorphic to an open subset of M^n, where $n = \dim(\mathcal{U})$.

A subset $X \subseteq \mathcal{U}$ is called *compatible* (see Edmundo [3]) if for every $Y \subseteq \mathcal{U}$ which is definable, the set $X \cap Y$ is also definable. It easily follows that X itself is also locally definable (namely, given as a countable union of definable subsets of $\mathcal{U}$). As was shown in [3], if $\mathcal{U}$ is locally definable and $\mathcal{H}$ is a normal compatible subgroup of $\mathcal{U}$, then there is a locally definable group $\mathcal{K}$ and a locally definable surjective homomorphism $f : \mathcal{U} \to \mathcal{K}$ whose kernel is $\mathcal{H}$. The converse is true as well; namely, if such a homomorphism exists, then $\mathcal{H}$ is necessarily compatible.

A locally definable group is called *connected* (see Baro and Edmundo [1]) if it has no compatible subset which is both closed and open, with respect to the group topology. As is shown in [2, Remark 4.3], a locally definable group $\mathcal{U}$ is connected if and only if it is path-connected; namely, for any two points $x, y \in \mathcal{U}$ there exists a *definable* continuous $\sigma : [0, 1] \to \mathcal{U}$ such that $\sigma(0) = x$ and $\sigma(1) = y$.

A typical example of a locally definable group is obtained by taking a definable subset of a definable group (say, of $\langle R^n, +\rangle$) and letting $\mathcal{U}$ be the subgroup generated by X. When the generating set is definably connected and contains the identity one obtains a connected locally definable group. We call a locally definable group $\mathcal{U}$ *definably generated* if it is generated, as a group, by some definable subset.

Definition 1.1 For $\mathcal{H} \subseteq \mathcal{U}$ a locally definable normal subgroup, we say that the *quotient $\mathcal{U}/\mathcal{H}$ is definable* if there exists a definable group G and a locally definable surjective homomorphism from $\mathcal{U}$ onto G, whose kernel is $\mathcal{H}$.

Definition 1.2 A locally definable normal subgroup $\Lambda \subseteq \mathcal{U}$ is called a *lattice in* $\mathcal{U}$ if $\dim(\Lambda) = 0$ and $\mathcal{U}/\Lambda$ is definable.

Notice that any countable group can be realized as a locally definable group, and therefore it is also a lattice in itself.

If $\mathcal{U}$ is the subgroup of R^n generated by the unit n-cube $[-1, 1]^n$, then $\mathbb{Z}^n$ is a lattice in $\mathcal{U}$. The quotient is definably isomorphic to the group H^n, where $H = [0, 1)$, with addition modulo 1.

In [4, Lemma 2.1] we prove the following equivalence.

Lemma 1.3 *Let $\mathcal{U}$ be a locally definable group in an o-minimal expansion of an ordered group, and let Λ be a locally definable normal subgroup of dimension zero. The following are equivalent.*

(1) Λ is a lattice in $\mathcal{U}$.

(2) Λ is compatible, and there exists a definable set $X \subseteq G$ such that $\Lambda \cdot X = \mathcal{U}$.

It is easy to see that every lattice in a locally definable group is countable. (The intersection with every definable set is finite.) We prove a stronger statement.

Lemma 1.4 *If Λ is a lattice in a locally definable connected group $\mathcal{U}$, then Λ is finitely generated as a group.*

Proof Let $\varphi : \mathcal{U} \to G$ be a locally definable surjective homomorphism onto a definable group G, with $\ker \varphi = \Lambda$. By compactness there exists a definable set $X \subseteq \mathcal{U}$ such that $\varphi(X) = G$. Because we have definable choice for subsets of $\mathcal{U}$ (see [3, Corollary 8.1]) we can find a definable section $s : G \to X$ (i.e., $\varphi \circ s = \mathrm{id}$), and so we replace X by the image of this section and call it X again. We may assume that $e \in X$.

Consider the topological closure (with respect to the group topology), $\mathrm{Cl}(X) \subseteq \mathcal{U}$.

Claim *There exists a finite set $F \subseteq \Lambda$ such that for every $g \in \Lambda$, if the intersection $g\,\mathrm{Cl}(X) \cap \mathrm{Cl}(X) \neq \emptyset$, then $g \in F$.*

Proof of Claim Let $X' \subseteq \mathcal{U}$ be any definable open set containing $\mathrm{Cl}(X)$. By saturation, there is a finite $F \subseteq \Lambda$, which we may assume is minimal, such that $X' \subseteq F \cdot X$. Because $gX \cap hX = \emptyset$ for every $g \neq h \in \Lambda$, if $gX \cap X' \neq \emptyset$, then necessarily $g \in F$. Now, if $g\,\mathrm{Cl}(X) \cap \mathrm{Cl}(X) \neq \emptyset$, then necessarily $gX \cap X' \neq \emptyset$, so $g \in F$. □

We now claim that F generates Λ; namely, every element of Λ is a finite word in F and F^{-1}.

Take $\lambda \in \Lambda$. Since $\mathcal{U}$ is path-connected, there exists a definable path $\gamma : [0,1] \to \mathcal{U}$, with $\gamma(0) = e$ and $\gamma(1) = \lambda$. Let $\Gamma \subseteq \mathcal{U}$ be the image of γ. Because Γ is definable it can be covered by finitely many Λ-translates of X. By taking a minimal number of translates, we obtain $\lambda_1, \dots, \lambda_k \in \Lambda$ (possibly with repetitions), such that $e \in \lambda_1 X$, $\lambda \in \lambda_k X$, and for $i = 1, \dots, k-1$, we have $\mathrm{Cl}(\lambda_i X) \cap \mathrm{Cl}(\lambda_{i+1} X) \neq \emptyset$.

By the claim, it follows that $\lambda_{i+1}^{-1}\lambda_i \in F$, for $i = 1, \dots, k-1$. But since $e \in X$, we must have $\lambda_1 = e$ and $\lambda_k = \lambda$, so $\lambda_1, \dots, \lambda_k$ are all in the group generated by F, and in particular, λ belongs to that group. □

We say that $\mathcal{U}$ *admits a lattice* if there is a lattice in $\mathcal{U}$. Note that not every locally definable group admits a lattice. For example, if $r \in R$ is larger than all elements of $\mathbb{N}$, then the subgroup of $\langle R, +\rangle$ given by $\bigcup [-r^n, r^n]$ does not admit any lattice.

