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Weight and Measure in NIP Theories

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Abstract We initiate an account of Shelah's notion of "strong dependence" in terms of generically stable measures, proving a measure analogue (for NIP theories) of the fact that a stable theory T is "strongly dependent" if and only if all (finitary) types have almost finite weight.

1 Introduction

Shelah [9] introduced the notion "T is strongly dependent" as an attempt to find an analogue of *superstability* for NIP theories. When T is stable, strong dependence is actually equivalent to "all finitary types have finite weight," rather than superstability (see Adler [1]). Here I give a version of this equivalence in the general NIP context using generically stable measures (see Theorem 1.1).

A strong influence on this work is a talk by Hrushovski in Oberwolfach in January 2010 where he presented some tentative notions of "finite weight" using orthogonality (in the sense of measure theory) and generically stable measures. Some connections between strong dependence and suitable notions of weight in the general NIP context also appear in Onshuus and Usvyatsov [7], but only for types (not measures).

In spite of the appearance of Theorem 1.1 below as a definitive *characterization* of strong dependence, we view it as a first and even tentative step, and we will state some problems and questions.

In the remainder of this introduction, I will give an informal description of the basic notions, referring to Section 2 for the precise definitions and further references, and then state the main result Theorem 1.1. I will assume a familiarity with stability theory, the "stability-theoretic" approach to NIP theories, as well as the notion of a Keisler measure. References are Pillay [8], Hrushovski, Peterzil, and Pillay [3], and Hrushovski and Pillay [4], as well as papers of Shelah such as [10]. We will also be referring to Adler's paper [1], which gives a nice treatment of the combinatorial

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notions around strong dependence and makes explicit the connection with weight in the stable case.

Concerning notation, we work in a very saturated model $\bar{M}$ of a complete first-order theory T in language L. There is no harm to work in $\bar{M}^{eq}$, except that at some point we might want to make definitions concerning a given sort: $x, y, z, \ldots$ usually denote finite tuples of variables. Likewise $a, b, c, \ldots$ usually denote finite tuples of elements, and $M_0, M, \ldots$ normally denote small elementary substructures of $\bar{M}$.

Recall that T has NIP (or is dependent) if for any indiscernible (over $\emptyset$) $(a_i : i < \omega)$, and formula $\varphi(x, b)$, the truth value of $\varphi(a_i, b)$ is eventually constant. I will make a blanket assumption, at least in this introduction, that T has NIP.

Our *working definition* of "T is strongly dependent" (or "strongly NIP") is that there do not exist formulas $\varphi_\alpha(x, \alpha)$, $k_\alpha < \omega$, and tuples b_i^α, for $\alpha < \omega$, $i < \omega$, such that for each α, $\{\varphi_\alpha(x, b_i^\alpha) : i < \omega\}$ is k_α-inconsistent (every subset of size k_α is inconsistent), and for each $\eta \in \omega^\omega$, $\{\varphi_\alpha(x, b_{\eta(\alpha)}^\alpha) : \alpha < \omega\}$ is consistent. This is equivalent to Shelah's original definition assuming that T has NIP (see Definition 2.1 and Fact 2.3).

When we speak of "global" types or measures we mean over $\bar{M}$. A global Keisler measure $\mu(x)$ is said to be *generically stable* if $\mu(x)$ is both finitely satisfiable in and definable over some "small" model M (see Definition 2.10). In fact it follows from [4] that one can choose M of "absolutely" small cardinality such as $2^{|T|}$. We call a Keisler measure $\mu(x)$ over a small model M generically stable if $\mu(x)$ has a global nonforking (M-invariant) extension $\mu'(x)$ which is generically stable (in which case μ' is both definable over and finitely satisfiable in M and is the unique global nonforking extension of $\mu(x)$; see Fact 2.11 and Definition 2.12). For $\mu(x)$ a generically stable measure over M we denote by $\mu|\bar{M}$ the unique global nonforking extension of μ. If $\lambda(y)$ is another generically stable measure over M, we can form *the nonforking amalgam* $\mu(x) \otimes \lambda(y)$, another generically stable measure (in variables (x, y)) over M, and we have symmetry $\mu(x) \otimes \lambda(y) = \lambda(y) \otimes \mu(x)$ (see Remark 2.13). We iterate this to form the nonforking amalgam of any set of generically stable measures. A measure (generically stable or not) $\omega(x, y)$ over M which extends $\mu(x) \cup \lambda(y)$ will be called a *forking amalgam* if it is not the nonforking amalgam. We will call $\omega(x, y)$ a *strong forking amalgam* of $\mu(x)$ and $\lambda(y)$, with respect to μ, if for some formula $\varphi(x, y)$ over M, $\omega(\varphi(x, y)) = 1$ but $(\mu|\bar{M})(\varphi(x, b)) = 0$ for all $b \in \bar{M}$ (see Definition 2.14). We will relate this notion to *orthogonality* of measures in Section 2, as well as asking about symmetry. But let me remark for now that if T is stable and $\omega(x, y)$ is a complete type over M realized by (a, b), then ω is a strong forking amalgam of $\mu(x)$ and $\lambda(y)$ with respect to μ if and only if $tp(a/bM)$ forks over M (iff $tp(b/aM)$ forks over M; see Remark 2.15).

Of course we have the notion of a generically stable measure $\omega(x_i : i \in I)$ over a small model M in maybe infinitely many variables x_i, and in fact ω will be generically stable if and only if every restriction of ω to finitely many variables is.

