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Negation-Free and Contradiction-Free Proof of the Steiner–Lehmus Theorem

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Abstract By rephrasing quantifier-free axioms as rules of derivation in sequent calculus, we show that the generalized Steiner–Lehmus theorem admits a direct proof in classical logic. This provides a partial answer to a question raised by Sylvester in 1852. We also present some comments on possible intuitionistic approaches.

1 Introduction

One of the late additions to the collection of elementary geometry theorems, first formulated in 1840 by Daniel Christian Ludolph Lehmus, and proved that same year by Jakob Steiner (but published only in 1844), states that a triangle with two congruent internal angle bisectors must be isosceles. Many proofs have been offered over the course of the more than 170 years since it was first stated, but all synthetic proofs, where one could meaningfully ask for a *direct* proof, appeared to be *indirect*, that is, to find some fault with a scalene triangle having two congruent internal bisectors.

As early as 1852, Sylvester [28] believed that all proofs of the theorem must be indirect:

> "My reader will now be prepared to see why it is that all the geometrical demonstrations given of this theorem [. . .] are indirect, I believe I may venture to say *necessarily* indirect. It is because the truth of the theorem depends on the necessary non-existence of real roots (between prescribed limits) of the analytical equation expressing the conditions of the question; and I believe that it may be safely taken as an axiom in geometrical method, that whenever this is the case no other form of proof than that of the *reductio ad absurdum* is possible in the nature of things. If this principle is erroneous, it must admit of an easy refutation in particular instances." (pp. 394–95)

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All arguments for and against this thesis brought right after Sylvester's paper (see Adamson [2]), all the way to the twenty-first century, suffer from the absence of a clear logical formulation of the problem, which ensures muddled conclusions "signifying nothing." In fact, the first serious analysis of the question of *indirectness* itself that I am aware of, in the context of propositional logic, can be found in Ekman [9], [10]. Indirect proofs and their relations to direct ones are also the subject of Orevkov [21, Chapter 4].

Our aim is to present a proof of the existence of a direct proof of a certain formal expression of a generalization of the Steiner–Lehmus theorem first inside an axiom system of dimension-free absolute geometry, then, based on the recent proof (see Pambuccian, Struve, and Struve [25]) in standard ordered metric planes, in a certain axiom system for a fragment of absolute geometry.

2 The Generalized Steiner–Lehmus Theorem in Tarski's Axiom System for Dimension-Free Absolute Geometry

2.1 The Skolemized axiom system with Makarios's changed five-segment axiom An axiom system with very few axioms for dimension-free absolute geometry (i.e., there are no dimension axioms, neither one stating that the dimension is at least n, nor one stating that the dimension is at most n, and there is no axiom stating that there is only one parallel from a point outside of a given line to that line) was provided by Tarski in Schwabhäuser, Szmielew, and Tarski [26]. One of Tarski's axioms was shown by Makarios [16] to be superfluous, if a small change is made to another one of Tarski's axioms. This brought their number down to six. The language in which the Skolemized axiom system is expressed has only one sort of individual variables, to be referred to as *points*, as well as: (i) two relation symbols, a ternary one B, for *betweenness*, with $B(abc)$ to be read as "point b lies between a and c" (and b may be a or c, and $a = b = c$ is also allowed), a quaternary one $\equiv$ for *equidistance*, with $ab \equiv cd$ to be read as "b is as distant from a as d is from c" (or "ab is congruent to cd"), and (ii) two operation symbols, arising from the Skolemization of the two axioms that contain existential quantifiers, a quaternary one S, with $S(abcd)$ producing, if $a \neq b$, the point on the ray opposite to $\overrightarrow{ba}$ for which $bS(abcd)$ is congruent to cd, and, if $a = b$, any point with that congruence property (thus S corresponds to a segment transport operation), and a 5-ary one, ι, with $\iota(ceadb)$ being, if $B(cea)$ and $B(dba)$, a point that is both between b and c and between e and d, else an arbitrary point. The axioms are:

T 1 $ab \equiv pq \wedge ab \equiv rs \rightarrow pq \equiv rs,$

T 2 $ab \equiv cc \rightarrow a = b,$

T 3 $B(qaS(qabc)) \wedge aS(qabc) \equiv bc,$

T 4 $B(abc) \wedge B(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ad \equiv a'd' \wedge bd \equiv b'd' \rightarrow (dc \equiv c'd' \vee a = b),$

T 5 $B(aba) \rightarrow a = b,$

T 6 $B(cea) \wedge B(dba) \rightarrow B(e\iota(ceadb)d) \wedge B(b\iota(ceadb)c).$

Axiom T4 is usually referred to as the *five-segment axiom* and can be considered to be, in case the points a, b, and d are noncollinear, and $c \neq b$, stating an angle-free

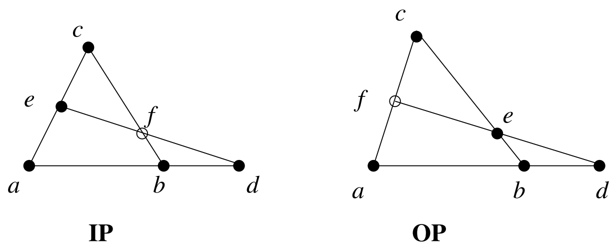

Figure 1 Inner and outer forms of the Pasch axiom, **IP** and **OP**.