As we point out in [4], there are many consequences, for a given group $\mathcal{U}$, of the fact that it admits a lattice. Hence, our main question is the following.

Question 1 Which locally definable groups in $\mathcal{M}$ admit a lattice?

We start with some basic observations.

Definition 1.5 A definable subset X of a locally definable group $\mathcal{U}$ is called *left generic in* $\mathcal{U}$ if there exists a bounded set $\Delta \subseteq \mathcal{U}$ (namely, $|\Delta| < \kappa$) such that $\mathcal{U} = \Delta \cdot X$. Equivalently, for every definable $Y \subseteq \mathcal{U}$ there is a finite set $F \subseteq \mathcal{U}$ such that $Y \subseteq F \cdot X$.

Lemma 1.3 immediately gives the following.

Lemma 1.6 *If a locally definable group $\mathcal{U}$ admits a lattice, then $\mathcal{U}$ contains a definable left generic set.*

Lemma 1.7 *Let $\mathcal{U}$ be a connected locally definable group which contains a left generic definable set X (e.g., if $\mathcal{U}$ admits a lattice). Then $\mathcal{U}$ is definably generated.*

Proof Let $X \subseteq \mathcal{U}$ be a definable, left generic set; namely, there is a bounded set $\Delta \subseteq \mathcal{U}$ such that $\Delta \cdot X = \mathcal{U}$. The group generated by X, call it $\mathcal{H}$, is therefore locally definable, of bounded index in $\mathcal{U}$ (since $\langle \Delta \rangle \cdot \mathcal{H} = \mathcal{U}$, where $\langle \Delta \rangle$ is the group generated by Δ). But then, if $Y \subseteq \mathcal{U}$ is a definable set, then $Y \cap \mathcal{H}$ and $Y \cap (\mathcal{U} \setminus \mathcal{H})$ are both bounded unions of definable sets. By saturation, this forces $Y \cap \mathcal{H}$ to be definable; hence $\mathcal{H}$ is compatible. It is easy to see that $\mathcal{H}$ is both closed and open so by connectedness of $\mathcal{U}$ must equal $\mathcal{U}$. □

It is now natural to ask the following question.

Question 2 Does every connected, definably generated group admit a lattice?

2 Lattices in Abelian Groups

We still work in a sufficiently saturated structure $\mathcal{M}$.

Recall that for a locally definable group $\mathcal{U}$, we say that $\mathcal{U}^{00}$ *exists* if there is a smallest type-definable normal subgroup of $\mathcal{U}$ of bounded index. (Note that a type-definable subgroup of $\mathcal{U}$ is necessarily contained in a definable subset of $\mathcal{U}$.) We denote that subgroup by $\mathcal{U}^{00}$.

One of the main results in [4] is the following. (The equivalence of the bottom three clauses is given in [4, Theorem 3.9]; the addition of clause (1) is obtained using Lemma 1.6.)

Theorem 2.1 *Let $\mathcal{U}$ be a connected, abelian, definably generated group. Then there is k so that the following are equivalent.*

(1) $\mathcal{U}$ admits a lattice.
(2) $\mathcal{U}$ admits a lattice, isomorphic to $\mathbb{Z}^k$.
(3) $\mathcal{U}$ contains a definable generic set.
(4) $\mathcal{U}^{00}$ exists, and $\mathcal{U}/\mathcal{U}^{00}$ is isomorphic to $\mathbb{R}^k \times K$, for some compact Lie group K.

In particular, we see that a connected, abelian, locally definable $\mathcal{U}$ admits a lattice if and only if it contains a definable generic set. Note that by (4), the above k is determined by $\mathcal{U}/\mathcal{U}^{00}$ and thus unique.

In [4] we made the conjecture that the conclusions of the above theorem are always true.

Conjecture A *Let $\mathcal{U}$ be an abelian, connected, definably generated group. Then $\mathcal{U}$ contains a definable generic set (so in particular admits a lattice).*

The number k in Theorem 2.1 can be viewed as a measure of how "nondefinable" the group $\mathcal{U}$ is. Namely, if $k = 0$, then $\mathcal{U}$ is outright definable, while if $k = \dim \mathcal{U} > 0$, then $\mathcal{U}$ will not contain any infinite definable subgroup. We prove the latter statement in Corollary 2.6 below.

In fact, we can define an invariant for every locally definable group $\mathcal{U}$ (not necessarily satisfying Conjecture A) which gives some indication as to how "nondefinable" $\mathcal{U}$ is.

Definition 2.2 The $\bigvee$*-dimension* of $\mathcal{U}$, denoted by $\text{vdim}(\mathcal{U})$, is the maximum k such that $\mathcal{U}$ contains a compatible subgroup isomorphic to $\mathbb{Z}^k$, if such k exists, and ∞, otherwise.

We prove in Theorem 2.8 below that Conjecture A is equivalent to the following.

Conjecture B *Let $\mathcal{U}$ be a connected, abelian, definably generated group. Then,*

(1) $\operatorname{vdim}(\mathcal{U}) \leq \dim(\mathcal{U})$*; in particular,* $\operatorname{vdim}(\mathcal{U})$ *is finite;*
(2) if $\mathcal{U}$ is not definable, then $\operatorname{vdim}(\mathcal{U}) > 0$.

In Section 3 we will prove Conjecture A for definably generated subgroups of $\langle R^n, +\rangle$, where R is a real closed field and $\mathcal{M}$ is an o-minimal expansion of R.

Unless otherwise stated, $\mathcal{U}$ denotes a connected, abelian, definably generated group.

We first prove the following.

Lemma 2.3 *Assume that $\mathcal{U}$ contains a definable group H. Then $\mathcal{U}$ admits a lattice Γ isomorphic to $\mathbb{Z}^k$ if and only if $\mathcal{U}/H$ (which is also definably generated) contains a lattice Δ isomorphic to $\mathbb{Z}^k$.*

Proof Let $\psi : \mathcal{U} \to \mathcal{U}/H$ be a locally definable surjective homomorphism.

