Notre Dame Journal of Formal Logic
Volume 60, Number 1, 2019

Π^0_1-Encodability and Omniscient Reductions

Benoit Monin and Ludovic Patey

Abstract A set of integers A is computably encodable if every infinite set of integers has an infinite subset computing A. By a result of Solovay, the computably encodable sets are exactly the hyperarithmetic ones. In this article, we extend this notion of computable encodability to subsets of the Baire space, and we characterize the Π^0_1-encodable compact sets as those which admit a nonempty Σ^1_1-subset. Thanks to this equivalence, we prove that weak weak König's lemma is not strongly computably reducible to Ramsey's theorem. This answers a question of Hirschfeldt and Jockusch.

1 Introduction

A set $A \subseteq \omega$ is *computably encodable* if every infinite set $X \subseteq \omega$ has an infinite subset computing A. Jockusch and Soare [16] introduced various notions of encodability, and Solovay [28] characterized the computably encodable sets as the hyperarithmetical ones. We extend the notion of computable encodability to collections of sets as follows. A set $\mathcal{C} \subseteq \omega^\omega$ is Π^0_1-*encodable* if every infinite set $X \subseteq \omega$ has an infinite subset Y such that $\mathcal{C}$ admits a nonempty Y-computably bounded $\Pi^{0,Y}_1$-subset $\mathcal{D} \subseteq \omega^\omega$. By this, we mean that $\mathcal{D} = [T]$ for some Y-computable tree T whose nodes are bounded by a Y-computable function. Our main result asserts that the compact sets that are Π^0_1-encodable are exactly those admitting a nonempty Σ^1_1-subset. This extends Solovay's theorem, as the members of the Σ^1_1-singletons and those of the computably bounded Π^0_1-singletons are exactly the hyperarithmetic ones (see Spector [29]) and the computable ones, respectively. Our motivations follow two axes.

First, the development of *mass problems* such as Muchnik and Medvedev degrees (see Hinman [11]) revealed finer computational behaviors than those captured by

Received March 3, 2016; accepted October 31, 2016
First published online January 18, 2019
2010 Mathematics Subject Classification: Primary 03D30; Secondary 03D65
Keywords: reverse mathematics, computable encodability, Ramsey's theory, computable reduction, higher recursion theory
 10.1215/00294527-2018-0020

the Turing degrees. For example, the cone avoidance basis theorem (see Jockusch and Soare [15]) asserts that the PA degrees are of no help in computing a single incomputable set of integers. However, it would be simplistic to deduce that PA degrees carry no computational power. For example, they enable one to compute separating sets given two computably inseparable c.e. sets. This work can therefore be seen as part of a program of extending core computability-theoretic theorems about Turing degrees to their generalized statements about mass problems.

Our second motivation comes from the reverse mathematics and computable analysis of Ramsey's theorem. Computable encodability is a very important feature of Ramsey's theorem, as for every k-coloring of $[\omega]^n$ and every infinite set X there is an infinite homogeneous subset contained in X. Computable encodability provides a formal setting to many intuitions about the computational weakness of Ramsey's theorem. In particular, we use this notion to answer a question asked by Hirschfeldt and Jockusch [12] about the link between variants of König's lemma and Ramsey's theorem over strong computable reducibility.

1.1 Reductions between mathematical problems A *mathematical problem* P is specified by a collection of *instances*, coming together with a collection of *solutions*. Many ordinary theorems can be seen as mathematical problems. For example, König's lemma (KL) asserts that every infinite, finitely branching tree admits an infinite path. In this setting, an instance of KL is an infinite, finitely branching tree T, and a solution to T is any infinite path $P \in [T]$.

There are many ways to compare the strength of mathematical problems. Among them, reverse mathematics study their logical consequences (see Simpson [26]). More recently, various notions of effective reductions have been proposed to compare mathematical problems, namely, Weihrauch reductions (see Brattka and Rakotoniaina [1] and Dorais, Dzhafarov, Hirst, Mileti, and Shafer [4]), computable reductions (see Hirschfeldt and Jockusch [12]), and computable entailment (see Shore [24]), among others. A problem P is *computably reducible* to another problem Q (written $\mathsf{P} \leq_c \mathsf{Q}$) if every P-instance I computes a Q-instance J such that every solution to J computes relative to I a solution to I. A problem P is *Weihrauch-reducible* to Q (written $\mathsf{P} \leq_W \mathsf{Q}$) if, moreover, this computable reduction is witnessed by two fixed Turing functionals. There exist strong variants of computable and Weihrauch reductions written $\mathsf{P} \leq_{\mathrm{sc}} \mathsf{Q}$ and $\mathsf{P} \leq_{sW} \mathsf{Q}$, respectively, where no access to the P-instance I is allowed in the backward reduction. In this article, we will focus on strong computable reduction.

Due to the range of potential definitions of effective reductions, there is a need to give a justification about the choices of the definition. An effective reduction from P to Q should reflect some computational aspect of the relationship between P and Q. The more precise the reduction is, the more insights it gives about the links between the two problems. As it happens, many proofs that P is not strongly computably reducible to Q actually produce a single P-instance I such that, for every Q-instance J, computable in I or not, there is a solution to J computing no solution to I. Such a relation suggests a deep structural difference between the problems P and Q, in that even with a perfect knowledge of I, there is no way to encode enough information in the Q-instance to solve I. We will therefore define P to be *strongly omnisciently computably reducible* to Q (written $\mathsf{P} \leq_{\mathrm{soc}} \mathsf{Q}$) if, for every P-instance I, there is a Q-instance J such that every solution to J computes a solution to I.

1.2 König's lemma and Ramsey's theorem König's lemma and Ramsey's theorem are core theorems from mathematics, both enjoying a special status in reverse mathematics.

Definition 1.1 (Various König lemmas) KL is the statement "Every infinite finitely-branching tree has an infinite path." WKL is the restriction of KL to binary trees. WWKL is the restriction of WKL to binary trees of positive measure (a binary tree $T \subseteq 2^{<\omega}$ has *positive measure* if $\lim_s \frac{|\{\sigma \in T : |\sigma| = s\}|}{2^s} > 0$).