Our main result is the following.

Theorem 1.1 *Suppose that T has NIP. Then the following are equivalent.*

(1) T is not strongly dependent.

(2) There is a model M_0 and generically stable measure $\omega(x, y_0, y_1, y_2, \ldots)$ over M_0 with the following properties:

(i) for each $\alpha < \omega$, $\omega_\alpha(x, y_\alpha)$ is a strong forking amalgam of $\lambda(x)$ and $\mu_\alpha(y_\alpha)$, with respect to μ_α, and

(ii) the restriction of ω to $(y_0, y_1, y_2, \ldots)$ is the nonforking amalgam $\bigotimes_\alpha \mu_\alpha(y_\alpha)$ of the $\mu_\alpha(y_\alpha)$,

where $\omega_\alpha(x, y_\alpha)$ is the restriction of ω to variables (x, y_α), $\mu_\alpha(y_\alpha)$ is the restriction of ω to variable y_α, and $\lambda(x)$ is the restriction of ω to variable x.

To make the connection with weight in stable theories, let us see what Theorem 1.1(2) means when T is stable and $\omega(x, y_0, y_1, \ldots)$ is a complete type over M_0 (which will of course be a generically stable type by stability of T). Let $(a, b_0, b_1, \ldots)$ be a realization of ω. Part (ii) of (2) says that $\{b_\alpha : \alpha < \omega\}$ is M_0-independent. And part (i) of (2) says (as remarked above) that $tp(a/b_\alpha M_0)$ forks over M_0 for each $\alpha < \omega$. Hence $tp(a/M_0)$ has infinite "preweight" in the strong sense that a forks over M_0 with each element of some infinite M_0-independent set. In fact in a stable theory T, no type having infinite preweight is equivalent to every type $p(x)$ having *finite weight* in the sense that there is a greatest n such that after possibly passing to a nonforking extension a realization of p can fork over the base with at most n elements of some independent sequence (see [8, Chapter 4, Proposition 3.10]).

So Theorem 1.1 is a generalization/analogue of the fact (see [1]) that a stable theory is strongly dependent if and only if every type has finite weight.

2 Preliminaries

The following definition is due to Shelah [9], and says that $\kappa_{\mathrm{ict}}(T) = \aleph_0$.

Definition 2.1 T is strongly dependent (or strongly NIP) if there do not exist formulas $\varphi_\alpha(x, y_\alpha) \in L$ for $\alpha < \omega$ and $(b_i^\alpha)_{\alpha<\omega, i<\omega}$ such that for every $\eta \in \omega^\omega$, the set of formulas $\{\varphi_\alpha(x, b^\alpha_{\eta(\alpha)}) : \alpha < \omega\} \cup \{\neg\varphi_\alpha(x, b_i^\alpha) : \alpha < \omega, i < \omega, i \neq \eta(\alpha)\}$ is consistent.

Remark 2.2

(i) T is strongly NIP; then T is NIP.

(ii) We can relativize the notion *strong NIP* to a sort S by specifying that the variable x in Definition 2.1 is of sort S.

(iii) In Definition 2.1 we could allow the φ^α to have parameters (by incorporating the parameters into the b^α).

Fact 2.3 Assume that T has NIP. The following are equivalent. (Also, sort by sort as far as the x variable is concerned.)

(1) T is strongly NIP in the sense of Definition 2.1.

(2) It is not the case that there exist formulas $\varphi_\alpha(x, y_\alpha)$ for $\alpha < \omega$, b_i^α for $\alpha < \omega$ and $i < \omega$, and $k_\alpha < \omega$ for each $\alpha < \omega$ such that

  (i) for each α, $\{\varphi_\alpha(x, b_i^\alpha) : i < \omega\}$ is k_α-inconsistent, and

  (ii) for each "path" $\eta \in \omega^\omega$, $\{\varphi_\alpha(x, b^\alpha_{\eta(\alpha)}) : \alpha < \omega\}$ is consistent.

(3) Just like (2) but with a further clause that

  (iii) for each α, the sequence $(b_i^\alpha : i < \omega)$ is indiscernible over $\bigcup_{\beta\neq\alpha}\{b_i^\beta : i < \omega\}$.

Proof This is contained in [1] (see Propositions 10, 13 there, and see [7] for (3)). Again, one can allow parameters in the formulas in (2) and (3). □

We now pass to Keisler measures and generically stable measures as well as notions specific to this paper. When we speak of a formula $\varphi(x)$ forking over a set of parameters we mean in the sense of Shelah; namely, $\varphi(x)$ implies a finite disjunction of formulas each of which divides over A.

A Keisler measure $\mu(x)$ (sometimes also written in earlier papers as μ_x) over A is a finitely additive probability measure on the Boolean algebra of formulas $\varphi(x)$ over A up to equivalence (or of A-definable sets in sort x). Such μ can be identified with a regular Borel probability measure on the Stone space $S_x(A)$ of complete types over A in variable x. By a global Keisler measure we mean a Keisler measure over $\bar{M}$.

Definition 2.4 Let $\mu(x)$ be a Keisler measure over B, and let $A \subseteq B$. We say that μ does *not* fork over A (or is a nonforking extension of $\mu|A$) if any formula $\varphi(x)$ over B with positive μ-measure does not fork over A.