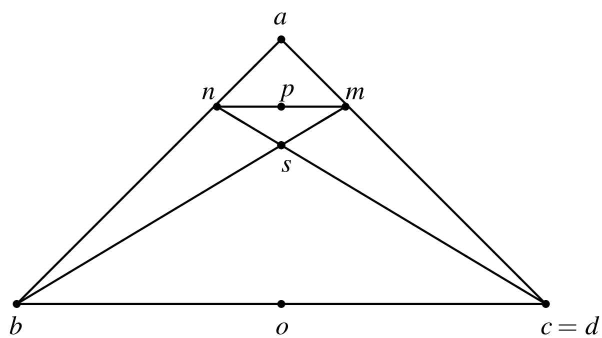

Figure 2 The generalized Steiner–Lehmus theorem.

variant of the side-angle-side congruence axiom, as triangles bcd and $b'd'c'$ have to be congruent if they have two congruent sides, $bc \equiv b'c'$ and $bd \equiv b'd'$, and the angles between those sides congruent (since there are no angle variables, this is expressed by means of the side-side-side congruence of triangles abd and $a'b'd'$).

Axiom T6 (see Figure 1) is a form of the Pasch axiom, referred to as the *inner form of the Pasch axiom* (**IP**), for it states, in case a, b, and c are three noncollinear points, that, if a line intersects the side ac and the extension, past b, of another side, ab, of triangle abc, then it must intersect the third side bc as well (in a point f, whose Skolemized name is $\iota(ceadb)$). The axiom also contains several degenerate cases, in case a, b, and c are collinear, or in case e does not lie strictly between a and c, or b does not lie strictly between a and d. These degenerate cases are essential in proving most universal properties of betweenness (enabling us to have only one universal betweenness axiom, namely T5).

2.2 The generalized Steiner–Lehmus theorem The generalized version of the Steiner–Lehmus theorem, that can be proved on the basis of these axioms (see [25]), can be stated as (see Figure 2; some of the points drawn are needed for a formulation we will encounter later on):

$$\neg\big(B(abc) \vee B(bca) \vee B(cab)\big) \wedge B(amc) \wedge B(anb) \wedge ad \equiv ab \wedge B(amd) \wedge sb \equiv sd \wedge B(bsm) \wedge m \neq c \wedge m \neq a \wedge B(csn) \wedge cn \equiv bm \wedge \to d = c. \quad (1)$$

It is a generalization of the Steiner–Lehmus theorem, as it states that, if $\overrightarrow{as}$ is the internal angle bisector of the angle $\widehat{bac}$ (which can be expressed by stating that s is equidistant from two distinct points on the rays $\overrightarrow{ab}$ and $\overrightarrow{ac}$ that are themselves equidistant from a (we chose those two points to be b, an already existing point on $\overrightarrow{ab}$, and d, a (new) point on $\overrightarrow{ac}$, so as not to needlessly increase the number of variables used)), s does not lie on bc (a condition expressed in the form $m \neq c$) and is different from a (a condition expressed by $m \neq a$), and if the segments bm and cn, the segments formed by intersecting the rays $\overrightarrow{bs}$ and $\overrightarrow{cs}$ with the sides ac and ab of the triangle abc, are congruent, then triangle abc must be isosceles; that is, ab must be congruent to ac. In the original Steiner–Lehmus theorem, $\overrightarrow{bs}$ and $\overrightarrow{cs}$ had to be the angle bisectors of $\widehat{abc}$ and $\widehat{acb}$.

2.3 Rules of derivation replace axioms In Negri and von Plato [18] (see also Negri and von Plato [19], [20]) axiom systems in *classical logic* (see [18, Proposition 2.6]) consisting of quantifier-free axioms are turned into (logic-free) rules of inference, in which from a certain *sequent* another sequent may be inferred, and a certain sequent in which no logical symbols appear is provable if and only if it can be derived as the endsequent of a chain of inferences, where only the inference rules corresponding to the axioms and two logical axioms (Γ and Δ are arbitrary multisets of formulas (repetitions are allowed))

$$P, \Gamma \Rightarrow \Delta, P \qquad \text{and} \qquad \bot, \Gamma \Rightarrow \Delta \tag{2}$$

—where P is any atomic formula, and $\bot$ is the logical symbol for falsity—are allowed to appear.

In our case, we first obtain two rules of inference, corresponding to the equality axioms, namely:

$$\frac{a = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}, \tag{3}$$

$$\frac{s = t, Q(s), Q(t), \Gamma \Rightarrow \Delta}{s = t, Q(s), \Gamma \Rightarrow \Delta}, \tag{4}$$

where s and t are terms, and $Q(a)$ is an atomic predicate from our language, in which the variable a occurs.

Next, the axioms T1–T6 are turned into rules of derivation according to the following principle. Each axiom

$$P_1 \wedge \cdots \wedge P_m \rightarrow (Q_1 \vee \cdots \vee Q_n) \tag{5}$$

becomes

$$\frac{P_1, \ldots, P_m, Q_1, \Gamma \Rightarrow \Delta \ \ \ldots \ \ P_1, \ldots, P_m, Q_n, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}. \tag{6}$$

They become the following rules of derivation:

$$\frac{ab \equiv pq, ab \equiv rs, pq \equiv rs, \Gamma \Rightarrow \Delta}{ab \equiv pq, ab \equiv rs, \Gamma \Rightarrow \Delta}, \tag{7}$$

$$\frac{ab \equiv cc, a = b, \Gamma \Rightarrow \Delta}{ab \equiv cc, \Gamma \Rightarrow \Delta}, \tag{8}$$

$$\frac{B(qaS(qabc)), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}, \tag{9}$$

$$\frac{aS(qabc) \equiv bc, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}, \tag{10}$$

$$\frac{\Sigma, dc \equiv c'd', \Gamma \Rightarrow \Delta \quad \Sigma, a = b, \Gamma \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta}, \tag{11}$$

where Σ stands for $B(abc)$, $B(a'b'c')$, $ab \equiv a'b'$, $bc \equiv b'c'$, $ad \equiv a'd'$, $bd \equiv b'd'$;

$$\frac{B(aba), a = b, \Gamma \Rightarrow \Delta}{B(aba), \Gamma \Rightarrow \Delta}, \tag{12}$$

$$\frac{B(cea), B(dba), B(e\iota(ceadb)d), \Gamma \Rightarrow \Delta}{B(cea), B(dba), \Gamma \Rightarrow \Delta}, \tag{13}$$

$$\frac{B(cea), B(dba), B(b\iota(ceadb)c), \Gamma \Rightarrow \Delta}{B(cea), B(dba), \Gamma \Rightarrow \Delta}. \tag{14}$$