Weak König's lemma captures compactness arguments and naturally arises from the study of ordinary theorems (see [26]). It is part of the so-called *Big Five* (see Montalbán [17]). On the other hand, weak weak König's lemma can be thought of as asserting the existence of randomness in the sense of Martin-Löf (see Downey and Hirschfeldt [5]). Although weak König's lemma is strictly weaker than König's lemma in reverse mathematics and over computable reducibility, the statements are trivially equivalent over strong omniscient computable reducibility. Indeed, given any problem P admitting an instance with at least one solution S, one can define a binary tree whose unique path is a binary coding of S. In particular, KL $\leq_{\mathrm{soc}}$ WKL. Weak weak König's lemma, for it, remains strictly weaker than König's lemma over strong omniscient computable reducibility, since the measure of the set of oracles computing a noncomputable set is null (see Sacks [21]). Therefore one can choose any tree with a unique incomputable path as an instance of König's lemma to show that KL $\not\leq_{\mathrm{soc}}$ WWKL.

Definition 1.2 (Ramsey's theorem) A subset H of ω is *homogeneous* for a coloring $f : [\omega]^n \to k$ (or f*-homogeneous*) if each n-tuples over H are given the same color by f. Moreover, RT^n_k is the statement "Every coloring $f : [\omega]^n \to k$ has an infinite f-homogeneous set," $\mathsf{RT}^n_{<\infty}$ is $(\forall k)\,\mathsf{RT}^n_k$, and RT is $(\forall n)\,\mathsf{RT}^n_{<\infty}$.

Ramsey's theorem received a lot of attention in reverse mathematics since it is one of the first examples of statements escaping the Big Five phenomenon. There is a profusion of literature around the reverse mathematics and computable analysis of Ramsey's theorem (see, e.g., Cholak, Jockusch, and Slaman [2], Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [13], Jockusch [14], Seetapun and Slaman [23]). In particular, RT^n_k is equivalent to KL in reverse mathematics for any standard $n \geq 3$ and $k \geq 2$ (see [26]), and RT^2_k is strictly in between RCA_0 and RT^3_k (see [23]). More recently, there have been studies of Ramsey's theorem under various notions of reducibility. Let SRT^2_k denote the restriction of RT^2_k to *stable* colorings, that is, functions $f : [\omega]^2 \to k$ such that $\lim_s f(x, s)$ exists for every x. In what follows, $k \geq 2$. Brattka and Rakotoniaina [1] and Hirschfeldt and Jockusch [12] studied the Weihrauch degrees of Ramsey's theorem and independently proved that $\mathsf{RT}^1_{k+1} \not\leq_W \mathsf{SRT}^2_k$ and $\mathsf{RT}^n_{<\infty} \leq_{sW} \mathsf{RT}^{n+1}_2$. Note that the reduction $\mathsf{RT}^1_k \leq_{sW} \mathsf{SRT}^2_k$ trivially holds.

From the point of view of omniscient reductions, the above discussion about weak König's lemma shows that RT $\leq_{\mathrm{soc}}$ WKL. Dzhafarov and Jockusch [6] proved that $\mathsf{SRT}^2_2 \not\leq_{\mathrm{soc}} \mathsf{RT}^1_{<\infty}$. Hirschfeldt and Jockusch [12] and Patey [20] independently proved that $\mathsf{RT}^1_{k+1} \not\leq_{\mathrm{soc}} \mathsf{RT}^1_k$, a result later strengthened by Dzhafarov, Patey, Solomon, and Westrick [7], who proved that $\mathsf{RT}^1_{k+1} \not\leq_{\mathrm{soc}} \mathsf{SRT}^2_k$ and that $\mathsf{RT}^2_2 \not\leq_{\mathrm{soc}} \mathsf{SRT}^2_{<\infty}$. Some differences between strong computable reducibility and strong omniscient computable reducibility are witnessed by Ramsey's theorem. For example, the second author [20] proved that $\mathsf{SRT}^2_{k+1} \not\leq_{\mathrm{sc}} \mathsf{RT}^2_k$, while we can prove the following theorem.

Theorem 1.3 $\mathsf{SRT}^2_{<\infty} \leq_{\mathrm{soc}} \mathsf{RT}^2_2$.

Proof Given a stable coloring $f : [\omega]^2 \to k$, and $x < y$, let $g(x, y) = 1$ if and only if $f(x, y) = \lim_s f(y, s)$.

We first claim that every infinite g-homogeneous set A is for color 1. Indeed, suppose it is for color 0. Let $x \in A$, and let $i, s_0 \in \omega$ be such that $f(x, s) = i$ for every $s > s_0$. For every $y \in A$ such that $y > s_0$, $f(x, y) \neq \lim_s f(y, s)$ since A is g-homogeneous for color 0. But $f(x, y) = i$ since $y > s_0$, so $\lim_s f(y, s) \neq i$, and this for almost every $y \in A$. By iterating the argument, we prove that f uses an unbounded number of colors, which is a contradiction.

We next claim that every A is homogeneous for f. Let $x \in A$, and let $i = \lim_s f(x, s)$. We will show that, for almost every $y \in A$, $\lim_s f(y, s) = i$. Indeed, for almost every $y \in A$, $f(x, y) = \lim_s f(x, s) = i$, and since A is g homogeneous for color 1, $f(x, y) = \lim_s f(y, s)$, and so $\lim_s f(y, s) = i$. Now suppose $\lim_s f(x, s) = j \neq i$ for some $x \in A$. The same argument shows that for almost every $y \in A$ we have $\lim_s f(y, s) = j \neq i$, which is a contradiction. Thus for every $x \in A$ we have $\lim_s f(x, s) = i$. As A is homogeneous for color 1, for every $x < y \in A$ we have $f(x, y) = \lim_s f(y, s) = i$. □

Hirschfeldt and Jockusch compared Ramsey's theorem and König's lemma over strong omniscient computable reducibility and proved that $\mathsf{RT}^1_2 \not\leq_{\mathrm{soc}} \mathsf{WWKL}$ and that $\mathsf{WKL} \not\leq_{\mathrm{soc}} \mathsf{RT}$. They asked whether weak weak König's lemma is a consequence of Ramsey's theorem over strong computable reducibility. We answer negatively by proving the stronger separation $\mathsf{WWKL} \not\leq_{\mathrm{soc}} \mathsf{RT}$.