Remark 2.5

(i) It is easy to show, as in the case of types, that if μ is a Keisler measure over B which does not fork over $A \subseteq B$, then μ has an extension over any $C \supseteq B$ (in particular, over $\bar{M}$) which does not fork over A.

(ii) If $\mu(x)$ is a Keisler measure over a model M, then μ does not fork over M and hence by (i) has a global nonforking extension.

Fact 2.6 (see [4]) Assume that T has NIP. If $\mu(x)$ is a global Keisler measure and M_0 is a small model, then the following are equivalent:

(i) μ does not fork over M_0,

(ii) μ is $\mathrm{Aut}(\bar{M}/M_0)$-invariant,

(iii) μ is Borel definable over M_0.

The meaning of (iii) is that for any L-formula $\varphi(x,y)$, and $b \in \bar{M}$, $\mu(\varphi(x,b))$ depends in a Borel way on $tp(b/M_0)$ in the sense that the function from $S_y(M_0)$ to $[0,1]$ taking $tp(b/M_0)$ to $\mu(\varphi(x,b))$ is Borel. A global measure μ_x satisfying (i) or (ii) or (iii) for some small M_0 is called *invariant*.

At this point we will make a blanket assumption that T has NIP.

Definition 2.7 Let $\mu(x)$ be a global invariant Keisler measure (so it comes equipped with a Borel defining schema over some small model M_0). Let $\lambda(y)$ be any global Keisler measure. Then $\mu(x) \otimes \lambda(y)$ denotes the following global Keisler measure (in variables xy). Let $\varphi(x,y)$ be a formula over $\bar{M}$. Let M be a small model containing M_0 and the parameters from φ; so μ is Borel definable over M. For any type $q(y) \in S(M)$, let $f_{\mu,\varphi}(q) = \mu(\varphi(x,b))$ for some (any) b realizing q. Then define $\mu(x) \otimes \lambda(y)(\varphi(x,y))$ to be $\int_{S_y(M)} f_\varphi(q)\, d(\lambda|M)$, where $\lambda|M$ is the restriction of $\lambda(y)$ to a Keisler measure over M which we identify with a regular Borel probability measure on $S_y(M)$. It is not hard to see that our definition of $(\mu(x) \otimes \lambda(y))(\varphi(x,y))$ above does not depend on the choice of the model M.

Remark 2.8 If $\mu(x)$ and $\lambda(y)$ are both global $\mathrm{Aut}(\bar{M}/M_0)$-invariant measures, then so are $\mu(x) \otimes \lambda(y)$ and $\lambda(y) \otimes \mu(x)$. Moreover from Hrushovski, Pillay, and Simon [6], if at least one of $\mu(x), \lambda(y)$ is generically stable, then $\mu(x) \otimes \lambda(y) = \lambda(y) \otimes \mu(x)$.

From Definition 2.7, we deduce the notion of a "Morley sequence" in μ where $\mu(x)$ is an invariant global type.

Definition 2.9 Let $\mu(x)$ be an invariant global type.

(i) Let $\mu^{(1)}(x_1) = \mu(x_1)$, and for $n >$ let $\mu^{(n)}(x_1, \ldots, x_n) = \mu(x_n) \otimes \mu^{(n-1)} \times (x_1, \ldots, x_{n-1})$.

(ii) Let $\mu^{(\omega)}(x_1, x_2, \ldots) = \bigcup_n \mu^{(n)}(x_1, \ldots, x_n)$.

Definition 2.10 Let $\mu(x)$ be a global Keisler measure, and let M_0 be a small model.

(i) μ is said to be *definable over* M_0 if $\mu(x)$ is $\mathrm{Aut}(\bar{M}/M_0)$-invariant and, moreover, for each $\varphi(x, y) \in L$ (or even in $L(M_0)$) the function taking $tp(b/M_0) \in S_y(M_0)$ to $\mu(\varphi(x, b)) \in [0, 1]$ is continuous.

(ii) $\mu(x)$ is said to be *finitely satisfiable* in M_0 if every formula $\varphi(x)$ with parameters from $\bar{M}$ which has positive μ-measure is realized by an element (i.e., tuple) from M_0.

(iii) $\mu(x)$ is said to be *generically stable* if for some small M_0, $\mu(x)$ is both definable over and finitely satisfiable in M_0.

Fact 2.11 (see [6])

(i) Suppose that $\mu(x)$ is a Keisler measure over a small model M_0 and that some global nonforking extension (i.e., $\mathrm{Aut}(\bar{M}/M_0)$-invariant global extension) $\mu'(x)$ of $\mu(x)$ is generically stable. Then $\mu'(x)$ is the unique global nonforking extension of μ', and μ' is both definable over and finitely satisfiable in M_0.

(ii) Suppose that $\mu(x)$ is a global generically stable Keisler measure. Then there is a model M_0 of cardinality at most $2^{|T|}$ such that μ does not fork over M_0.

Definition 2.12 Let $\mu(x)$ be a Keisler measure over a small model M_0.

(i) We will say that $\mu(x)$ is *generically stable* if some global nonforking extension $\mu'(x)$ is generically stable.

(ii) Suppose that $\mu(x)$ is generically stable (as in (i)), and suppose that $\lambda(y)$ is any Keisler measure over M_0. We define the Keisler measure $\mu(x) \otimes \lambda(y)$ (over M_0 and in variables xy) as follows: For any formula $\varphi(x, y)$ over M_0, $\mu(x) \otimes \lambda(y)(\varphi(x, y)) = \int_{S_y(M_0)} f_{\mu',\varphi}(q)\, d\lambda$ where μ' is the unique global nonforking extension of μ (given by Fact 2.11(i)), and as in Definition 2.7, $f_{\mu',\varphi}(q) = \mu'(\varphi(x, b))$ for some (any) realization b of q.