The generalized Steiner–Lehmus theorem (1) becomes, when written as a sequent,

$$B(amc), B(anb), ad \equiv ab, B(amd), sb \equiv sd, B(bsm), B(csn), cn \equiv bm$$
$$\Rightarrow d = c, m = c, m = a, B(abc), B(bca), B(cab). \tag{15}$$

That (1) holds in absolute geometry was first noticed by Tarry [30]. In fact, the proof presented in Descube [8] can be turned into one true in a geometry weaker than that axiomatized by T1–T6, as shown in [25]. Thus, given the equivalence (see [18]–[20]) of the two formalisms (proof of a quantifier-free sentence from quantifier-free axioms and proof from the axioms (2) of a sequent using rules of derivation inside the sequent calculus), we have the following.

Theorem 2.1 *The sequent (15) can be deduced from the axioms (2) by using (3), (4), and (7)–(14) as rules of derivation.*

3 Direct and Indirect Proofs

When deriving a sequent from the axioms (2) by using rules of derivation, there are never negation symbols, as our sequents are *logic-free*; that is, there are neither logical connectives nor quantifiers. So, how do we decide whether a proof is direct or indirect?

Indirectness is commonly understood to arise when the contrapositive of a desired sentence is proved or when one arrives at a contradiction from alternatives to the statement to be proved.

The notion of "contrapositive" makes no sense in our sequent calculus setting, given that each proposition gets turned in an *essentially unique* manner into a rule of derivation or a sequent, none of which includes the negation sign. So, one cannot make any tricks with the sequent to be proved. The only variation allowed is in the *order* in which the terms occurring in the multisets are listed.

Contradictions, on the other hand, can appear in the sequent to be proved (which may have a $\bot$ after the $\Rightarrow$ sign), do appear in all axioms of the second type in (2), and in some rules of derivation with whose help we are to prove the given sequent. A sequent corresponding to a sentence to be proved will have a $\bot$ after the $\Rightarrow$ sign only if the sentence cannot be written solely in terms of the connectors $\wedge$, $\vee$, and $\rightarrow$. A rule of derivation reflects a contradiction only if it holds with an *empty premise*.

For example, suppose that we had the lower dimension axiom in our axiom system, which states that three individual constants a_0, a_1, a_2 (which belong to an extended language) are such that $\neg(B(a_0a_1a_2) \vee B(a_1a_2a_0) \vee B(a_2a_0a_1))$ holds. To appear as in (5), the axiom needs to be first broken up into three axioms, namely, $\neg B(a_0a_1a_2)$, $\neg B(a_1a_2a_0)$, and $\neg B(a_2a_0a_1)$, and each of these needs to be expressed without the negation symbol as $B(a_0a_1a_2) \to\perp$, $B(a_1a_2a_0) \to\perp$, and $B(a_2a_0a_1) \to\perp$. The rules of derivation that would correspond to them, would, according to (6), be (for $i = 0, 1, 2$, addition modulo 3)

$$\frac{B(a_ia_{i+1}a_{i+2}), \perp, \Gamma \Rightarrow \Delta}{B(a_ia_{i+1}a_{i+2}), \Gamma \Rightarrow \Delta}. \tag{16}$$

However, given the second class of axioms in (2), the premises in (16) are true, so (16) holds with an *empty premise*; that is, it actually is

$$\frac{}{B(a_ia_{i+1}a_{i+2}), \Gamma \Rightarrow \Delta}.$$

A rule of derivation with an empty premise acts like an additional axiom, beyond those in (2). Had we needed, for example, the lower dimension axiom in our proof (e.g., as it is *needed* in the proofs employing Pasch's axiom of certain betweenness properties in Hilbert [13, Section 4]), then the proof would have been *indirect*. We do not, however, need it, given that the generalized Steiner–Lehmus theorem mentions that the triangle *abc* ought to be isosceles only if it is a nondegenerate triangle; that is, in case there is no noncollinear triple available, the theorem is vacuously true.

If we were to write the lower dimension axiom as a sequent to be proved, then we would get $B(a_ia_{i+1}a_{i+2}) \Rightarrow\perp$.

Indirectness in the proof of a sequent without $\perp$ in its multisets can thus occur only if one of the following takes place: (i) $\perp, \Gamma \Rightarrow \Delta$ is used as an axiom, or (ii) a rule of derivation with empty upper sequent is used.

In general, we conclude that each quantifier-free axiom not containing $\perp$ that can be written only with $\wedge$, $\vee$, and $\to$ as logical connectors can be transformed into a rule of derivation with a nonempty premise, and each sentence which can be thus written can be transformed into a sequent without an occurrence of $\perp$. Thus, if a quantifier-free axiom system AS and a quantifier-free sentence σ which can be proved from AS are such that all can be written solely with $\wedge$, $\vee$, and $\to$, then the proof of σ from AS is a *direct proof*.

To see that this holds in the special case of the result in Theorem 2.1, note that an axiom of the form $\perp, \Gamma \Rightarrow \Delta$ cannot appear in the derivation of (15), given that, in all our rules, $\perp$ can appear only in Γ, and Γ carries over from the upper sequent to the lower sequent in every rule of derivation, so $\perp$ would have to be present in the antecedent of (15), which is not the case. No rule of derivation has an empty premise, so we have the following result.

Theorem 3.1 *The deduction of (15) from axioms of the first kind in (2) by using (3), (4), and (7)–(14) as rules of derivation is a direct one.*

However, this kind of directness applies to a very large class of universal statements that are theorems of absolute geometry, so within the direct-indirect dichotomy the vast majority of theorems belong to the *direct* camp.