Assume that $\mathcal{U}$ contains a lattice $\Gamma \simeq \mathbb{Z}^k$. Because H is definable the intersection $\Gamma \cap H$ is finite and so must equal $\{0\}$. Let $\Delta = \psi(\Gamma) \simeq \mathbb{Z}^k$. To see that Δ is compatible in $\mathcal{U}/H$, take a definable $Y \subseteq \mathcal{U}/H$, and find a definable $X \subseteq \mathcal{U}$ such that $\psi(X) = Y$. Our goal is to show that $Y \cap \Delta$ is finite. But $Y \cap \Delta = \psi((X + H) \cap \Gamma)$, and since Γ is compatible its intersection with $X + H$ is finite. Thus $Y \cap \Delta$ is finite, and so Δ is compatible in $\mathcal{U}/H$.

Let $\varphi : \mathcal{U} \to G$ be a locally definable surjective homomorphism onto a definable group, with $\Gamma = \ker \varphi$. Notice that $\varphi(H)$ is a definable subgroup of G. To see that Δ is a lattice in $\mathcal{U}/H$, we note that

$$(\mathcal{U}/H)/\Delta \simeq \mathcal{U}/(H + \Gamma) \simeq G/\varphi(H),$$

and therefore $(\mathcal{U}/H)/\Delta$ is definable.

Assume now that $\mathcal{U}/H$ admits a lattice $\Delta \simeq \mathbb{Z}^k$. We can find $u_1, \ldots, u_k \in \mathcal{U}$ with $\varphi(u_1), \ldots, \varphi(u_k)$ generators of Δ. Let $\Gamma \subseteq \mathcal{U}$ be the group generated by the u_i's.

We first show that Γ is compatible. Because Δ is torsion-free, φ is injective on Γ. Therefore, if $X \subseteq \mathcal{U}$ is definable the intersection $X \cap \Gamma$ must be finite, or else $\varphi(X) \cap \Delta$ is infinite, contradicting the compatibility of Δ. To see that Γ is a lattice it is sufficient, by Lemma 1.3, to see that $\mathcal{U}$ contains a definable set X with $X + \Gamma = \mathcal{U}$. We first find a definable $Y \subseteq \mathcal{U}/H$ such that $Y + \Delta = \mathcal{U}/H$, then a definable $X' \subseteq \mathcal{U}$ with $\psi(X') = Y$, and finally take $X = X' + H$. It is easy to verify that $X + \Gamma = \mathcal{U}$. □

Lemma 2.4 *Assume that $\mathcal{U}$ contains a definable generic set. Then $\mathcal{U}$ is definable if and only if* $\operatorname{vdim}(\mathcal{U}) = 0$.

Proof One direction is obvious, for if $\mathcal{U}$ is definable, then it cannot contain any infinite zero-dimensional compatible subgroup. For the converse, assume that $\mathcal{U}$ is not definable.

By Theorem 2.1, the group $\mathcal{U}^{00}$ exists, and for some $k \in \mathbb{N}$, we have $\mathcal{U}/\mathcal{U}^{00} \simeq R^k \times K$, for a compact Lie group K. We claim that $k > 0$. Indeed, if $k = 0$, then $\mathcal{U}/\mathcal{U}^{00} = K$ is compact. But then, by [4, Lemma 3.3], the preimage of K would be contained in a definable subset of $\mathcal{U}$, and thus $\mathcal{U}$ would be definable, a contradiction.

If we now apply Theorem 2.1 (4) $\Rightarrow$ (2), we see that $\mathcal{U}$ admits a lattice isomorphic to $\mathbb{Z}^k$, so $\operatorname{vdim}(\mathcal{U}) \geq k > 0$. □

Proposition 2.5 *Assume that $\mathcal{U}$ admits a lattice.*

(i) If Λ is a zero-dimensional, compatible subgroup of $\mathcal{U}$, then $\Lambda \simeq \mathbb{Z}^l + F$, with $l \leq \mathrm{vdim}(\mathcal{U})$ and F a finite subgroup of $\mathcal{U}$.

(ii) $\mathrm{vdim}(\mathcal{U}) \leq \dim(\mathcal{U})$.

(iii) If Λ is a lattice in $\mathcal{U}$, then $\Lambda \simeq \mathbb{Z}^l + F$, with $l = \mathrm{vdim}(\mathcal{U})$ and F a finite subgroup of $\mathcal{U}$.

(iv) If $\mathcal{U}$ is torsion-free and generated by a definably compact set, then every lattice in $\mathcal{U}$ is isomorphic to $\mathbb{Z}^l$, with $l = \dim(\mathcal{U}) = \mathrm{vdim}(\mathcal{U})$.

Proof By [4, Claim 3.4], there exists a definable torsion-free subgroup $H \subseteq \mathcal{U}$ such that the group $\mathcal{U}/H$ is generated by a definably compact set.

By [4, Theorem 3.9], there exists a unique k such that $\mathcal{U}/H$ admits a lattice isomorphic to $\mathbb{Z}^k$, and, moreover, because $\mathcal{U}/H$ is generated by a definably compact set, we have $k \leq \dim(\mathcal{U}/H)$ and hence $k \leq \dim(\mathcal{U})$. Also, by Lemma 2.3, the group $\mathcal{U}$ also admits a lattice isomorphic to $\mathbb{Z}^k$, so $k \leq \mathrm{vdim}(\mathcal{U})$. Our proof below implies that $k = \mathrm{vdim}(\mathcal{U})$.

Again, by [4, Theorem 3.9], the group $\mathcal{U}/\mathcal{U}^{00}$ is isomorphic to $\mathbb{R}^k \times K$, where K is a compact Lie group. The rest of the argument is extracted from the proof of [4, Lemma 3.7].

(i) Assume that $\Lambda \subseteq \mathcal{U}$ is a zero-dimensional compatible subgroup. Consider $\varphi : \mathcal{U} \to \mathcal{U}/\Lambda$. We claim that $\ker(\varphi) \cap \mathcal{U}^{00} = \{0\}$. Indeed, take any definable set $X \subseteq \mathcal{U}$ containing $\mathcal{U}^{00}$. Then, since $\varphi \restriction X$ is definable, the intersection $\ker(\varphi) \cap \mathcal{U}^{00} \subseteq \ker(\varphi) \cap X$ is finite. However, by [4, Proposition 3.5], the group $\mathcal{U}^{00}$ is torsion-free, so $\ker(\varphi) \cap \mathcal{U}^{00} = \{0\}$.