1.3 Background in higher computability We use in the article several tools from higher computability or effective descriptive set theory that we sum up here. More details on the following well-known definitions and theorems can be found in Sacks [22], Chong and Yu [3], and Moschovakis [18], among others.

Definition 1.4 A subset of ω^ω is Σ^1_1 if it is definable by a formula of arithmetic with quantification over ω or over ω^ω, such that the quantifications over ω^ω are only existential (and not preceded by a negation).

Theorem 1.5 (Kleene normal form) *A set $\mathcal{A} \subseteq \omega^\omega$ is Σ^1_1 if and only if there exists a computable predicate $R \subseteq \omega^\omega \times \omega^\omega \times \omega$ such that*

$$\mathcal{A} = \{X : \exists Y \, \forall z \, R(X, Y, z)\}.$$

In the following definition, W_e denotes the eth c.e. set, and $\langle \, , \rangle$ denotes a computable pairing function.

Definition 1.6 We denote by Kleene's O the set of codes e such that the relation $n <_e m$ if and only if $\langle n, m\rangle \in W_e$ is a well-order. An ordinal is computable if it is the order-type of such a well-order defined by some W_e.

For any set X, it is possible to iterate the jump of X in the Turing degrees, through the computable ordinals in a rather straightforward way. For a computable ordinal α, let $X^{(\alpha)}$ denote the Turing degree of the αth iteration of the Turing jump of X.

Definition 1.7 A set $A \subseteq \omega$ is *hyperarithmetic* if it is Turing below $\emptyset^{(\alpha)}$ for some computable ordinal α. It is hyperarithmetic in X if it is below $X^{(\alpha)}$ for some ordinal α computable in X.

A *basis* for the Σ^1_1-sets is a collection of sets $\mathcal{B} \subseteq \omega^\omega$ such that $\mathcal{B} \cap \mathcal{D} \neq \emptyset$ for every nonempty Σ^1_1-set $\mathcal{D} \subseteq \omega^\omega$. The following two basis theorems are well known.

Theorem 1.8 (Gandy, Kreisel, and Tait [9]) *If A is not hyperarithmetic, then every nonempty Σ^1_1-set $D \subseteq \omega^\omega$ has a member X such that A is not hyperarithmetic in X.*

Theorem 1.9 *The sets Turing below Kleene's O are a basis for the nonempty Σ^1_1-subsets of ω^ω.*

In particular, we will prove and use in this article (see Theorem 2.3) an extension of Theorem 1.8. Finally, the *Gandy–Harrington topology on ω^ω* is the topological space ω^ω whose basic open sets are the Σ^1_1-sets. We will use the following theorem.

Theorem 1.10 *The space ω^ω with the Gandy–Harrington topology is a Baire space; that is, a countable intersection of dense open sets is dense.*

1.4 Notation Given a set A and some integer $n \in \omega$, we let $[A]^n$ denote the collection of all unordered subsets of A of size n. Accordingly, we let $A^{<\omega}$ and $[A]^\omega$ denote the collection of all finite and infinite subsets of A, respectively. Given $a \in [\omega]^{<\omega}$ and $X \in [\omega]^\omega$ such that $\max a < \min X$, we let $\langle a, X\rangle$ denote the set of all $B \in [\omega]^\omega$ such that $a \subseteq B \subseteq a \cup X$. The pairs $\langle a, X\rangle$ are called *Mathias conditions* and form, together with $\emptyset$, the basic open sets of the *Ellentuck topology*.

Given a function $f \in \omega^\omega$ and an integer $t \in \omega$, we write f^t for the set of all strings $\sigma \in \omega^{<\omega}$ of length t such that $(\forall x < t)\sigma(x) \leq f(x)$. Accordingly, we write $f^{<\omega}$ for $\bigcup_{t\in\omega} f^t$.

2 Main Result

A function $f \in \omega^\omega$ is a *Π^0_1-modulus* of a set $\mathcal{C} \subseteq \omega^\omega$ if $\mathcal{C}$ has a nonempty g-computably bounded $\Pi^{0,g}_1$-subset for every function $g \geq f$. A function $f \in \omega^\omega$ is a modulus of a set $A \in \omega^\omega$ if $g \geq_T A$ for every $g \geq f$. Note that the notion of Π^0_1-modulus of the singleton $\{A\}$ coincides with the existing notion of modulus of the set A since the members of computably bounded Π^0_1-singletons are computable. The purpose of this section is to prove the following main theorem.

Theorem 2.1 *Fix a set $\mathcal{C} \subseteq \omega^\omega$ compact in the product topology. The following are equivalent:*

(i) *$\mathcal{C}$ is Π^0_1-encodable,*
(ii) *$\mathcal{C}$ admits a Π^0_1-modulus,*
(iii) *$\mathcal{C}$ has a nonempty Σ^1_1-subset.*

Proof (ii)$\Rightarrow$(i): Let f be a Π^0_1-modulus of $\mathcal{C}$. For every set $X \in [\omega]^\omega$, there is a set $Y \in [X]^\omega$ such that $p_Y \geq f$, where $p_Y(x)$ is the xth element of Y in increasing order. In particular, $\mathcal{C}$ has a nonempty $\Pi^{0,Y}_1$-subset. (iii)$\Rightarrow$(ii): Let $R(X, Y, z)$ be a computable predicate such that $\mathcal{D} = \{X \in \omega^\omega : (\exists Y \in \omega^\omega)(\forall z)R(X, Y, z)\}$ is a nonempty subset of $\mathcal{C}$. Since $\mathcal{D} \neq \emptyset$, there are some $X, Y \in \omega^\omega$ such that $R(X, Y, z)$ holds for every $z \in \omega$. We claim that the function f defined by $f(x) = \max(X(x), Y(x))$ is a Π^0_1-modulus of $\mathcal{C}$. To see this, pick any function