A potential critique of this approach to directness would be one emphasizing the fact that not (1) was proved, a sentence with a very simple, indeed atomic, conclusion $d = c$, but rather (15), whose conclusion is a disjunction of no less than six atoms. Although the transformation of (1) into (15) happened along algorithmic lines, required for the transformation of Hilbert-style expressions into Gentzen-style expressions, the transformation itself is sound only because it happens within classical logic. Is the transformation itself not obscuring the very essence of directness, is it not unduly enlarging the set of directly provable sentences?

One remedy we have for this state of affairs is to drop classical logic altogether, and turn to intuitionism, one of whose main critiques was precisely the incorrect way several theorems are phrased in mathematics, giving an impression of a positive achievement when only a negative result has been actually proved. This appears to be the case for the Steiner–Lehmus theorem, in which the statement is very neat, stating that a certain congruence implies another congruence, yet the existing proofs prove that a certain segment inequality implies another such inequality.

Turning to intuitionist reasoning for an answer, we are faced with some major problems. One approach, very far removed from Brouwer's intentions, is to work within intuitionistic logic, regardless of whether the primitives and axioms are intuitionistically justifiable. This is, for example, the approach through which one arrives at intuitionistic Zermelo–Fraenkel.

If we were to follow that route, then the answer is simple. First, let us recall that a formula in the language of first-order logic is called *geometric* if it does not contain $\rightarrow$ or $\forall$ (i.e., if it can be written using $\wedge$, $\vee$, and $\exists$ only—since $\neg\varphi$ is considered to be an abbreviation of $\varphi \rightarrow \bot$). A *geometric implication* is a sentence of the form $(\forall \overline{x})\, A \rightarrow B$, where A and B are geometric formulas, and $\forall \overline{x}$ stands for $(\forall x_1)(\forall x_2)\cdots(\forall x_n)$. A *geometric theory* is a theory axiomatized by geometric implications. Equivalently, a theory is geometric if its axioms are sentences of the form

$$(\forall \overline{x}) \Big(\bigwedge_{i=1}^{m} \varphi_i(\overline{x}) \rightarrow \bigvee_{j=1}^{n} (\exists \overline{y_j}) \bigwedge_{k=1}^{l_j} \psi_{j,k}(\overline{x}, \overline{y_j}) \Big), \tag{17}$$

where the formulas φ_i and $\psi_{j,k}$ are atomic formulas (including $\top$ and $\bot$), the $\overline{x}$ and $\overline{y}_j$ or the antecedent itself could be empty (i.e., $m = 0$) and the $\overline{y_j}$ are not free in the φ_i.

A Glivenko-style theorem (see Palmgren [22, Lemma 2.2, p. 298], Negri [17, Theorem 6, p. 399] for proofs) tells us that if $\mathcal{T}$ is a geometric theory and θ a geometric implication such that $\mathcal{T} \vdash_c \theta$, then $\mathcal{T} \vdash_i \theta$, where $\vdash_c$ and $\vdash_i$ designate the fact that the underlying logic used for the derivation is classical (resp., intuitionistic). Now the axioms T1–T6 are all geometric implications, and (1) becomes one if we rewrite it as

$$\begin{aligned} B(amc) \wedge B(anb) \wedge ad \equiv ab \wedge B(amd) \wedge sb \equiv sd \wedge B(bsm) \wedge B(csn) \wedge cn \\ \equiv bm \wedge \rightarrow \big(d = c \vee B(abc) \vee B(bca) \vee B(cab) \vee m = c \vee m = a\big). \end{aligned} \tag{18}$$

So, from the well-known fact that $\{T1–T6\} \vdash_c$ (18), we conclude that $\{T1–T6\} \vdash_i$ (18). This means that, in the form (18), the generalized Steiner–Lehmus theorem is provable in intuitionistic logic.

If we now look at a genuinely intuitionistic axiomatization of elementary geometry, we find that the only paper in which Brouwer referred to elementary geometry was [7], in which he showed that one cannot conclude intuitionistically that two lines that are known to be distinct and not parallel must intersect, and that Heyting axiomatized only projective and affine geometry. The first axiom system for plane Euclidean geometry that claimed to be intuitionistic in the choice of the primitive and the axioms as well as the underlying logic, was put forward by Lombard and Vesley [15]. It has only one six-place relation *Dist*, with points as the only sort of individual variables. The meaning of $\text{Dist}(a,b,c,d,e,f)$ is that the sum of the lengths of the segments ab and cd is positively greater than that of the segment cd, that is, that one can find a natural number n such that $ab + |cd| > |ef| + 2^{-n}$. A four-place relation δ is then defined by setting $\delta(a,b,c,d)$ to be an abbreviation of $\text{Dist}(a,b,b,b,c,d)$; that is, $\delta(a,b,c,d)$ stands for the length of ab being positively longer than that of cd. Segment congruence $\equiv$ is then defined as $ab \equiv cd$ if and only if $\neg\delta(a,b,c,d)$ and $\neg\delta(c,d,a,b)$. Since the usual proofs of the generalized Steiner–Lehmus theorem show that if $|ac|\circ > |ab|$, then $|cn|\circ > |bm|$, that is, that if $\delta(a,c,a,b)$, then $\delta(c,n,b,m)$, and, by symmetry, also that if $\delta(a,b,a,c)$, then $\delta(b,m,c,n)$. Since $(p \to q) \to (\neg q \to \neg p)$ is valid intuitionistically, the usual proof, going back to [8], thus proves that $\neg\delta(c,n,b,m)$ implies $\neg\delta(a,c,a,b)$, as well as that $\neg\delta(b,m,c,n)$ implies $\neg\delta(a,b,a,c)$. With the above definition of $\equiv$, the usual proof shows intuitionistically that $bm \equiv cn$ implies $ab \equiv ac$, so that Sylvester's problem never existed to start with!