Consider the map $\pi_{\mathcal{U}} : \mathcal{U} \to \mathbb{R}^k \times K$, and let Γ be the image of $\ker(\varphi)$ under π_U. We just showed that Γ is isomorphic to $\Lambda = \ker(\varphi)$. We claim that Γ is discrete. Indeed, using X as above we can find another definable set X' whose image $\pi_{\mathcal{U}}(X')$ contains an open neighborhood of zero and no other elements of Γ, so Γ is discrete.

Now, since K is compact, the projection Γ' of Γ into $\mathbb{R}^k$ has a finite kernel $F \subseteq K$. Furthermore, Γ' is a discrete subgroup of $\langle \mathbb{R}^k, + \rangle$, and hence $\Gamma' \simeq \mathbb{Z}^l$, for some $l \leq k$. Therefore, $\Gamma \simeq \mathbb{Z}^l + F$, so $\Lambda \simeq \mathbb{Z}^l + F$. In particular, if $\Lambda \simeq \mathbb{Z}^l$, then $l \leq k$, which implies $\mathrm{vdim}(\mathcal{U}) \leq k$. Since $\mathcal{U}$ does contain a compatible copy of $\mathbb{Z}^k$, it follows that $k = \mathrm{vdim}(\mathcal{U})$, so $l \leq \mathrm{vdim}(\mathcal{U})$, as required.

(ii) Since $k \leq \dim(\mathcal{U})$, we have $\mathrm{vdim}(\mathcal{U}) \leq \dim(U)$.

(iii) Assume now that $\Lambda \simeq \mathbb{Z}^l + F$ is a lattice in $\mathcal{U}$. Namely, $\mathcal{U}/\Lambda$ is a definable group G. We proceed to show that $l = k$. Let $X \subseteq \mathcal{U}$ be a definable set so that $\varphi(X) = G$. Then $X + \ker(\varphi) = \mathcal{U}$. Thus, $\pi_{\mathcal{U}}(X) + \Gamma = \mathbb{R}^k \times K$. Let Y, F', and Γ' be the projections of $\pi_{\mathcal{U}}(X)$, F, and Γ, respectively, into $\mathbb{R}^k$. We have $Y + \Gamma' = \mathbb{R}^k$. Since X is definable, the set $\pi_{\mathcal{U}}(X)$ is compact and so Y is also compact.

We let $\lambda_1, \ldots, \lambda_l$ be the generators of $\ker(\varphi)$ and let $v_1, \ldots, v_l \in \mathbb{R}^k$ be their images in Γ'. If $H \subseteq \mathbb{R}^k$ is the real subspace generated by $v_1, \ldots, v_l$, then $Y + H + F' = \mathbb{R}^k$, and therefore, since Y is compact and F' finite, we must have $H = \mathbb{R}^k$. This implies that $l = k$.

(iv) By [4, Proposition 3.8], if $\mathcal{U}$ is generated by a definably compact set and is torsion-free, then $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^{\dim(\mathcal{U})}$, so by Theorem 2.1 every lattice is isomorphic to $\mathbb{Z}^{\dim U}$. By (iii), $\dim(\mathcal{U}) = \mathrm{vdim}(\mathcal{U})$. □

We can now see better why $\mathrm{vdim}(\mathcal{U})$ gives an indication as to how "nondefinable" $\mathcal{U}$ is.

Corollary 2.6 *Assume that $\mathcal{U}$ admits a lattice and H is a definable subgroup of $\mathcal{U}$. Then*

(i) $\operatorname{vdim}(\mathcal{U}) = \operatorname{vdim}(\mathcal{U}/H)$*;*
(ii) if $\operatorname{vdim}(\mathcal{U}) = \dim(\mathcal{U})$*, then H must be finite;*
(iii) if $\mathcal{U}$ is torsion-free, and H has maximal dimension among all definable subgroups of $\mathcal{U}$, then $\dim H = \dim(\mathcal{U}) - \operatorname{vdim}(\mathcal{U})$.

Proof (i) By Theorem 2.1, $\mathcal{U}$ admits a lattice isomorphic to $\mathbb{Z}^k$, and by Proposition 2.5(iii), $k = \operatorname{vdim}(\mathcal{U})$. By Lemma 2.3, $\mathcal{U}/H$ also admits a lattice isomorphic to $\mathbb{Z}^k$, and so by again by the same proposition, we have $\operatorname{vdim}(\mathcal{U}/H) = k$.

(ii) Assume that H is an infinite definable subgroup of $\mathcal{U}$. Then by (i), we have $\operatorname{vdim}(\mathcal{U}/H) = \operatorname{vdim}(\mathcal{U}) = \dim(\mathcal{U}) > \dim(\mathcal{U}/H)$, which contradicts Proposition 2.5(ii) for $\mathcal{U}/H$.

(iii) If $\dim H$ has maximal dimension among the definable subgroups of $\mathcal{U}$, then, as we already noted, $\mathcal{U}/H$ is generated by a definably compact set. Because H is torsion-free, as a subgroup of $\mathcal{U}$, it must be definably connected and therefore divisible. It follows that $\mathcal{U}/H$ is torsion-free as well. By Proposition 2.5(iv), $\operatorname{vdim}(\mathcal{U}/H) = \dim(\mathcal{U}/H)$. But then, by (i) we have

$$\dim H = \dim(\mathcal{U}) - \dim(\mathcal{U}/H) = \dim(\mathcal{U}) - \operatorname{vdim}(\mathcal{U}). \qquad \square$$

The torsion-free condition in (iii) above is necessary. For example, the group $\bar{G}$ in Eleftheriou and Peterzil [5, Example 6.2] does not contain any nontrivial definable subgroups, yet $\dim(\overline{G}) = 2$ and $\operatorname{vdim}(\overline{G}) = 1$. We describe below a general method to obtain a locally definable group $\mathcal{V}$, generated by a definably compact set, such that $\mathcal{V}$ has no infinite definable subgroups and $\operatorname{vdim}(\mathcal{V}) < \dim(\mathcal{V})$.