$g \geq f$. The set $\{X \leq g : (\forall z \in \omega)(\exists \rho \in g^z)(\forall y < z)R(X, \rho, y)\}$ is clearly $\Pi^{0,g}_1$. It is nonempty as it contains X. Also, if Z is in the above set, then by compactness of the set $\{f \in \omega^\omega : f \leq g\}$ we have that Z is in $\mathcal{C}$. It follows that the above set is a nonempty subset of $\mathcal{C}$, bounded by g. The remainder of this section will be dedicated to the proof of (i)⇒(iii). □

Corollary 2.2 (Solovay [28], Spector [29], Groszek and Slaman [10]) *Fix a set $A \in \omega^\omega$. The following are equivalent:*

(i) *A is computably encodable,*
(ii) *A admits a modulus,*
(iii) *A is hyperarithmetic.*

Proof By Theorem 2.1, it suffices to prove that A is computably encodable, admits a modulus, and is hyperarithmetic if and only if $\{A\}$ is Π^0_1-encodable, admits a Π^0_1-modulus, and has a nonempty Σ^1_1-subset, respectively.

By Spector [29], a set $A \in \omega^\omega$ is hyperarithmetic if and only if it is the unique member of a Σ^1_1-singleton set $\mathcal{C} \subseteq \omega^\omega$. Therefore, A is hyperarithmetic if and only if $\{A\}$ has a nonempty Σ^1_1-subset. Every modulus of $A \in \omega^\omega$ is a Π^0_1-modulus of $\{A\}$. Conversely, if $\{A\}$ admits a Π^0_1-modulus f, then, for every $g \geq f$, $\{A\}$ is a g-computably bounded $\Pi^{0,g}_1$-singleton, and so A is g-computable. Therefore f is a modulus of A. If A is computably encodable, then $\{A\}$ is Π^0_1-encodable since every X-computable set is an X-computably bounded $\Pi^{0,X}_1$-singleton. Conversely, suppose that $\{A\}$ is Π^0_1-encodable. Then, for every set $X \in [\omega]^\omega$, there is a set $Y \in [X]^\omega$ such that $\{A\}$ is a Y-computably bounded Π^0_1-class. In particular, Y computes A. □

Recall the basis theorem of Gandy, Kreisel, and Tait [9], who proved that whenever a set $A \in \omega^\omega$ is non-hyperarithmetic, every nonempty Σ^1_1-set $\mathcal{D} \subseteq \omega^\omega$ has a member X such that A is not hyperarithmetic in X. We now need to extend this basis theorem by replacing non-hyperarithmetic sets by compact sets with no nonempty Σ^1_1-subsets in order to prove the remaining direction of Theorem 2.1. Note that when we apply Theorem 2.3 with $\mathcal{C} = \{A\}$ for some non-hyperarithmetic set A, we get back the non-hyperarithmetic basis theorem of Gandy, Kreisel, and Tait.

Theorem 2.3 (Σ^1_1-immunity basis theorem) *For every compact set $\mathcal{C} \subseteq \omega^\omega$ with no nonempty Σ^1_1-subset, and every nonempty Σ^1_1-set $\mathcal{D} \subseteq \omega^\omega$, there is some $X \in \mathcal{D}$ such that $\mathcal{C}$ has no nonempty $\Sigma^{1,X}_1$-subset.*

Theorem 2.3 is an easy consequence of the following lemma.

Lemma 2.4 *Fix a compact set $\mathcal{C} \subseteq \omega^\omega$ with no nonempty Σ^1_1-subset and a Σ^1_1-predicate $P(X, Y)$. Every nonempty Σ^1_1-set $\mathcal{D} \subseteq \omega^\omega$ has a nonempty Σ^1_1-subset $\mathcal{E}$ such that $\{Y \in \omega^\omega : P(X, Y)\}$ is not a nonempty subset of $\mathcal{C}$ for every $X \in \mathcal{E}$.*

Proof We reason by case analysis. In the first case, $\{Y \in \omega^\omega : P(X, Y)\} \nsubseteq \mathcal{C}$ for some $X \in \mathcal{D}$. Let $Y \notin \mathcal{C}$ be such that $P(X, Y)$ holds. By closure of $\mathcal{C}$, there is some finite initial segment $\sigma \prec Y$ such that $[\sigma] \cap \mathcal{C} = \emptyset$. The Σ^1_1-set $\mathcal{E} = \{X \in \mathcal{D} : (\exists Y \succ \sigma)P(X, Y)\}$ is nonempty and satisfies the desired properties. In the second case, for every $X \in \mathcal{D}$, $\{Y \in \omega^\omega : P(X, Y)\} \subseteq \mathcal{C}$. Then $\{Y \in \omega^\omega : (\exists X \in \mathcal{D})P(X, Y)\}$ is a Σ^1_1-subset of $\mathcal{C}$ and therefore must be empty. We can simply choose $\mathcal{E} = \mathcal{D}$. □

Proof of Theorem 2.3 Let us consider for any Σ^1_1-predicate $P(X,Y)$, the union $\mathcal{U}_P$ of all the Σ^1_1-sets $\mathcal{E}$ such that $\{Y \in \omega^\omega : P(X,Y)\} \not\subseteq \mathcal{C}$ for every $X \in \mathcal{E}$. By Lemma 2.4, each $\mathcal{U}_P$ is dense for the Gandy–Harrington topology (where open sets are those generated by the Σ^1_1-sets). Also, ω^ω with the Gandy–Harrington topology is a Baire space (see Theorem 1.10). It follows that $\bigcap_P \mathcal{U}_P$ is dense. In particular, it has a nonempty intersection with any Σ^1_1-set. Moreover, it is clear by the definition of $\mathcal{U}_P$ that $\mathcal{C}$ contains no $\Sigma^{1,X}_1$-subset for any $X \in \bigcap_P \mathcal{U}_P$. □

We will now prove the core lemma from which we will deduce the last direction of Theorem 2.1. To do so we will need the Galvin–Prikry theorem, which states that every Borel set $\mathcal{A} \subseteq 2^\omega$ is *Ramsey*; that is, there exists a set $X \in [\omega]^\omega$ such that $[X]^\omega \subseteq \mathcal{A}$ or $[X]^\omega \subseteq 2^\omega - \mathcal{A}$. In our case, we will need a slightly stronger version of the theorem. This stronger version was already used in a similar way by Soare [27] to build an infinite set which contains no subset of higher Turing degree.