Two other intuitionistic axiomatizations, that have been put forward in Beeson [4], [5], adopt "Markov principles" for equality, betweenness, and equidistance; that is, $\neg\neg x = y \to x = y$, $\neg\neg B(abc) \to B(abc)$, and $\neg\neg ab \equiv cd \to ab \equiv cd$. By [5, Theorem 11.1], if a statement θ is deducible by using classical logic from the axiom system of theory $\mathcal{T}$, formulated inside intuitionistic logic with Markov principles axiomatized in [5], then the double negation of θ is deducible inside $\mathcal{T}$ itself. This renders the generalized Steiner–Lehmus theorem in almost its (1) form, which is (1) with $\neg B(abc) \wedge \neg B(bca) \wedge \neg B(cab) \wedge b \neq c$ instead of $\neg(B(abc) \vee B(bca) \vee B(cab))$, which is equivalent to its double negation, provable in Beeson's $\mathcal{T}$ (in Beeson's $\mathcal{T}$, the predicate B is defined by the axioms for strict betweenness, that is, $B(abc)$ holds if b is strictly between a and c, which is why we have to add $\wedge b \neg c$ to $\neg B(abc) \wedge \neg B(bca) \wedge \neg B(cab)$; we may also drop $m \neq c \wedge m \neq a$ from (1), as these are implied by $B(amc)$, which is part of the hypothesis). Such a proof would still be suspicious, as one could claim that the indirectness got swept under the rug with the use of Markov principles.

However, one can prove the existence of a direct proof of the generalized Steiner–Lehmus theorem, expressed as in (1), but with $0 < abc < \pi$ instead of $\neg(B(abc) \vee B(bca) \vee B(cab)) \wedge m \neq c \wedge m \neq a$—where $0 < abc < \pi$ should be read as abc is a positive angle and is positively different from a straight angle, both notions being defined in [6] in terms of B—from an axiom system for plane Euclidean geometry in terms of B and $\equiv$, with B to be interpreted as "strict betweenness," with all axioms intuitionistically meaningful, without the use of any Markov principle (thus inside purely intuitionistic logic)! This follows directly from: (i) the syntactic form of the generalized Steiner–Lehmus theorem, (ii) the fact established above that it is deducible inside Euclidean geometry *with* the Markov principles allowed in $\mathcal{T}$, and (iii) Theorem 14 of [6]. This represents proof of the existence of an *entirely unobjectionable direct proof of the generalized Steiner–Lehmus theorem*. This result and Theorem 3.1 thus present possible solutions to the question raised by

Sylvester. These are solutions "in principle," as we do not present line-by-line proofs of the generalized Steiner–Lehmus theorem, we just know that such proofs must exist, both in the classical and in the intuitionistic setting. The actual line-by-line formal proofs would likely contain thousands of lines.

Although our main problem has thus found a solution, we will also indicate the availability of a direct proof from even weaker assumptions.

When asking ourselves what the likely minimal requirements are under which the generalized Steiner–Lehmus theorem holds, we find that there is no need for free mobility, such as the ability to lay off a segment on any ray, or for the existence of the midpoint of any segment, both of which are true in Tarski's absolute geometry. What is needed to express the generalized Steiner–Lehmus theorem is a notion of segment congruence and one of betweenness. The axioms that they need to satisfy so that we can prove (15) were the subject of [25]. It thus turns out that there is a direct proof of the generalized Steiner–Lehmus theorem from a significantly weaker set of axioms.

4 The Axiom System for Metric Planes

4.1 Quantifier-free axioms for metric planes The concept of a *metric plane* grew out of the work of Hessenberg, Hjelmslev, and Schmidt and was provided with a simple group-theoretic axiomatics by Bachmann [3, Section 3.2, p. 33]. It can be understood as the common orthogonality core of plane Euclidean, hyperbolic, and elliptic geometry (see Pambuccian [24]).

For the purpose of this note we will choose among the many possible axiomatizations of nonelliptic metric planes one that is quantifier-free and was first proposed in Pambuccian [23]. It is expressed in a language $\mathcal{L}$ with only one sort of individual variables, to be interpreted as "points," three individual constants a_0, a_1, a_2 to be interpreted as three noncollinear points, and with two operation symbols, F and π. $F(abc)$ is the foot of the perpendicular from c to the line ab, if $a \neq b$, and a itself if $a = b$, and $\pi(abc)$ is the fourth reflection point whenever a, b, c are collinear points with $a \neq b$ and $b \neq c$, and arbitrary otherwise (although it could have been defined even in case $a = b$ or $b = c$, the axioms do not specify a value in this case, as there is no need to specify one). By *fourth reflection point* we mean the following. If we designate by σ_x the mapping defined by $\sigma_x(y) = \sigma(xy)$, that is, the reflection of y in the point x, then, if a, b, c are three collinear points, by [3, Section 3.9, Satz 24b], the composition $\sigma_c\sigma_b\sigma_a$ is the reflection in a point, which lies on the same line as a, b, c. That point is designated by $\pi(abc)$. Had midpoints existed, then $\pi(abc)$ would have been the reflection of b in the midpoint of ac (see Figure 3).

To improve the readability of the axioms, we will use the following abbreviations:

$$\sigma(ab) := \pi(aba), \tag{19}$$

$$R(abc) := \sigma\big(F(abc)c\big), \tag{20}$$

$$L(abc) :\leftrightarrow F(abc) = c \lor a = b, \tag{21}$$

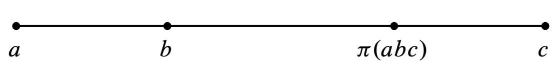

Figure 3 The meaning of $\pi(abc)$ when a, b, and c are three collinear points.

where σ has the same meaning as above, $R(abc)$ stands for the reflection of c in ab (a line if $a \neq b$, the point a if $a = b$), and $L(abc)$ stands for "the points a, b, c are collinear (but not necessarily distinct)."