Example 2.7 Let G be a k-dimensional definably compact abelian group which has no proper definable subgroups of positive dimension, and let $\mathcal{U}$ be the universal covering of G, so $\dim(\mathcal{U}) = k$. Let $\Gamma \simeq \mathbb{Z}^k$ be the kernel of the covering map, so Γ is compatible in $\mathcal{U}$. Write $\Gamma = \Gamma_1 \oplus \Gamma_2$ with $\Gamma_1 \simeq \mathbb{Z}^m$, $\Gamma_2 \simeq \mathbb{Z}^{k-m}$, and $0 < m < k$. Obviously, Γ_1 is still compatible in $\mathcal{U}$, and therefore $\mathcal{V} = \mathcal{U}/\Gamma_1$ is a locally definable group with $\dim(\mathcal{V}) = \dim(\mathcal{U})$. It is not hard to see that the covering map $\mathcal{U} \to G$ factors through $\mathcal{V}$ and hence $\mathcal{V}$ cannot have any proper definable subgroup of positive dimension. We claim that $\operatorname{vdim}(\mathcal{V}) = k - m$.

Let $\varphi : \mathcal{U} \to \mathcal{V}$ be a locally definable projection. The image of Γ under φ is a group $\Delta \simeq \mathbb{Z}^{k-m}$ which we claim to be compatible in $\mathcal{V}$. We start with $Y \subseteq \mathcal{V}$ definable and claim that $Y \cap \Delta$ is finite.

Let $\pi_2 : \Gamma \to \Gamma_2$ be the projection with respect to the direct sum decomposition. For every $W \subseteq \Gamma$, $\varphi(W)$ is in bijection with $\pi_2(W)$, so it is enough to prove that $\pi_2(\varphi^{-1}(Y \cap \Delta)) = \pi_2(\varphi^{-1}(Y) \cap \Gamma)$ is finite.

If we choose a definable $X \subseteq \mathcal{U}$ such that $\varphi(X) = Y$, then $\varphi^{-1}(Y) = X + \Gamma_1$. But $(X + \Gamma_1) \cap \Gamma = (X \cap \Gamma) + \Gamma_1$, and because Γ is compatible the set $X \cap \Gamma$ is finite. It follows that

$$\pi_2\big(\varphi^{-1}(Y) \cap \Gamma\big) = \pi_2\big((X \cap \Gamma) + \Gamma_1\big) = \pi_2(X \cap \Gamma)$$

is finite so $Y \cap \Delta$ is finite, showing that Δ is compatible in $\mathcal{V}$. Hence, $\operatorname{vdim}(\mathcal{V}) \geq k - m$.

For the opposite inequality, assume $\mathcal{V}$ contains a compatible subgroup Δ isomorphic to $\mathbb{Z}^r$, and choose $u_1, \ldots, u_r \in \mathcal{U}$ so that $\varphi(u_1), \ldots, \varphi(u_r)$ are generators of Δ.

It is not hard to see that $\Gamma_1+\mathbb{Z}u_1+\cdots+\mathbb{Z}u_r$ is a compatible subgroup of $\mathcal{U}$, isomorphic to $\mathbb{Z}^{m+r}$, so necessarily $m+r\leq k$. Hence, $r\leq k-m$, so $\mathrm{vdim}(\mathcal{V})=k-m$.

Note that $\mathcal{V}$ has nontrivial torsion since any $a\in\mathcal{U}$ for which $na\in\Gamma_1$ will be mapped to an n-torsion element of $\mathcal{V}$.

We end by noting that the two conjectures mentioned above are equivalent.

Theorem 2.8 *Conjecture A is equivalent to Conjecture B. More precisely,*

(i) if $\mathcal{U}$ admits a definable generic set, then $\mathcal{U}$ satisfies clauses (1), (2) of Conjecture B;

(ii) Conjecture B implies Conjecture A.

Proof (i) This is by Proposition 2.5 and Lemma 2.4.

(ii) Let $\Lambda\simeq\mathbb{Z}^k$ be a compatible subgroup of $\mathcal{U}$ with $k=\mathrm{vdim}(\mathcal{U})$. We will prove that the locally definable group $\mathcal{U}/\Lambda$ is actually definable.

Assume that $\mathcal{U}/\Lambda$ is not definable. By Conjecture B(2) (applied to $\mathcal{U}/\Lambda$), there exists some $a\in\mathcal{U}/\Lambda$ such that $\mathbb{Z}a$ is a compatible subgroup of $\mathcal{U}/\Lambda$, and for every n, $na\neq 0$. Let $b\in\mathcal{U}$ be an element that projects via $\varphi:\mathcal{U}\to\mathcal{U}/\Lambda$ to a. Clearly, $\mathbb{Z}b\cap\Lambda=\{0\}$. We claim that $\Lambda+\mathbb{Z}b$ is a compatible subgroup of $\mathcal{U}$, contradicting $k=\mathrm{vdim}(\mathcal{U})$. Let $X\subseteq\mathcal{U}$ be definable. The image of $X\cap(\Lambda+\mathbb{Z}b)$ under φ is contained in $\varphi(X)\cap\mathbb{Z}a$. Since φ is locally definable, $\varphi(X)$ is definable. Therefore $\varphi(X)\cap\mathbb{Z}a$ is finite, by compatibility of $\mathbb{Z}a$. The preimage of this finite set under π is a union of sets $\Lambda+x$, $x\in B$, for some finite $B\subseteq\mathbb{Z}b$. So $X\cap(\Lambda+\mathbb{Z}b)$ is equal to the finite union of the sets $X\cap(\Lambda+x)$, $x\in B$, each of which is finite, because so is $(X-x)\cap\Lambda$ by compatibility of Λ. Hence $X\cap(\Lambda+\mathbb{Z}b)$ is finite, and thus $\Lambda+\mathbb{Z}b$ is compatible. □

3 Locally Definable Subgroups of $\langle R^n,+\rangle$

We assume here that $\mathcal{M}$ is an o-minimal expansion of a real closed field R.