Theorem 2.5 (Galvin and Prikry [8, Theorem 2]) *Let $\mathcal{A}$ be a Borel subset of 2^ω. For any $X \in [\omega]^\omega$, there must exist $Y \in [X]^{<\omega}$ such that $[Y]^\omega \subseteq \mathcal{A}$, or there must exist $Y \in [X]^{<\omega}$ such that $[Y]^\omega \subseteq 2^\omega - \mathcal{A}$.*

This stronger version follows from the proof of Galvin–Prikry's theorem. It is also explicitly stated by Silver [25], who shows in particular that Galvin–Prikry holds for Σ^1_1-sets.

In what follows, we assume the functional Γ has the purpose of computing a tree of the Baire space. In particular, we will consider only $\{0,1\}$-valued functionals. A computation $\Gamma^X : \omega^{<\omega} \to \{0,1\}$ is considered valid if Γ^X is total and if $\{\tau \in \omega^{<\omega} \mid \Gamma^X(\tau) \downarrow = 1\}$ is a tree of the Baire space. In this case, $[\Gamma^X]$ denotes the set of infinite paths of this tree.

Lemma 2.6 *Fix a set $X \in [\omega]^\omega$ and a compact set $\mathcal{C} \subseteq \omega^\omega$ with no nonempty $\Sigma^{1,X}_1$-subset. For every functional Γ and every $t \in \omega$, there is a set $Y \in [X]^\omega$ such that, for every $G \in [Y]^\omega$, if Γ^G is a tree, then either $[\Gamma^G] \cap \mathcal{C} = \emptyset$ or $\Gamma^G(\sigma) \downarrow = 1$ for some string $\sigma \in \omega^{<\omega}$ of length at least t such that $\mathcal{C} \cap [\sigma] = \emptyset$. Moreover, we can choose Y so that $\mathcal{C}$ has no $\Sigma^{1,Y}_1$-subset.*

Proof For every $\sigma \in \omega^{<\omega}$, let

$$\mathcal{Q}_\sigma = \left\{Y \in [X]^\omega \ : \ \forall v \in [Y]^{<\omega} \ \ \Gamma^v(\sigma) \uparrow \ \text{or} \ \Gamma^v(\sigma) \downarrow = 1\right\}.$$

Note that each $\mathcal{Q}_\sigma$ is $\Pi^{0,X}_2$ uniformly in σ (in particular, it is $\Sigma^{1,X}_1$ uniformly in σ). Also note that, for every $Y \in \mathcal{Q}_\sigma$ and for every $Z \in [Y]^{<\omega}$, if Γ^Z computes a tree, then σ is a node of this tree.

Suppose first that, for every $\ell \in \omega$, there is some $\sigma \in \omega^{<\omega}$ of length ℓ such that $\mathcal{Q}_\sigma \neq \emptyset$. If $\mathcal{Q}_\sigma \neq \emptyset$ for some $\sigma \in \omega^{<\omega}$ of length at least t such that $\mathcal{C} \cap [\sigma] = \emptyset$, then, by Theorem 2.3, there is some $Y \in \mathcal{Q}_\sigma$ such that $\mathcal{C}$ has no nonempty $\Sigma^{1,Y}_1$-subset. Such a Y and σ satisfy the desired properties. If $\mathcal{C} \cap [\sigma] \neq \emptyset$ for every $\sigma \in \omega^{<\omega}$ of length at least t such that $\mathcal{Q}_\sigma \neq \emptyset$, then, by compactness of $\mathcal{C}$, the set

$$\{h \in \omega^\omega : \forall \sigma \prec h \ \text{ with } |\sigma| \geq t \text{ we have } Q_\sigma \neq \emptyset\}$$

is a nonempty $\Sigma^{1,X}_1$-subset of $\mathcal{C}$, contradicting our hypothesis.

Suppose now that there is some $\ell \in \omega$ such that $\mathcal{Q}_\sigma = \emptyset$ for every $\sigma \in \omega^{<\omega}$ of length ℓ. Let $\sigma_0, \dots, \sigma_{n-1}$ be the finite sequence of all $\sigma \in \omega^\ell$ such that $\mathcal{C} \cap [\sigma] \neq \emptyset$. This sequence is finite by compactness of $\mathcal{C}$. Let

$$\mathcal{E} = \left\{Y \in [X]^\omega \ : \ \forall v \in [Y]^{<\omega} \ \forall i < n \ \Gamma^v(\sigma_i) \uparrow \text{ or } \Gamma^v(\sigma_i) \downarrow = 0\right\}.$$

Note that $\mathcal{E}$ is $\Pi^{0,X}_2$ and in particular $\Sigma^{1,X}_1$. Also note that, for every $Y \in \mathcal{E}$ and for every $Z \in [Y]^\omega$, if Γ^Z computes a tree, then $[\Gamma^Z] \cap \mathcal{C} = \emptyset$.