The axiom system consists of the following axioms:

C 1 $F(aab) = a$,

C 2 $\sigma(aa) = a$,

C 3 $\sigma(a\sigma(ab)) = b$,

C 4 $\sigma(ax) = \sigma(bx) \rightarrow a = b$,

C 5 $L(aba)$,

C 6 $L(abc) \rightarrow L(cba) \wedge L(bac)$,

C 7 $L(ab\sigma(ab))$,

C 8 $L(abF(abc))$,

C 9 $a \neq b \wedge F(abx) = F(aby) \rightarrow L(xyF(abx))$,

C 10 $a \neq b \wedge c \neq d \wedge F(abc) = c \wedge F(abd) = d \rightarrow F(abx) = F(cdx)$,

C 11 $\neg L(abx) \wedge F(xF(abx)y) = y \rightarrow F(abx) = F(aby)$,

C 12 $a \neq b \wedge a \neq c \wedge F(abc) = a \rightarrow F(acb) = a$,

C 13 $a \neq x \wedge x \neq y \wedge F(axy) = x \rightarrow F(a\sigma(ax)\sigma(ay)) = \sigma(ax)$,

C 14 $\sigma(\sigma(xa)\sigma(xb)) = \sigma(x\sigma(ab))$,

C 15 $u \neq v \wedge a \neq b \wedge F(abc) = a \rightarrow F(R(uva)R(uvb)R(uvc)) = R(uva)$,

C 16 $\neg L(oba) \wedge \neg L(obc)$
$\rightarrow \sigma(F(xR(ocR(obR(oax)))o)x) = R(ocR(obR(oax)))$,

C 17 $\neg L(oba) \wedge \neg L(obc) \wedge \sigma(mx) = R(ocR(obR(oax))) \wedge$
$\sigma(ny) = R(ocR(obR(oay))) \rightarrow L(omn)$,

C 18 $a \neq b \wedge b \neq c \wedge F(abc) = c \wedge a \neq a' \wedge b \neq b' \wedge c \neq c' \wedge F(aba') = a$
$\wedge F(bab') = b \wedge F(cbc') = c$
$\rightarrow \sigma(F(xR(cc'R(bb'R(aa'x)))\pi(abc))x) = R(cc'R(bb'R(aa'x)))$
$\wedge F(\pi(abc)cF(xR(cc'R(bb'R(aa'x)))\pi(abc))) = \pi(abc)$,

C 19 $\neg L(a_0a_1a_2)$.

The axioms make the following statements. C1 defines the value of $F(abc)$ when $a = b$—it is an axiom with no geometric function (we could have opted to leave it undefined, but that would have lengthened the statements of the axioms C16 and C18); C2: the point a is a fixed point of the reflection σ_a; C3: reflections in points are involutory transformations (or the identity); C4: reflections of a point in two different points do not coincide; C5: a lies on the line determined by a and b; C6: collinearity of three points is a symmetric relation; C7: the reflection of b in a is collinear with a and b; C8: for $a \neq b$, the foot of the perpendicular from c to the line ab lies on that line; C9 states the uniqueness of the perpendicular to the line ab in the point $F(abx)$; C10: the foot of the perpendicular from x to the line ab does not depend on the particular choice of points a and b that determine the line ab; C11: if x is a point outside of the line ab, and y is a point on the perpendicular from x to ab, then the feet of the perpendiculars of x and y to the line ab coincide; C12 states that perpendicularity is a symmetric relation (if ca is perpendicular to ab, then ba is perpendicular to ac); C13: if yx is perpendicular to xa, then so are $\sigma_a(y)\sigma_a(x)$ and $\sigma_a(x)a$; C14 states a certain preservation of the operation σ under reflections in points; C15: reflections in lines preserve the orthogonality relation; C16 and C17 together state the three reflections theorem for lines having a point in common; C18 is the three reflections theorem for lines having a common perpendicular; C19: a_0, a_1, a_2 are three noncollinear points.

As shown in [23], with $\Sigma = \{\text{C1–C19}\}$, we have: Σ *is an axiom system for nonelliptic metric planes; in every model of* Σ, *the operations* F *and* π *have the intended interpretations.*

The theory of nonelliptic metric planes can also be axiomatized with *points* as the only individual variables, in terms of the notions of collinearity, L, and equidistance $\equiv$, as shown in Sörensen [27]. One could thus ask for a definition of $\equiv$ in terms of the notions of the axiom system Σ. A generally valid definition is not known and will likely be rather involved. It is easy, however, to define $ab \equiv cd$ if any of the segments ac, bc, ac, or ad have a midpoint. One first defines

$$ab \equiv ac :\Leftrightarrow b = c \vee \sigma\big(F(bca)b\big) = c, \tag{22}$$

and then, for the case in which, say, ac has a midpoint o, one has

$$ab \equiv cd :\Leftrightarrow c\sigma(ob) \equiv cd, \quad \text{provided that } \sigma(oa) = c. \tag{23}$$

The cases in which bc, ac, or ad have a midpoint are treated analogously. All congruent segments appearing in the generalized Steiner–Lehmus theorem will have midpoints for some pairing of endpoints, and so the absence of a general definition of segment congruence presents, for our purposes, no problem.