Our goal is to prove Conjecture A for subgroups of $\langle R^n,+\rangle$, but in fact we prove a stronger result (as was suggested to us by a referee).

Theorem 3.1 *Let $\mathcal{U}$ be a connected definably generated subgroup of $\langle R^n,+\rangle$ of dimension k. Then there are linearly independent one-dimensional R-subspaces $R_1,\ldots,R_k$ and intervals $I_i=(-a_i,a_i)\subseteq R_i$ (with a_i possibly ∞) such that $\mathcal{U}$ is generated by the set $X=I_1+\cdots+I_k$. The set X is generic in $\mathcal{U}$.*

Proof Recall that for $X\subseteq R^n$, we write $X(m)$ for the addition of $X-X$ to itself m times. If $0\in X$, then $X\subseteq X(m)$.

Definition 3.2 A subset of R^n is called *convex with respect to R (or R-convex)* if for all $x,y\in X$, the line segment connecting x and y is also in X.

The *R-convex hull* of X is the smallest R-convex subset of R^n containing X. It consists of all finite combinations $\sum_{i=1}^m t_ix_i$, where the x_i's are in X, all $t_1\geq 0$, and $\sum t_i=1$.

Lemma 3.3 *If $X\subseteq R^n$ is definable, then the R-convex hull of X is also definable.*

Proof More precisely, we claim that the following set equals the R-convex hull of X:

$$X'=\Big\{\sum_{i=1}^{n+1}t_ix_i : t_1+\cdots+t_{n+1}=1, t_i\in[0,1], x_i\in X\Big\}.$$

Indeed, by Carathéodory's theorem, every convex combination of any number of points from X can also be realized as a combination of $n+1$ of these points; hence the R-convex hull of X equals X'. (Note that although Carathéodory's theorem is usually proved over the reals, the same proof works over any ordered field. Alternatively, the statement over the real numbers implies, by transfer, the same result over any real closed field.) □

Lemma 3.4 *Assume that $X \subseteq R^n$ is a definably connected set containing zero. Then there is m such that $X(m)$ (in the sense of the additive group $\langle R, +\rangle$) contains the R-convex hull of X.*

Proof Given $f : X \to Z$, the fiber power of X is defined as

$$X \times_f X = \big\{\langle x, y\rangle \in X \times X : f(x) = f(y)\big\}.$$

Clearly, the diagonal Δ is contained in $X \times_f X$.

Note that for $\langle x_1, x_2\rangle, \langle y_1, y_2\rangle \in X \times_f X$, there is a continuous definable path in $X \times_f X$, connecting the two points if and only if there are definable continuous curves $\gamma_1, \gamma_2 : [0, 1] \to X$ such that $\gamma_i(0) = x_i$, $\gamma_i(1) = y_i$, and for every $t \in [0, 1]$ we have $f(\gamma_1(t)) = f(\gamma_2(t))$.

Claim 3.5 *For $X \subseteq R^n$, consider the projection $\pi : R^n \to R$ onto the first coordinate. Assume that $\pi(x_1) = \pi(x_2)$, $\pi(y_1) = \pi(y_2)$ (in particular, $\pi_1(x_1 - x_2) = \pi(y_1 - y_2) = 0$). Assume further that $\langle x_1, x_2\rangle$ and $\langle y_1, y_2\rangle$ are in the same connected component of $X \times_\pi X$. Then the elements $x_1 - x_2$ and $y_1 - y_2$ are in the same connected component of the set $(X - X) \cap \{0\} \times R^{n-1}$.*

Proof Note that the image of $X \times_\pi X$ under the binary map $\langle x, y\rangle \mapsto x - y$ is contained in the set $\{0\} \times R^{n-1}$. Consider the restriction of this map to the connected component of $X \times_\pi X$ which contains $\langle x_1, x_2\rangle$ and $\langle y_1, y_2\rangle$. The image is connected and clearly contains $x_2 - x_1$ and $y_2 - y_1$. □

Claim 3.6 *Assume that $x, y \in X$, $\pi(x) = \pi(y)$ and that there is a curve*

$$\gamma = (\gamma_1, \ldots, \gamma_n) : [0, 1] \to X$$

connecting x and y inside X. (Note that $\gamma_1(0) = \gamma_1(1)$.) Let Γ be the image of γ.

(1) If γ_1 is constant on $[0, 1]$, then $\Gamma \times_\pi \Gamma$ is definably connected. In particular, for every $x, y, z \in \Gamma$, $\langle x, y\rangle$ and $\langle z, z\rangle$ are in the same definably connected component of $X \times_\pi X$.

(2) If for some $a \in (0, 1)$, γ_1 is increasing on $(0, a)$ and decreasing on $(a, 1)$, then $y - x$ and zero are in the same connected component of $(X - X) \cap \{0\} \times R^{n-1}$.

(3) If for some $a_1 < a_2$ in $(0, 1)$, γ_1 is increasing on $(0, a_1)$, constant on (a_1, a_2), and decreasing on $(a_2, 1)$, then $y - x$ and zero are in the same connected component of $(X - X) \cap \{0\} \times R^{n-1}$.

Proof (1) By assumption the map π is constant on Γ, and therefore $\Gamma \times_\pi \Gamma = \Gamma \times \Gamma$, which is clearly definably connected.

(2) Let $[b_1, b_2]$ be the image of γ under π. By assumptions, $\pi(\gamma_1(0)) = \pi(\gamma_1(1)) = b_1$, $\pi(\gamma_1(a)) = b_2$, and the restrictions of π to the pieces $\gamma([0, a])$ and $\gamma([a, 1])$ are both injective. Let α_1, α_2 be their inverse maps, respectively (so these are maps from $[b_1, b_2]$ into Γ). We have $\alpha_1(b_1) = x$, $\alpha_2(b_1) = y$, $\alpha_1(b_2) = \alpha_2(b_2) = \gamma(a)$. Moreover, for every $t \in [b_1, b_2]$ we have $\pi(\alpha_1(t)) =$

$\pi(\alpha_2(t)) = t$. It follows that $\langle x, y\rangle$ and $\langle \gamma(a), \gamma(a)\rangle$ are in the same component of $X \times_\pi X$, so by Claim 3.5, $y - x$ and zero are in the same component of $(X - X) \cap \{0\} \times R^{n-1}$.