We claim that $\mathcal{E}$ is nonempty. To see this, let

$$\mathcal{A}_0 = \left\{Y \in [X]^\omega : \ \Gamma^Y(\sigma_0) \downarrow = 0\right\}.$$

By the Galvin–Prikry theorem (Theorem 2.5), there must exist a set $Y \in [X]^\omega$ such that $[Y]^\omega \subseteq \mathcal{A}_0$ or such that $[Y]^\omega \subseteq [\omega]^\omega - \mathcal{A}_0$. But, as $\mathcal{Q}_\sigma = \emptyset$, it means that $\forall Y \in [X]^\omega \ \exists Z \in [Y]^\omega \ \ Z \in \mathcal{A}_0$. Thus there can be no $Y \in [X]^\omega$ such that $[Y]^\omega \subseteq [\omega]^\omega - \mathcal{A}_0$. Thus there must be some $Y_0 \in [X]^\omega$ such that $[Y_0]^\omega \subseteq \mathcal{A}_0$. We can now repeat this same argument iteratively for every $i < n$, with the sets $\mathcal{A}_{i+1} = \{Y \in [Y_i]^\omega : \ \Gamma^Y(\sigma_i) \downarrow = 0\}$, to argue the existence of a set $Y_{i+1} \in [Y_i]^\omega$ and $[Y_{i+1}]^\omega \subseteq \mathcal{A}_{i+1}$. At the end, we obtain a set $Y_n \in [X]^\omega$ with $Y_n \in \mathcal{E}$.

Now by Theorem 2.3, there is some $Y \in \mathcal{E}$ such that $\mathcal{C}$ has no nonempty $\Sigma^{1,Y}_1$-subset. Such a Y satisfies the desired conditions. This completes the proof. □

Lemma 2.7 *Fix a Mathias condition $\langle a, X\rangle$ and a compact set $\mathcal{C} \subseteq \omega^\omega$ with no nonempty $\Sigma^{1,X}_1$-subset. For every functional Γ and every $t \in \omega$, there is a condition $\langle a, Y\rangle \subseteq \langle a, X\rangle$ such that, for every $G \in \langle a, Y\rangle$ and every $H \in [G]^\omega$ such that Γ^H is a tree, either $\mathcal{C} \cap [\Gamma^H] = \emptyset$ or $\Gamma^H(\sigma) \downarrow = 1$ for some string $\sigma \in \omega^{<\omega}$ of length at least t such that $\mathcal{C} \cap [\sigma] = \emptyset$. Moreover, we can choose Y so that $\mathcal{C}$ has no $\Sigma^{1,Y}_1$-subset.*

Proof Let $a_0, \dots, a_{n-1}$ be the finite listing of all subsets of a, and, for every $i < n$, let Γ_i be the functional defined by $\Gamma^Z_i = \Gamma^{a_i \cup Z}$. By iterating Lemma 2.6 on each Γ_i, we obtain a set $Y \in [X]^\omega$ such that $\mathcal{C}$ has no nonempty $\Sigma^{1,Y}_1$-subset, and, for every $Z \in [Y]^\omega$ and every $i < n$, either $\mathcal{C} \cap [\Gamma^Z_i] = \emptyset$ or $\Gamma^Z_i(\sigma) \downarrow = 1$ for some string $\sigma \in \omega^{<\omega}$ of length at least t such that $\mathcal{C} \cap [\sigma] = \emptyset$.

We claim that $\langle a, Y\rangle$ satisfies the desired properties. Fix any $G \in \langle a, Y\rangle$ and $H \in [G]^\omega$. In particular, $H = a_i \cup Z$ for some $i < n$ and $Z \in [Y]^\omega$. Therefore, either $\mathcal{C} \cap [\Gamma^H] = \mathcal{C} \cap [\Gamma^Z_i] = \emptyset$, or $\Gamma^H(\sigma) \downarrow = \Gamma^Z_i(\sigma) = 1$ for some string $\sigma \in \omega^{<\omega}$ of length at least t such that $\mathcal{C} \cap [\sigma] = \emptyset$. □

Proof of Theorem 2.1 (i)⇒(iii): We now prove that if a compact set $\mathcal{C} \subseteq \omega^\omega$ has no nonempty Σ^1_1-subset, then there is a set $Y \in [\omega]^\omega$ such that, for every $G \in [Y]^\omega$, every G-computably bounded $\Pi^{0,G}_1$-set is not included in $\mathcal{C}$.

By iterating Lemma 2.7, build an infinite sequence of Mathias conditions $\langle \emptyset, \omega\rangle = \langle a_0, X_0\rangle \supseteq \langle a_1, X_1\rangle \supseteq \dots$ such that, for every $i \in \omega$, $\mathcal{C}$ has no nonempty Σ^{1,X_i}_1-subset, $|a_{i+1}| \geq i$, and, for every $G \in \langle a_{i+1}, X_{i+1}\rangle$, every $H \in [G]^\omega$, and every $j < i$ such that Φ^H_j is a tree, either $\mathcal{C} \cap [\Phi^H_j] = \emptyset$ or $\Phi^H_j(\sigma) \downarrow = 1$ for some string $\sigma \in \omega^{<\omega}$ of length at least i such that $\mathcal{C} \cap [\sigma] = \emptyset$. Take $Y = \bigcup_i a_i$ as the desired set. By construction, for every $G \in [Y]^\omega$ and every $j \in \omega$ such that Φ^G_j is a tree, either $\mathcal{C} \cap [\Phi^G_j] = \emptyset$ or $\{\sigma \in \omega^{<\omega} : \Phi^G_j(\sigma) \downarrow = 1 \text{ and } \mathcal{C} \cap [\sigma] = \emptyset\}$ is infinite.

Now, if Φ^G_j is not computably bounded, then we are done. Otherwise, it is compact and then, by compactness, there is an infinite path of $[\Phi^G_j]$ which is not in $\mathcal{C}$. □

Corollary 2.8 WWKL $\not\leq_{\text{soc}}$ RT.

Proof Let $T \subseteq 2^{<\omega}$ be a tree of positive measure such that $[T]$ has no nonempty Σ^1_1-subset. For example, take T to be a tree whose infinite paths are the elements of a $\Pi^{0,\mathcal{O}}_1$-set of Martin-Löf randoms relative to Kleene's O (more on algorithmic randomness can be found in [5] or Nies [19]). Theorem 1.9 says that the sets that are Turing below Kleene's O are a basis for the Σ^1_1-subsets of 2^ω; thus $[T]$ cannot have any Σ^1_1-subset.