These axioms become the following rules of derivation (we omit the rule corresponding to axiom C19, as it will not be needed in the proof of the Steiner–Lehmus theorem):

$$\frac{F(aab) = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}, \tag{24}$$

$$\frac{\sigma(aa) = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}, \tag{25}$$

$$\frac{\sigma(a\sigma(ab)) = b, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}, \tag{26}$$

$$\frac{\sigma(ax) = \sigma(bx), a = b, \Gamma \Rightarrow \Delta}{\sigma(ax) = \sigma(bx), \Gamma \Rightarrow \Delta}, \tag{27}$$

$$\frac{F(aba) = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}, \tag{28}$$

$$\frac{F(baa) = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}, \tag{29}$$

$$\frac{F(abc) = c, F(cba) = a, \Gamma \Rightarrow \Delta \quad F(abc) = c, c = b, \Gamma \Rightarrow \Delta}{F(abc) = c, \Gamma \Rightarrow \Delta}, \tag{30}$$

$$\frac{F(abc) = c, F(bac) = c, \Gamma \Rightarrow \Delta}{F(abc) = c, \Gamma \Rightarrow \Delta}, \tag{31}$$

$$\frac{F(ab\sigma(ab)) = \sigma(ab), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}, \tag{32}$$

$$\frac{F(abF(abc)) = F(abc), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}, \tag{33}$$

$$\frac{\Gamma_1 \Rightarrow \Delta \quad \Gamma_2 \Rightarrow \Delta \quad \Gamma_3 \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta}, \tag{34}$$

where Γ_1 stands for $\Sigma, a = b, \Gamma$; Γ_2 for $\Sigma, F(xyF(abx)) = F(abx), \Gamma$; Γ_3 for $\Sigma, x = y, \Gamma$, and Σ for $F(abx) = F(aby)$;

$$\frac{\Gamma_1 \Rightarrow \Delta \quad \Gamma_2 \Rightarrow \Delta \quad \Gamma_3 \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta}, \tag{35}$$

where Γ_1 stands for $\Sigma, a = b, \Gamma$; Γ_2 stands for $\Sigma, c = d, \Gamma$; Γ_3 for $\Sigma, F(abx) = F(cdx), \Gamma$, and Σ for $F(abc) = c, F(abd) = d$;

$$\frac{\Gamma_1 \Rightarrow \Delta \quad \Gamma_2 \Rightarrow \Delta \quad \Gamma_3 \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta}, \tag{36}$$

where Γ_1 stands for $\Sigma, F(abx) = x, \Gamma$; Γ_2 for $\Sigma, a = b, \Gamma$; Γ_3 for $\Sigma, F(abx) = F(aby), \Gamma$; and Σ for $F(xF(abx)y) = y$;

$$\frac{F(abc) = a, F(acb) = a, \Gamma \Rightarrow \Delta}{F(abc) = a, \Gamma \Rightarrow \Delta}, \tag{37}$$

$$\frac{F(axy) = x, F(a\sigma(ax)\sigma(ay)) = \sigma(ax), \Gamma \Rightarrow \Delta}{F(axy) = x, \Gamma \Rightarrow \Delta}, \tag{38}$$

$$\frac{\sigma(\sigma(xa)\sigma(xb)) = \sigma(x\sigma(ab)), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}, \tag{39}$$

$$\frac{\Gamma_1 \Rightarrow \Delta \quad \Gamma_2 \Rightarrow \Delta}{F(abc) = a, \Gamma \Rightarrow \Delta}, \tag{40}$$

where Γ_1 stands for $F(abc) = a, F(R(uva)R(uvb)R(uvc)) = R(uva), \Gamma$; and Γ_2 for $F(abc) = a, u = v, \Gamma$;

$$\frac{\Gamma_1 \Rightarrow \Delta \quad \Gamma_2 \Rightarrow \Delta \quad \Gamma_3 \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}, \tag{41}$$

where Γ_1 stands for $\Sigma, F(oba) = a, \Gamma$; Γ_2 for $\Sigma, F(obc) = c, \Gamma$; Γ_3 for $\Sigma, o = b, \Gamma$; and Σ for $\sigma(F(xR(ocR(obR(oax)))o)x) = R(ocR(obR(oax)))$;

$$\frac{\Gamma_1 \Rightarrow \Delta \quad \Gamma_2 \Rightarrow \Delta \quad \Gamma_3 \Rightarrow \Delta \quad \Gamma_4 \Rightarrow \Delta \quad \Gamma_5 \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta}, \tag{42}$$

where Γ_1 stands for $\Sigma, F(oba) = a, \Gamma$; Γ_2 for $\Sigma, F(obc) = c, \Gamma$; Γ_3 for $\Sigma, o = b, \Gamma$; Γ_4 for $\Sigma, F(omn) = n, \Gamma$; Γ_5 for $\Sigma, o = m, \Gamma$; and Σ for $\sigma(mx) = R(ocR(obR(oax))), \sigma(ny) = R(ocR(obR(oay)))$; and, for $i = 1, 2$

$$\frac{\Gamma_{1i} \Rightarrow \Delta \quad \Gamma_{2i} \Rightarrow \Delta \quad \Gamma_{3i} \Rightarrow \Delta \quad \Gamma_{4i} \Rightarrow \Delta \quad \Gamma_{5i} \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta}, \tag{43}$$

where Γ_{1i} stands for $\Sigma, \Xi_i, a = b, \Gamma$; Γ_{2i} for $\Sigma, \Xi_i, b = c, \Gamma$; Γ_{3i} for $\Sigma, \Xi_i, a = a', \Gamma$; Γ_{4i} for $\Sigma, \Xi_i, b = b', \Gamma$; Γ_{5i} for $\Sigma, \Xi_i, c = c', \Gamma$; Σ for $F(abc) = c$, $F(aba') = a$, $F(bab') = b$, $F(cbc') = c$;

Ξ_1 for $\sigma(F(xR(cc'R(bb'R(aa'x)))\pi(abc))x) = R(cc'R(bb'R(aa'x)))$; and
Ξ_2 for $F(\pi(abc)cF(xR(cc'R(bb'R(aa'x)))\pi(abc))) = \pi(abc)$.