Remark 2.13 Suppose that M_0 is a small model, $\mu(x)$ is a generically stable measure over M_0 (in the sense of Definition 2.12 (i)), and $\lambda(y)$ is an arbitrary Keisler measure over M_0.

(i) $\mu(x) \otimes \lambda(y)$ (as defined in 2.12(ii)) coincides with $(\mu' \otimes \lambda'(y))|M_0$ (in the sense of Definition 2.7) where μ' is the unique global nonforking extension of μ and λ' is any global extension of λ.

(ii) If $\lambda(y)$ is also generically stable, then $\mu(x) \otimes \lambda(y) = \lambda(y) \otimes \mu(x)$. (This uses Remark 2.8.)

Here is the main new notion in this section.

Definition 2.14 Let M_0 be a small model, let $\mu(x)$ be a generically stable measure over M_0, let $\lambda(y)$ be an arbitrary measure over M_0, and let $\omega(x, y)$ be a measure over M_0 whose restrictions to the x-variables and y-variables, respectively, are $\mu(x)$, $\lambda(y)$. Let $\mu'(x)$ be the unique global nonforking extension of $\mu(x)$. We say that $\omega(x,y)$ is a *strong forking amalgam* of $\mu(x)$ and $\lambda(y)$ *with respect to* $\mu(x)$ if, for some formula $\varphi(x,y)$ over M_0, $\omega(\varphi(x,y)) = 1$, but $\mu'(\varphi(x,b)) = 0$ for all $b \in \bar{M}$.

Let us first remark that for types in stable theories, a strong forking amalgam is simply a forking amalgam (and the reader can check that this also goes through for generically stable types).

Remark 2.15 Suppose that T is stable. Let $p(x), q(y)$, and $r(x,y) \supset p(x) \cup q(y)$ be complete types over a model M_0. Let (a,b) realize $r(x,y)$. Then $r(x,y)$ is a strong forking amalgam of $p(x)$ and $q(y)$ with respect to $p(x)$ if and only if it is a strong forking amalgam of $p(x)$ and $q(y)$ with respect to $q(y)$ if and only if $tp(a/M_0b)$ forks over M_0 (if and only if $tp(b/M_0a)$ forks over M_0).

Proof If $tp(a/M_0b)$ forks over M_0, then $tp(a/M_0b) \neq p|M_0b$ (the unique non-forking extension of p over M_0b), so for some formula $\varphi(x,y)$ over M_0, $\models \varphi(a,b)$ but $\neg\varphi(x,b) \in p|M_0b$. Let $\psi(y)$ over M_0 be the $\varphi(x,y)$-definition of p. So $\models \neg\psi(b)$, whereby the formula $\chi(x,y) : \varphi(x,y) \wedge \neg\psi(y)$ is in $r(x,y)$, and for each $b' \in \bar{M}$, $\neg\chi(x,b') \in p(x)|\bar{M}$. □

Another observation is that in the last clause of Definition 2.14 it suffices to assume that $\mu(\varphi(x,b)) = 0$ for all $b \in M_0$.

Remark 2.16 Let $\mu'(x)$ be a global Keisler measure which is definable over the small model M_0. Let $\varphi(x,y)$ be over M_0. Suppose $\mu'(\varphi(x,b)) = 0$ for all $b \in M_0$; then $\mu'(\varphi(x,b)) = 0$ for all $b \in \bar{M}$.

Proof Suppose for a contradiction that $\mu'(\varphi(x,b)) = r > 0$ for some $b \in \bar{M}$. Let $0 < s < r$. Then $\{b' \in \bar{M} : \mu(\varphi(x,b')) > s\}$ is defined by a disjunction $\bigvee \psi_i(y)$ where the ψ_i are over M_0. Now b satisfies some ψ_i; hence there is $b' \in M_0$ satisfying ψ_i, a contradiction. □

Let us briefly make the connection with the notion of orthogonality of (sets) of measures from Berger [2]. For simplicity fix a topological space X, and let $\mathcal{M}(X)$ be the family of Borel probability measures on X. If $M_1, M_2 \subset \mathcal{M}(X)$ are disjoint, then M_1 is said to be *orthogonal* to M_2 if for some Borel subset B of X, $\mu(B) = 0$ for all $\mu \in M_1$ and $\mu(B) = 1$ for all $\mu \in M_2$. One could restrict one's attention to rather special B such as open, closed, and then say that M_1 and M_2 are orthogonal with respect to open, closed, and so on.

Remark 2.17 Let $\mu(x), \lambda(y)$ be Keisler measures over M_0 with $\mu(x)$ generically stable, and let $\omega(x,y)$ over M_0 extend $\mu(x) \cup \lambda(y)$. Then $\omega(x,y)$ is a strong forking amalgam with respect to $\mu(x)$ if and only if $\{\omega(x,y)\}$ is orthogonal with respect to clopens to the set $\{\mu(x) \otimes \epsilon(y) : \epsilon(y)$ any generically stable measure over $M_0\}$.