(3) As in (2), let $[b_1, b_2]$ be the image of γ under π. It is easy to see that $\gamma_1(t) = b_2$ for all $t \in [a_1, a_2]$. Similarly to the proof of (2), $\langle x, y\rangle$ and $\langle \gamma(a_1), \gamma(a_2)\rangle$ are in the same component of $X \times_\pi X$. Using (1), we see that $\langle \gamma(a_1), \gamma(a_2)\rangle$ is in the same component as $\langle z, z\rangle$ for some $z \in \gamma([a_1, a_2])$. Applying Claim 3.5, we conclude that $x - y$ and zero are in the same component of $(X - X) \cap \{0\} \times R^{n-1}$. □

We now return to the proof of Lemma 3.4. So, X is a definably connected subset of R^n containing zero, and we want to show that for some m, the convex hull of X is contained in $X(m)$.

We will use induction on n. If $n = 1$, then X is already convex. So, we assume that the result is true for $X \subseteq R^n$ and prove it for $X \subseteq R^{n+1}$. We take $x, y \in X$ and first want to show that for some m the line segment $[x, y]$ (i.e., the line connecting x and y in R^{n+1}) is contained in $X(m)$.

Using a linear automorphism of R^n, we may assume that $\pi(x) = \pi(y) = 0$. Since X is definably connected, there exists a definable curve $\gamma : [0, 1] \to X$ connecting x and y. Let $\Gamma \subseteq X$ be the image of γ, and again let $\gamma_1 = \pi \circ \gamma$.

Notation For $f : [0, 1] \to R$ continuous, let $k = k(f)$ be the minimal natural number so that there are $0 = a_0 < a_1 < \cdots < a_k = 1$, and f is either constant or strictly monotone on $[a_i, a_{i+1}]$.

We consider the map $\gamma_1 : [0, 1] \to R$ and prove the result by subinduction on $k(\gamma_1)$.

Assume first that $k(\gamma_1) = 1$, namely, that γ_1 is constant on $[0, 1]$. In this case, Γ is contained in $\{0\} \times R^n$, so we can work in R^n and use the inductive hypothesis to conclude that the line segment $[x, y]$ is contained in $\Gamma(m)$ for some m. Clearly, $\Gamma(m) \subseteq X(m)$, so we are done.

Assume then that $k(\gamma_1) > 1$, so γ_1 is not constant. Without loss of generality, γ_1 takes some positive value on $(0, 1)$, so let $a \in (0, 1)$ be a point where γ_1 takes its maximum value in $[0, 1]$.

Case 1. Assume first that γ_1 is not locally constant at a.

Then there are $a_1 < a < a_2$ such that γ_1 is increasing on (a_1, a), decreasing on (a, a_2), $\gamma_1(a_1) = \gamma_1(a_2)$, and furthermore, either a_1 or a_2 are local minima for γ_1. Indeed, we take $a_1' < a$ to be the minimum of all points t such that γ_1 is increasing on (t, a), and we take $a_2' > a$ be the maximum of all $t > a$ such that γ_1 is decreasing on (a, t). (In this case, a_1' and a_2' are local minima for γ_1.) We then compare $\gamma_1(a_1')$ and $\gamma_1(a_2')$. If $\gamma_1(a_1') > \gamma_1(a_2')$, then we take $a_1 := a_1'$ and take a_2 to be the unique element of the interval (a, a_2') such that $\gamma_1(a_2) = \gamma_1(a_1)$. Otherwise, we do the opposite.

Let $x_1 = \gamma(a_1)$ and $x_2 = \gamma(a_2)$. Consider now the curve Γ' which is the image of $[a_1, a_2]$ under γ. By Claim 3.6(2), $x_2 - x_1$ and zero are in the same connected component of $(\Gamma' - \Gamma') \cap \{0\} \times R^n$. But then, we can view this component as living in R^n, so by inductive hypothesis there exists m such that the line segment connecting zero and $x_2 - x_1$ is contained in $(\Gamma' - \Gamma')(m)$. By adding x_1 to both sides, we see that the line segment connecting x_1 and x_2 is contained in $(X - X)(m + 1)$. Hence, after replacing X with $X(m)$, we can also replace the original curve Γ with a new curve Γ'', in which the piece $\gamma([a_1, a_2])$ was replaced by a linear segment all of whose points project to the same point $\pi(x_1)$. Let $\gamma'' : [0, 1] \to X$ be the map whose image is Γ'' (so $\gamma'' = \gamma$ everywhere, except on $[a_1, a_2]$, in which the image is linear and γ''_1 is constant). Because a_1 or a_2 is a local minimum of γ_1, it is easy to see that $k(\gamma''_1) = k(\gamma_1) - 1$. By subinductive hypothesis, the line connecting x and y is contained in some $X(m')$.

Case 2. Assume that γ_1 is locally constant at a.

So, there are $a'_1 \leq a \leq a'_2$ such that γ_1 is constant on $[a'_1, a'_2]$ and this is a maximal such interval. As in Case 1, we can find $a_1 < a'_1$ and $a_2 > a'_2$ such that γ_1 is increasing on $[a_1, a'_1]$, decreasing on $[a'_2, a_2]$, $\gamma_1(a_1) = \gamma_1(a_2)$, and furthermore, either a_1 or a_2 is a local minimum of γ_1.

Let Γ' be the piece of Γ connecting $\gamma(a_1)$ and $\gamma(a_2)$. Then, by Claim 3.6(3), the points $\gamma(a_2) - \gamma(a_1)$ and zero are in the same component of $(\Gamma' - \Gamma') \cap \{0\} \times R^n$. Again, by inductive hypothesis, the line segment connecting zero and $\gamma(a_2) - \gamma(a_1)$ is contained in $(\Gamma' - \Gamma')(m)$ for some m, so the line segment connecting $\gamma(a_1)$ and $\gamma(a_2)$ is contained in $X(m + 1)$. As in case (1), we can replace Γ by Γ'', in which the piece $\gamma([a_1, a_2])$ is replaced by the line segment connecting $\gamma(a_1)$ and $\gamma(a_2)$. Again, the map $\gamma'' : [0, 1] \to X$ whose image is Γ'' now satisfies $k(\gamma''_1) = k(\gamma_1) - 2$ (because we replaced three pieces by one). By subinductive hypothesis, the line connecting x and y is in some $X(m')$.