5 Order and the Steiner–Lehmus Theorem in Standard Ordered Metric Planes

Since the Steiner–Lehmus theorem (and thus its generalized version) is no longer true if the points m and n lie just on the *lines* ac and ad rather than on the respective *segments*, for the theorem does not hold if bm and cn were to be *external* angle bisectors (as shown in Abu-Saymeh and Hajja [1], Hajja [11], Henderson [12], Kharazishvili [14], van Yzeren [31]), we do need an order notion for it to hold.

The order axioms are expressed in terms of the ternary relation B of *betweenness* we encountered earlier. A 5-ary operation symbol ω, providing the point of intersection stipulated by the outer form of the Pasch axiom is also added to the language.

Since our lines cannot have exactly four points on them, given the result of Szmielew [29, Section 7.2], it suffices to add to the axioms for metric planes the axiom T5 and

A 1 $B(aab)$,

A 2 $F(abc) = c \to (B(abc) \vee B(bca) \vee B(cab))$,

A 3 $B(abc) \to (a = b \vee F(abc) = c)$,

A 4 $B(ba\sigma(ab))$,

A 5 $B(abc) \to B(cba)$,

A 6 $B(acb) \wedge B(abd) \to B(cbd)$,

A 7 $B(abd) \wedge B(bec) \to (B(a\omega(abcde)c)) \wedge B(de\omega(abcde))$,

A 8 $F(oab) = o \wedge o \neq b \wedge o \neq a \wedge B(\sigma(ab)ac) \wedge F(abo) = c \to B(acb)$.

A1 and A2 state that one of three (not necessarily distinct) collinear points must be between the other two, A3 that if b lies between a and c, then the points a, b, and c must be collinear, A4 that the midpoint of a segment lies between its endpoints, A5 is a symmetry axiom, A6 a transitivity axiom, A7 is the outer form of the Pasch axiom (**OP**; see Figure 1) stating that, if a line intersects the side bc and the extension, past b, of the side ab of triangle abc, then it must intersect the third side, ac, as well (in a point f, whose Skolemized name is $\omega(abcde)$), and A8 states that the foot of the altitude to the hypotenuse lies between the endpoints of the hypotenuse.

In this language, the generalized Steiner–Lehmus theorem becomes

$$\neg L(abc) \wedge B(amc) \wedge B(anb) \wedge ad \equiv ab \wedge B\big(\sigma(ac)ad\big) \wedge sb \equiv sd \wedge B(bsm) \wedge B(csn) \wedge cn \equiv c\sigma(om) \wedge \sigma(pm) = n \wedge \sigma(ob) = c \to d = c. \quad (44)$$

There are several differences between (44) and (1): (i) mn is now said to have a midpoint p; (ii) bc is now said to have a midpoint o; (iii) instead of $bm \equiv cn$, we now have $cn \equiv c\sigma(om)$. The raison d'être of (i) and (ii) is that, while in Tarski's absolute geometry the existence of all midpoints is guaranteed (if three noncollinear points exist; if such points do not exist, then (1) vacuously holds), no such existence can be proved in standard ordered metric planes, and our proof in [25] needed those midpoints, so they became part of the hypothesis (they are probably not needed, but we do not have a proof which avoids their use). That (iii) does not represent an actual change can be seen from our definition (23) of $\equiv$ for segments without a common endpoint.

As shown in [25], (44) follows from C1–C18 and A1–A8.

When turning the order axioms into rules of derivation, we get

$$\frac{B(aab), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}, \tag{45}$$

$$\frac{\varphi, B(abc), \Gamma \Rightarrow \Delta \quad \varphi, B(bca), \Gamma \Rightarrow \Delta \quad \varphi, B(cab), \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}, \tag{46}$$

where φ stands for $F(abc) = c$,

$$\frac{B(abc), a = b, \Gamma \Rightarrow \Delta \quad B(abc), F(abc) = c, \Gamma \Rightarrow \Delta}{B(abc), \Gamma \Rightarrow \Delta}, \tag{47}$$

$$\frac{B(ba\sigma(ab)), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}, \tag{48}$$

$$\frac{B(abc), B(cba), \Gamma \Rightarrow \Delta}{B(abc), \Gamma \Rightarrow \Delta}, \tag{49}$$

$$\frac{B(acb), B(abd), B(cbd), \Gamma \Rightarrow \Delta}{B(acb), B(abd), \Gamma \Rightarrow \Delta}, \tag{50}$$

$$\frac{B(abd), B(bec), B(a\omega(abcde)c), \Gamma \Rightarrow \Delta}{B(abd), B(bec), \Gamma \Rightarrow \Delta}, \tag{51}$$

$$\frac{B(abd), B(bec), B(de\omega(abcde)), \Gamma \Rightarrow \Delta}{B(abd), B(bec), \Gamma \Rightarrow \Delta}, \tag{52}$$

$$\frac{\Sigma, o = b, \Gamma \Rightarrow \Delta \quad \Sigma, o = a, \Gamma \Rightarrow \Delta \quad \Sigma, B(acb), \Gamma \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta}, \tag{53}$$

where Σ stands for $F(oab) = o, B(\sigma(ab)ac), F(abo) = c$.

In this language, (44) turns into the following sequent:

$$\begin{aligned} &B(amc), B(anb), ad \equiv ab, B\big(\sigma(ac)ad\big), sb \equiv sd, B(bsm), B(csn),\\ &\quad cn \equiv c\sigma(om), \sigma(pm) = n, \sigma(ob) = c\\ &\Rightarrow B(abc), B(bca), B(cab), d = c. \end{aligned} \tag{54}$$

Given the equivalence of the Hilbert-style and the Gentzen-style formalisms mentioned earlier (see [18]–[20]), we conclude with the following result.

Theorem 5.1 *The sequent (54) can be deduced from the axioms (2) by using (3), (4), (24)–(43), (12), and (45)–(53) as rules of derivation. The deduction is a direct one.*

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