Proof That left implies right is immediate. Suppose that $\omega(x,y)(\varphi(x,y) = 1$ but $\mu'(\varphi(x,b)) = 0$ for all $b \in \bar{M}$ (where μ' is the unique global nonforking extension of μ). Then $f_{\mu',\varphi}(q) = 0$ for all $q \in S_y(M_0)$, so from Definition 2.12(ii) we see that $(\mu(x) \otimes \epsilon(y))(\varphi(x,y)) = 0$ for any $\epsilon(y)$ over M_0, generically stable or not.

Conversely, suppose that $\omega(\varphi(x,y)) = 1$ but $\mu(x) \otimes \epsilon(y)(\varphi(x,y)) = 0$ for all generically stable measures $\epsilon(y)$ over M_0. In particular, considering $\epsilon(y)$ of the form $tp(b/M_0)$ for $b \in M_0$, it follows (from Definition 2.12) that $\mu(\varphi(x,b)) = 0$

for all $b \in M_0$. By Remark 2.16 this implies that $\mu'(\varphi(x,b)) = 0$ for all $b \in \bar{M}$, so $\omega(x,y)$ is a strong forking amalgam with respect to $\mu(x)$. □

We are not sure of the status of the following question. A positive answer would make the theory we develop here more robust.

Question 2.18 Suppose that $\mu(x), \lambda(y)$, and $\omega(x,y) \supset \mu(x) \cup \lambda(y)$ are all generically stable measures over M_0. Is it the case that $\omega(x,y)$ is a strong forking amalgam of $\mu(x)$ and $\lambda(y)$ with respect to $\mu(x)$ if and only if $\omega(x,y)$ is a strong forking amalgam of $\mu(x)$ and $\lambda(y)$ with respect to $\lambda(y)$?

Finally in this section we state a couple of results which will play important roles in the proof of Theorem 1.1. First recall the notion *weakly random*.

Definition 2.19 Let $\mu(x)$ be a Keisler measure over M (where now x may be an infinite tuple of variables, and M may be the "monster model" $\bar{M}$).

(i) A complete type $p(x) \in S_x(M)$ is said to be *weakly random for* $\mu(x)$ if every formula in p has positive μ-measure.
(ii) Assuming that M is a small model, then a tuple c (of appropriate length) is said to be *weakly random over* M *for* μ if $tp(c/M)$ is weakly random for μ.

The first result is the following lemma.

Lemma 2.20 *Suppose that* $\mu(x)$ *is a global generically stable measure and* $\varphi(x,y)$ *is a formula over* $\bar{M}$. *Then the following are equivalent:*

(i) $\mu(\varphi(x,b)) = 0$ *for all* $b \in \bar{M}$,
(ii) *for some* n, $\mu^{(n)}\big(\exists y\ (\varphi(x_1,y) \wedge \cdots \wedge \varphi(x_n,y))\big) = 0$,
(iii) *for some* n, *for any weakly random type* $p(x)$ *for* μ, $p^{(n)}(x_1,\ldots,x_n)$ *implies* $\neg\exists y\ (\varphi(x_1,y) \wedge \cdots \wedge \varphi(x_n,y))$.

Proof That (i) implies (ii) is due to Hrushovski, Pillay, and Simon [5, Proposition 2.1]. That (ii) implies (iii) is [5, Lemma 1.2]. And that (iii) implies (i) is immediate. (If $\mu(\varphi(x,b)) > 0$, let $p(x)$ be a weakly random type for μ containing $\varphi(x,b)$. Then $\varphi(x_1,b) \wedge \cdots \varphi(x_n,b) \in p^{(n)}(x_1,\ldots,x_n)$; hence (iii) fails.) □

The second result is the following proposition.

Proposition 2.21 *Suppose that* $\mu_1(y_1),\ldots,\mu_n(y_n)$ *are global Keisler measures, all invariant over a small model* M_0. *Let* $\mu(y_1,\ldots,y_n)$ *be the nonforking product* $\mu_1 \otimes \cdots \otimes \mu_n$. *Let* $B(y_1,\ldots,y_n)$ *be a Borel set over* M_0 *with* μ*-measure* 1. *Then there are sequences* $I_\alpha = (b_i^\alpha : i < \omega)$ *for* $\alpha = 1,\ldots,n$ *such that*

(i) *each* I_α *is weakly random for* $(\mu_\alpha)^{(\omega)}|M_0$,
(ii) *for all* $(c_1,\ldots,c_n) \in I_1 \times \cdots \times I_n$, $(c_1,\ldots,c_n) \in B$.

Proof We argue by induction on n. For $n = 1$, let x be the variable y_1. Then the intersection of all the $B(x_i)$ for $i < \omega$ and the closed set consisting of the intersection of all M_0-definable sets of $\mu_1^{(\omega)}$-measure 1, is a Borel subset of the type space over M_0 in variables $(x_1,x_2,\ldots)$ of $\mu_1^{(\omega)}$-measure 1 and hence contains a point, and any realization is the required I_1.