We therefore showed that for every $x, y \in X$, there exists m such that the line segment $[x, y]$ is contained in $X(m)$. To see that we can find a uniform m for all $x, y \in X$, we use logical compactness (writing a type $p(x, y)$, which says that the line segment $[x, y]$ is not contained in any $X(m)$). This ends the proof of Lemma 3.4. □

Question It is interesting to ask what is the required m in the above result. The argument suggests that it depends on the possible number of "twistings" of the curve connecting two points in X. But maybe this is just an effect of the proof, and one can find a uniform such m which depends only on the ambient R^n.

Next, we show that $\mathcal{U} \subseteq R^n$ can be generated by a sum of intervals in linearly independent one-dimensional spaces. By Lemma 3.4 we can assume that it is generated by a definably connected convex set $X \ni 0$. In particular, $\mathcal{U}$ is convex. Since $\mathcal{U}$ is closed, we may replace X by its closure, which is still convex, and assume that X is closed. We may also assume that $-X = X$. (Otherwise, we replace it with $X - X$.)

We prove the result by induction on n. When $\mathcal{U}$ is a subset of R, then any convex subset of R is an interval (possibly equaling the whole of R), so we are immediately done.

We now consider the case $\mathcal{U} \subseteq R^{n+1}$.

Assume first that X is bounded. Consider all line segments contained in X, and let J_0 be such segment of maximal length. (It exists by o-minimality and the fact that X is closed.) Since we work in a field we may assume that J_0 is parallel to the x_{n+1}-coordinate and furthermore that $0 \in J_0$ divides it exactly into two equal parts. We can therefore write $J_0 = (-a_{k+1}, a_{k+1})$. Let $\pi(X)$ be the projection onto the first n coordinates. By induction, there are linearly independent one-dimensional spaces $R_1, \ldots, R_k \subseteq R^n$, and in each R_i an interval $I_i = (-a_i, a_i)$ (with a_i possibly ∞) such that the sum $Y = I_1 + \cdots + I_k$ generates the same group as $\pi(X)$. In particular, there is an $m \in \mathbb{N}$ such that $Y \subseteq \pi(X)(m)$. Our goal is to show that $Y + J_0$ generates the group $\mathcal{U}$. It is thus sufficient to prove the following.

Claim *We have $X \subseteq Y + J_0 \subseteq X(2m)$.*

Proof Consider $\langle \bar{x}, y\rangle \in X$, with $\bar{x} \in \pi(X)$. Note that $|y| \leq a_{k+1}/2$, because if $y > a_{k+1}/2$, then the length of the line segment connecting $\langle \bar{x}, y\rangle$ to zero is then greater than $a_{k+1}/2$. Because X is symmetric, the point $\langle -\bar{x}, -y\rangle$ is also in X, and thus the line segment connecting $\langle -\bar{x}, -y\rangle$ and $\langle \bar{x}, y\rangle$ is longer than $a_{k+1} = |J_0|$, a contradiction. We therefore showed that $y \leq a_{k+1}/2$ and hence

$$\langle \bar{x}, y\rangle \in \big\{\langle \bar{x}, 0\rangle\big\} + J_0 \subseteq \pi(X) + J_0 \subseteq Y + J_0.$$

For the opposite inclusion, take $\langle \bar{x}, y\rangle \in Y + J_0$. Since $Y \subseteq \pi(X)(m) = \pi(X(m))$, there exists $y' \in R$ such that $\langle \bar{x}, y'\rangle \in X(m)$. Because $\max\{|y| : \langle \bar{x}, y\rangle \in X\} = a_{k+1}/2$, we have $|y'| \leq m a_{k+1}/2$. But then

$$\langle \bar{x}, y\rangle \in X(m) + mJ_0 \subseteq X(2m).$$

This ends the proof of the claim and the case where the generating set X is bounded. $\square$

In the general case, we first find a definable subgroup H such that $\mathcal{U}/H$ is generated by a definably compact set. Since all definable subgroups of R^n are R-vector spaces, the group H is linear. Without loss of generality, $H = R^k$, for $k \leq n$, identified with the first k coordinates. Let $\pi_1 : \mathcal{U} \to R^{n-k}$ be the projection onto the last $n-k$ coordinates, and let $\mathcal{V} = \pi_1(\mathcal{U})$. We claim that $\mathcal{U} = H + \mathcal{V}$.

Indeed, assume that $\langle \bar{x}, \bar{y}\rangle \in \mathcal{U}$. Since $\mathcal{U}$ is convex, the line segments which connect $\langle \bar{x}, \bar{y}\rangle$ to arbitrary large points in R^k belong to $\mathcal{U}$. Hence we can approach every point on the affine space $R^k \times \{\bar{y}\}$ by points inside $\mathcal{U}$. Since $\mathcal{U}$ is closed, we have that $H + \{\langle \bar{0}, \bar{y}\rangle\}$ is contained in $\mathcal{U}$. This shows that $H + \mathcal{V}$ is contained in $\mathcal{U}$. The converse is immediate. This ends the proof that $\mathcal{U}$ is generated by a sum of intervals in linearly independent one-dimensional spaces.

Our final goal is to show that $Y_0 = I_1 + \cdots + I_k + J_0$ is generic in $\mathcal{U}$. We have $I_i = (-a_i, a_i)$ and $J_0 = (-a_{k+1}, a_{k+1})$. If we let $\mathcal{V}_i$ be the one-dimensional group generated by $(-a_i, a_i)$, then we have $\mathcal{U} = \mathcal{V}_1 + \cdots + \mathcal{V}_{k+1}$. Each $(-a_i, a_i)$ is generic in $\mathcal{V}_i$, so it is easy to verify that Y_0 is generic in $\mathcal{U}$. This ends the proof of Theorem 3.1. $\square$

As noted in the above proof, $\mathcal{U}$ is convex in R^n. This immediately implies that $\mathcal{U}$ is divisible. In [4], we prove more generally that Conjecture A implies that every connected definably generated abelian group is divisible.

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