Fix an RT-instance f, and suppose that every infinite f-homogeneous set H computes an infinite path through T. In particular, $[T]$ has a nonempty $\Pi^{0,H}_1$-subset. Since for every set $X \in [\omega]^\omega$ there is an f-homogeneous set $Y \in [X]^\omega$, $[T]$ is Π^0_1-encodable. Therefore, by Theorem 2.1, $[T]$ admits a nonempty Σ^1_1-subset, contradicting our hypothesis. □

Note that we make an essential use of compactness in Theorem 2.1. Actually, there exist Π^0_1-encodable closed sets $\mathcal{C} \subseteq \omega^\omega$ with no Σ^1_1-subset , as witnessed by the following lemma.

Lemma 2.9 *Let $Z \subseteq \omega$ be a set with no infinite subset Turing below Kleene's O, in either it or its complement. The set*

$$\mathcal{C}_Z = \left\{Y \in [\omega]^\omega : Y \subseteq Z \vee Y \subseteq \overline{Z}\right\}$$

is Π^0_1-encodable and has no nonempty Σ^1_1-subset.

Proof For any $X \in [\omega]^\omega$, either $X \cap Z$, or $X \cap \overline{Z}$ is infinite and therefore belongs to $\mathcal{C}_Z$. Thus $\mathcal{C}_Z$ is Π^0_1-encodable. Also using Theorem 1.9, the sets Turing below Kleene's O are a basis for the Σ^1_1-subsets of 2^ω, and thus $\mathcal{C}_Z$ cannot have a nonempty Σ^1_1-subset. □

3 Summary and Open Questions

In this last section, we summarize the relations between variants of Ramsey's theorem and of König's lemma over strong omniscient computable reducibility, and we state two remaining open questions.

In Figure 1, a plain arrow from P to Q means that Q $\leq_{\text{soc}}$ P. A dotted arrow indicates a hierarchy between the statements. Except for the open arrow from RT^2_2 to RT, the missing arrows are all known separations and can be derived from Section 1.2. The remaining questions are of two kinds: whether the number of colors and the size of the tuples has a structural impact reflected over strong omniscient computable reducibility.

Question 3.1 Is $\mathsf{RT}^n_{k+1} \leq_{\text{soc}} \mathsf{RT}^n_k$ whenever $n, k \geq 2$?

Question 3.2 Is $\mathsf{RT}^{n+1}_k \leq_{\text{soc}} \mathsf{RT}^n_k$ whenever $n, k \geq 2$?

Note that a negative answer to Question 3.1 would give a negative answer to Question 3.2 since $\mathsf{RT}^n_{<\infty} \leq_{sW} \mathsf{RT}^{n+1}_2$ (see either [1] or [12]).

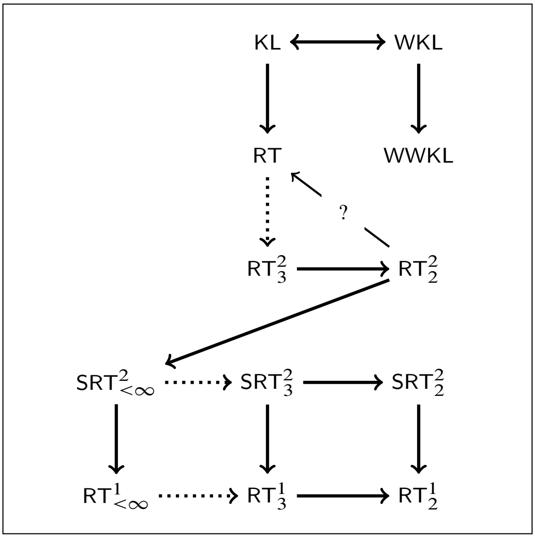

Figure 1 Versions of RT and KL under $\leq_{\text{soc}}$.

References

[1] Brattka, V., and T. Rakotoniaina, "On the uniform computational content of Ramsey's theorem," *Journal of Symbolic Logic*, vol. 82 (2017), pp. 1278–316. Zbl 08624519. MR 3743611. 2, 3, 9

[2] Cholak, P. A., C. G. Jockusch, Jr., and T. A. Slaman, "On the strength of Ramsey's theorem for pairs," *Journal of Symbolic Logic*, vol. 66 (2001), pp. 1–55. Zbl 0977.03033. MR 1825173. DOI 10.2307/2694910. 3

[3] Chong, C. T., and L. Yu, *Recursion Theory: Computational Aspects of Definability*, vol. 8 of *De Gruyter Series in Logic and its Applications*, De Gruyter, Berlin, 2015. Zbl 1334.03003. MR 3381097. DOI 10.1515/9783110275643. 4

[4] Dorais, F. G., D. D. Dzhafarov, J. L. Hirst, J. R. Mileti, and P. Shafer, "On uniform relationships between combinatorial problems," *Transactions of the American Mathematical Society*, vol. 368 (2016), pp. 1321–59. Zbl 06560459. MR 3430365. DOI 10.1090/tran/6465. 2

[5] Downey, R. G., and D. R. Hirschfeldt, *Algorithmic Randomness and Complexity*, Springer, New York, 2010. Zbl 1221.68005. MR 2732288. DOI 10.1007/978-0-387-68441-3. 3, 9

[6] Dzhafarov, D. D., and C. G. Jockusch, Jr., "Ramsey's theorem and cone avoidance," *Journal of Symbolic Logic*, vol. 74 (2009), pp. 557–78. Zbl 1166.03021. MR 2518811. DOI 10.2178/jsl/1243948327. 3

[7] Dzhafarov, D. D., L. Patey, D. R. Solomon, and L. B. Westrick, "Ramsey's theorem for singletons and strong computable reducibility," *Proceedings of the American Mathematical Society*, vol. 145 (2017), pp. 1343–55. Zbl 06668907. MR 3589330. DOI 10.1090/proc/13315. 3

[8] Galvin, F., and K. Prikry, "Borel sets and Ramsey's theorem," *Journal of Symbolic Logic*, vol. 38 (1973), pp. 193–98. Zbl 0276.04003. MR 0337630. DOI 10.2307/2272055. 7