Assume this is true for n. Let $B(y_1,\ldots,y_{n+1})$ be a Borel set over M_0 of μ measure 1, where $\mu = \mu_1 \otimes \cdots \otimes \mu_{n+1}$. By Borel definability of invariant measures, and the definition of the nonforking product measure, $\{(c_2,\ldots,c_{n+1}) : \mu_1(B(y_1,c_2,$

$\ldots, c_{n+1})) = 1\}$ is a Borel set $C(y_2, \ldots, y_{n+1})$ over M_0 of $(\mu_2 \otimes \cdots \otimes \mu_{n+1})$-measure 1. By induction hypothesis we find $I_2, \ldots, I_{n+1}$ satisfying (i) and (ii) of the proposition for C in place of B. Now again let x be the variable y_1. Consider the countable set of conditions $B(x_i, c_2, \ldots, c_{n+1})$ for $i < \omega$ and $(c_2, \ldots, c_{n+1}) \in I_2 \times \cdots \times I_{n+1}$. The intersection of all of these is a Borel set in variables $(x_1, x_2, \ldots)$ which has $\mu_1^{(\omega)}$-measure 1. The intersection of this with the set of all formulas over M_0 of $\mu_1^{(\omega)}$-measure 1 again has a point, which is the required I_1. □

3 Average Measures

One direction of the proof of Theorem 1.1 will make heavy use of a special class of generically stable measures, which we call *average measures* and which were introduced in [6]. So we will give the definition again here and record a few facts concerning nonforking products (or amalgams) which will be needed later.

Definition 3.1 By an indiscernible segment we mean something of the form $\{a_i : i \in [0,1]\}$ which is indiscernible with respect to the usual ordering on $[0,1]$.

As pointed out in [6], such an indiscernible segment I gives rise to a global generically stable measure μ_I. For any formula (with parameters) $\varphi(x)$ the set of $i \in [0,1]$ such that $\models \varphi(a_i)$ is a finite union of intervals and points and so has a Lebesgue measure, which we define to be $\mu_I(\varphi(x))$. Noting that μ_I is both finitely satisfiable in and definable over I, we see that μ_I is a global generically stable measure, which is, moreover, by [6, Proposition 3.3], the unique nonforking extension of $\mu_I|I$.

Definition 3.2

(i) By a global average measure we mean something of the form μ_I for I an indiscernible segment.

(ii) For M_0 a small model, by an average measure over M_0 we mean something of the form $\mu_I|M_0$ where μ_I is a global average measure which does not fork over M_0 (or is $\mathrm{Aut}(\bar{M}/M_0)$-invariant).

Remark 3.3 A generically stable *type* is the same thing as an average measure which happens to be a type.

We now introduce some data and notation relevant for the proposition below. Let us suppose that for $\alpha < \kappa$, $I_\alpha = (b_i^\alpha : i \in [0,1])$ is an indiscernible segment and that the I_α's are *mutually indiscernible* in the sense that each I_α is indiscernible over $\bigcup_{\beta \neq \alpha} I_\beta$. For $i \in [0,1]$ let c_i be the sequence $(b_i^\alpha : \alpha < \kappa)$. It is then easy to see that $K = (c_i : i \in [0,1])$ is also an indiscernible segment (of possibly infinite tuples if $\kappa \geq \omega$). So we have the average measure μ_K, as well as the average measures μ_{I_α} for each α. As one might expect, with these assumptions and notation we have the following.

Proposition 3.4 *μ_K (in variables $(x_\alpha : \alpha < \kappa)$) is the nonforking product $\bigotimes_{\alpha<\kappa} \mu_{I_\alpha}(x_\alpha)$ of the $\mu_{I_\alpha}(x_\alpha)$.*

Proof It is clearly enough to prove the proposition when $\kappa = 2$ (e.g., by finite character together with induction). So let us rename I_0 as I and I_1 as J, as well as renaming x_0 as x and x_1 as y. Also let us write I as $(a_i : i \in [0,1])$, and let $J = (b_i : i \in [0,1])$. We still let c_i denote (a_i, b_i).

We aim to prove that $\mu_K(x, y)|K$ coincides with $(\mu_I(x) \otimes \mu_J(y))|K$. As both global measures μ_K and $\mu_I \otimes \mu_J$ are generically stable and K-invariant, it will then follow from [4, Proposition 3.3] that $\mu_K = \mu_I \otimes \mu_J$.

So let us fix a formula $\varphi(x, y, c)$ over K where c witnesses the parameters in φ and without loss of generality $c = (c_{i_1}, \ldots, c_{i_k})$ with $i_1 < i_2 < \cdots < i_k \in [0, 1]$.

Claim 1 *Let $i \neq i_1, \ldots, i_k$. Then either*

(a) *for all $j \in [0, 1]$ except possibly $i_1, \ldots, i_k$ we have $\models \varphi(a_j, b_i, c)$, or*
(b) *for all $j \in [0, 1]$ except possibly $i_1, \ldots, i_k$ we have $\models \neg\varphi(a_j, b_i, c)$.*

Proof This holds by indiscernibility of I over J. □

Claim 2 $\mu_J(\{b \in \bar{M} : 0 < \mu_I(\varphi(x, b, c)) < 1\}) = 0.$

Proof Note that by definability of μ_I over I, $\{b \in \bar{M} : 0 < \mu(\varphi(x, b, c)) < 1\}$ is defined by a disjunction $\bigvee_{\theta\in\Theta} \theta(y)$ of formulas $\theta(y)$ over I. If by way of contradiction some $\theta \in \Theta$ has μ_J-measure > 0, then by definition of μ_J, there are infinitely many $i \in [0, 1]$ such that $\models \theta(b_i)$. For each such i, $\mu_I(\varphi(x, b_i, c)) \neq 0, 1$. On the other hand, we know that $\mu_I(\varphi(x, b_i, c))$ is the Lebesgue measure of $\{j \in [0, 1] :\models \varphi(a_j, b_i, c)\}$. We clearly have a contradiction to Claim 1. □

By Claim 2 and the definition of the product measure, $(\mu_I \otimes \mu_J)(\varphi(x, y, c)) = \mu_J(\{b \in \bar{M} : \mu_I(\varphi(x, b, c)) = 1\})$. Now $Z = \{b \in \bar{M} : \mu_I(\varphi(x, b, c)) = 1\}$ is type definable over Ic by definability of μ_I over I, say, by $\bigwedge_{\psi\in\Psi} \psi(y)$, where each $\psi(y)$ is over Ic. Now $\mu_J(\psi(y))$ is the Lebesgue measure of $\{i \in [0, 1] :\models \psi(b_i)\}$, and by indiscernibility of J over I, for $i \neq j_1, \ldots, j_k$, whether or not $\models \psi(b_i)$ depends on the order type of i with respect to $j_1, \ldots, j_k$ in $[0, 1]$. In any case, we see that $\mu_J(Z)$ equals the Lebesgue measure of $\{i \neq j_1, \ldots, j_k : \mu(\varphi(x, b_i, c)) = 1\}$ which is moreover a union of intervals with endpoints from $0, j_1, \ldots, j_k, 1$. By Claim 1, this coincides with the Lebesgue measure of $\{i \neq j_1, \ldots, j_k :\models \varphi(a_i, b_i, c)\}$ which by definition of $\mu_K(x, y)$ is precisely $\mu_K(\varphi(x, y, c))$. We have shown that $\mu_K|K$ coincides with $(\mu_I \otimes \mu_J)|K$, which proves the proposition. □

Finally, for the record we note the obvious.

Lemma 3.5 *Suppose that $I = (a_i : i \in [0, 1])$ is an indiscernible segment over A. Let $\varphi(x, y)$ be a formula over A. Then the following are equivalent:*

(i) $\mu_I(\varphi(x, b)) = 0$ *for all* $b \in \bar{M}$,
(ii) *for some n, for some (any) distinct $i_1, \ldots, i_n \in [0, 1]$,* $\models \neg\exists y(\varphi(a_{i_1}, y) \wedge \cdots \wedge \varphi(a_{i_n}, y))$.

Proof (ii) implies (i): If for some b, $\mu_I(\varphi(x, b)) > 0$, then for infinitely many $i \in [0, 1]$, $\models \varphi(a_i, b)$, so clearly (ii) fails.

(i) implies (ii): If (ii) fails, then by compactness there is b such that $\models \varphi(a_i, b)$ for infinitely many i; hence $\mu_I(\varphi(x, b)) > 0$. □

4 Proof of Theorem 1.1

We start with the following.

Proof of (1) implies (2) Assume (1). By Fact 2.3, there are $\varphi_\alpha(x, y_\alpha) \in L$ for $\alpha < \omega$, b_i^α for $\alpha < \omega$ and $i < \omega$, and $k_\alpha < \omega$ for each $\alpha < \omega$ such that

(i) for each α, $\{\varphi_\alpha(x, b_i^\alpha) : i < \omega\}$ is k_α-inconsistent,
(ii) for each "path" $\eta \in \omega^\omega$, $\{\varphi_\alpha(x, b_{\eta(\alpha)}^\alpha) : \alpha < \omega\}$, and
(iii) for each α, the sequence $(b_i^\alpha : i < \omega)$ is indiscernible over $\bigcup_{\beta\neq\alpha}\{b_i^\beta : i < \omega\}$.

By compactness we may find b_i^α for $\alpha < \omega$ and $i \in [0, 1]$ satisfying the analogues of (i), (ii), and (iii). So in (i) we now have $\eta \in [0, 1]^\omega$, and in (iii) we have mutually indiscernible *segments*. For each $i \in [0, 1]$ let c_i be the sequence $(b_i^\alpha : \alpha < \omega)$. So $(c_i : i \in [0, 1])$ is an indiscernible segment (of infinite tuples). For each i let d_i realize $\{\varphi_\alpha(x, b_i^\alpha) : \alpha < \omega\}$, and let e_i be the sequence $(d_i, b_i^\alpha)_\alpha$ (i.e., $(d_i, b_i^0, b_i^1, \ldots)$). Clearly we man assume that $(e_i : i \in [0, 1])$ is also an indiscernible segment (of infinite tuples).

Now let I_α denote $(b_i^\alpha : i \in [0, 1])$, let K denote $(c_i : i \in [0, 1])$, and let J denote $(e_i : i \in [0, 1])$. Let M_0 be any model containing J. Let $\omega(x, y_0, y_1, \ldots) = \mu_J$, let $\nu(y_0, y_1, \ldots) = \mu_K$, and for each $\alpha < \omega$ let $\mu_\alpha = \mu_{I_\alpha}$. These are all global average (so generically stable) measures, which are M_0-invariant. Clearly the restriction of ω to $(y_0, y_1, \ldots)$ is ν and the restriction of ν to each y_α is μ_α. Let $\lambda(x)$ be the restriction of ω to x, and for each α let $\omega_\alpha(x, y_\alpha)$ be the restriction of ω to (x, y_α).