[9] Gandy, R. O., G. Kreisel, and W. W. Tait, "Set existence," *Bulletin L'Académie Polonaise des Science, Série des Sciences Mathématiques, Astronomiques et Physiques*, vol. 8 (1960), pp. 577–82. Zbl 0207.30102. MR 0159747. 5, 6

[10] Groszek, M. J., and T. A. Slaman, "Moduli of computation," conference lecture on *Logic, Computability and Randomness*, Buenos Aires, Argentina, 2007. 6

[11] Hinman, P. G., "A survey of Mučnik and Medvedev degrees," *Bulletin of Symbolic Logic*, vol. 18 (2012), pp. 161–229. Zbl 1248.03063. MR 2931672. DOI 10.2178/bsl/1333560805. 1

[12] Hirschfeldt, D. R., and C. G. Jockusch, Jr., "On notions of computability-theoretic reduction between Π^1_2 principles," *Journal of Mathematical Logic*, vol. 16 (2016), no. 1, art. ID 1650002. Zbl 1373.03068. MR 3518779. DOI 10.1142/S0219061316500021. 2, 3, 9

[13] Hirschfeldt, D. R., C. G. Jockusch, Jr., B. Kjos-Hanssen, S. Lempp, and T. A. Slaman, "The strength of some combinatorial principles related to Ramsey's theorem for pairs," pp. 143–61 in *Computational Prospects of Infinity, Part II: Presented Talks*, vol. 15 of *Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore*, World Scientific, Hackensack, NJ, 2008. Zbl 1167.03009. MR 2449463. DOI 10.1142/9789812796554_0008. 3

[14] Jockusch, C. G., Jr., "Ramsey's theorem and recursion theory," *Journal of Symbolic Logic*, vol. 37 (1972), pp. 268–80. Zbl 0262.02042. MR 0376319. DOI 10.2307/2272972. 3

[15] Jockusch, C. G., Jr., and R. I. Soare, "Π^0_1 classes and degrees of theories," *Transactions of the American Mathematical Society*, vol. 173 (1972), pp. 33–56. Zbl 0262.02041. MR 0316227. DOI 10.2307/1996261. 2

[16] Jockusch, C. G., Jr., and R. I. Soare, "Encodability of Kleene's O, *Journal of Symbolic Logic*, vol. 38 (1973), pp. 437–40. Zbl 0279.02025. MR 0363842. DOI 10.2307/2273040. 1

[17] Montalbán, A., "Open questions in reverse mathematics," *Bulletin of Symbolic Logic*, vol. 17 (2011), pp. 431–54. Zbl 1233.03023. MR 2856080. DOI 10.2178/bsl/1309952320. 3

[18] Moschovakis, Y. N., *Descriptive Set Theory*, 2nd edition, vol. 155 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, 2009. Zbl 1172.03026. MR 2526093. DOI 10.1090/surv/155. 4

[19] Nies, A., *Computability and Randomness*, vol. 51 of *Oxford Logic Guides*, Oxford University Press, Oxford, 2009. Zbl 1169.03034. MR 2548883. DOI 10.1093/acprof:oso/9780199230761.001.0001. 9

[20] Patey, L., "The weakness of being cohesive, thin or free in reverse mathematics," *Israel Journal of Mathematics*, vol. 216 (2016), pp. 905–55. Zbl 1368.03018. MR 3557471. DOI 10.1007/s11856-016-1433-3. 3, 4

[21] Sacks, G. E., *Degrees of Unsolvability*, vol. 55 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, 1963. Zbl 0143.25302. MR 0186554. 3

[22] Sacks, G. E., *Higher Recursion Theory*, *Perspectives in Mathematical Logic*, Springer, Berlin, 1990. Zbl 0716.03043. MR 1080970. DOI 10.1007/BFb0086109. 4

[23] Seetapun, D., and T. A. Slaman, "On the strength of Ramsey's theorem," *Notre Dame Journal of Formal Logic*, vol. 36 (1995), pp. 570–82. Zbl 0843.03034. MR 1368468. DOI 10.1305/ndjfl/1040136917. 3

[24] Shore, R. A., "Reverse mathematics: The playground of logic," *Bulletin of Symbolic Logic*, vol. 16 (2010), pp. 378–402. Zbl 1218.03006. MR 2731250. 2

[25] Silver, J., "Every analytic set is Ramsey," *Journal of Symbolic Logic*, vol. 35 (1970), pp. 60–64. Zbl 0216.01304. MR 0332480. DOI 10.2307/2271156. 7

[26] Simpson, S. G., *Subsystems of Second Order Arithmetic*, 2nd edition, *Perspectives in Logic*, Cambridge University Press, Cambridge, 2009. Zbl 1181.03001. MR 2517689. DOI 10.1017/CBO9780511581007. 2, 3

[27] Soare, R. I., "Sets with no subset of higher degree," *Journal of Symbolic Logic*, vol. 34 (1969), pp. 53–56. Zbl 0182.01602. MR 0263627. DOI 10.2307/2270981. 7

[28] Solovay, R. M., "Hyperarithmetically encodable sets," *Transactions of the American Mathematical Society*, vol. 239 (1978), pp. 99–122. Zbl 0411.03039. MR 0491103. DOI 10.2307/1997849. 1, 6

[29] Spector, C., "Recursive well-orderings," *Journal of Symbolic Logic*, vol. 20 (1955), pp. 151–63. Zbl 0067.00303. MR 0074347. DOI 10.2307/2266902. 1, 6

Acknowledgments

The authors thank the anonymous referee for numerous comments that improved the readability of the article. The second author is funded by the John Templeton Foundation project "Structure and Randomness in the Theory of Computation." The opinions expressed in this publication are those of the author(s) and do not necessarily reflect the views of the John Templeton Foundation.

Monin
LACL, Département d'Informatique
Faculté des Sciences et Technologie
94010 Créteil Cedex
France
benoit.monin@computability.fr
http://www.lacl.fr/~benoit.monin

Patey
Department of Mathematics
University of California, Berkeley
Berkeley, California 94720-3840
USA
ludovic.patey@computability.fr
http://ludovicpatey.com