Claim 1 *For each α, $\omega(\varphi_\alpha(x, y_\alpha)) = 1$, and hence $\omega_\alpha(\varphi_\alpha(x, y_\alpha)) = 1$.*

Proof This holds because $\models \varphi_\alpha(d_i, b_i^\alpha)$ for all i. □

Claim 2 *For each $\alpha < \omega$, $\mu_\alpha(\varphi(d, y_\alpha) = 0$ for all $d \in \bar{M}$.*

Proof This is by Lemma 3.5 and the fact that $\{\varphi(x, b_i^\alpha) : i \in [0, 1]\}$ is k_α-inconsistent. □

Claim 3 *We have that $\nu(y_0, y_1, \ldots)$ is $\bigotimes_\alpha \mu_\alpha(y_\alpha)$ (and this is also true for the restrictions of these measures to M_0).*

Proof This holds by Proposition 3.4 and the mutual indiscernibility of the I_α's. □

By Claims 1 and 2, for each $\alpha < \omega$, $\omega|M_0$ is a strong forking amalgam (of $\lambda|M$ and $\mu_\alpha|M$) with respect to $\mu_\alpha|M$. Together with Claim 3, this yields Theorem 1.1(2).

Proof of (2) implies (1) Let M_0, $\omega(x, y_0, y_1, \ldots)$, $\mu_\alpha(y_\alpha)$, and so on, be as in the statement of Theorem 1.1(2). For each $\alpha < \omega$ let $\varphi_\alpha(x, y_\alpha)$ be a formula over M_0 witnessing that $\omega_\alpha(x, y_\alpha)$ is a strong forking amalgam of $\lambda(x)$ and $\mu_\alpha(y_\alpha)$ with respect to μ_α; namely, $\omega(\varphi_\alpha(x, y_\alpha)) = 1$, but $(\mu_\alpha|\bar{M})(\varphi_\alpha(d, y_\alpha)) = 0$ for all $d \in \bar{M}$ (or, equivalently, for all $d \in M_0$).

The assumption that (2) holds gives generically stable measures $\mu_\alpha(y_\alpha)$ over M_0 for $\alpha < \omega$ and $\omega(x, y_0, y_1, \ldots)$ over M_0 extending $\bigotimes_\alpha \mu_\alpha$ such that the restriction ω_α of ω to (x, y_α) is a strong forking extension of $\mu_\alpha(y_\alpha)$ for all α. By Lemma 2.20, for each $\alpha < \omega$ let $k_\alpha < \omega$ be such that

(*) $\mu_\alpha^{(k_\alpha)}\big(\exists x\ (\varphi_\alpha(x, y_{\alpha,1}) \wedge \cdots \wedge \varphi_\alpha(x, y_{\alpha,k_\alpha}))\big) = 0$.

Let us now fix $N < \omega$. Let $\mu(y_0, \ldots, y_N)$ be the restriction of ω to $y_0, \ldots, y_N$ which we know to be $\mu_0(y_0) \otimes \cdots \otimes \mu_N(y_N)$. So as $\omega(\varphi_0(x, y_0) \wedge \cdots \wedge \varphi_N(x, y_N)) = 1$, it follows that $\mu\big(\exists x\ (\varphi_0(x, y_0) \wedge \cdots \wedge \varphi_N(x, y_N))\big) = 1$. By Proposition 2.21 (where here the Borel set $B(y_0, \ldots, y_N)$ is the one defined by

$\exists x\ (\varphi_0(x, y_0) \wedge \cdots \wedge \varphi_N(x, y_N)))$, there are weakly random $I_\alpha = (b_i^\alpha : i < \omega)$ for μ_α over M_0, for $\alpha = 0, \dots, N$ such that

(**) for all $(c_0, \dots, c_N) \in I_0 \times \cdots \times I_N$ we have $\models \exists x(\varphi_0(x, c_1) \wedge \cdots \wedge \varphi_N(x, c_N))$.

By (*) we have

(***) for each $\alpha = 0, \dots, N$, $\varphi_\alpha(x, b_{i_1}^\alpha) \wedge \cdots \wedge \varphi_\alpha(x, b_{i_{k_\alpha}}^\alpha)$ is inconsistent, for all $i_1 < \cdots < i_{i_{k_\alpha}}$.

Now (**), (***), and compactness yield the failure of Fact 2.3(2), whereby T is not strongly dependent. This completes the proof of Theorem 1.1.

5 Final Remarks and Questions

A weakness in the theory developed here is the status of "strong forking amalgams" and in particular that Question 2.18 probably has a negative answer. Nevertheless, the theory as it stands gives rise to obvious notions of preweight and weight for a generically stable measure $\lambda(x)$. For $\lambda(x)$, a generically stable measure over a model M_0, the preweight of λ is defined to be the supremum of κ such that there exists generically stable $\omega(x, y_\alpha)_{\alpha<\kappa}$ over M_0 such that the restriction of ω to $(y_\alpha)_{\alpha<\kappa}$ is the nonforking product of the restrictions μ_α to each y_α and where we have strong forking of $\omega_\alpha(x, y_\alpha)$ with respect to y_α (with the obvious notation).

Question 5.1 Suppose that T is strongly dependent. Does every generically stable measure have finite weight?

Another obvious question raised by the work concerns the relationship between generically stable measures and average measures in a NIP theory. In the stable case, any Keisler measure is a weighted average of some of its weakly random types. (Strictly speaking we should consider here rather φ-measures, for $\varphi(x, y)$ a fixed L-formula.) Is there a similar relation between a generically stable measure and various average measures obtained from its weakly random types?

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